The multi-layer free boundary problem for the p-Laplacian in convex domains

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Abstract

The main result of this paper concerns existence of classical solutions to the multi-layer Bernoulli free boundary problem with nonlinear joining conditions and the p-Laplacian as governing operator. The present treatment of the 2-layer case involves technical refinements of the one-layer case, studied earlier by two of the authors. The existence treatment of the multi-layer case is largely based on a reduction to the two-layer case, in which uniform separation of the free boundaries plays a key role.

1 Introduction and statement of the problem

1.1 The mathematical setting

In this paper we continue the study of the free boundary problem arising in connection with potential flow with power-law nonlinearity (see [HS1–4] for backgrounds). Mathematically, our starting point is an annular region, bounded by two convex surfaces in $\mathbb{R}^N$ $(N \geq 2)$:

$$K = K_{m+2} \setminus K_1 \quad \text{with } K_1, K_{m+2} \quad \text{convex and } K_1 \subset\subset K_{m+2}.$$

The aim is to show that for given positive integer m and data

$$\lambda_i \in (-1, 1), \qquad F_i(x, p, q) : K \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}, \qquad (i = 1, \cdots, m)$$

with $\lambda_i > \lambda_{i+1}$, one can find convex domains

$$K_1 \subset\subset K_2 \subset\subset \cdots \subset\subset K_{m+1} \subset\subset K_{m+2}$$

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such that the p-capacitary potential u_i for each annular convex region $K_{i+1} \setminus \overline{K_i}$ satisfies a nonlinear joining Bernoulli condition (see the main theorems, Theorem 4.1, and Theorem 6.1)

$$F_i(x, |\nabla u_i(x)|, |\nabla u_{i+1}(x)|) = 0 \quad \text{on } \partial K_{i+1}, \quad (i = 1, \cdots, m). \tag{1}$$

The p-capacitary potential refers to the solution of the following Dirichlet problem

$$\begin{cases} \Delta_p u = 0 & \text{in} \quad K_{i+1} \setminus \overline{K_i} \\ u = \lambda_i & \text{on} \quad \partial K_i \\ u = \lambda_{i+1} & \text{on} \quad \partial K_{i+1}, \end{cases}$$

where $\Delta_p,\ 1 < p < \infty$ is the p-Laplace operator defined by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

1.2 Applications

The above described problem appears in several physical situations and can be appropriately interpreted in many industrial applications. A general way of interpreting the above problem is to consider u as the potential function of several adjacent flows in convex rings with prescribed pressure on the free streamlines.

A more interesting application, however, is related to the so-called Stefan problem, for large time. In this connection, the two phase model describes crystallization (freezing) or melting of some physical substance. Multi-phase Stefan problem refers to materials capable of assuming any of three or more different states (solid, liquid, gaseous, in particular). We expand this in more details for the two phase case.

Let us consider a cylindrical container with the horizontal cut as the domain $K = K_3 \setminus K_1$ (this is the two dimensional case). The exterior wall ∂K_3 is kept at temperature $u = -1$, and the interior wall ∂K_1 at temperature $u = 1$. The container is also filled with liquid, and the temperature of the liquid is assumed to be known initially.

Suppose the material (liquid) solidifies at temperature $-1 < \lambda < 1$. For simplicity we take $\lambda = 0$. By continuity of the temperature for positive times, we know that there must be a curve $\Gamma(x, t)$ (for each time t) on which $u(x, t) = 0$. Hence on the subregion $\{u > 0\}$ the material is in liquid form and on the subregion $\{u < 0\}$ the material is in solid form. Let us also assume that the temperature u (depending on the material) also satisfies the (nonlinear) heat equation

$$\Delta_p u - D_t u = 0, \qquad \text{in} \qquad K \setminus \{u = 0\}. \tag{2}$$

On the transition phase $\Gamma(x, t)$ the Stefan condition (Bernoulli condition), which follows from the energy conservation law, gives the dynamic equation of the moving curve

$$|\nabla u_1| = g(x, |\nabla u_2|, V), \tag{3}$$

where u_1 and u_2 represent the function u on $\{u > 0\}$ and on $\{u < 0\}$ respectively. Here V is the normal velocity of the curve $\Gamma(x, t)$, and the nonlinear joining condition (3) may depend on the density of the heat source over the inter-phase boundary (due for instance to an extra super-heating).

For large time, the heat flux tends to stabilize and becomes stationary. Hence u_t, and V both become approximately zero. Therefore the realistic model for the stationary problem is the one given by

$$\Delta_p u = 0, \qquad \text{in} \qquad K \setminus \{u = 0\}, \qquad |\nabla u_1| = g(x, |\nabla u_2|, 0) \qquad \text{on} \qquad \{u = 0\}. \tag{4}$$

It is noteworthy that the p-Laplace operator constitutes a subclass of a larger class of operators, appearing in many modeling problems in industrial applications, due to non-Newtonian behavior of fluids.

For further applications, and backgrounds in the case $p = 2$, we refer the reader to [A1,2], and the references therein.

1.3 Main result

We prove existence and C^1 regularity of the free boundary in the two-phase case. More precisely, the main result of this paper is the following:

Theorem 1.1 *(two phases) Let K_1, K_3 be two convex domains, such that $K_1 \subset\subset K_3$ and $g : (K_3 \setminus K_1) \times \mathbb{R}_+ \to \mathbb{R}_+$ a continuous positive function, non-decreasing with respect to its second argument and satisfying some concavity property (see Definition 2.3 for a precise statement). Then there exists a convex C^1 domain ω, $K_1 \subset\subset \omega \subset\subset K_3$, which is a classical solution of the two-layer free boundary problem, the latter means that the p-capacitary potentials u_1 and u_2 of the sets $\omega \setminus \overline{K_1}$ and $K_3 \setminus \overline{\omega}$ respectively, i.e. solutions of*

$$\left\{ \begin{array}{lll} \Delta_p u_1 = 0 & \text{in} & K_2 \setminus \overline{K_1} \\ u_1 = 1 & \text{on} & \partial K_1 \\ u_1 = 0 & \text{on} & \partial\omega \end{array} \right. , \qquad \left\{ \begin{array}{lll} \Delta_p u_2 = 0 & \text{in} & K_3 \setminus \overline{\omega} \\ u_2 = -1 & \text{on} & \partial K_3 \\ u_2 = 0 & \text{on} & \partial\omega \end{array} \right.$$

with Δ_p, $1 < p < +\infty$ the p-Laplace operator, satisfy

$$\lim_{\substack{z \to x \\ z \in \omega \setminus \overline{K_1}}} |\nabla u_1(z)| = \lim_{\substack{y \to x \\ y \in K_3 \setminus \overline{\omega}}} g(y, |\nabla u_2(y)|) \qquad \forall\ x\ \in \partial\omega\ .$$

Section 2 is devoted to describe the possible nonlinear joining conditions we are able to handle. In section 3, we give some useful auxiliary results. Section 4 is devoted to the proof of the main theorem, section 5 is the separation result and section 6 describes extension to the multi-phase case.

2 The nonlinear joining condition

In this section, we discuss what could be the nonlinear joining condition involving ∇u_i and ∇u_{i+1} at the interface $\gamma_i = \partial K_{i+1}$ between the two phases. We recall that this condition is written in the general form

$$F_i(x, |\nabla u_i(x)|, |\nabla u_{i+1}(x)|) = 0, \qquad (i = 1, \cdots, m) \tag{5}$$

with $F_i : K \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$.

We will always assume that $F_i(x, p, q)$ is a continuous function on $K \times \mathbb{R}_+ \times \mathbb{R}_+$, and that $F_i(x, p, q)$ is strictly increasing regarding as a function of variable p for all x, q.

This assumption and the implicit function theorem allows us to write the joining condition (5) in the following equivalent form

$$|\nabla u_i(x)| = g_i(x, |\nabla u_{i+1}(x)|), \tag{6}$$

where $g_i : K \times \mathbb{R}_+ \to \mathbb{R}_+$ are given functions.

An important tool, used in the proof of our main theorem and due to [LS] and [A4], is the following property: if γ_i contains a line segment I then $x \mapsto 1/|\nabla u_i(x)|$ is a convex function while $x \mapsto 1/|\nabla u_{i+1}(x)|$ is a concave function on I (see Lemma 3.12). This in conjunction with concavity assumption on the function $x \mapsto 1/g_i(x, |\nabla u_{i+1}(x)|)$ underlies one of the main techniques in the proof of our main result. Therefore, the property that g_i must satisfy is the following:

$$x \mapsto 1/g_i(x, q(x)) \text{ is a concave function as soon as } 1/q(x) \text{ is a concave function.} \tag{7}$$

For general functions g we cannot expect to have convexity of the level sets of the solution. In fact the first author (see [A3]) obtained an example of the convex two-layer problem in the plane for which no convex solution exists corresponding to the joining condition in the form: $|\nabla u_2(x)|^2 - |\nabla u_1(x)|^2 = \lambda^2$ (compare with eq. (8)). Laurence and Stredulinsky ([LS]) gave an example of the convex two-layer problem with the same joining condition such that the natural variational minimizer is not convex.

It is also an open question whether there exist classical solutions at all, for general (regular) functions g. In two space dimensions this was settled by H.W. Alt and L.A. Caffarelli [AC]. So the question to be raised is what are the “necessary and sufficient” conditions to have existence of convex classical solutions. Our condition (7) may seem somewhat artificial, but it is the only working condition at this moment. Let us remark that a similar ”convexity condition” was assumed in [A2] and [A5].

Example : The classical nonlinear joining condition, see e.g. [A1], [LS], is given by

$$|\nabla u_i(x)|^\alpha - |\nabla u_{i+1}(x)|^\alpha = a_i(x)^\alpha, \tag{8}$$

where $\alpha \geq 1$ and $a_i(x) > 0$. This joining condition satisfies the ”convexity condition” (7) provided that the function $1/a_i$ is concave, as will follow from Lemma 2.1 below. (Regarding applicability of other convexity conditions, we refer to [A2], Example 2.9 and [A5], Example 4.7.)

