FREE BOUNDARY REGULARITY FOR A PROBLEM ARISING IN SUPERCONDUCTIVITY

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ABSTRACT. This paper concerns regularity properties of a free boundary arising in the mean-field theory of superconductivity. The problem is reminiscent of the one studied earlier by two of the authors and L. Karp in connection with potential theory. The difficulty introduced in this paper is the existence of several patches, where on each patch the solution to the problem may have different constant values. However, using a refined analysis, we reduce the problem to the case of one-patch; at least locally near 'regular' free boundary points. Using a monotonicity formula, due to Georg S. Weiss, we characterize global solutions of a related equation. Hence earlier regularity results apply and we conclude the C^1-regularity of the free boundary.

1. INTRODUCTION

In analyzing the evolution of vortices arising in the mean-field model of penetration of the magnetic field into super-conducting bodies, one ends up with a degenerate parabolic-elliptic system (see [?] for details). A simplified stationary model of this problem (in a local setting), where the scalar stream function admits a functional dependence on the scalar magnetic potential, reduces to finding u such that

$$\Delta u = u\chi_{\{|\nabla u|>0\}}, \quad u \geq 0, \quad \text{in } B_\rho(x_0), \tag{1.1}$$

where $B_\rho(x_0)$ denotes the ball of radius ρ centered at $\zeta \in \mathbb{R}^n$), the equation is in the sense of distribution, and appropriate boundary data are fulfilled.

Related problems have been studied in [?], [?], [?], [?]; see also the references therein. However, less attention has been paid to the regularity nature of the solution function u and the free boundary $\partial\{|\nabla u| > 0\}$.

Existence of solutions of the Dirichlet Problem associated with this equation was studied in [?], where the $C^{1,1}$-interior regularity and the local finiteness of the $(n-1)$-dimensional Hausdorff measure of the free boundary was established.

In this paper, it is our prime goal to analyze the above problem in the context of regularity theory. Using a refined analysis inspired by techniques introduced in [?], we reduce the problem to the case of one-patch, near 'regular' free boundary points. From here the (by now) classical results can be applied, to obtain the desired regularity properties of the free boundary.

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2. Definitions and Known Results

In most of the paper, we deal with functions $u \in C^{1,1}(B_\rho(x_0))$, $0 < \rho \leq \infty$, $x_0 \in \mathbb{R}^n$, which satisfy the differential equation

$$\Delta u = \chi_{\{|\nabla u|>0\}}, \tag{2.1}$$

with

$$|\nabla u(x_0)| = 0\,, \quad |u(x) - u(x_0)| \leq c(1+|x|^2), \tag{2.2}$$

where c is a fixed positive constant. We denote by $\mathcal{P}(c, \rho, x_0)$ this class of functions. The differential equation (2.1) is interpreted in the sense of distributions.

Also global solutions to (2.1)-(2.2) are denoted by $\mathcal{P}$, i.e.

$$\mathcal{P} := \bigcup_{c>0} \bigcap_{\rho>0} \mathcal{P}(c, \rho, x_0)$$

for any $x_0 \in \mathbb{R}^n$.

For the purpose of this paper, the most relevant results from the regularity theory developed in [?], [?], is the following theorem.

Theorem 2.1. *([?] and [?]) For $u \in \mathcal{P}(c, \rho, x_0)$, the following uniform $C^{1,1}$-estimate holds:*

$$\sup_{B_{\rho/2}\cap\{|\nabla u|>0\}} |D_{ij}u(x)| \leq C,$$

where C is a constant depending only on c and n.

Moreover, the free boundary $\partial\{|\nabla u| > 0\}$ has locally finite $(n-1)$-Hausdorff measure.

The proof given in [6] should be slightly changed (cf. [5]) to yield uniformly for the class. However, the proof given in [5] works perfectly also in this case.

By Theorem 2.1, for all $u \in \mathcal{P}(c, \rho, x_0)$, $\Delta u = 1$, in the classical sense, in the interior of the closure of $\{|\nabla u| > 0\}$. In particular it will be convenient to define Ω^c as the closure of the interior of $\{|\nabla u| = 0\}$, and $\Omega = \mathbb{R}^n \setminus \Omega^c$.

3. Regularity of the Free Boundary

Before establishing our main result, we need the following definition.

Definition 3.1. The minimal diameter of a bounded set $D \subset \mathbb{R}^n$, denoted $\mathrm{MD}(D)$, is the infimum of distances between pairs of parallel planes such that D is contained in the strip determined by the planes.

