Generalized mean value property for caloric functions

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Abstract

In this note we discuss a generalized mean value property

$$\int\int_{\Omega} f(x,t)h(x,t)\;dxdt \;=\int\int h(x,t)\mu(x,t)\;dxdt\;,$$

for given functions f and μ, and all integrable caloric functions h in Ω.

The particular case of $f \equiv 1$ and μ being the Dirac delta function supported in $\overline{\Omega}$ would give rise to the corresponding mean value property for harmonic functions, provided the above integral identity holds for some $\Omega \subset \mathbf{R}^{n+1}$ and caloric functions h in Ω.

1 Introduction

1.1 Problem setting

The mean-value property for harmonic functions

$$\int_{B_1(0)} h(x)\;dx \;= Ch(0),$$

is a well-known fact. A similar phenomenon for caloric functions

$$\int\int_{\Omega} h(x,t)\;dxdt \;= Ch(0,0),$$

is as well as unknown, i.e., it is unknown whether one can find $\Omega \subset \mathbf{R}^{n+1}$ such that the above integral identity holds for all functions $h \in L^1(\Omega)$ such that $\Delta h - D_t h = 0$ in Ω.

Although, with some efforts, using variational inequalities, one may hope to prove[a] the existence of a bounded domain $\Omega \subset \mathbf{R}^{n+1}$ with the above mean-value property for caloric functions, it seems that this type of question is untouched in the literatures.

In this note we will discuss different aspects of a generalized form of this problem. Namely for non-negative functions f, and μ, where the latter is also allowed to be a measure, we want to study the problem of finding a bounded domain $\Omega \subset \mathbf{R}^{n+1}$ such that $\mathrm{supp}(\mu) \subset \overline{\Omega}$ and

$$\int\int_{\Omega} f(x,t)h(x,t)\;dxdt \;=\int\int h(x,t)\mu(x,t)\;dxdt\;, \tag{1}$$

[a] To the authors knowledge H.Hedenmalm (Lund) has studied the existence question to some extend. However, our discussion goes much further than the simple mean value property.

for all integrable caloric functions h in Ω. The particular case of μ being the Dirac measure at (x^0, t^0) gives the mean-value property for caloric functions with respect to the point (x^0, t^0). One also observes immediately that, by (1) $\Omega \subset \mathbf{R}^n \times (-\infty, t^0)$.

1.2 *Heat balls*

A, probably, well-known identity of the type (1) with $f = |x|^2/t^2$ and μ a multiple of the Dirac mass is given by the so-called heat balls, which are the level sets of the fundamental solution to the adjoint heat operator $\Delta + D_t$. For a nice presentation of the mean value property for heat balls see the reader-friendly book by L.C. Evans [E].

1.3 *Reformulation*

In order to reformulate the problem to one more "handy" let us reflect the domain Ω in $\{t = 0\}$ so that the identity will now hold for anti-caloric functions. So from now on we assume that

$$\int\int_\Omega f(x,t)h(x,t)\ dxdt\ = \int\int h(x,t)\mu(x,t)\ dxdt\ , \tag{2}$$

for all integrable anti-caloric functions h in Ω; here we have used the same notation Ω for the reflected domain.

Let us also for simplicity make a couple of assumptions. First we assume that $\Omega \subset \mathbf{R}^{n+1}_+ := \mathbf{R}^n \times (0, \infty)$. Secondly we split the measure μ into two parts $\mu = g + g_0$ where $\text{supp}(g) \subset \overline{\Omega} \cap \{t > 0\}$ and $\text{supp}(g_0) \subset \{t = 0\}$. In this way we separate the initial data g_0 and the heat source g.

Now by applying formula (1) to the fundamental solution

$$K(z,\tau) = (4\pi\tau)^{-n/2} \exp\left(-|z|^2/(4\tau)\right), \qquad \tau > 0,$$

and $K(z,\tau) = 0$ for $\tau \le 0$, one verifies that the function

$$u(x,t) = \int\int_\Omega K(x-y,t-s)\mu(y,s)dyds - \int\int K(x-y,t-s)f(y,s)dyds,$$

where $(x,t) \in \Omega$ will satisfy the following

$$\begin{cases} \Delta u - D_t u = f\chi_\Omega - g & \text{in } \mathbf{R}^{n+1}_+, \\ u = 0 & \text{in } \mathbf{R}^{n+1}_+ \setminus \Omega, \\ u(x,0) = g_0, \\ \text{supp}(g) \subset \overline{\Omega}, \end{cases} \tag{3}$$

where

$$\mathbf{R}^{n+1}_+ = \mathbf{R}^n \times (0,\infty),$$

where differential equation is in the weak or distributional sense.

2 Existence, a discussion

The question of existence of solutions to (3) is a very delicate problem and indeed not an easy one. In this part we'll give a fair overview of the technique that seems to apply to this situation. It is however, outside the scope of the present notes to enter into any detailed existence theory. The scheme presented below will give a good understanding of the approach we suggest. We hope to come back to this question in the future.

