FUTURE STABILITY OF THE EINSTEIN-NON-LINEAR SCALAR FIELD SYSTEM

HANS RINGSTRÖM

ABSTRACT. We consider the question of future global non-linear stability in the case of Einstein's equations coupled to a non-linear scalar field. The class of potentials V to which our results apply is defined by the conditions $V(0) > 0$, $V'(0) = 0$ and $V''(0) > 0$. Thus Einstein's equations with a positive cosmological constant represents a special case, obtained by demanding that the scalar field be zero. In that context, there are stability results due to Helmut Friedrich, the methods of which are, however, not so easy to adapt to the presence of matter. The goal of the present paper is to develop methods that are more easily adaptable. Due to the extreme nature of the causal structure in models of this type, it is possible to prove a stability result which only makes local assumptions concerning the initial data and yields global conclusions in time. To be more specific, we make assumptions in a set of the form $B_{4r_0}(p)$ for some $r_0 > 0$ on the initial hypersurface, and obtain the conclusion that all causal geodesics in the maximal globally hyperbolic development that start in $B_{r_0}(p)$ are future complete. Furthermore, we derive expansions for the unknowns in a set that contains the future of $B_{r_0}(p)$. The advantage of such a result is that it can be applied regardless of the global topology of the initial hypersurface. As an application, we prove future global non-linear stability of a large class of spatially locally homogeneous spacetimes with compact spatial topology.

1. Introduction

1.1. **Background, previous results.** This paper is concerned with cosmological solutions to Einstein's equations with accelerated expansion, one motivation being that, at present, physicists use such solutions to model the universe. The problem we wish to study is that of stability. What we mean by stability here is the question of whether a cosmological solution, which is future causally geodesically complete, has the property that if we make small perturbations of the initial data of this solution, then the resulting spacetimes are also future causally geodesically complete. In other words, we do not concern ourselves with the question of whether the perturbed solution decays to the background solution, though we shall derive asymptotic expansions in the cases where we prove future causal geodesic completeness. The first results on stability were obtained by Helmut Friedrich in [15] in the $3+1$-dimensional case. These were later extended by Michael Anderson in [1] to the $n+1$-dimensional case for n odd. The specific situation the above authors consider is Einstein's equations with a positive cosmological constant, though the methods can also be generalized to include matter of Maxwell and Yang-Mills type, cf. [17]. Let us try to sketch the ideas on which [15] is based. The first important result is that a solution to Einstein's vacuum equations with a positive cosmological constant Λ in 4 spacetime dimensions,

$$\mathrm{Ric}[\tilde{g}] = \Lambda\tilde{g}, \tag{1}$$

can be considered to be a solution to the *conformal field equations*, developed by Helmut Friedrich, the variables of which include a *conformal factor* Ω, and conversely, a solution to the conformal field equations gives a solution to Einstein's equations on the region where $\Omega > 0$. Given the metric g produced by the conformal field equations, the metric solving (1) is given by $\tilde{g} = \Omega^{-2}g$. In fact, the set defined by $\Omega = 0$ corresponds to "infinity" from the point of the Lorentz manifold with metric $\tilde{g}$ in some suitable sense, cf. [15], [16] and references cited therein. On the other hand, from the point of view of the conformal field equations, there are no complications associated with Ω being zero. The de Sitter metric

$$-dt^2 + \cosh^2(t)g_1, \tag{2}$$

where $t \in \mathbb{R}$ and g_1 is the standard metric on $\mathbb{S}^3$, allows a rescaling by a conformal factor so that it becomes $-ds^2 + g_1$ where $s \in (-\pi/2, \pi/2)$, cf. [1]. The question of stability of the de Sitter metric, which from a PDE point of view would seem to be a global in time question, can then be reduced to local in time stability of a specific solution to the conformal field equations. Since the local stability follows from the fact that the conformal field equations, after suitable gauge choices, form a symmetric hyperbolic system, the stability of de Sitter space follows immediately. Another very interesting result which follows from [15] is that one can specify data at "infinity", and that there are no topological restrictions for doing so. Thus one gets a large family of solutions to (1) with arbitrary spatial topology which are future causally geodesically complete and, furthermore, future stable. Note that in the case of 0 cosmological constant, there are no results of this type, and perhaps one should not even expect the corresponding statement to be true. To conclude, the method is very elegant and geometric in nature and makes it possible to avoid proving global existence of solutions to a non-linear hyperbolic PDE. However, it does suffer from a lack of robustness. A first indication of this is the fact that the conformal field equations developed by Friedrich are specific to $3+1$ dimensions. However, in the $n+1$ dimensional case, n odd, Anderson [1] showed that the equation $\mathcal{H} = 0$, where $\mathcal{H}$ is the ambient obstruction tensor of Fefferman and Graham, cf. [13], has properties analogous to the conformal field equations. Consequently, for constants $H > 0$, metrics of the form