Lemma 2.1 *Let a and q be positive functions defined on $\mathbb{R}^N$ and such that $1/a$ and $1/q$ are concave. Then, for $\alpha \geq 1$, the function $x \mapsto \dfrac{1}{(a(x)^\alpha + q(x)^\alpha)^{1/\alpha}}$ is concave.*

Proof : It is sufficient to do the proof for C^1 functions a and q, since the result will follow for less regular functions by a simple density argument just using pointwise convergence.

Let us set

$$f(x) := \frac{1}{(a(x)^\alpha + q(x)^\alpha)^{1/\alpha}}\ ,$$

which is a C^1 function. We want to prove the following inequality:

$$\forall x, y \in \mathbb{R}^N \quad (\nabla f(x), y - x) \geq f(y) - f(x). \tag{9}$$

Now

$$(\nabla f(x), y-x) = -\frac{1}{(a(x)^\alpha + q(x)^\alpha)^{1+1/\alpha}}\ \left[a(x)^{\alpha-1}(\nabla a(x), y-x) + q(x)^{\alpha-1}(\nabla q(x), y-x)\right]\ . \tag{10}$$

By concavity of $1/a$ we have

$$(\nabla\left(\frac{1}{a}\right)(x), y-x) = -\frac{1}{a^2(x)}\,(\nabla a(x), y-x) \geq \frac{1}{a(y)} - \frac{1}{a(x)}$$

A similar inequality holds for $1/q$. Putting these in (10) yields

$$(\nabla f(x), y-x) \geq \frac{1}{(a(x)^\alpha + q(x)^\alpha)^{1+1/\alpha}}\ \left[a(x)^{\alpha+1}\left(\frac{1}{a(y)} - \frac{1}{a(x)}\right) + q(x)^{\alpha+1}\left(\frac{1}{q(y)} - \frac{1}{q(x)}\right)\right]$$

that is

$$(\nabla f(x), y-x) \geq \frac{1}{(a(x)^\alpha + q(x)^\alpha)^{1+1/\alpha}}\ \left(\frac{a(x)^{\alpha+1}}{a(y)} + \frac{q(x)^{\alpha+1}}{q(y)}\right) - f(x)\,.$$

So, to prove (9), it remains to prove the following inequality

$$\frac{1}{(a(x)^\alpha + q(x)^\alpha)^{1+1/\alpha}}\ \left(\frac{a(x)^{\alpha+1}}{a(y)} + \frac{q(x)^{\alpha+1}}{q(y)}\right) \geq \frac{1}{(a(y)^\alpha + q(y)^\alpha)^{1/\alpha}}\ . \tag{11}$$

Let us set $x_1 = \frac{a(x)}{a(y)}$, $x_2 = \frac{q(x)}{q(y)}$, $t_1 = a(x)^\alpha$, $t_2 = q(x)^\alpha$. Inequality (11) can be rewritten

$$x_1^\alpha x_2^\alpha \frac{\frac{1}{t_1} + \frac{1}{t_2}}{\frac{x_1^\alpha}{t_1} + \frac{x_2^\alpha}{t_2}} \leq \left(\frac{t_1 x_1 + t_2 x_2}{t_1 + t_2}\right)^\alpha$$

or

$$\frac{t_1+t_2}{t_1 x_2^\alpha + t_2 x_1^\alpha} \leq \left(\frac{\frac{t_1}{x_2}+\frac{t_2}{x_1}}{t_1+t_2}\right)^\alpha . \tag{12}$$

Now, the mean-value inequality between harmonic and arithmetic means yields

$$\left(\frac{t_1+t_2}{\frac{t_1}{x_2}+\frac{t_2}{x_1}}\right)^\alpha \leq \left(\frac{t_1 x_2 + t_2 x_1}{t_1+t_2}\right)^\alpha$$

and the inequality (12) follows immediately using the convexity property of the function $x \mapsto x^\alpha$. $\square$

We present now another class of functions g_i which satisfy the above-mentioned property.

Lemma 2.2 *Assume that the function $g : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfies the following set of hypothesis:*

(H1): *g is concave,*

(H2): *g satisfies the following inequality*

$$\forall (\xi_1, \xi_2, \eta_1, \eta_2) \in \mathbb{R}_+^4 \qquad g(\xi_1, \eta_1) g(\xi_2, \eta_2) \geq g^2(\sqrt{\xi_1 \xi_2}, \sqrt{\eta_1 \eta_2}). \tag{13}$$

Now if $a : \mathbb{R}^N \to \mathbb{R}_+$, $q : \mathbb{R}^N \to \mathbb{R}_+$ are two given functions such that $1/a$ and $1/q$ are concave, then $x \mapsto \dfrac{1}{g(a(x), q(x))}$ is concave.

Proof : It suffices to prove this for C^1-functions g, a and q, since the result will follow for less regular functions by a simple density argument just using pointwise convergence.

Let us set

$$f(x) := \frac{1}{g(a(x), q(x))} ,$$

which is a C^1 function. We prove the following inequality (cf. (9))

$$\forall x, y \in \mathbb{R}^N \quad (\nabla f(x), y - x) \geq f(y) - f(x) . \tag{14}$$

Now

$$(\nabla f(x), y-x) = \frac{-1}{g^2(a(x), q(x))} \left[\frac{\partial g}{\partial \xi}(a(x), q(x))\, (\nabla a(x), y-x) + \frac{\partial g}{\partial \eta}(a(x), q(x))\, (\nabla q(x), y-x)\right] . \tag{15}$$

and by concavity assumption on $1/a$ and $1/q$ (see the proof of Lemma 2.1) we'll have

$$\begin{aligned}(\nabla f(x), y-x) \geq \tfrac{1}{g^2(a(x),q(x))} &\left[\tfrac{\partial g}{\partial \xi}(a(x), q(x))\, a^2(x) \left(\tfrac{1}{a(y)} - \tfrac{1}{a(x)}\right) + \right.\\ &\left. + \tfrac{\partial g}{\partial \eta}(a(x), q(x))\, q^2(x) \left(\tfrac{1}{q(y)} - \tfrac{1}{q(x)}\right)\right]\end{aligned}$$

Next, using concavity of g we arrive at

$$\frac{\partial g}{\partial \xi}(\xi_1, \eta_1)(\xi_2 - \xi_1) + \frac{\partial g}{\partial \eta}(\xi_1, \eta_1)(\eta_2 - \eta_1) \geq g(\xi_2, \eta_2) - g(\xi_1, \eta_1),$$

where $(\xi_1, \eta_1) = (a(x), q(x))$ and $(\xi_2, \eta_2) = (\frac{a^2(x)}{a(y)}, \frac{q^2(x)}{q(y)})$. Inequality (14) now follows immediately from this and inequality (13). $\square$

Examples

1. The function $g(\xi, \eta) := (\xi^\alpha + \eta^\alpha)^{1/\alpha}$ which corresponds to the classical nonlinear joining condition already mentioned does not fall into the above framework when $\alpha \geq 1$ (because g is convex and not concave), but for $0 \leq \alpha < 1$!. Indeed, in this case assumption (H1) is easily verified by proving that the Hessian of g is negative on $\mathbb{R}^2_+$. As for (H2), it follows immediately from the inequality

$$(\xi_1\eta_2)^\alpha + (\xi_2\eta_1)^\alpha \geq 2(\xi_1\xi_2\eta_1\eta_2)^{\alpha/2}$$

which gives

$$[(\xi_1\xi_2)^\alpha + (\eta_1\eta_2)^\alpha + (\xi_1\eta_2)^\alpha + (\xi_2\eta_1)^\alpha]^{1/\alpha} \geq \left[(\xi_1\xi_2)^\alpha + (\eta_1\eta_2)^\alpha + 2(\xi_1\xi_2\eta_1\eta_2)^{\alpha/2}\right]^{1/\alpha}$$

which is inequality (13).

2. The function $g(\xi, \eta) := \xi^\alpha \eta^\beta$ with $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta \leq 1$ can also be considered. Conditions (H1), (H2) are readily verified in this case.
3. More generally, we can consider a function like $g(\xi, \eta) := \sum_{i \in I} a_i \xi^{\alpha_i} \eta^{\beta_i}$ (finite or infinite sum) with $a_i \geq 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$ and $\alpha_i + \beta_i \leq 1$. Assumption (H1) is elementary (g is a combination of concave functions with positive coefficients) while inequality (13) is obtained from the expansion of

$$\sum_{i_1 \in I} \sum_{i_2 \in I} a_{i_1} a_{i_2} \left(\xi_1^{\alpha_{i_1}/2} \xi_2^{\alpha_{i_2}/2} \eta_1^{\beta_{i_1}/2} \eta_2^{\beta_{i_2}/2} - \xi_1^{\alpha_{i_2}/2} \xi_2^{\alpha_{i_1}/2} \eta_1^{\beta_{i_2}/2} \eta_2^{\beta_{i_1}/2}\right)^2 \geq 0.$$

Definition 2.3 *Define $\mathcal{G}$ to be the family of all functions $g : K \times \mathbb{R}_+ \to \mathbb{R}_+$, satisfying the following conditions*

(A1): *g is continuous and $\exists\, c_0 > 0$ such that $g(x, 0) \geq c_0$ for all $x \in K$,*

(A2): *g is non-decreasing with respect to second argument,*

(A3): *g satisfies the following concavity property: $x \mapsto \dfrac{1}{g(x, q(x))}$ is concave whenever q is a given function such that $1/q$ is concave, and*

(A4): *for any given value $y_0 > 0$, there exist constants $0 < C_1 < C_2$ such that $C_1 \leq (g(x, y)/y) \leq C_2$, uniformly for all $x \in K$ and all $y \geq y_0$.*

Hence forward we will always consider the following nonlinear joining condition

$$|\nabla u_i(x)| = g_i(x, |\nabla u_{i+1}(x)|), \tag{16}$$

with $g_i \in \mathcal{G}$.

3 Preliminary results

In this section we will sum up some of the auxiliary results used in this paper. We remark that the usual comparison and maximum principle for elliptic partial differential equations, is one of the basic tools here; see [T].

