$C^{1,1}$-regularity in semilinear elliptic problems

Henrik Shahgholian

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden

Abstract

In this paper we give an astonishing-simple proof of $C^{1,1}$−regularity in elliptic theory. Our technique yields both new simple proofs of old results as well as new optimal results.

The setting we'll consider is the following. Let u be a solution to

$$\Delta u = f(x, u), \qquad \text{in} \quad B,$$

where B is the unit ball in $\mathbf{R}^n$, $f(x,t)$ is a bounded Lipschitz function in x and f'_t is bounded from below. Then we prove that $u \in C^{1,1}(B_{1/2})$. Our method is a simple corollary to a recent monotonicity argument due to Caffarelli-Jerison-Kenig.

1 The problem

Let B_1 be the unit ball in $\mathbf{R}^n$ $(n \geq 2)$, and $f(x,t)$ a bounded Lipschitz function in x with $f'_t \geq -M$. Let moreover u be a solution to the stationary reaction-diffusion equation

$$\Delta u = f(x, u) \quad \text{in } B_1.$$

Then we are interested in the optimal regularity for the solution function u. In a recent work (see [CKS]) the author, L. Caffarelli, and L. Karp proved this result for the specific case $f(x,u) = \chi_\Omega$, where Ω is a domain characterized by $B_1 \setminus \Omega = \{u = |\nabla u| = 0\}$. This, in an informal way, can be rewritten as $f(x,u) = H(u) + H(-u)$, where H is the Heaviside function $H(t) = \chi_{(t>0)}$; see Example 1 below. Using a recent monotonicity argument due to Caffarelli-Jerison-Kenig, we present a simple proof of $C^{1,1}$-regularity of the solution u. We remark that solutions to the above equation are already in $W^{2,p} \cap C^{1,\alpha}(B_1)$ for $p < \infty$ and $\alpha < 1$ (since f is bounded), and that results concerning $\alpha = 1$ are rare.

To formulate our main result we need to introduce some definitions and conditions.

Conditions on $\mathbf{f(x,t)}$: The function $f(x,t)$ is assumed to satisfy

$$|f(x,t)| \leq M, \qquad |f(x,t) - f(y,t)| \leq M|x-y|, \qquad f'_t(x,t) \geq -M, \quad \text{(weakly)}$$

for all $x \in B_1$.

To appear in Comm. on Pure App. Mathematics,

DEFINITION 1.1 *(The class $P(M,n)$) We say a function $u \in W^{2,p}(B_1)$ $(p > n)$ belongs to the class $P := P(M,n)$ if u satisfies:*

- $\Delta u = f(x,u)$ *in* B_1, *(f as above)*
- $\|u\|_{W^{2,p}(B_1)} \le M$.

THEOREM 1.2 *(**Main**) There exists a universal constant $C = C(M,n)$ such that for $u \in P(M,n)$ we have*

$$|\nabla u(x) - \nabla u(y)| \le C|x-y| \qquad \forall\, x,y \in B_{1/2}.$$

The main tool in this paper will be the monotonicity lemma of Caffarelli-Jerison-Kenig, defined as follows. For given functions $h_1, h_2 \in W^{1,p}$ $(p > n)$ one sets

$$\varphi(r) := \varphi(r,h_1,h_2,x^0) := \frac{1}{r^4} I(r,h_1,x^0) I(r,h_2,x^0),$$

where

$$I(r,h,x^0) = \int_{B(x^0,r)} \frac{|\nabla h(x)|^2 dx}{|x-x^0|^{n-2}}.$$

Then one has the following result.

LEMMA 1.3 (CJK; Theorem 1.3) *Let h_1, h_2 be two non-negative continuous sub-solutions of $\Delta u = -C_1$ in $B(x^0,R)$ $(R>0)$. Assume further that $h_1 h_2 = 0$, $h_1(x^0) = h_2(x^0) = 0$. Then*

$$\varphi(r) \le C_2 \left(1 + I(R,h_1,x^0) + I(R,h_2,x^0)\right),$$

where $r < R$, $C_2 = C_2(n, C_1)$.

We refer to Lemma 1.3 as "monotonicity lemma".