$$-dt^2 + \cosh^2(Ht)g_\Sigma \tag{3}$$

on $\mathbb{R} \times \Sigma$, where (Σ, g_Σ) is a compact odd dimensional Riemannian manifold with $\mathrm{Ric}[g_\Sigma] = (n-1)H^2 g_\Sigma$, are stable solutions to Einstein's vacuum equations with a positive cosmological constant $\Lambda = n(n-1)H^2/2$. One can also specify data at "infinity" on odd dimensional manifolds of arbitrary topology.

1.2. **Motivation.** The question then arises why any further consideration of the question of stability in the context of accelerated expansion should be of interest. The answer lies in the lack of robustness of the above method; if one wants to go beyond Einstein's equations with a positive cosmological constant, possibly coupled to Maxwell or Yang-Mills type matter, the method does not give clear indications concerning how to proceed. At present, many different mechanisms that yield accelerated expansion are being considered, the simplest one being a positive cosmological constant. Other mechanisms of interest involve a scalar field with a non-linear potential, whence the desire to understand the stability properties of such models. Furthermore, in order to be able to say something concerning the models of the universe physicists consider, one does in the end need to study the stability of models which include matter. The motivation for developing the methods described in this paper is the hope that they will be of use in the treatment of the above mentioned problems.

The formulation of the main result is non-standard in the sense that we do not make assumptions concerning an entire initial hypersurface, but only concerning a subset. Let us motivate this type of formulation. Consider the metric

$$-dt^2 + e^{2Ht} g_0 \tag{4}$$

on $M = \mathbb{R} \times \mathbb{R}^n$ (or $\mathbb{R} \times \mathbb{T}^n$), where g_0 is the ordinary Euclidean metric. This is a solution to Einstein's equations with a positive cosmological constant $\Lambda = n(n-1)H^2/2$. Consider a future directed causal curve $\gamma : [0, a] \to M$ (we always assume ∂_t to be future oriented) such that $\gamma(0) \in \{t_0\} \times \mathbb{R}^n$. The length of the projection of this curve to $\mathbb{R}^n$, measured with respect to the Riemannian metric induced on $\{t_0\} \times \mathbb{R}^n$, is bounded from above by H^{-1}. Say now, for the sake of argument, that we wish to determine a solution to the wave equation with respect to the metric (4) in the causal future of $\{t_0\} \times B_{H^{-1}}(p)$, where the radius of the ball is measured with respect to the Riemannian metric induced on $\{t_0\} \times \mathbb{R}^n$. Then all we need to know is what the initial data look like on the set $\{t_0\} \times B_{3H^{-1}}(p)$. In other words

$$J^+[\{t_0\} \times B_{H^{-1}}(p)] \subseteq D^+[\{t_0\} \times B_{3H^{-1}}(p)], \tag{5}$$

using the notation of Subsection 3.2. This should be compared with Minkowski space, for which it is only possible to determine a solution to the wave equation on the causal future of a point if one controls the initial data on an entire Cauchy hypersurface. The above example indicates that it might be enough to make local assumptions concerning the initial data in order to get global in time conclusions concerning the solution. The advantage of such a formulation is that it could be applied regardless of the topology of the initial hypersurface.

The above observations suggest that in the context of accelerated expansion, the connection between the global topology of the Cauchy hypersurfaces and the dynamics is less strong than in the context of non-accelerated expansion. In fact, it is tempting to make the following conjecture. Let (M, g) be a globally hyperbolic and future causally geodesically complete Lorentz manifold. We shall say that *late time observers are oblivious to topology* if there is a Cauchy hypersurface Σ such that there is no causal curve whose past contains Σ and we shall say that *late time observers are not oblivious to topology* if for every Cauchy hypersurface Σ there is a causal curve whose past contains Σ. A stronger version would be to say that *late time observers in M are completely oblivious to topology* if there is a Cauchy hypersurface Σ such that for every causal curve γ, the intersection of the causal past of γ with Σ is contained in a coordinate chart on Σ, the domain of which is diffeomorphic to a ball in $\mathbb{R}^n$. The conjecture is then that if (M, g) is a future causally geodesically complete vacuum solution to Einstein's equations with a positive cosmological constant and compact Cauchy hypersurfaces, then late time observers in M are oblivious to topology. That the corresponding conjecture with the word "oblivious" replaced by "completely oblivious" is incorrect follows by an important example which is due to an anonymous referee. The example is given by the metric