Lemma 3.1 *(Exterior Barrier) Let D be a convex domain in $\mathbb{R}^N$ and suppose u is a continuous non-negative function on $B(x^0, r)$, p-harmonic in $B(x^0, r) \cap D$, with $x^0 \in \partial D$. Let also $u = 0$ on $\partial D \cap B(x^0, r)$. If ∂D is not C^1 at x^0, i.e. D has (at least) two supporting planes at x^0, then*

$$\lim_{x \to x^0} |\nabla u(x)| = 0, \quad x \in D.$$

Lemma 3.2 *(Interior Barrier) Let D be a convex domain in $\mathbb{R}^N$ and suppose u is a continuous non-negative function on $B(x^0, r)$, p-harmonic in $B(x^0, r) \setminus D$, with $x^0 \in \partial D$. Let also $u = 0$ on $\partial D \cap B(x^0, r)$. If*

$$|\nabla u| \leq C_0 \quad in \quad B(x^0, r) \setminus D,$$

then $\partial D \cap B(x^0, r/2)$ is C^1 with a uniform C^1 norm, i.e. there exists a constant $C_1 = C_1(C_0, N)$ such that

$$|\nabla\psi(x) - \nabla\psi(y)| \leq C_1,$$

where ψ is a map that represents ∂D near $x^0 \in \partial D$.

The proofs of these lemmas follow from standard theory using barriers at conical boundary points. The existence of such barriers are proven in [Do], see also [K].

Remark 3.3 *By Lemmas 3.1, and 3.2, if $C_0^{-1} \leq |\nabla u| \leq C_0$ in D, ∂D must be C^1 with C^1-norm depending on C_0.*

Definition 3.4 *(Blow-up) For the functions $u^j : B(x^j, 1) \to \mathbb{R}$ and for a sequence of non-negative numbers $\{r_j\}$ ($r_j \to 0$) we define the scaled functions on $B(0, 1/r_j)$ by*

$$\tilde{u}^j(x) = \frac{u^j(r_j x + x^j) - u^j(x^j)}{r_j}.$$

Obviously, if all functions u^j are Lipschitz-continuous in $B(x^j, 1)$ with the same constant, then $\tilde{u}^j$ are uniformly Lipschitz in $B(0, R)$ ($R < 1/r_j$). Thus, there exists a subsequence n_k, such that $\tilde{u}^{n_k}$ converges locally in $C^\alpha(\mathbb{R}^N)$ to a function u^0. Moreover, if u^j are p-harmonic, then so is u^0 in $\{u^0 > 0\}$ and $u^0(0) = 0$.

Lemma 3.5 *Let $S(C_0)$ be the set of all C^1 domains $D \subset \mathbb{R}_+ \times \mathbb{R}^{N-1}$, such that $B(0, 1) \cap D$ is convex, $0 \in \partial D$ and $||\partial D||_{C^1(B(0,1/2))} \leq C_0$.*

Then any blow-up of a sequence $D_j \in S(C_0)$ converge to a half space, i.e., if $r_j \downarrow 0$ and $D_j \in S(C_0)$, then for $\tilde{D}_j := \frac{1}{r_j} D_j = \{x : r_j x \in D_j\}$ we have

$$\limsup \tilde{D}_j = \overline{\mathbb{T}},$$

where $\mathbb{T} = \{x_1 > 0\}$, and $\limsup$ means the set of all limit points of sequences $\{x^j\}$ with $x^j \in \tilde{D}_j$.

Lemma 3.6 *Let u^j be the p-capacitary potential of an annular domain $D_j = D_j^2 \setminus D_j^1$ with convex uniform C^1 boundaries. Suppose moreover the gradient of u^j satisfy*

$$|\nabla u^j(x)| \leq \Lambda_0 < \infty,$$

uniformly both in j and $x \in D_j$. Then any convergence blow-up $\tilde{u}^{r_j}$ at any boundary point gives a linear function $u^0 = \alpha x_1^+$, after suitable rotation and translation. In particular, for any boundary point $x^j \in \partial D_j$

$$u^j(y + x^j) = u^j(x^j) + \alpha y_1^+ + o(r_j)$$

in $B(0, r_j)$, in some rotated system.

The proof of this lemma is just the same as the proof of Lemma 2.4 in [HS2]. The uniformity in norms are crucial.

Using these lemmas, we can prove the following (cf. Theorem 1.3 [HS4]).

Lemma 3.7 *Let D_1 and D_2 be two nested open convex domains ($\overline{D}_1 \subset D_2$), and u denote the p-capacitary potential of $D = D_2 \setminus \overline{D}_1$. Then for $x \in \partial D$*

$$\lim_{y \to x} |\nabla u(y)| \qquad exists$$

non-tangentially (with values in $[0, \infty]$). In particular $|\nabla u|$ can be defined (with values in $[0, \infty]$) up to the boundary ∂D as non-tangential limit. Moreover, $|\nabla u|$ is upper semi-continuous up to ∂D_2 and lower semi-continuous up to ∂D_1.

Proof: Since the problem is local, depending on whether we are close to ∂D_1 or ∂D_2, we may start with point $x^0 \in \partial D_1$. In case $|\nabla u|$ is bounded in a neighborhood of x^0 the proof was given in Theorem 1.3 in [HS4]. So suppose there exists a sequence $x^j \in D$ with

$$\lim_j x^j = x^0, \qquad |\nabla u(x^j)| \to \infty \qquad c_0|x^j - x^0| \le \operatorname{dist}(x^j, D_1),$$

for some $c_0 > 0$, where the last condition means non-tangential approach of x^j to x^0. Obviously it suffices to show that for any such sequence $\{x^j\}$ we have

$$|\nabla u(y^j)| \quad \to \quad \infty \qquad\qquad \forall\, y^j \in B(x^j, d_j),$$

where $8d_j = \operatorname{dist}(x^j, D_1)$. To show this, we scale the function u by

$$u_j(x) := \frac{u(d_j x + x^j)}{u(x^j)} \qquad \text{in } B_8(0).$$

Since

$$\Delta_p u_j = 0, \qquad u_j > 0 \qquad \text{in } B_8(0),$$

we have, by Harnack's inequality,

$$\sup_{B_4} u_j \le C \inf_{B_4} u_j \le C u_j(0) = C.$$

In particular, u_j is a bounded sequence in B_4. Hence by standard elliptic theory, a subsequence of u_j converges to a solution u_0 in B_4, satisfying

$$\Delta_p u_0 = 0, \qquad u_0(0) = 1, \qquad u_0 > 0 \qquad \text{in } B_4.$$

Moreover, the level sets of u_0 are convex, since they are convex for all u_j.

Now suppose $|\nabla u(x^j)| > j$. Then by uniform $C^{1,\alpha}$ estimates

$$C_0 \ge |\nabla u_j(0)| = \frac{d_j |\nabla u(x^j)|}{u(x^j)} \ge \frac{j d_j}{u(x^j)}.$$

Hence

$$u(x^j) \ge \frac{j d_j}{C_0}. \tag{17}$$

Now if for some

$$y^j = d_j \tilde{y}^j + x^j \;\in\; B(x^j, d_j), \qquad \left(\tilde{y}^j \in B_1\right),$$

we have $|\nabla u(y^j)| \le C_1$ for some $C_1 > 0$, then

$$|\nabla u_j(\tilde{y}^j)| = \frac{d_j |\nabla u(d_j \tilde{y}^j + x^j)|}{u(x^j)} = \frac{d_j |\nabla u(y^j)|}{u(x^j)} \le \frac{d_j C_1}{u(x^j)} \le \frac{C_1 C_0}{j},$$

where in the last inequality we have used (17). Hence it follows that $|\nabla u_0(\tilde{y})| = 0$, where $\tilde{y} = \lim \tilde{y}^j \in B_1$, for an appropriate subsequence.

To summarize, we have a positive p-harmonic function u_0 in B_4, with convex level sets, and with the further property that for some $\tilde{y} \in B_1$ there holds $\nabla u_0(\tilde{y}) = 0$. This contradicts the Hopf's boundary point lemma (see [T]). And the proof is completed in this case.

The second case $x^0 \in \partial D_2$ is treated similarly, with reversed argument. We sketch some details. So we may start as we did in the previous case, and assuming now

$$|\nabla u(x^j)| < 1/j, \qquad \text{and} \qquad |\nabla u(y^j)| \ge C_0 > 0,$$

with y^j as before

$$y^j = d_j \tilde{y}^j + x^j \;\in\; B(x^j, d_j), \qquad \left(\tilde{y}^j \in B_1\right).$$

Again all the above arguments are in order and we have the limit function u_0 and the limit point $\tilde{y}$ in B_1. Let us see what more information we can deduce. Indeed, on the one side, by elliptic estimates,

$$C_1 \geq |\nabla u_j(\tilde{y}^j)| = \frac{d_j |\nabla u(y^j)|}{u(x^j)} \geq \frac{d_j C_0}{u(x^j)},$$

and on the other side

$$|\nabla u_j(0)| = \frac{d_j |\nabla u(x^j)|}{u(x^j)} \leq \frac{d_j}{j u(x^j)}.$$

Upon combining these estimates, we arrive at

$$|\nabla u_j(0)| \leq \frac{C_0 C_1}{j}.$$

As j tends to infinity we'll have $|\nabla u_0(0)| = 0$. And again the Hopf's principle is violated.

The lower and upper semi-continuity properties follow in the same way as in the proof of Theorem 1.3 in [HS4]. ◻

Lemma 3.8 *Let u be a solution to $\Delta_p u = 0$ in a domain Ω, and introduce the linear elliptic operator L_u defined everywhere, except at critical points of u, by*

$$L_u := |\nabla u|^{p-2}\Delta + (p-2)|\nabla u|^{p-4} \sum_{k,l=1}^{N} \frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_l}\frac{\partial^2}{\partial x_k \partial x_l}.$$

Then $L_u(|\nabla u|^p) \geq 0$ in Ω.

This lemma is essentially proved, though stated differently, in the papers of Payne and Philippin, [PP1] and [PP2], see also the discussion in [HS2].

For two nested convex sets $D_1 \subset D_2$, and for $x \in \partial D_1$ we denote by $T_{x,a}$ the supporting hyperplane at x with the normal a pointing away from D_1. Obviously, $T_{x,a}$ is not necessarily unique, depending on the geometry of ∂D_1. Now for each $x \in \partial D_1$ there corresponds a point y_x (not necessarily unique) on $\partial D_2 \cap \{z : \ a\cdot(z-x) > 0\}$ and such that $a\cdot(y_x - x) = \max a\cdot(z-x)$, where the maximum has been taken over all $z \in \partial D_2 \cap \{z : \ a\cdot(z-x) > 0\}$.