Definition 3.2. The density function of Ω^c (at the origin) is defined by

$$\delta_\rho(u) = \frac{\mathrm{MD}(\Omega^c \cap B_\rho)}{\rho}.$$

Theorem 3.3. *There exists a modulus of continuity σ ($\sigma(0^+) = 0$) such that if $\delta_{\rho_0}(u) > \sigma(\rho_0)$ for some $\rho_0 < 1/2$, then for $u \in \mathcal{P}(c, 1, x_0)$, the boundary $\partial\{|\nabla u| > 0\}$ is a C^1 graph in $B(0, c_0{\rho_0}^2)$. Here c_0 is a universal constant, depending only on c, and the dimension n.*

This theorem will be a consequence of lemmas presented in the following sections, combined with Theorem III in [?].

It is, however, noteworthy that the regularity theory of almost all free boundaries studied so far in the existing literatures share a common feature. Namely, one starts with classifying the so-called global solutions. These solutions come from the local ones after rescaling for a closer look at the micro-local structure of the unknown set (the free boundary). In order to scale the solutions in our problem, we need $C^{1,1}$-regularity of the solution function u (to (1.1)). This property keeps the solution invariant, i.e., the rescaled function is also a solution to (1.1), with Ω replaced by a scaled version of it. More important is though that the supremum norm remains unchanged up to a (universal) multiplicative constant.

The classification of global solutions, in turn, is a non-easy task. Our prime goal, will be to show that global solutions to (1.1) actually have only one patch, i.e. the set $\{\nabla u = 0\}$ has just one component. Once we have done this we can use [CKS] to classify global solutions.

The novelty in this paper, besides treating a new problem with a wider application, is the use of two completely different type of monotonicity formulas.

To apply Theorem 3.3 to solutions to (1.1) we need to use the following lemma.

Lemma 3.4. *([?], Lemma 10) Assume that u is a non negative solution to (1.1) and $x_0 \in \partial\{|\nabla u| > 0\}$. Then $u > 0$.*

Although in Theorem 3.3 we consider the case of constant right hand side (constant in the set $\{|\nabla u| > 0\}$), our analysis works perfectly for solutions of equation (1.1). The constant c_0 in the theorem will then depend on $u(x_0)$ as well. We leave the small changes needed in this case to the reader, and continue our analysis in the rest of this paper for the case of equation (2.1).

4. Further definitions and preliminary results

Given a function $u \in \mathcal{P}(c,\rho,x_0)$, let us define the r-scaled function of u at x_0 as

$$u_r(x) := \frac{u(x_0 + rx) - u(x_0)}{r^2}, \qquad 0 < r \le \rho,\ x \in B_1(0). \tag{4.1}$$

Set also

$$\Omega_r = \{x \in \mathbb{R}^n\ ;\ x_0 + rx \in \Omega\}.$$

By Theorem 2.1, $\{u_r\}_{0<r\le\rho}$ is a relatively compact family. By the Ascoli-Arzela theorem, given any sequence of positive numbers $\{r\}$ tending to zero, there is a subsequence $r_k \to 0$ such that u_{r_k} converges uniformly on compact sets to a globally defined function

$$u_0 = \lim_{k\to\infty} u_{r_k} \in \mathcal{P}. \tag{4.2}$$

We refer to this function as blow-up of u at x_0 (with respect to the sequence $\{r_k\}$).

Although blow-ups at a fixed point $x_0 \in \mathbb{R}^n$ might depend a priori on the sequence $r_k \to 0$, we denote any blow-up at x_0 by u_0. This will cause no confusion since we do not use several blow ups at the same time. Besides, by the analysis that follows and Weiss' energy decay estimate (Theorem 3 in [?]), we will eventually show that for all x_0 fixed, u_r is indeed convergent when $r \to 0$.

If $u \in \mathcal{P}$, u_r is defined for all $r > 0$, since the family $\{u_r\}_{r>0}$ is also relatively compact, we may consider blowing up at ∞. We denote any blow-up at ∞ by u_∞. As above, this will cause no confusion and eventually $u_r \to u_\infty$ when $r \to \infty$.

Lemma 4.1. *If $u \in \mathcal{P}(c,\rho,x_0)$, then any blow-up u_0 at a free boundary point is either:*

- *a half space solution, i.e.,*

$$u_0(x) = \frac{1}{2}[(x \cdot \nu_0)^+]^2,$$

for some $\nu_0 \in S^{n-1}$, *or*
- *a Homogeneous, degree two polynomial* $P(x)$ *with* $\Delta P = 1$.

