Topology, geometry and the Einstein flow

- Objects of study
- Questions
- Some basic topology/geometry
- Classification of 3-manifolds by geometry
- Connections with the expanding direction of cosmological spacetimes

Cosmological vacuum spacetimes

Definition 1 A *cosmological vacuum spacetime* is a globally hyperbolic Lorentz mfd with $\mathrm{Ric} = 0$ and compact spatial Cauchy surfaces.

Definition 2 A Lorentz mfd is *globally hyperbolic* if and only if there is a hypersurface Σ such that all inextendible causal curves intersect Σ exactly once.

The topology of a globally hyperbolic spacetime is $I \times \Sigma$, where I is an interval.

Expanding direction: there is a causally geodesically complete direction.

Questions

Does the Einstein evolution in some sense homogenize and isotropize?

Are there attractors of the Einstein flow?

Note that some of these questions only make sense with respect to a foliation. Unless otherwise specified, I will have a constant mean curvature (CMC) foliation in mind.

Characterizing topology by geometry

Dream: Given that a manifold admits a Riemannian metric, and that this metric has certain local properties, I would like to determine the manifold up to diffeomorphism.

Problem 1: $\mathbb{R}^2$ and $\mathbb{R}^2 - \{0\}$ with the standard metrics are the same locally, but they're topologically not the same.

Solution 1: $\mathbb{R}^2 - \{0\}$ is not geodesically complete. There are geodesics which end artificially at the origin.

Problem 2: Consider $\mathbb{R}$ with the standard metric, and the Riemannian manifold obtained by identifying points at integer distance from each other in this manifold (i.e. S^1).

Solution 2: Demand simple connectedness.

Basic topology

Definition 3 A manifold M is said to be *simply connected* if any loop, i.e. continuous function $\gamma : [a, b] \to M$, $\gamma(a) = \gamma(b)$, can be continuously deformed to a point, i.e. there is a continuous

$$F : [a, b] \times [0, 1] \to M, \quad F(t, 0) = \gamma(t),$$

$$F(t, 1) = \gamma(a), \quad F(0, s) = F(1, s) = \gamma(a).$$

Ex: S^k, $k \geq 2$ and $\mathbb{R}^n$ are simply connected. S^1, $\mathbb{R}^2 - \{0\}$ and T^n are not.

Universal covering space

Given M, there is an $\tilde{M}$ and a map $\pi : \tilde{M} \to M$ (covering map):

- $\tilde{M}$ is simply connected
- Given $p \in M$, there is an open $U \ni p$ such that $\pi^{-1}(U) = \cup_\alpha U_\alpha$, with different U_α disjoint and $\pi|_{U_\alpha}$ a diffeomorphism to U

$\tilde{M}$ is called the universal covering space.

Ex: The universal covering space of S^1 is $\mathbb{R}$, and the covering map $\pi : \mathbb{R} \to S^1$ is given by

$$\pi(t) = e^{2\pi i t}.$$

Deck transformations

Let $\tilde{M}$ be the universal covering space of M and π be the covering projection.

Definition 4 A *deck transformation* is a diffeo–morphism ϕ of $\tilde{M}$ such that $\pi \circ \phi = \pi$.

Ex: $\phi_n(x) = x + n$ acting on $\mathbb{R}$.

The deck transformations form a nice group of diffeomorphisms Γ.

Conversely, given Γ (nice) acting on $\tilde{M}$ (simply connected), there is an M and a π such that $\tilde{M}$ is the universal covering space of M with covering projection π.

Notation: $M = \tilde{M}/\Gamma$. Ex. $S^1 = \mathbb{R}/\mathbb{Z}$.

Connections to standard cosmology

Topology: $I \times \Sigma$, $\Sigma(t) = \{t\} \times \Sigma$.

Isotropy etc. $\rightarrow \Sigma(t)$ has constant curvature, i.e. ${}^3R_{abcd} = Kh_{c[a}h_{b]d}$, K constant.

Theorem 1 *If (Σ^n, g) is a connected, simply connected and geodesically complete Riemannian mfd of constant curvature, then (Σ^n, g) is isometric to hyperbolic n-space, $\mathbb{R}^n$ or the n-sphere.*

Compact, orientable 2-mfds

Classification strategy:

- Each orientable, compact 2-dimensional mfd M has a constant scalar curvature metric
- $\Rightarrow \tilde{M}$ is $\mathbb{R}^2$, S^2 or H^2
- Classify the nice subgroups of the isometry group

3-mfds?

In the 3-dim case, there are eight "canonical" geometries - the Thurston geometries. Universal covers: $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ and S^3 (topologically).

For topological reasons, most manifolds will not allow a geometry.

Idea: cut the manifold into pieces, each of which allows a canonical geometry.

Connected sum

M_1, M_2, n-mfds. Cut out n-balls in each of them and glue them together along the boundary. Result: connected sum of M_1 and M_2, denoted

$$M_1 \# M_2.$$

Note: S^3 special.

A 3-mfd is prime if it is not the 3-sphere and cannot be written as a non-trivial connected sum of closed 3-mfds.

Prime decomposition

Theorem 2 *Let M be a closed, oriented 3-mfd, which is not S^3. Then M has a finite decomposition as a connected sum*

$$M = M_1 \# \cdots \# M_k,$$

where each M_i is prime. The collection $\{M_i\}$ is unique up to permutation of the factors.