Lemma 3.9 *Let D_1 and D_2 be two nested convex domains ($\overline{D_1} \subset D_2$) and denote by u the p-capacitary potential of $D_2 \setminus D_1$, i.e. the solution of*

$$\begin{cases} \Delta_p u = 0 & \text{in} \quad D_2 \setminus \overline{D_1} \\ u = c_1 & \text{on} \quad \partial D_1 \\ u = c_2 & \text{on} \quad \partial D_2 \end{cases} \tag{18}$$

where c_1 and c_2 are two given constants $c_1 > c_2 \geq 0$. Then

$$\limsup_{\substack{z\to x \\ z \in D_2\setminus\overline{D_1}}} |\nabla u(z)| \geq \limsup_{\substack{z\to y_x \\ z \in D_2\setminus\overline{D_1}}} |\nabla u(z)| \qquad \forall\ x \in \partial D_1, \tag{19}$$

where y_x is the point indicated in the discussion preceding this lemma.

For a proof of this lemma see [HS2], [HS3].

Definition 3.10 *(Extremal points) For a domain $D \in \mathbb{R}^N$ we say a point $x \in \partial D$ is an extremal point if there exists a supporting plane to D touching ∂D at x only. We denote the set of all extremal points of D by E_D.*

Lemma 3.11 *Retain the hypothesis in Lemma 3.9 and suppose also that ∂D_1 and ∂D_2 are C^1. Then*

$$|\nabla u(x)| \geq \inf_{y \in E_{D_2}} |\nabla u(y)|, \quad \text{for all} \quad x \in D_2 \setminus \overline{D_1}.$$

This lemma is a consequence of Lemma 3.9 and geometric considerations.

The next lemma was an important tool in the variational existence treatment of the multi-layer problem by P. Laurence and E Stredulinsky [LS]:

Lemma 3.12 *(See [LS], Lemma 4.1 and [A4], Thm. 1.) Retain the hypothesis in Lemma 3.9. Suppose moreover ∂D_i $(i = 1, 2)$ contains a line segment l_i, and that $|\nabla u| \geq c_0 > 0$. Then $|\nabla u|^{-1}$ is convex on l_2 and it is concave on l_1.*

4 The two-layer problem

4.1 Main result

Let us consider two bounded convex domains K_1 and K_3 in $\mathbb{R}^N$, such that K_3 strictly contains K_1 (i.e. $K_1 \subset\subset K_3$). We look for a convex domain K_2, such that

$$K_1 \subset\subset K_2 \subset\subset K_3$$

and the p-capacitary potentials u_1 and u_2 of the sets $K_2 \setminus \overline{K_1}$ and $K_3 \setminus \overline{K_2}$ respectively, i.e. solutions of

$$\left\{ \begin{array}{lll} \Delta_p u_1 = 0 & \text{in} & K_2 \setminus \overline{K_1} \\ u_1 = 1 & \text{on} & \partial K_1 \\ u_1 = 0 & \text{on} & \partial K_2 \end{array} \right. , \qquad \left\{ \begin{array}{lll} \Delta_p u_2 = 0 & \text{in} & K_3 \setminus \overline{K_2} \\ u_2 = -1 & \text{on} & \partial K_3 \\ u_2 = 0 & \text{on} & \partial K_2 \end{array} \right. \tag{20}$$

satisfy a nonlinear joining condition like

$$|\nabla u_1(x)| = g(x, |\nabla u_2(x)|) \quad \text{on } \partial K_2. \tag{21}$$

We have the following result.

Theorem 4.1 *(two phases) Let K_1, K_3 be two convex domains, such that K_3 strictly contains K_1, and $g \in \mathcal{G}$. Then there exists a convex C^1 domain ω, $K_1 \subset\subset \omega \subset\subset K_3$, which is a classical solution of the two-layer free boundary problem. The latter means that the p-capacitary potentials u_1 and u_2 of the sets $\omega \setminus \overline{K_1}$ and $K_3 \setminus \overline{\omega}$ respectively (i.e. solutions of (20) with $K_2 = \omega$) satisfy*

$$\lim_{\substack{z \to x \\ z \in \omega \setminus \overline{K_1}}} |\nabla u_1(z)| = \lim_{\substack{y \to x \\ y \in K_3 \setminus \overline{\omega}}} g(y, |\nabla u_2(y)|) \qquad \forall \ x \ \in \partial\omega \, . \tag{22}$$

4.2 Notations, definitions

4.2.1 p-capacitary potentials

For every subdomain ω such that $K_1 \subset\subset \omega \subset\subset K_3$, we set $\omega_1 = \omega \setminus \overline{K_1}$ and $\omega_2 = K_3 \setminus \overline{\omega}$. We introduce the p-capacitary potentials u_1^ω (respectively u_2^ω) or more simply u_1 (respectively u_2) when there is no possible confusion, the solutions of the boundary value problems

$$\left\{ \begin{array}{lll} \Delta_p u_1 = 0 & \text{in} & \omega_1 \\ u_1 = 1 & \text{on} & \partial K_1 \\ u_1 = 0 & \text{on} & \partial\omega \end{array} \right. \qquad \left\{ \begin{array}{lll} \Delta_p u_2 = 0 & \text{in} & \omega_2 \\ u_2 = -1 & \text{on} & \partial K_3 \\ u_2 = 0 & \text{on} & \partial\omega \end{array} \right. \tag{23}$$

In the sequel, we will refer to u_1 as the inner potential and to u_2 as the outer potential of the set ω. We want to find a domain Ω satisfying a joining condition written

$$|\nabla u_1(x)| = g(x, |\nabla u_2(x)|)$$

as explained in the previous subsection. For that purpose, we introduce the following classes of domains:

4.2.2 Subsolutions, supersolutions

An open set ω (such that $K_1 \subset\subset \omega \subset\subset K_3$) is called a subsolution (of the problem) if its p-capacitary potentials u_1 and u_2 satisfy:

$$\liminf_{\substack{z\to x\\ z\in\omega_1}} \ |\nabla u_1(z)| \geq \limsup_{\substack{y\to x\\ y\in\omega_2}} \ g(y, |\nabla u_2(y)|) \qquad \forall\ x\ \in \partial\omega\ . \tag{24}$$

An open set ω (such that $K_1 \subset\subset \omega \subset\subset K_3$) is called a supersolution (of the problem) if its p-capacitary potentials u_1 and u_2 satisfy:

$$\limsup_{\substack{z\to x\\ z\in\omega_1}} \ |\nabla u_1(z)| \leq \liminf_{\substack{y\to x\\ y\in\omega_2}} \ g(y, |\nabla u_2(y)|) \qquad \forall\ x\ \in \partial\omega\ . \tag{25}$$

4.2.3 The classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$

We are going to work only with convex domains, so let us denote by

$$\mathcal{C} = \{\omega \text{ convex bounded open subset of } \mathbb{R}^N,\ K_1 \subset\subset \omega \subset\subset\ K_3\}.$$

Then, we will denote by $\mathcal{A}$ the class of convex subsolutions and $\mathcal{B}$ the class of convex supersolutions:

$$\mathcal{A} = \{\omega \in \mathcal{C} : \liminf_{\substack{z\to x\\ z\in\omega_1}} \ |\nabla u_1(z)| \geq \limsup_{\substack{y\to x\\ y\in\omega_2}} \ g(y, |\nabla u_2(y)|) \qquad \forall\ x\ \in \partial\omega\ \}.$$

$$\mathcal{B} = \{\omega \in \mathcal{C} : \limsup_{\substack{z\to x\\ z\in\omega_1}} \ |\nabla u_1(z)| \leq \liminf_{\substack{y\to x\\ y\in\omega_2}} \ g(y, |\nabla u_2(y)|) \qquad \forall\ x\ \in \partial\omega\ \}.$$

A **classical solution** of the two-phase free-boundary problem is obviously a domain $\Omega \in\ \mathcal{A} \cap \mathcal{B}$.

4.3 Stability results for the class $\mathcal{B}$

First we show that the class $\mathcal{B}$ is closed under intersection.

Lemma 4.2 *Let* ω^1, ω^2 *be in* $\mathcal{B}$. *Then* $\omega^1 \cap \omega^2 \in \mathcal{B}$.

Proof: As the intersection of two convex domains is convex, we need to prove the condition on the gradients for $u_1 := u_1^{\omega^1\cap\omega^2}$ and $u_2 := u_2^{\omega^1\cap\omega^2}$ at the boundary of $\omega^1 \cap \omega^2$. By comparison principle, $0 \leq u_1 \leq \min(u_1^{\omega^1}, u_1^{\omega^2})$, which implies that for $x \in \partial(\omega^1 \cap \omega^2) \subset \partial\omega^1 \cup \partial\omega^2$ we have (for example, we choose the case where $x \in \partial\omega^1$):

$$u_1(x) = u_1^{\omega^1}(x) = 0 \text{ and } \limsup_{\substack{y\to x\\ y\in(\omega^1\cap\omega^2)_1}} \ |\ \nabla u_1(y)\ | \leq \limsup_{\substack{y\to x\\ y\in\omega^1_1}} \ |\ \nabla u_1^{\omega^1}(y)\ |$$

while, since $u_2 \leq \min(u_2^{\omega^1}, u_2^{\omega^2}) \leq 0$, we have

$$u_2(x) = u_2^{\omega^1}(x) = 0 \text{ and } \liminf_{\substack{y\to x\\ y\in(\omega^1\cap\omega^2)_2}} \ |\ \nabla u_2(y)\ | \geq \liminf_{\substack{y\to x\\ y\in\omega^1_2}} \ |\ \nabla u_2^{\omega^1}(y)\ |\ .$$

Now, by monotonicity of g with respect to its second argument, and the fact that ω^1 belongs to $\mathcal{B}$:

$$\limsup_{\substack{y\to x\\ y\in(\omega^1\cap\omega^2)_1}} \ |\ \nabla u_1(y)\ | \leq \limsup_{\substack{y\to x\\ y\in\omega^1_1}} \ |\ \nabla u_1^{\omega^1}(y)\ | \leq$$

$$\leq \liminf_{\substack{y\to x\\ y\in\omega^1_2}} \ g(y, |\nabla u_2(y)|) \leq \liminf_{\substack{y\to x\\ y\in(\omega^1\cap\omega^2)_2}} \ g(y, |\nabla u_2(y)|)$$

□

Now, the technical and more difficult point is to prove that $\mathcal{B}$ is stable, in some sense, for decreasing sequences of convex domains. Indeed, our aim is to construct a solution to the free boundary problem, by taking a minimal element (for inclusion) in the class $\mathcal{B}$. So, we need some stability of $\mathcal{B}$ by the constructing process that we are going to use.