Remark 4.2. Due to the non-degeneracy of solutions (see [?]), i.e.

$$\sup_{B_r(y)} u \geq ar^2 + u(y), \qquad y \in \partial\{|\nabla u| > 0\},$$

blow-ups don't vanish identically. Here the constant a depends only on the space dimension.

Proof. (of Lemma 4.1) We apply the monotonicity formula of Alt, Caffarelli and Friedman [?] to the positive and negative parts of a directional derivative of u. To fix the notation, set

$$\varphi(r, \nu, u) = \frac{1}{r^4} \int_{B_r(x_0)} \frac{|\nabla (D_\nu u)^+|^2}{|x - x_0|^{n-2}} \int_{B_r(x_0)} \frac{|\nabla (D_\nu u)^-|^2}{|x - x_0|^{n-2}},$$

where $r > 0$ and $(D_\nu u)^\pm$ denotes the positive and negative parts (respectively) of the directional derivative of u in the direction $\nu \in S^{n-1}$. Now the strong form of the monotonicity lemma says:

Remark 4.3. (See [5]) For φ, u, and ν as above we have

$$\varphi'(r, \nu, u) \geq 0.$$

More exactly, if any of the sets $\text{supp}(D_\nu u)^\pm \cap \partial B_r(x^0)$ digresses from a spherical cap by a positive area, then either $\varphi'(r) > 0$ or $\varphi(r) = 0$.

Fix a sequence $r_k \to 0$ such that

$$u_0 = \lim_{k \to \infty} u_{r_k}.$$

Since u_{r_k} converges in $W^{2,p}$,

$$\varphi(s, \nu, u_0) = \lim_{k \to \infty} \varphi(s, \nu, u_{r_k}).$$

Using a change of variable we readily verify that

$$\varphi(1, \nu, u_r) = \varphi(r, \nu, u).$$

This and the monotonicity lemma imply

$$\varphi(s, \nu, u_0) = \lim_{k \to \infty} \varphi(sr_k, \nu, u) := \varphi(0^+, \nu, u).$$

Hence, for any blow-up u_0, $\varphi(r, \nu, u_0)$ is constant with respect to r.

It thus follows from the strong form of the monotonicity formula (see above) that either $\varphi(r, \nu, u_0) = 0$ for all r and directions ν, where in this case we immediately conclude that u_0 is a half space solution, or $\Omega^c = \emptyset$. The latter in combination with Liouville's theorem (applied to the second order partial derivatives of u_0) implies that u_0 is a polynomial of degree two. Finally, the homogeneity comes from the fact that $u_0(0) = |\nabla u_0(0)| = 0$. □

In the same way we can prove the following lemma.

Lemma 4.4. *If $u \in \mathcal{P}$, then any blow-up u_∞ at ∞ is either: a half space solution or a homogeneous, degree-two polynomial.*

Using Remark 4.3 and the following

$$\varphi(r,\nu,u_0) = \varphi(0^+,\nu,u) \leq \varphi(r,\nu,u) \leq \varphi(\infty^-,\nu,u) = \varphi(r,\nu,u_\infty),$$

we can conclude as in the next proposition.

Proposition 4.5. *For functions $u \in \mathcal{P}$ the following hold.*

- *A blow-up u_0 at the origin is a half space solution if and only if*
$$\varphi(0^+,\nu,u) = 0, \quad \forall \nu \in S^{n-1}.$$
- *If some blow-up u_0 at the origin is a half space solution, then any blow-up at the origin is a half space solution.*
- *If some blow-up u_∞ of u at ∞ is a half space solution, then $u - u(x_0)$ itself is a half space solution, $x_0 \in \Omega^c$.*

5. The local structure of the patches

The main result of this section is the following theorem.

Theorem 5.1. *Let $u \in \mathcal{P}(c,1,x_0)$. Suppose that there is a sequence $r_k \to 0$ such that the blow-up u_0 of u, with respect to r_k, is a half space solution. Then, there is a $\rho = \rho(u) > 0$ such that*

$$\Omega^c \cap B_\rho(x_0) \subset \{u = u(x_0)\}.$$

Recall that the blow-up u_0 was defined in (4.2).

The proof of this theorem will be divided into several lemmas. The first of them is inspired by Lemma 4.2 in [?].