$$g_R = -dt^2 + \cosh^2(Ht)dx^2 + H^{-2}g_{\mathbb{S}^2} \tag{6}$$

on $M_R = \mathbb{R}\times\mathbb{S}^1\times\mathbb{S}^2$, where $\Lambda > 0$, $H = \Lambda^{1/2}$ and $g_{\mathbb{S}^2}$ is the standard metric on the unit 2-sphere. Then (M_R, g_R) is causally geodesically complete, satisfies Einstein's vacuum equations with a cosmological constant Λ and if Σ is an arbitrary Cauchy hypersurface in (M_R, g_R) and γ is an arbitrary inextendible causal curve, then the intersection of the causal past of γ with Σ is not contained in a subset of Σ homeomorphic to a 3-ball, cf. Lemma 21. The spacetime (M_R, g_R) is sometimes referred to as the Nariai spacetime. It is also of interest to note that the conjecture that late time observers are oblivious to topology is false in the class of solutions to the Einstein-Maxwell equations with a positive cosmological constant (again, the example is due to an anonymous referee). In fact, let $\Lambda > 0$, $\Lambda_0 = 2\Lambda$, $\gamma_0 = (2\Lambda)^{1/2}$ and

$$g_{\mathfrak{R}} = -dt^2 + dx^2 + \Lambda_0^{-1}g_{\mathbb{S}^2}, \quad F = \gamma_0(dt\otimes dx - dx\otimes dt). \tag{7}$$

Then $g_{\mathfrak{R}}$ is a Lorentz metric and F a 2-form on M_R. Furthermore, one can compute that

$$\nabla_\alpha F_{\mu\nu} = 0,$$

where ∇ is the Levi-Civita connection associated with $g_{\mathfrak{R}}$. Consequently, F satisfies Maxwell's equations without sources:

$$\nabla^\alpha F_{\alpha\beta} = 0, \quad \nabla_{[\alpha}F_{\mu\nu]} = 0,$$

cf. [32], p. 70. Furthemore $g_{\mathfrak{R}}$ and F satisfy Einstein's equations with a positive cosmological constant Λ:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta},$$

where the stress energy tensor T is given by

$$T_{\alpha\beta} = F_{\alpha\mu}F_{\beta}{}^{\mu} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}.$$

To conclude, $(M_R, g_{\mathfrak{R}}, F)$ is a globally hyperbolic and geodesically complete solution to the Einstein-Maxwell equations with a positive cosmological constant Λ but late time observers are not oblivious to topology, cf. Corollary 57, p. 89, of [24] and the arguments presented in the proof of Lemma 21.

In view of the above conjecture, there is reason to expect that the global topology should not play an important role when studying the stability of future causally geodesically complete vacuum models with a positive cosmological constant, since one can restrict one's attention to the future of any fixed Cauchy hypersurface when doing the analysis. Nevertheless, the example (6) should be kept in mind; if the Cauchy hypersurfaces are such that they allow a metric of positive scalar curvature, the situation might be more complicated. If one instead considers vacuum solutions to Einstein's equations without a cosmological constant, the situation is, however, quite different, cf. [14], [2] and references cited therein. In fact, the existing conjectures, with which all understood solutions conform, imply that in the vacuum context without a cosmological constant, the global spatial topology plays a crucial role in the asymptotic behaviour. Furthermore, it is natural to conjecture that if (M, g) is a future causally geodesically complete vacuum solution to

Einstein's equations with a vanishing cosmological constant and compact Cauchy hypersurfaces, then late time observers in M are not oblivious to topology.