Theorem 4.3 *Let $\omega^1 \supset \omega^2 \supset \cdots$, be a decreasing sequence of convex domains in $\mathcal{B}$, and suppose $\omega = \overset{\circ}{\overline{\cap \omega^n}}$ (the interior of the closure) belongs to $\mathcal{C}$. Then $\omega \in \mathcal{B}$.*

Proof: Since the domains involved are convex and they all contain K_1, they are uniformly Lipschitz. In particular, by standard up to boundary regularity (see [K]), the p-capacitary potentials u_1^n, u_2^n are C^α (α depending on the uniform cone property of ω^n) in the entire space $\mathbb{R}^N$ (after appropriate extension). Since also u_1^n, u_2^n are decreasing sequences we have a limit functions u_1, u_2 which are the p-capacitary potentials of $\omega_1 = \omega \setminus K_1$, and $\omega_2 = K_3 \setminus \overline{\omega}$, respectively. Moreover, by local $C^{1,\alpha}$ regularity (see [Le]), convergence takes place also for the gradients on every compact subset of ω_1 and ω_2 respectively.

We need to show $\omega \in \mathcal{B}$. Let $\epsilon > 0$ be small enough and fix $x^0 \in K_1$. Now for each $y \in \partial\omega$ let us denote by $R(x^0, y)$ the ray emanating from x^0 and traveling through y. Then, by the choice of x^0, and the convexity of the sets ω, ω^n we can choose unique points $x = x(y,n) \in \partial\omega^n \cap R(x^0, y)$ and $x^\epsilon \in \{u_2^n = -\delta_\epsilon\} \cap R(x^0, y)$, where $\delta_\epsilon > 0$ is to be chosen later. It follows that

$$\lim_{n\to\infty} x(y,n) = y, \qquad \lim_{\epsilon\to 0} x^\epsilon = x(y,n) \qquad \text{non-tangentially.}$$

Next denote by v_n the solution of the following boundary-value problem:

$$\begin{cases} L_{u_1^n}(v_n) = 0 & \text{in } \{0 < u_1^n < \frac{1}{2}\} \\ v_n(x) = |\nabla u_1^n(x)|^p & \text{on } \{u_1^n = \frac{1}{2}\} \\ v_n(x) = G_{n,\epsilon}(x) & \text{on } \partial\omega^n \end{cases},$$

where

$$G_{n,\epsilon}(x) = \min\left(2M^p, g^p(x, |\nabla u_2^n(x^\epsilon)| + \epsilon)\right), \qquad \text{with } M = \sup_{\{0<u_1^n<\frac{1}{2}\}} |\nabla u_1^n|,$$

and $L_{u_1^n}$ is defined in Lemma 3.8. Observe that the boundedness of M follows by simple (linear) barrier argument.

Fix a point $y \in \partial\omega$. Then two possibilities may arise (see Lemma 3.7)
Case 1) $|\nabla u_2(y)| = \infty$ non-tangentially,
Case 2) $|\nabla u_2(y)| = M_1$ non-tangentially.
In both cases we'll have

$$|\nabla u_2^n(x^\epsilon)| \approx |\nabla u_2^n(x)|.$$

In *Case 1* we obtain

$$g^p(x, |\nabla u_2^n(x^\epsilon)| + \epsilon) > 2M^p \geq |\nabla u_1^n(x)|^p,$$

i.e., $G_{n,\epsilon}(x) \geq |\nabla u_1^n(x)|^p$.

In *Case 2* we have (by non-tangential continuity of $|\nabla u_2^n|$)

$$|\nabla u_2^n(x)| \leq |\nabla u_2^n(x^\epsilon)| + \epsilon,$$

provided δ_ϵ is small enough. And by non-decreasing property of g we have

$$g^p(x, |\nabla u_2^n(x^\epsilon)| + \epsilon) \geq g^p(x, |\nabla u_2^n(x)|) \geq |\nabla u_1^n(x)|^p,$$

Hence

$$G_{n,\epsilon}(x) \geq |\nabla u_1^n(x)|^p.$$

Therefore upon applying the comparison principle (for the operator $L_{u_1^n}$; see Lemma 3.8) we can obtain

$$v_n(x) \geq |\nabla u_1^n(x)|^p \quad \text{in} \quad \{0 < u_1^n < 1/2\}.$$

Now as $n \to \infty$,

$$v(x) := \lim_n v_n(x) \geq |\nabla u_1(x)|^p \quad \text{in} \quad \{0 < u_1 < 1/2\}.$$

Since x^ϵ is compactly inside ω_2 and $\nabla u_2^n(x^\epsilon) \to \nabla u_2(x^\epsilon)$ in C^α-norm (see [Le]) we have a uniform convergence for

$$v_n|_{\partial\omega_1^n} = G_{n,\epsilon}(x) = \min\left(2M^p, g^p(x, |\nabla u_2^n(x^\epsilon)| + \epsilon)|\right),$$

to

$$G_\epsilon(x) = \min\left(2M^p, g^p(x, |\nabla u_2(x^\epsilon)| + \epsilon)|\right).$$

Therefore for $z \in B(x, r_\epsilon) \cap \omega_1$ and $x \in \partial\omega$

$$|\nabla u_1|^p(z) \leq v(z) \leq G_\epsilon(x) + \epsilon \leq g^p(x, |\nabla u_2(x^\epsilon)| + \epsilon)| + \epsilon,$$

provided r_ϵ is small enough. By Lemma 3.7, and continuity of g (as $\epsilon \to 0$) we get

$$\limsup_{\substack{z \to x \\ z \in \omega_1}} |\nabla u_1(z)| \leq \liminf_{\substack{y \to x \\ y \in \omega_2}} g(y, |\nabla u_2(y)|).$$

Hence $\omega \in \mathcal{B}$. $\square$

4.4 Proof of Theorem 4.1

1st step *Existence of subsolutions and supersolutions.*
Let us consider the solution u of the boundary value problem (p-capacitary potential)

$$\begin{cases} \Delta_p u = 0 & \text{in} \quad K_3 \setminus \overline{K_1} \\ u = 1 & \text{on} \quad K_1 \\ u = -1 & \text{on} \quad \overline{\mathbb{R}^N \setminus K_3} \end{cases} \tag{26}$$

For any $-1 < \alpha < 1$, let $\omega_\alpha = \{u(x) > \alpha\}$. Also define $u_{1,\alpha}(x) = ((u(x) - \alpha)/(1 - \alpha))$ in the closure of $\omega_{1,\alpha} = \{\alpha < u(x) < 1\}$ and $u_{2,\alpha}(x) = ((u(x) - \alpha)/(1 + \alpha))$ in the closure of $\omega_{2,\alpha} = \{-1 < u(x) < \alpha\}$. Then ω_α is a supersolution (resp. subsolution) if

$$\frac{|\nabla u(x)|}{1 - \alpha} = |\nabla u_{1,\alpha}(x)| < (>) g(x, |\nabla u_{2,\alpha}(x)|) = g(x, \frac{|\nabla u(x)|}{1 + \alpha})$$

for all $x \in \partial\omega_\alpha$. But a comparison argument involving the p-capacitary potential in any slab between parallel boundary planes tangent to $\partial\omega_\alpha$ and ∂K_3 shows that $|\nabla u(x)| \geq ((\alpha + 1)/M) \geq ((1 + \alpha)/R)$ for all $x \in \partial\omega_\alpha$, where $M = \sup\{\text{dist}(x, \omega_\alpha) : x \in \partial K_3\}$. Therefore, ω_α is a supersolution (resp. subsolution) provided that

$$\frac{1 + \alpha}{1 - \alpha} < (>) \frac{g(x, y)}{y}$$

for all $x \in \partial\omega_\alpha$ and all $y \geq y_0 = (1/R)$. Applying Assumption (A4), we see that ω_α is a supersolution if $((1 + \alpha)/(1 - \alpha)) \leq C_1$ (true for α sufficiently close to -1), and that ω_α is a subsolution if $((1 + \alpha)/(1 - \alpha)) \geq C_2$ (true for α sufficiently close to 1).

We remark that K_1 and K_3 are regular, so that $|\nabla u(x)|$ is both uniformly bounded and uniformly positive in $K_3 \setminus \overline{K_1}$, then the above argument obtains supersolutions and subsolutions without involking Assumption (A4) (one can replace it by the much weaker assumption that $g(x, y) \to \infty$ as $y \to \infty$ uniformly over $x \in K$).

In the sequel Ω_0 will denote a given subsolution and Ω_1 a given supersolution.

2nd step *Construction of a minimal element in the class $\mathcal{B}$.*
We introduce the class $\mathcal{S} := \{\omega \in \mathcal{B}, \quad \text{with} \quad \Omega_0 \subset \omega \subset \Omega_1\}$. Let I be the intersection of all domains in the class $\mathcal{S}$ and set $\Omega = \overset{\circ}{\overline{I}}$ (the interior of the closure, which is still convex). To prove $\Omega \in \mathcal{B}$, we select a sequence of domains $\{\omega_n\}_{n=1}^{\infty}$ in $\mathcal{S}$ such that $\bigcap_{n\geq 1} \omega_n = I$ and we consider the sequence of domains $\{\Omega_n\}_{n=1}^{\infty}$ defined by $\Omega_1 = \omega_1$ and $\Omega_{n+1} = \Omega_n \cap \omega_{n+1}$ $(n \geq 1)$. By Lemma 4.2 each Ω_n is convex and belongs to $\mathcal{B}$. Hence, since $\Omega_{i+1} \subset \Omega_i$, Theorem 4.3 gives the desired result.