For the purpose of the next lemma we need the following definition. Given $\rho > 0$, $0 < \delta < 1$, $\nu \in S^{n-1}$, define

$$C(\rho,\delta,\nu) := \left\{ x \in \mathbb{R}^n;\ 0 < |x| \leq \rho,\ \frac{x}{|x|} \cdot \nu \leq -1 + \delta \right\}.$$

Lemma 5.2. *Assume that $u \in \mathcal{P}(c,1,x_0)$ and there exists a blow-up u_0 at x_0 that is a half space solution, i.e.,*

$$u_0(x) = \frac{1}{2}[(x \cdot \nu_0)^+]^2.$$

Then, there is $\rho > 0$ such that for all $\nu \in C(1,\frac{1}{2},-\nu_0) \cap S^{n-1}$,

$$2D_\nu u - |\nabla u|^2 \geq 0 \quad \textit{in } B_\rho(x_0).$$

In particular, u is non-decreasing in $B_\rho(x_0)$, in the direction of ν, for all $\nu \in C(1,\frac{1}{2},-\nu_0) \cap S^{n-1}$.

Proof. It follows from the hypothesis that

$$D_\nu u_0 = (x \cdot \nu_0)^+ \nu_0 \cdot \nu \quad \text{and} \quad |\nabla u_0|^2 = [(x \cdot \nu_0)^+]^2.$$

Hence, for all $\nu \in C(1,\frac{1}{2},-\nu_0) \cap S^{n-1}$,

$$2D_\nu u_0 - |\nabla u_0|^2 \geq 0 \quad \text{in } B_1.$$

Fix a sequence $r_k \to 0$ such that

$$u_0 = \lim_{k \to \infty} u_{r_k}.$$

By uniform convergence, for all $\epsilon > 0$, there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, and $\nu \in C(1, \frac{1}{2}, -\nu_0) \cap S^{n-1}$ we have

$$2D_\nu u_{r_k} - |\nabla u_{r_k}|^2 \geq -\epsilon \quad \text{in } B_1.$$

Since

$$-\Delta(2D_\nu u_{r_k} - |\nabla u_{r_k}|^2) = |\nabla^2 u_{r_k}|^2 \geq \frac{1}{n^2} \quad \text{in } B_1 \cap \Omega_{r_k}\,,$$

we are in position to apply the same argument as the one in the proof of Lemma 4.2 in [?] to conclude that there is a universal constant ϵ_0 (independent of u_{r_k} and ν) such that $\epsilon \leq \epsilon_0$ implies

$$2D_\nu u_{r_k} - |\nabla u_{r_k}|^2 \geq 0\,, \quad \forall x \in B_{\frac{1}{2}}\,,\ \forall \nu \in C(1, \frac{1}{2}, -\nu_0) \cap S^{n-1},$$

as soon as we choose k big enough. Taking $\rho = \frac{r_k}{2}$, the lemma is proved. □

Lemma 5.3. *Under the assumptions and notations of Lemma 5.2, for ρ as in Lemma 5.2 we have*

$$x_0 + C(\rho, \frac{1}{2}, \nu_0) \subset \{u \leq u(x_0)\}\,.$$

Proof. For all (fixed) $x \in C(\rho, \frac{1}{2}, \nu_0)$, the function

$$\tau \to u\,(x_0 + (1-\tau)x) \qquad \tau \in (0,1)$$

is non-decreasing. Indeed,

$$\frac{\partial}{\partial \tau}(u(x_0 + (1-\tau)x)) = -x \cdot \nabla u(x_0 + (1-\tau)x) = |x| D_\nu u(x_0 + (1-\tau)x),$$

where

$$\nu = \frac{-x}{|x|} \in C(1, \frac{1}{2}, -\nu_0) \cap S^{n-1}.$$

By Lemma 5.2, the derivative is non-negative and the lemma is proved. □

Lemma 5.4. *Under the assumptions and notations of Lemma 5.2, there is $\rho' > 0$ such that*

$$x_0 + C(\rho', \frac{1}{3}, \nu_0) \subset \Omega^c.$$

Proof. Suppose there is a sequence $\{x_k\} \subset C(1, \frac{1}{3}, \nu_0) \cap \Omega$ such that $\rho_k = |x_k| \to 0$. Fix a constant $\tau > 0$, such that for all $k \in \mathbb{N}$,

$$B_{\tau\rho_k}(x_k) = \{\, y;\ |y - x_k| \leq \tau\rho_k \,\} \subset C(1, \frac{1}{2}, \nu_0).$$

By the quadratic growth of u in Ω, there is a sequence $\{y_k\}$ such that

$$y_k \in B_{\tau\rho_k}(x_k), \qquad \forall k,$$

and

$$u_{\rho_k}({\rho_k}^{-1} y_k) - u_{\rho_k}({\rho_k}^{-1} x_k) \geq \gamma, \tag{5.1}$$

for some constant $\gamma > 0$, independent of k.