A 3-mfd M is irreducible if every smoothly embedded 2-sphere in M bounds a 3-ball.

Fact 1: Irreducible $\Rightarrow$ prime.

Fact 2: Prime+orientable $\Rightarrow$ irreducible or $S^2 \times S^1$.

Prime decomposition - geometries

If M is not S^3, then

$M = (K_1 \# \cdots \# K_p) \# (L_1 \# \cdots \# L_q) \# (\#_1^r S^2 \times S^1)$

K_i: closed, irreducible, aspherical.

L_i: closed, irreducible, finitely covered by homotopy 3-spheres.

A Thurston geometry - universal covering space $\mathbb{R}^3$, S^3 or $S^2 \times \mathbb{R}$

$\mathbb{R}^3 \to$ only one K_i factor.

$S^3 \to S^3$ or only one L_i factor.

$S^2 \times \mathbb{R} \to S^2 \times S^1$.

Seifert fibred spaces

A 3-manifold is said to be a Seifert fibered space if it satisfies the following two conditions:

1. It can be written as a disjoint union of circles;

2. Each circle fiber has an open neighbourhood U satisfying:

   - U can be written as a disjoint union of circle fibers,

   - U is isomorphic either to a solid torus or a cylinder where the ends have been identified after a rotation by a rational angle.

Geometrization of a K factor

Conjecture:

$$K = H \cup S,$$

where H is a finite collection of complete connected hyperbolic mfds of finite volume embedded in K, and S is a finite collection of Seifert fibered spaces. The union is along 2-tori in some canonical way.

Topology - Bianchi

If M is not S^3, then

$M = (K_1 \# \cdots \# K_p) \# (L_1 \# \cdots \# L_q) \# (\#_1^r S^2 \times S^1)$

Bianchi:

Kantowski-Sachs: $S^2 \times S^1$.

IX: S^3 or quotient.

V, VII_h: hyperbolic.

VI_0 (Sol): Graph.

Rest: Seifert fibered.

Cosmology, setup

Consider a cosmological vacuum spacetime (M, g) with a Cauchy surface Σ.

Assume that the prime decomposition of Σ consists of one K factor.

Assume there is a CMC foliation exhausting the interval $[H_0, 0)$.

Geometry of the leaves

Σ_τ - hypersurface of constant mean curvature τ.

$\widehat{g}_\tau$ - Riemannian metric induced on Σ_τ.

One expects the volume of $(\Sigma_\tau, \widehat{g}_\tau)$ to tend to infinity and the metric $\widehat{g}_\tau$ to become more and more flat.

Solution: Rescale.

Rescaling

Σ fixed Cauchy surface.

$$\hat{t}(\Sigma_\tau) = \sup_\gamma \int_0^1 [-\langle \gamma', \gamma' \rangle]^{1/2} ds,$$

where the supremum is taken over all timelike curves γ with $\gamma(0) \in \Sigma$ and $\gamma(1) \in \Sigma_\tau$.

Define

$$g_\tau = \hat{t}^{-2}(\Sigma_\tau)\hat{g}_\tau.$$

The foliation is $I \times \Sigma$, and I will identify $\Sigma_\tau = \{\tau\} \times \Sigma$ with Σ.

Object of study:

$$(\Sigma, g_\tau).$$

Conjecture

Conjecture: $\Sigma = K$ can be written as $\Sigma = H \cup S$, where

- On each component of H, $g_\mathcal{T}$ converges to a complete hyperbolic metric of finite positive volume.

- On S, $g_\mathcal{T}$ collapses in the sense that the volume of S with respect to $g_\mathcal{T}$ converges to zero, and the length of the circle fibers tends to zero.

Cf. Collins and Hawking

Collins and Hawking (1973):

"We show that the set of spatially homogeneous cosmological models which approach isotropy at infinite times is of measure zero in the space of all spatially homogeneous models."

General spatially homogeneous cosmological models: VI_h, VII_h, VIII and IX.

Support?

Lars Andersson and Vincent Moncrief considered perturbations of initial data corresponding to a spacetime of the form:

$$\bar{g} = -dt^2 + t^2\gamma, \quad \bar{M} = (0, \infty) \times M$$

where (M, γ) is a compact hyperbolic mfd.

For small data:

1) Future global foliation by CMC hypersurfaces and future causal geodesic completeness.

2) After rescaling, the metric and second fundamental form converge to the corresponding objects for the standard model.

U(1)

Yvonne Choquet-Bruhat and Vincent Moncrief considered a situation in which one has $U(1)$ symmetry. Topology:

$$\Sigma \times U(1) \times \mathbb{R},$$

Σ compact higher genus surface. For small data with $U(1)$ symmetry:

1) Future global CMC foliation and future causal geodesic completeness.

2) Collapse of the circle fibers after rescaling.

T^3-Gowdy

The Gowdy spacetimes with T^3 topology have a $U(1) \times U(1)$ isometry group.

1) Future global existence in CMC and areal time coordinate and future causal geodesic completeness.

2) Collapse of the Seifert fibres (in fact collapse along 2-tori) and collapse of the volume in the areal time coordinate.

Conclusions

- There is a general picture for the expanding direction of cosmological spacetimes if the spatial topology is a simple K-factor.

- Everything known to me fits into this picture.

- The methods available makes the study of most topologies impossible at this time.