3rd step *On E_Ω, extremal points of Ω, we have* $\limsup |\nabla u_1(z)| = \liminf g(y, |\nabla u_2(y)|)$.

This property can be proved in the same way as in [HS2], but since it is slightly more complicated and for sake of completeness, we give here the complete proof. Suppose the property fails. Then, there exists $X_0 \in E_\Omega$ such that

$$\limsup_{\substack{z\to X_0\\ z\in\Omega_1}} |\nabla u_1(z)| = \liminf_{\substack{y\to X_0\\ y\in\Omega_2}} g(y, |\nabla u_2(y)|)(1-4\alpha), \quad \text{with } \alpha > 0. \tag{27}$$

We denote by

$$l_1 = \limsup_{\substack{z\to X_0\\ z\in\Omega_1}} |\nabla u_1(z)|$$

and

$$l_2 = \liminf_{\substack{y\to X_0\\ y\in\Omega_2}} |\nabla u_2(y)|.$$

Note that since g is continuous and non decreasing, $\liminf_{y\to X_0} g(y, |\nabla u_2(y)|) = g(X_0, l_2)$. Therefore assumption (27) can be written as

$$l_1 = g(X_0, l_2)(1-4\alpha). \tag{28}$$

We assume first $l_2 < +\infty$. Hence for some small neighborhood $\mathcal{V}$ of X_0 there holds

$$|\nabla u_1(z)| \leq l_1(1+\alpha) \qquad \forall z \in \mathcal{V} \cap \Omega_1 \tag{29}$$

and

$$|\nabla u_2(y)| \geq l_2(1-\alpha) \qquad \forall y \in \mathcal{V} \cap \Omega_2. \tag{30}$$

Let us fix a hyperplane T_d, parallel to a supporting plane at X_0, with $dist(X_0, T_d) = d$ and such that $T_d \cap \Omega \subset \mathcal{V}$. This is possible due to the extremal property of X_0.

By rotation and translation, we assume X_0 is the origin and $T_d = \{x_1 = d\}$. Let now $T_\delta = \{x_1 = \delta\}$ and set $\Omega^\delta = \Omega \setminus \{x_1 \leq \delta\}$. Then by comparison principle the (inner) p-capacitary potential u_1^δ of Ω_1^δ satisfies

$$0 \leq u_1^\delta \leq u_1 \qquad \text{in } \Omega_1^\delta, \tag{31}$$

while the (outer) potential satisfies

$$u_2^\delta \leq u_2 \leq 0 \qquad \text{in } \Omega_2, \tag{32}$$

which implies that on points x belonging to $\partial\Omega \cap \partial\Omega_\delta$:

$$\limsup_{\substack{y\to x\\ y\in\Omega_1^\delta}} |\nabla u_1^\delta(y)| \leq \limsup_{\substack{y\to x\\ y\in\Omega_1}} |\nabla u_1(y)| \leq \liminf_{\substack{y\to x\\ y\in\Omega_2}} g(y, |\nabla u_2(y)|) \leq \liminf_{\substack{y\to x\\ y\in\Omega_2^\delta}} g(y, |\nabla u_2^\delta(y)|).$$

Now by (29) and (31)

$$\max_{T_d} u_1^\delta \leq \max_{T_d} u_1 \leq d \sup_{\{0\leq\ x_1\leq d\}} |\nabla u_1| \leq\ l_1(1+\alpha)d. \tag{33}$$

Define

$$v := u_1^\delta + \frac{l_1(1+\alpha)d}{d-\delta}\,(d-x_1).$$

Since the second derivatives of v and u_1^δ coincide, we have

$$L_{u_1^\delta} v = L_{u_1^\delta} u_1^\delta = 0 \qquad \text{in } \Omega_1^\delta \cap \{x_1 < d\}.$$

Therefore in $\Omega_1^\delta \cap \{\ x_1 < d\}$, v takes its maximum on the boundary. By inspection and (33), it is easy to see that on $\partial(\Omega_\delta \cap \{\ x_1 < d\}) \subset T_d \cup T_\delta \cup (\partial\Omega \cap \{\delta < x_1 < d\})$,

$$v \le l_1(1+\alpha)d,$$

with equality on T_δ. Hence $\dfrac{\partial v}{\partial x_1} \le 0$ on T_δ, i.e.,

$$|\nabla u_1^\delta| \le \frac{l_1(1+\alpha)d}{d-\delta} \qquad \text{on } T_\delta. \tag{34}$$

Now, it remains to estimate $|\nabla u_2^\delta|$ on T_δ. For that purpose, let us introduce a part of a level set of u_2^δ contained in the neighborhood $\mathcal{V} \cap \Omega_2$ and consider, on that level set, one point, say x_δ where the supporting hyperplane is parallel to T_δ. By lemma 3.9, we have

$$\forall y \in T_\delta \quad |\nabla u_2^\delta(y)| \ge |\nabla u_2^\delta(x_\delta)|. \tag{35}$$

Now, by continuity of g, we can choose ε and δ small enough such that

$$\forall y \in T_\delta \quad g(X_0, l_2) \le g(y, l_2(1-\varepsilon))(1+\alpha)\,. \tag{36}$$

Now, by uniform convergence of $|\nabla u_2^\delta|$ to $|\nabla u_2|$ when $\delta \to 0$ on the level set, we can choose δ small enough such that

$$|\nabla u_2^\delta(y)| \ge |\nabla u_2^\delta(x_\delta)| \ge l_2(1-\varepsilon).$$

Replacing in (36) and using(28), (34) and the monotonicity of g yields

$$\forall y \in T_\delta \quad |\nabla u_1^\delta(y)| \le \frac{g(y, |\nabla u_2^\delta(y)|)(1+\alpha)^2(1-4\alpha)d}{d-\delta}\,.$$

Now, it suffices to choose δ even smaller so that $\dfrac{(1+\alpha)^2(1-4\alpha)d}{d-\delta} \le 1$, which in turn implies $\Omega_\delta \in \mathcal{B}$. Since $\Omega_\delta \subset \Omega$ we have reached a contradiction.

Now, if $l_2 = +\infty$, we can choose the neighborhood $\mathcal{V}$ in such a way that

$$|\nabla u_2(y)| \ge 2M \qquad \forall y \in \mathcal{V} \cap \Omega_2$$

where $M = \sup_{x\in\Omega_1} |\nabla u_1(x)|$. Then, we reach a contradiction exactly in the same way, by choosing δ small enough such that Ω_δ will be in the class $\mathcal{B}$.

4th step *The boundary of* Ω *is* C^1.
It suffices to show that at each boundary point there exists a unique tangent plane. Suppose the latter fails. Let $x^0 \in \partial\Omega$, with two supporting planes Π_1, Π_2 at x^0. Then by barrier arguments (Lemma 3.1–3.2)

$$\lim_{\Omega_1 \ni y \to \partial\Omega\cap\Pi_1\cap\Pi_2.} |\nabla u_1(y)| = 0 \quad \text{and} \quad \lim_{\Omega_2 \ni z \to \partial\Omega\cap\Pi_1\cap\Pi_2.} |\nabla u_2(z)| = +\infty.$$

Let Π_3 be a third plane supporting $\partial\Omega$ at x^0 and such that $\Pi_3 \cap \partial\Omega \subset \Pi_1 \cap \Pi_2$, i.e., Π_3 does not touch any other boundary points of Ω than those on the intersection of the planes Π_1 and Π_2. Now, move Π_3 towards the interior of Ω such that it cuts off Ω a small cap; it may well be a tub-like region. Then a similar argument as in the previous step will prove that this new domain is still in the class $\mathcal{B}$. This contradicts the minimal property of Ω.

5th step *The nonlinear joining condition holds on E_Ω.*

Let $x \in E_\Omega$ be fixed. On one hand, we have the following chain of (in)-equalities (here *n.t.* means non-tangentially, see Lemma 3.7 for details):

$$\begin{array}{l} |\nabla u_1(x)| := \lim_{\substack{z\to x \ n.t. \\ z\in\Omega_1}} |\nabla u_1(z)| \le \limsup_{\substack{z\to x \\ z\in\Omega_1}} |\nabla u_1(z)| \le \\ \liminf_{\substack{y\to x \\ y\in\Omega_2}} g(y, |\nabla u_2(y)|) \le \lim_{\substack{y\to x \ n.t. \\ y\in\Omega_2}} g(y, |\nabla u_2(y)|) := g(x, |\nabla u_2(x)|) \end{array} \tag{37}$$

where the first and last equalities are due to Lemma 3.7, the second and fourth inequalities come from the definition of a liminf and limsup (we also use the continuity of g) and the third inequality comes from the fact that Ω belongs to the class $\mathcal{B}$. On the other hand, we have the following chain of (in)-equalities:

$$|\nabla u_1(x)| \ge \lim \sup_{\substack{z\to x \\ z\in\Omega_1}} |\nabla u_1(z)| = \lim \inf_{\substack{y\to x \\ y\in\Omega_2}} g(y, |\nabla u_2(y)|) \ge g(x, |\nabla u_2(x)|) \tag{38}$$

where the first inequality is the upper semi-continuity of u_1 at x, the equality is step 3 and the second inequality is the lower semi-continuity of u_2 at x. Now, (37) and (38) together give the desired result.

6th step *The nonlinear joining condition holds at every boundary point.*
According to step 5, it remains to prove the equality $|\nabla u_1(x)| = g(x, |\nabla u_2(x)|)$ on maximal line segments in $I = [a, b] \subset \partial\Omega$. For any such line segment one readily verifies that $a, b \in \overline{E}_\Omega$. Also at the points a, b we have equation (38) verified. In view of assumption **(A3)** for the function g in conjunction with Lemma 3.12 we claim that the function

$$x \mapsto \frac{1}{|\nabla u_1(x)|} - \frac{1}{g(x, |\nabla u_2(x)|)}$$

is convex, non-negative. The latter depends on the fact that Ω belongs to the class $\mathcal{B}$ and it vanishes at the extremities of any segment (by step 5 and n.t.-continuity). Therefore, this function vanishes identically. This completes the proof.