Now, by Proposition 4.5, any blow up of u at x_0 is a half space solution. In particular

$$\liminf_{r \to 0} u_r(x) \geq 0\,, \quad \forall x \in \mathbb{R}^n.$$

On the other hand, by Lemma 5.3,

$$\limsup_{r \to 0} u_r(x) \leq 0\,, \quad \forall x \in C(\infty, \frac{1}{2}, \nu_0).$$

Hence, for all $x \in C(\infty, \frac{1}{2}, \nu_0)$,

$$\lim_{r\to 0} u_r(x) = 0.$$

Since $\{\rho_k{}^{-1}y_k\}$ and $\{\rho_k{}^{-1}x_k\}$ are two bounded sequences contained in $C(2, \frac{1}{2}, \nu_0)$,

$$\lim_{k\to\infty} u_{\rho_k}(\rho_k{}^{-1}y_k) - u_{\rho_k}(\rho_k{}^{-1}x_k) = 0,$$

which is a contradiction to (5.1). □

Proof. (of Theorem 5.1) Let $\rho'' = \frac{2}{3}\min(\rho, \rho')$, with ρ as in Lemma 5.2 and ρ' as in Lemma 5.4.

We claim that any point $x \in \Omega^c \cap B_{\rho''}$ can be joined to $C(\rho', \frac{1}{3}, \nu_0)$ with a segment parallel to ν_0 and contained in Ω^c. In fact, if the claim is not true, then we can find two points, $x_1 \in \Omega \cap B_\rho$ and $x_2 \in \Omega^c \cap B_\rho$ such that

$$x_2 - x_1 = |x_2 - x_1|\nu_0.$$

Now, take a small ball $B_\epsilon(x_1) \subset \Omega$ and denote by C the cone generated by x_2 and $B_\epsilon(x_1)$. Move $B_\epsilon(x_1)$ from x_1 to x_2 along the axis of C, reducing its radius to fit in C, until we touch for the first time Ω^c. Let ζ_0 be a point of contact. There is $\varrho > 0$ and $0 < \delta \le \frac{1}{2}$ such that

$$\zeta_0 + C(\varrho, \delta, \nu_0) \subset B_\rho(x_0) \cap \Omega. \tag{5.2}$$

Since, by Lemma 5.2, $D_\nu u \ge 0$ in $B_{\rho(x_0)}$, for all $\nu \in C(1, \frac{1}{2}, -\nu_0) \cap S^{n-1}$, then

$$\zeta_0 + C(\varrho, \delta, \nu_0) \subset \{u \le u(\zeta_0)\}$$

and

$$\zeta_0 + C(\varrho, \delta, -\nu_0) \subset \{u \ge u(\zeta_0)\}\,.$$

Let $\hat{u}_0$ be a blow up of u at ζ_0. Then $\hat{u}_0$ can not be a polynomial solution since $\hat{u}_0 \le 0$ in $C(\varrho, \delta, \nu_0)$ and $\hat{u}_0 \ge 0$ in $C(\varrho, \delta, -\nu_0)$ and homogeneous polynomials of degree two are even. Then, $\hat{u}_0$ should be a half space solution. In that case, the same argument as the one in Lemma 5.4 shows that there is $\varrho' > 0$ and $\delta' > 0$ such that

$$\zeta_0 + C(\varrho', \delta', \nu_0) \subset \Omega^c.$$

This contradiction with (5.2) proves the theorem. □

The results of this section show that the regularity problem reduces to the one treated in [?], at least near the seemingly-regular boundary points, i.e., boundary points ζ_0 such that a blow-up u_0 at ζ_0 is a half space solution. Hence, earlier regularity results apply (see [?]) and we conclude the C^1 regularity of the free boundary claimed in Theorem 3.3.

6. Global solutions with compact Ω^c

In this and the next section, we characterize global solutions. This characterization is useful for a further study of convexity properties. It is shown that any global solution to equation (2.1), with quadratic growth, either solves

$$\Delta u = \chi_{\{u>a\}}\,, \quad \text{in } \mathbb{R}^n, \tag{6.1}$$
$$u \ge a$$

for some $a \in \mathbb{R}$, or it is a degree two polynomial.