2 Proof of the main result

To start with, we note that since $u \in W^{2,p}(B_1)$ with $1 < p < \infty$ we'll have that u is twice differentiable a.e. in the unit ball. Now fix any point $y \in B_{1/2}$ where u is twice differentiable. Define $v(x) := D_e u(x)$ where e is any vector orthogonal to $\nabla u(y)$. Our aim is to estimate $D_{je}u(y)$ for all e orthogonal to $\nabla u(y)$ and $j = 1, \cdots, n$. Now if $\nabla v(y) = 0$, then $D_{je}u(y) = 0$ for all $j = 1, \cdots, n$. Therefore it is more interesting to consider the nontrivial case $\nabla v(y) \neq 0$. Set now $\nu := \nabla v(y)$. Then one trivially sees (see [CK; page 435]) that

$$v(x) = \nu \cdot (x-y) + (x-y)o(1),$$

as $x \to y$. Hence on a fixed appropriate (truncated) cone K in the half space $\nu \cdot (x-y) > 0$ we have $\nu \cdot (x-y) \geq |\nu||x-y|/2$, and in the opposite cone $-K$ we have $\nu \cdot (x-y) \leq -|\nu||x-y|/2$. From here, we may infer Fatou's lemma, to deduce that

$$|\nabla v(y)|^4 \leq C_3 \lim_{r\to 0} \varphi(r, v^+, v^-, y), \qquad C_3 = C_3(n).$$

Finally to apply the monotonicity lemma to the functions $v^{\pm}$ we observe that these function satisfy the hypothesis in the monotonicity lemma. Indeed,

$$\Delta v^{\pm} = \Delta(D_e u)^{\pm} \geq -|D_e f(x,u)| + f_t'(x,u)(D_e u)^{\pm} \geq -C_4 = -C_4(M,n),$$

and the rest of the hypotheses follow readily. Hence applying the monotonicity lemma we arrive at

$$|\nabla v(y)|^4 \leq C_3 \lim_{r\to 0} \varphi(r, v^+, v^-, y) \leq C_3 C_2 (1 + I(1/2, v^+, y) + I(1/2, v^-, y)) \leq C_5,$$

where $C_5 = C_5(M,n)$. Observe that if $\nabla u(y) = 0$ then we may take e arbitrary and arrive at $|D_{ij}u(y)| \leq C_6 = C_6(M,n)$. So suppose $\nabla u(y) \neq 0$. Then we rotate the system of coordinates around the point y and assume $\nabla u(y)$ is parallel to e_1-direction and e is any of the remaining coordinates $e_2, \cdots, e_n$. The above estimate will then be reduced to

$$|D_{ij}u(y)| \leq C_6, \qquad i = 2, \cdots, n, \quad j = 1, \cdots, n.$$

But then using the equation $\Delta u = f$ we'll have

$$|D_{11}u(y)| \leq \sum_{i=2}^{n} |D_{ii}u(y)| + M \leq C.$$

Now to complete the proof we note that the second derivatives of u exist a.e. in $B_{1/2}$ and they are uniformly bounded. Hence a standard argument will close up the proof.

3 Examples

Next we give some examples of situations where the non-linearity f has shown to be nontrivial to handle. These examples are related to free boundary problems. However, even for standard elliptic problems with the right hand side f as described above, our result is new.

Example 1. As already mentioned in the beginning of the paper the case $f(x,u) = \chi_\Omega$ (where $B_1 \setminus \Omega = \{u = |\nabla u| = 0\}$) was treated in [CKS] with a slightly complicated technique in combination with the monotonicity formula. To see how our technique applies we compute

$$\Delta u_e^{\pm} \geq (\delta(u) - \delta(-u))\, u_e^{\pm},$$

which is zero in $\Omega = \{\Delta u = 1\}$. Also on the boundary $\partial\Omega$ we have $u_e = 0$.

Example 2. The second example concerns the two phase obstacle problem

$$f(x,u) = \lambda_+ H(u) - \lambda_-(1 - H(u)), \qquad \lambda_+ + \lambda_- \geq 0 \qquad \text{(see [U], [SUW]).}$$

For $\lambda_- = 0$ the above is reduced to the usual obstacle problem with zero obstacle.

Example 3. Our last example is related to problems in super-conductivity (see [CS], [CSS])

$$f(x,u) = g(x)H(|\nabla u|),$$

with g Lipschitz. Observe that even though the right hand side depends on ∇u, the equation satisfies the requirements in the proof. Indeed, one can see easily that

$$\Delta v^{\pm} = \Delta(D_e u)^{\pm} \geq -|D_e g(x)|H(|\nabla u|) + g(x)H'(|\nabla u|)(D_e u)^{\pm} \geq -C.$$

We would like to remark that the result can be improved to yield in any compact sub-domain Ω' of a given domain $\Omega \subset \mathbf{R}^n$. Obviously the constant in the estimate will then also depend on the distance between Ω' and $\mathbf{R}^n \setminus \Omega$. It seems also plausible that with some efforts one can extend the estimate up to the boundary of the given domain Ω with some regularity assumptions on the boundary $\partial\Omega$.

Finally we mention that the parabolic case (see [CPS]), which can be solved for specific type of problems such as that in Example 1, seems to introduce certain difficulties. We leave as an open question whether one can improve the above proof to adapt to the parabolic case.

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