5 Uniform separation estimate

Theorem 5.1 *(compare to [A2], Lemma 4.4) Let H denote the set of all configurations (K_1, ω, K_3) such that K_1, ω, K_3 are convex,*

$$B_\rho(0) \subset K_1 \subset\subset \omega \subset\subset K_3 \subset B_R(0),$$

and ω is a supersolution relative to K_1 and K_3. Then there exists a value $\eta > 0$ such that

$$\mathrm{dist}(\partial K_1, \partial\omega) \ge \eta \mathrm{dist}(\partial K_1, \partial K_3) \tag{39}$$

uniformly for all $(K_1, \omega, K_3) \in H$.

This result follows directly from lemmas 5.2 and 5.3, which follow.

Lemma 5.2 *For any $(K_1, \omega, K_3) \in H$, let*

$$\alpha = \max\{u(x) : x \in \partial\omega\} \in (-1, 1),$$

where u solves the Dirichlet problem (26). Then there exists a value $\alpha_0 \in (-1, 1)$ such that $\alpha \le \alpha_0$ uniformly over all $(K_1, \omega, K_3) \in H$.

Proof: It suffices to consider only configurations in H such that $\alpha \in (0, 1)$. Given such a configuration (and the corresponding value α),let u_1, u_2, ω_1, ω_2 be as defined in (23). Define the p-harmonic functions $u_{1,\alpha}(x) = ((u(x) - \alpha)/(1 - \alpha))$ and $u_{2,\alpha}(x) = ((u(x) - \alpha)/(1 + \alpha))$, both in the closure of the set $\Omega := K_3 \setminus \overline{K_1}$. Then $u_1 = u_{1,\alpha} = 1$ on ∂K_1 and $u_{1,\alpha} \leq 0 = u_1$ on $\partial\omega$. It follows by the comparison principle for p-harmonic functions that $u_{1,\alpha} \leq u_1$ in ω_1. Similarly, we have $u_2 = u_{2,\alpha} = -1$ on ∂K_3 and $u_{2,\alpha} \leq 0 = u_2$ on $\partial\omega$, from which it follows by the comparison principle that $u_{2,\alpha} \leq u_2$ in ω_2. We choose a point $x_0 \in \partial\omega$ such that $u(x_0) = \alpha$. Clearly the function u is regular near $x_0 \in \Omega$ (and therefore so are $u_{1,\alpha}$ and $u_{2,\alpha}$). For small $\delta > 0$, let $x_\delta = x_0 + \delta\nu_0 \in \omega_1$, where ν_0 denotes the unit vector with direction opposite $\nabla u(x_0)$. Also let $\gamma_\delta \subset \omega_1$ denote the directed line-segment of length δ joining x_0 to x_δ. Clearly

$$(\partial u_1(x)/\partial\nu_0) \leq |\nabla u_1(x)| \leq \sup_{x\in\gamma_\delta} |\nabla u_1(x)|$$

and

$$|(\partial u_{1,\alpha}(x)/\partial\nu_0) - |\nabla u_{1,\alpha}(x_0)|| \leq z(\delta),$$

both on γ_δ, where $z(\delta) \to 0$ as $\delta \to 0$. Therefore

$$\begin{aligned} 0 \leq u_1(x_\delta) - u_{1,\alpha}(x_\delta) &= \int_{\gamma_\delta} (\partial/\partial\nu_0)(u_1(x) - u_{1,\alpha}(x))ds \\ &\leq (\sup_{x\in\gamma_\delta} |\nabla u_1(x)| - |u_{1,\alpha}(x_0)| + z(\delta))\delta, \end{aligned}$$

from which it follows that

$$\limsup_{\omega_1\ni x\to x_0} |\nabla u_1(x)| \geq |\nabla u_{1,\alpha}(x_0)| = \frac{|\nabla u(x_0)|}{1-\alpha} \tag{40}$$

For small $\delta > 0$, Let γ_δ denote a directed arc of steepest ascent of u_2 of length δ, joining a point $x_\delta \in \omega_2$ to the point x_0. Since $\partial u_2(x)/\partial\nu = |\nabla u_2(x)|$ on γ_δ, where ν denotes the forward unit tangent vector to the arc, we have

$$\begin{aligned} 0 \geq u_{2,\alpha}(x_\delta) - u_2(x_\delta) &= \int_{\gamma_\delta} (\partial/\partial\nu)(u_2(x) - u_{2,\alpha}(x))ds \\ &\geq \int_{\gamma_\delta} (|\nabla u_2(x)| - |\nabla u_{2,\alpha}(x)|)ds, \end{aligned}$$

from which it follows that

$$\inf_{x\in\gamma_\delta} |\nabla u_2(x)| \leq |\nabla u_{2,\alpha}(x_0)| + z(\delta),$$

and therefore that

$$\liminf_{\omega_2\ni x\to x_0} |\nabla u_2(x)| \leq |\nabla u_{2,\alpha}(x_0)| = \frac{|\nabla u(x_0)|}{1+\alpha}. \tag{41}$$

In view of the definition of an outer solution (see), it follows from (40) and (41) that

$$(|\nabla u(x_0)|/(1-\alpha)) \leq g(x_0, (|\nabla u(x_0)|/(1+\alpha))). \tag{42}$$

A simple comparison argument involving the p-capacitary potential in a slab bounded by parallel planes, one tangent to the surface $\{u(x) = \alpha\}$ at x_0, the other tangent to ∂K_3, shows that $|\nabla u(x_0)| \geq ((\alpha+1)/M) \geq ((\alpha+1)/R)$, where $M = \sup_{x\in\partial K_3} \mathrm{dist}(x, \{u(x) = \alpha\})$. It follows from (42) and Assumption (A4) that

$$(1/(1-\alpha)) \leq ((1+\alpha)/(1-\alpha)) \leq (g(x_0, y)/y) \leq C_2, \tag{43}$$

where we set $y = (|\nabla u(x_0)|/(1+\alpha)) \geq y_0 = (1/R)$, and where C_2 depends only on R, y_0, and the function g. The assertion follows, since (43) cannot be satisfied unless $\alpha \leq \alpha_0 = (1 - (1/C_2))$.

Lemma 5.3 *In the context of Lemma 5.2, there is a constant $\eta > 0$ such that*

$$\mathrm{dist}(\partial K_1, \{u(x) = \alpha_0\}) \geq \eta\,\mathrm{dist}(\partial K_1, \partial K_3)$$

for any convex sets K_1 , K_3 such that $B_\rho(0) \subset K_1 \subset\subset K_3 \subset B_R(0)$.

Proof: For any $r \in (0,1]$ and unit vector ν, let $E(r,\nu) = \{x \in \mathbb{R}^N : \mathrm{dist}(x, D(r,\nu)) < r\}$, where $\lambda = (\rho^2/R)$ and $D(r,\nu)$ denotes the closure of the convex hull of the set $\{0\} \cup B_{\lambda r}(-r\rho\nu)$. Let $u_{r,\nu}(x)$ denote the p-harmonic function in the annular domain $\Omega(r,\nu) = E(r,\nu) \setminus D(r,\nu)$ whose continuous extension to the closure satisfies $u_{r,\nu}(\partial D(r,\nu)) = 1$, $u_{r,\nu}(\partial E(r,\nu)) = -1$. Then $\mathrm{dist}(0, \{u_{r,\nu}(x) = \alpha_0\}) = r\eta$, where $\eta = \mathrm{dist}(0, \{u_{1,\nu}(x) = \alpha_0\}) > 0$, since $u_{r,\nu}(x) = u_{1,\nu}(x/r)$. For $r = \min\{1, \mathrm{dist}(\partial K_1, \partial K_3)\}$ and any point $x_0 \in \partial K_1$, we have $x_0 + D(r, (x_0/|x_0|)) \subset \overline{K}_1$ and $x_0 + E(r, (x_0/|x_0|)) \subset K_3$. By the comparison principle, we have $u(x) \geq u_{r,\nu}(x - x_0)$ in $\Omega \cap (x_0 + \Omega_{r,\nu})$, where $\nu = (x_0/|x_0|)$. It follows that $B_{r\eta}(x_0) \subset K_1 \cup \{u(x) < \alpha_0\}$ for all $x_0 \in K_1$, from which the assertion follows.

6 The multi-layer case

Let us recall the problem. We are given two strictly nested convex domains $K_1 \subset K_{m+2}$, real numbers $-1 \leq \lambda_i \leq 1$, $(i = 1, 2, \cdots, m+1)$ with $\lambda_i > \lambda_{i+1}$, and continuous functions $g_i : (K_{m+2} \setminus K_1) \times \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 2, \cdots, m+1)$. We are looking for a sequence of nested convex domains

$$K_1 \subset\subset K_2 \subset\subset \cdots \subset\subset K_{m+1} \subset\subset K_{m+2}$$

such that the p-capacitary potentials $u_i(x)$ of the sets $K_{i+1} \setminus \overline{K_i}$, i.e. solutions of

$$\begin{cases} \Delta_p u_i = 0 & \text{in } K_{i+1} \setminus \overline{K_i} \\ u_i = \lambda_i & \text{on } \partial K_i \\ u_i = \lambda_{i+1} & \text{on } \partial K_{i+1}, \end{cases} \tag{44}$$

satisfy the following joining conditions:

$$|\nabla u_i(x)| = g_i(x, |\nabla u_{i+1}(x)|) \quad \text{on } \partial K_{i+1} \quad (i = 1, \cdots, m).$$

For simplicity we set $\lambda_1 = 1$ and $\lambda_{m+1} = -1$. The following is our main result in this paper.