Equation (6.1) was treated by Caffarelli in relation with the obstacle problem, see [?] and the reference there. The above shows that, as far as global solutions are concerned, (2.1) reduces to the one patch problem treated in [?]: Given $u \in \mathcal{P}$, there is $a \in \mathbb{R}$, such that

$$\Delta u = \chi_\Omega, \quad \text{in } \mathbb{R}^n$$
$$u = a\,,\ |\nabla u| = 0\,, \quad \text{in } \Omega^c.$$

Theorem 6.1. *Let $u \in \mathcal{P}$ and assume that Ω^c is non empty and compact. Set*

$$\sup_{x\in\Omega^c} u(x) = a.$$

Then

$$u(x) \geq a, \quad \forall x \in \mathbb{R}^n.$$

In particular, according to [?], $\Omega^c = \{u = a\}$ is convex.

Before we prove the theorem, we need several lemmas. Changing u by $u - a$, we can assume, without loss of generality, that $a = 0$.

Lemma 6.2. *Let $u \in \mathcal{P}$ and assume that Ω^c is a non empty compact set. Furthermore assume*

$$\sup_{x\in\Omega^c} u(x) = 0.$$

Then for a suitable choice of the origin, for all $x \neq 0$ fixed, the function

$$r \longrightarrow \frac{u(rx)}{r^2} \tag{6.2}$$

is nondecreasing.

Proof. Denote by V the Newtonian potential of Ω^c, i.e.

$$V(x) = \int_{\Omega^c} \frac{c_n}{|x-y|^{n-2}} dy\,.$$

V is a bounded super-harmonic function in $\mathbb{R}^n$. Since V is harmonic in Ω, due to the maximum principle there is at least one point $\zeta_0 \in \Omega^c$ such that

$$V(\zeta_0) \geq V(x)\,, \quad \forall x \in \mathbb{R}^n.$$

Choose the origin at ζ_0.

Since

$$\Delta(u - V) = 1$$

in the sense of distributions and all second order partial derivatives of $u - V$ are bounded harmonic functions, the Hessian of $u - V$ is a constant matrix, by Liouville's theorem.

Hence $u - V$ is a polynomial of degree two. Set

$$P(x) = u(x) - V(x) - u(0) + V(0).$$

Note that $|\nabla V(0)| = |\nabla u(0)| = 0$. Hence $P(0) = |\nabla P(0)| = 0$, this implies that P is homogeneous.

Now consider the function

$$h(x) = x \cdot \nabla u(x) - 2u(x)\,.$$

h is continuous in $\mathbb{R}^n$ and for all $x \neq 0$ fixed,

$$\frac{d}{dr}\left(\frac{u(rx)}{r^2}\right) = \frac{1}{r^3} h(rx).$$

We will show that h is non-negative in $\mathbb{R}^n$. In fact

$$h(x) = -2u(x) \geq 0\,, \quad \forall x \in \Omega^c\,.$$

On the other hand, by the homogeneity of P,

$$h(x) = x \cdot \nabla V(x) - 2V(x) + 2V(0) - 2u(0)$$

then

$$\lim_{|x|\to\infty} h(x) = 2V(0) - 2u(0) \geq 0\,.$$

Since h harmonic in Ω, by the minimum principle, h is positive in Ω. □

Corollary 6.3. *Under the hypothesis of Lemma 6.2, for all $\kappa \geq 0$, the set $\{u \leq \kappa\}$ is star-shaped with respect to the origin.*

Remark 6.4. The family $u_r(x) = u(rx)/r^2$ indexed by r is not relatively compact, since a priori $u(0) \neq 0$. Therefore, the monotonicity given by Lemma 6.2 doesn't mean that the blow up at the origin is convergent.

Lemma 6.5. *Let $u \in \mathcal{P}$ and assume (6.2).*
Then any blow up u_0 of u at $x_0 \in \partial\Lambda_0$, where $\Lambda_0 := \Omega^c \cap \{u = 0\}$, is a half space solution.

Before we prove this lemma, we need a result concerning a balanced energy functional, introduced by G.S. Weiss in [?]. We shall use a slightly different version of Weiss' formula. Define

$$\Phi(r, u, x_0) = r^{-n-2}\int_{B_r(x_0)} \left(|\nabla u|^2 + 2u\right) - r^{-n-3}\int_{\partial B_r(x_0)} 2u^2.$$

The following result is basically due to G.S. Weiss, see [?]. For the reader's convenience, we also give a proof.

Lemma 6.6. *(Weiss) Let $u \in \mathcal{P}$ and assume (6.2).*
Then for all $x_0 \in \mathbb{R}^n$, $\Phi(r, u, x_0)$ is non decreasing with respect to r.