The first observation one should make is the compatibility of the data f, g, and g_0. Indeed, from (2), applied to $h \equiv 1$, we obtain

$$\int_{\mathbf{R}^n} g_0(x)dx = \int\int (f\chi_\Omega - g)\ dx\,dt\ ;$$

observe that Ω is assumed to be bounded and g to be supported in Ω. A simple example such as

$$g_0 = 0, \qquad f = 1, \qquad g = \alpha\chi_D,$$

where D is a bounded domain in $\mathbf{R}^{n+1}$ will give

$$\int_0^\infty \int_{\mathbf{R}^n} (\chi_\Omega - \alpha\chi_D)\ dx\,dt\ = 0\ . \tag{4}$$

Then taking into account that $\text{supp}(g) = \chi_D \subset \overline{\Omega}$ is a requirement in the problem then for $\alpha < 1$ we'll have a contradiction to (4).

By relaxing the condition $\text{supp}(g) \subset \overline{\Omega}$ in (3) and by approximating the functions g, g_0 by smooth functions, one can use variational formulation technique, to prove the existence of a unique solution to (3), for any given data. The difficulty in our problem lies in showing that the solution we find, through variational problem, or any other technique for that matter, should fulfill the requirement $\text{supp}(\mu) \subset \overline{\Omega}$. This is the core difficulty in the existence theory for the equation in (3).

This type of questions for the elliptic case has been considered earlier by M. Sakai [S], B. Gustafsson [G] and Gustafsson and the second author [GS]. In these papers one can find general structure conditions on the function μ (which is actually allowed to be measure) in order to have solutions to (3) in the elliptic case.

We'll now present, heuristically, and without entering into the details, a general existence theory. This should be possible through a standard penalization of a variational formulation of the problem

$$\int_{\mathbf{R}^n} u_t(v-u)\ dx + \int_{\mathbf{R}^n} \nabla u \cdot \nabla(v-u)\ dx \geq \int_{\mathbf{R}^n} (g-f)(v-u)\ dx$$

$$\text{a.e. in } t \in (0,\infty), \text{ and}$$

$$\forall\ v \in \mathcal{K} := \left\{ v \in W^{1,2}(\mathbf{R}^{n+1}_+) :\ v \geq 0,\ v(x,0) = g_0, \right\}.$$

From here one hopes that the standard penalization will give a solution to the problem. The penalization itself is as follows. We look for a solution u to

$$\Delta u - D_t u = \beta_\epsilon(u) + f - g, \qquad u(x,0) = g_0 \quad \text{in } \mathbf{R}^{n+1}, \tag{5}$$

where

$$\beta_\epsilon \in C^\infty(\mathbf{R}), \qquad \beta'_\epsilon(s) \geq 0,$$
$$\beta_\epsilon(s) \to -\infty \qquad \text{if } s < 0, \quad \beta_\epsilon(s) \to 0 \qquad \text{if } s > 0, \quad \text{and} \quad \epsilon \to 0,$$
$$\beta_\epsilon(s) \leq C_0, \qquad \text{and} \qquad \beta_\epsilon(0) \geq -C_0,$$

and C_0 is independent of ϵ.

Next, using standard techniques one may show that problem (5) has a solution u_ϵ, which in passing to the limit $\epsilon \to 0$ will fulfill equation (3), with the mere difference that the condition $\mathrm{supp}(\mu) \subset \Omega$ is not necessarily fulfilled. At this point one needs to introduce new techniques and ideas to solve the problem of the inclusion of the support of g in $\overline{\Omega}$. Observe also that a by-product of this technique is the fact that $u \geq 0$ and $\Omega = \{u > 0\}$, when the solution exists. This property, a priori and in general, is not verified when a solution to (3) is given.

The main remaining difficulty is to impose conditions on the ingredients g, f and g_0 such that $\mathrm{supp}(g) \subset \overline{\Omega}$. The standard way of tackling this problem is to find barrier functions $v = v_y$ for each $y \in \mathrm{supp}(g)$ in relation to simplified versions of the given ingredients. Now, the simplified models of the ingredients should be chosen in such a way as to force $0 \leq v_y \leq u$. From here then it would follow the support condition, we've asked for.

In the elliptic case, usually, one can find explicit solutions, that act as barriers, for simple data such as $g = a\chi_D \subset \mathbf{R}^n$, and $f = b$, for appropriate constants a, b. In the parabolic case the explicit examples with compact support are more rare.

The above technique has successfully been applied to the elliptic case; see [GS] and the references therein.

Before closing this discussion, we would like to mention that the particular case of $g = 0$ and g_0 being a constant multiple of the Dirac mass function, discussed earlier, can be obtained by the following procedure. Suppose that we already have a procedure of finding solutions to simple cases such as $g = a\chi_D$, and $f = b$; this has to be done rigorously before any further steps. Then one lets $f = 1$, $g_0 = a_m\chi_{B(0,1/m)}$. Now for each m there is a solution u_m. By appropriately choosing a_m and letting $m \to \infty$, one can obtain a limit solution u_0 with the desired properties. Of course there are several steps of verification of the existence of the limit solution and the convergence of $a_m\chi_{B(0,1/m)}$ to the Dirac mass function.

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