Theorem 6.1 *(multi-layer) Let K_1, K_{m+2} be two bounded convex domains, such that K_{m+2} strictly contains K_1, $\lambda_i \in (-1,1), i = 2, \cdots, m+1$ are arbitrary real numbers with $\lambda_i > \lambda_{i+1}$, and $g_i \in \mathcal{G}, i = 1, \cdots, m$. Then there exists a sequence of convex C^1 domains $\{K_i : 1 < i < m+2\}$, such that*

$$K_1 \subset\subset K_2 \subset\subset \cdots \subset\subset K_{m+1} \subset\subset K_{m+2},$$

and which is a classical solution of the multilayer free boundary problem. The latter means that the p-capacitary potentials u_i of the sets $K_{i+1} \setminus \overline{K_i}, i = 1, \cdots, m$, i.e. solutions of (44) satisfy

$$\lim_{\substack{z \to x \\ z \in K_{i+1} \setminus \overline{K_i}}} |\nabla u_i(z)| = \lim_{\substack{y \to x \\ y \in K_{i+2} \setminus \overline{K_{i+1}}}} g(y, |\nabla u_{i+1}(y)|) \qquad \forall\ x\ \in \partial K_{i+1} \quad i = 1, \cdots, m. \tag{45}$$

Definitions: We let $\mathcal{B}$ denote the family of all ordered $m-1$-tuples $\omega := (\omega_2, \omega_3, \cdots, \omega_{m+1})$, such that

$$K_1 \subset\subset \omega_2 \subset\subset \cdots \subset\subset \omega_{m+1} \subset\subset K_{m+2},$$

every ω_i is a convex domain and each domain ω_i is a supersolution of the two-layer problem relative to ω_{i-1} and ω_{i+1}. The latter means that for every $i = 1, \cdots, m$, we have

$$\limsup_{\substack{z \to x \\ z \in \omega_{i+1} \setminus \overline{\omega_i}}} |\nabla u_i(z)| \leq \liminf_{\substack{y \to x \\ y \in \omega_{i+2} \setminus \overline{\omega_{i+1}}}} g_i(y, |\nabla u_{i+1}(y)|) \qquad \forall\ x\ \in \partial\omega_{i+1}, \tag{46}$$

where we take $\omega_1 := K_1, \omega_{m+2} := K_{m+2}$ and define u_i to be the solution or (44) with K_i replaced by ω_i for each $i = 2, \cdots, m+1$.

1st step: modified 2-layer existence result

In view of Theorem 5.1, the proof of Theorem 4.1 actually proves the following result for the 2-layer problem: (a)There exists a unique absolute minimizer among all outer solutions. (b) This absolute minimizer solves the 2-layer problem (in the same sense as in Theorem 4.1).

2nd step: $\mathcal{B}$ is not empty.

Under our assumptions, an outer $(m-1)$-surface outer solution can be easily obtained in the form: $\omega_i = \{u(x) > \alpha_i\}$, $i = 2, \cdots, m+1$, where u denotes the solution of (26) with K_3 replaced by K_{m+2}, and where the values α_i are appropriately chosen so that each ω_i is a supersolution relative to its neighbors ω_{i-1} and ω_{i+1} (same argument as in the first step in the proof of Theorem 4.1).

3rd step Uniform separation in $\mathcal{B}$

Let $\omega := (\omega_2, \cdots, \omega_{m+1}) \in \mathcal{B}$. Then for each $i = 2, \cdots, m+1$, ω_i is a supersolution of the 2-layer problem relative to ω_{i-1}, ω_{i+1}, and the function g_i. Thus, by Theorem 5.1, we have

$$\mathrm{dist}(\partial\omega_{i-1}, \partial\omega_i) \geq \eta \mathrm{dist}(\partial\omega_{i-1}, \partial\omega_{i+1}) \geq \eta \mathrm{dist}(\partial\omega_i, \partial\omega_{i+1})$$

for all $i = 2, \cdots, m+1$. It follows that

$$\mathrm{dist}(\partial\omega_{i-1}, \partial\omega_i) \geq \eta^{m+2-i}\ \mathrm{dist}(\partial\omega_{m+1}, \partial K_{m+2}) \geq \eta^m \mathrm{dist}(\partial\omega_{m+1}, \partial K_{m+2})$$

for all $i = 2, \cdots, m+1$. Thus, if ω^n, $n = 1, 2, \cdots$, is a weakly decreasing sequence of elements of $\mathcal{B}$ (so that the corresponding sequence of $(m-1)$-st components is also weakly decreasing and thus uniformly bounded away from ∂K_{m+2}), then there exists a value $\delta > 0$ such that for all $n = 1, 2, \cdots$, the surface components of ω^n are separated from each other (and from ∂K_1 and ∂K_{m+2} by a distance of at least δ. Therefore the componentwise intersection has the same property.

4th step pairwise intersection; minimal sequence in $\mathcal{B}$.

$\mathcal{B}$ is closed under the operation of componentwise intersection. In fact, given $\omega^1, \omega^2 \in \mathcal{B}$, let $\omega = \omega^1 \cap \omega^2$ be the componentwise intersection. Then $\partial\omega \subset \partial\omega^1 \cup \partial\omega^2$, and it is easy to see (using the standard comparison principle) that $u^\omega \leq u^{\omega^j}$, $j = 1, 2$, componentwise in the common domains of the component p-capacitary potentials. By repeated application of componentwise intersections, one defines a (componentwise) weakly decreasing minimal sequence of supersolutions $\omega^n = (\omega_2^n, \cdots, \omega_{m+1}^n)$, $n = 1, 2, \cdots$, where the latter means that for any $i = 2, \cdots, m-1$ and any $x \in \mathbb{R}^N$ such that $x \notin \omega_i$ for some supersolution $\omega \in \mathcal{B}$, we have $x \notin \omega_i^n$ for all sufficiently large n.

5th step minimal element in $\mathcal{B}$

For each fixed $i = 2, \cdots, m+1$, the sequence of domains ω_i^n, $n = 1, 2, 3, \cdots$, is weakly decreasing under set inclusion and therefore convergent to a domain $\Omega_i \supset K_1$ ($\Omega_i :=$ the interior of the infinite intersection of the domains ω_i^n, $n = 1, 2, 3, \cdots$). Clearly the domains Ω_i are strictly ordered by inclusion, and in fact by step 3, we have $\mathrm{dist}(\partial\Omega_i, \partial\Omega_{i+1}) \geq \delta$ for all $i = 1, \cdots, m+1$ (where we set $\Omega_1 := K_1$ and $\Omega_{m+2} := K_{m+2}$). Since $\Omega_i \subset \omega_i^n$ for all i, each ω_i^n, $i = 2, \cdots, m+1$, is actually a supersolution of the 2-layer problem relative to Ω_{i-1}, Ω_{i+1}, and g_i. Therefore, Ω_i (the interior of the infinite intersection of the ω_i^n) is also a supersolution of the same 2-layer problem, due to Theorem 4.3. Therefore $\Omega \in \mathcal{B}$. In fact Ω is, by construction, the minimal supersolution in $\mathcal{B}$.

6th step Ω solves the multi-layer problem

Since Ω is a minimal element in $\mathcal{B}$, each component Ω_i of Ω must be the minimal supersolution of the 2-layer problem relative Ω_{i-1}, Ω_{i+1}, and g_i. Therefore, by step 1, Ω_i is a solution of this 2-layer problem in the sense of Theorem 4.1. Thus Theorem 6.1 is proved.

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References

[A1] A. Acker, On the multi-layer fluid problem: regularity, uniqueness, convexity and successive approximation of solutions *Comm. Part. Diff. Eq.*, **16** (1991), 647-666.

[A2] A. Acker, On the existence of convex classical solutions for multi-layer free boundary problems with general nonlinear joining conditions. *Trans. Amer. Math. Soc.* **350** (1998), no. 8, 2981–3020.

[A3] A. Acker, On the non-convexity of solutions in free-boundary problems arising in plasma physics and fluid dynamics. *Comm. Pure Appl. Math.* **42** (1989), 1165-1174. (Addendum: *Comm. Pure Appl. Math.* **44** (1991), no. 7, 869-872.)

[A4] A. Acker, Qualitative properties of the boundary derivative of the capacity potential for special classes of annular domains. *Math. Meth. in the Appl. Sci.* **7** (1985), 251-259.

[A5] A. Acker, On 2-layer free boundary problems with generalized joining conditions. In: *Comparison Methods and Stability Theory* (edited by Xinzhi Liu and David Siegel). Lecture notes in pure and applied mathematics, Vol. 162, Marcel Dekker, Inc. 1994.

[AC] H.W. Alt and L.A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.*, **325** (1981), 105-144.

[Do] Dobrowolski M. On quasilinear elliptic equations in domains with conical boundary points. *J. Reine Angew. Math 394 (1989), 186–195.*

[HS1] A. Henrot and H. Shahgholian, Convexity of free boundaries with Bernoulli type boundary condition, Nonlinear Analysis T.M.A., **28**, n° 5, 1997, p. 815-823.

[HS2] A. Henrot and H. Shahgholian, Existence of classical solutions to a free boundary problem for the p-Laplace operator: (I) the exterior convex case. J. Reine Angew. Math. 521 (2000), 85–97.

[HS3] A. Henrot and H. Shahgholian, Existence of classical solutions to a free boundary problem for the p-Laplace operator: (II) the interior convex case. Indiana Univ. Math. J., v.49, no. 1 (2000), 311-323.

[HS4] A. Henrot and H. Shahgholian, The one-phase free boundary problem for the p-Laplacian with non-constant Benoulli boundary condition. Trans. Amer. Math. Soc. 354(2002), no. 6, 2399-2416.

[K] I.N. Krol, The solutions of the equation $D_{x_i}(Du^{p-2}D_{x_i}u) = 0$ with a singularity at a boundary point, Boundary value problems of mathematical physics, 8. Trudy Mat. Inst. Steklov. 125 (1973), p. 127-139.

[Le] J.L. Lewis, Capacitary functions in convex rings, Arch. Rat. Mech. Anal. 66(1977), pp. 201 - 224.

[LS] P. Laurence, E. Stredulinsky, Existence of regular solutions with convex levels for semilinear elliptic equations with nonmonotone L^1 nonlinearities. Part (I). Indiana Univ. Math. J. vol. 39 (4) (1990) pp. 1081–1114.

[PP1] L.E. Payne and G. Philippin, On the conformal capacity problem, *Geometry of solutions to Partial Differential Equations*, Symposia Mathematica, G. Talenti ed., Academic Press (1989), New York, p. 119-136.

[PP2] L.E. Payne and G. Philippin, On gradient maximum principles for quasilinear elliptic equations, *Nonlinear Analysis T.M.A.*, **23**, n° 3 (1994), p. 387-398.

[T] P. Tolksdorf On the Dirichlet problem for quasilinear equations in domains with conical boundary points. Comm. in Part. Diff. Equations, 8(7), 773-817 (1983).