The hypothesis (6.2) is crucial for the proof of Lemma 6.6. The lemma fails if we replace u by $u - u(x_0)$, unless $u(x_0) \geq 0$. We can use this lemma in conjunction with blow-ups only when $x_0 \in \Lambda_0$. Nevertheless, it is convenient to write $\Phi(r, u, x_0)$ in terms of

$$u_r(x) = \frac{u(x_0 + rx)}{r^2}\,, \qquad r > 0.$$

In that case

$$\Phi(r, u, x_0) = \int_{B_1(0)} \left(|\nabla u_r|^2 + 2u_r\right) - \int_{\partial B_1(0)} 2{u_r}^2.$$

Proof. *(of Lemma 6.6)* We shall prove that the derivative of $\Phi(r, u, x_0)$ with respect to r is non-negative.

Indeed,

$$\Phi'(r) = \int_{B_1} (2\nabla u_r \cdot \nabla {u_r}' + 2{u_r}') - \int_{\partial B_1} 4u_r{u_r}'\,,$$

where

$${u_r}'(x) = \frac{d}{dr}u_r(x) = \frac{1}{r}\left(\nabla u_r \cdot x - 2u_r\right).$$

Using integration by parts,

$$\int_{B_1} 2\nabla u_r \cdot \nabla {u_r}' = \int_{\partial B_1} 2(\nabla u_r \cdot \eta){u_r}' - \int_{B_1} 2\Delta u_r {u_r}' .$$

Since

$$\nabla u_r \cdot x = r{u_r}'(x) + 2u_r$$

and $\eta = x$ on ∂B_1,

$$\Phi'(r) = \int_{\partial B_1} 2r({u_r}')^2 + \int_{B_1} 2(1-\chi_{\Omega_r}){u_r}'.$$

The first integrand above is non-negative. The second one is also non-negative since we have assumed $u(x) \le 0$ for all $x \in \Omega^c$. ☐

Since ${u_r}' \equiv 0$ if and only if u is homogeneous of degree two, the above expression leads to the following important conclusion, already found in Weiss' paper for the obstacle problem.

Corollary 6.7. *(Weiss) Under the hypothesis of Lemma 6.6, the function*

$$v(x) = u(x_0 + x)$$

is homogeneous of degree two if and only if $\Phi(r, u, x_0)$ *is constant with respect to* r.

Remark 6.8. Let P be a degree-two homogeneous polynomial whose Laplacian is 1. Then $\Phi(r, P, 0)$ does not depend on r nor on P. Hence we set

$$\alpha_n = \Phi(r, U, 0) = \frac{1}{2}\int_{B_1} x_1^2 .$$

where x_1 is the first coordinate component of $x \in B_1$.

This is twice the value of $\Phi(r, U, 0)$, when U is a half space solution,

$$\Phi(r, U, 0) = \frac{\alpha_n}{2}.$$

Remark 6.9. Let $u \in \mathcal{P}$ and $x_0 \in \partial\Omega$. Since by uniform convergence,

$$\Phi(r, u_0, 0) = \Phi(0^+, u, x_0) \le \Phi(r, u, x_0) \le \Phi(\infty^-, u, x_0) = \Phi(r, u_\infty, 0)$$

and any blow up or blow down is homogeneous, we are left with only three possibilities

$$\Phi(0^+, u, x_0) = \Phi(\infty^-, u, x_0) = \frac{\alpha_n}{2},$$

or

$$\Phi(0^+, u, x_0) = \frac{\alpha_n}{2} \quad \text{and} \quad \Phi(\infty^-, u, x_0) = \alpha_n,$$

or

$$\Phi(r, u_0, 0) = \Phi(\infty^-, u, x_0) = \alpha_n.$$

Proof. *(of Lemma 6.5)* If u_0 is a polynomial, then $\Phi(0^+, u, x_0) = \alpha_n$. Since α_n is the maximum value of $\Phi(r, u, x_0)$,

$$\Phi(r, u, x_0) = \alpha_n, \quad \forall r > 0.$$

This means that u itself is a polynomial. This contradicts the assumption $\Omega^c \neq \emptyset$. ☐

Proof. *(of Theorem 6.1)* If $0 \in \partial\{u \le 0\}$, then $0 \in \partial\Lambda_0$. By Lemma 6.2 and Lemma 6.5,
$$\lim_{r\to 0} u_r(x) = \inf_{r>0} u_r(x) = u_0 \ge 0.$$
The theorem is proved in this case.

If 0 is an interior point of $\{u \le 0\}$, the interior of $\{u \le 0\}$ is connected, by Corollary 6.3.

By Lemma 6.5 and Lemma 5.4, Λ_0 contains a truncated cone. Since u is subharmonic in the interior of $\{u \le 0\}$ and the interior of $\Lambda_0 \neq \emptyset$, by the maximum principle,
$$\{u \le 0\} = \{u = 0\}.$$
This completes the prove of the theorem. □

7. Global solutions with unbounded Ω^c

Theorem 7.1. *Let $u \in \mathcal{P}$ such that Ω^c is non empty and unbounded. Then, there is $a \in \mathbb{R}$ such that $u \ge a$ and $\Omega^c = \{u = a\}$. In particular, according to* [**?**], *Ω^c is convex.*

Proof. Suppose that some blow up u_∞ of u at infinity is a half space solution. Then, by Proposition 4.5, iii), $u - u(x_0)$ is a half space solution, for any $x_0 \in \Omega^c$. And the theorem follows in this case.

Now, if no blow-up at infinity is a half-space solution, then by Lemma 4.4 we may assume that any blow-up u_∞ is a polynomials. The assumption $\Omega^c \neq \emptyset$ prevents u of being a polynomial (by Liouville's theorem).

Since Ω^c is unbounded, there exists a sequence $x^j \in \partial\Omega$ tending to ∞. In this case we may scale by $R_j = |x^j|$ so as to obtain, in the limit, a global solution with a free boundary $\tilde{x}$ on the unit sphere. By homogeneity then the ray $r\tilde{x}$ must lie in the free boundary. It thus follows that $D_e u_\infty \equiv 0$, for $e = \tilde{x}/|\tilde{x}|$. Hence
$$0 \le \varphi(r, e, u) = \varphi(\infty, e, u) = \varphi(1, e, u_\infty) = 0,$$
and we conclude that $D_e u$ doesn't change sign for $e = \tilde{x}/|\tilde{x}|$, we assume $D_e u \ge 0$, otherwise we replace e by $-e$. Now similar to analysis in [**?**] we translate $u(x) - u(x_0)$ (for some x_0 with $B_r(x_0) \subset \{|\nabla u| = 0\}$) in the direction e and obtain an $(n-1)$ dimensional problem; we let u_∞ now denote the translated limit function.

First, suppose the lower dimensional function u_∞ is either a half space solution, or falls into the hypotheses of Theorem 6.1. Then the lower dimensional solution is convex and non-negative. Since $D_e u \ge 0$ we conclude $u \ge 0$ (or more correctly $u(x) - u(x_0) \ge 0$).

Due to the convexity of u_∞, positivity of u, and that $D_e u \ge 0$ we must have $\{|\nabla u| = 0\}$ is connected. Hence we are reduced to the case of $u = u(x_0)$ in the set $\{|\nabla u| = 0\}$, and we can apply [5] to conclude u is convex.

Next, if the lower dimensional solution, u_∞, is neither of the above it must fall into the third category analyzed above. Hence we repeat our argument and translate u_∞ again in a new direction and reduce the dimension further. Finally, by induction, we need to classify the one dimensional solutions. However, the one dimensional problem is solved by $x_1^2/2$, $(\max(0, x_1))^2/2$, or two separated solutions of the latter. Obviously any rotation of these are also possible solutions. And these are all nonnegative solutions. □

8. Proof of Theorem 3.3

Now the proof of Theorem 3.3 follows in the same way as that of Theorem III in [5]. In what follows we'll explain the minor changes needed in the proof of Theorem III in [5] so as to adapt it to our case.

By the classification of global solutions in our case, we already have proven that the global solutions of our problem coincide (in nature) with that of [5] and hence Lemma 6.2 in [5] follows in our case too. A similar statement as that of Lemma 6.2 in [5] works also for our case with the minor change of [5; (6.2)] to

$$sD_e u - |\nabla u|^2;$$

cf. Lemma 4.2 in that paper and its proof. It is noteworthy that all the above analysis ($\S 4 - \S 7$) seem to reduce the problem to show that Lemma 5.2 in our paper holds in a uniform fashion for the class. Once we have this we now that u does not change sign, the set $\{|\nabla u| = 0\}$ is connected and the free boundary $\partial\{|\nabla u| > 0\}$ is locally (in a uniform neighborhood, depending on the modulus of continuity $\sigma(r)$) a Lipschitz graph and the rest now follows as in [5]. □

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