1.3. **Matter models, initial value problem.** Let us be more specific concerning the models we wish to study. We are interested in Einstein's equations

$$G_{\mu\nu} = T_{\mu\nu}, \tag{8}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} S g_{\mu\nu},$$

$R_{\mu\nu}$ is the Ricci tensor of a Lorentz metric g on an $n+1$ dimensional manifold M, and S is the associated scalar curvature. Concerning the stress energy tensor, we assume it to be of the form

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \left[\frac{1}{2}\nabla^\gamma \phi \nabla_\gamma \phi + V(\phi)\right] g_{\mu\nu}, \tag{9}$$

where $\phi \in C^\infty(M)$ and $V \in C^\infty(\mathbb{R})$ is a function such that $V(0) = V_0 > 0$, $V'(0) = 0$ and $V''(0) > 0$. For the sake of future convenience, let us define H to be the positive solution to

$$nH^2 = \frac{2}{n-1} V_0 \tag{10}$$

and define χ by

$$\chi = V''(0)/H^2. \tag{11}$$

By assumption $H, \chi > 0$. With this choice of H, (3) and (4) are solutions to (8) with $\phi = 0$ since they are both solutions to Einstein's equations with a positive cosmological constant $\Lambda = n(n-1)H^2/2$. Note that (8) is equivalent to

$$R_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi + \frac{2}{n-1} V(\phi) g_{\mu\nu}. \tag{12}$$

It should of course be coupled to a matter equation for ϕ, which is given by

$$\nabla^\mu \nabla_\mu \phi - V'(\phi) = 0. \tag{13}$$

As a consequence, $\nabla^\mu T_{\mu\nu} = 0$, i.e. $T_{\mu\nu}$ is divergence free. Thus this equation ensures the compatibility of a stress energy tensor of the form (9) with (8), since the Bianchi identities imply that $\nabla^\mu G_{\mu\nu} = 0$. Note, however, that $\nabla^\mu T_{\mu\nu} = 0$ does not imply (13), since if $\phi = \phi_0$ is a constant such that $V'(\phi_0) \neq 0$, then $\nabla^\mu T_{\mu\nu} = 0$, but (13) is not satisfied. The system of equations of interest is thus (12)-(13). The main motivation for studying these equations comes from the $n = 3$ case, but since the dimension does not play any significant role in the arguments, we shall only assume $n \geq 3$ in what follows.

Let us recall some basic facts concerning spacelike hypersurfaces in Lorentz manifolds that we shall need in what follows. Assume we have an $n+1$-dimensional Lorentz manifold (M, g) and a scalar function ϕ satisfying (12)-(13). If Σ is a spacelike hypersurface in M, and the future directed unit normal to this surface is N, then

$$N^\mu N^\nu G_{\mu\nu} = \frac{1}{2}[r - k_{ij}k^{ij} + (\mathrm{tr} k)^2], \tag{14}$$

where all the objects that appear on the right hand side are intrinsic to the hypersurface Σ: if h is the Riemannian metric induced on Σ by g, then r is the scalar curvature of h, k is the second fundamental form, defined by

$$k(X,Y) = \langle \nabla_X N, Y \rangle,$$

for vectors X, Y tangent to the surface Σ (where ∇ is the Levi-Civita connection associated with g), and indices are raised and lowered by h. For a derivation of (14), see [32]. Combining (14) with (8), we obtain

$$\frac{1}{2}[r - k_{ij}k^{ij} + (\mathrm{tr}k)^2] = \frac{1}{2}[(N\phi)^2 + D^i\phi D_i\phi] + V(\phi) \tag{15}$$

where D is the Levi-Civita connection on Σ induced by h. We refer to (15) as the *Hamiltonian constraint*. For any vector X tangent to Σ, we have

$$X^\mu N^\nu G_{\mu\nu} = [D^j k_{ji} - D_i(\mathrm{tr}k)]X^i,$$

cf. [32]. Combining this with (8), we obtain

$$D^l k_{li} - D_i(\mathrm{tr}k) = N(\phi)D_i\phi, \tag{16}$$

the so-called *momentum constraint*. This leads to the following initial value problem.

Definition 1. *Initial data* for (12)-(13) consist of an n dimensional manifold Σ, a Riemannian metric h, a covariant 2-tensor k and two functions ϕ_0 and ϕ_1 on Σ, all assumed to be smooth and to satisfy

$$r - k_{ij}k^{ij} + (\mathrm{tr}k)^2 = \phi_1^2 + D^i\phi_0 D_i\phi_0 + 2V(\phi_0), \tag{17}$$

$$D^j k_{ji} - D_i(\mathrm{tr}k) = \phi_1 D_i\phi_0, \tag{18}$$

where D is the Levi-Civita connection of h, r is the associated scalar curvature and indices are raised and lowered by h. Given initial data, the *initial value problem* is that of finding an $n+1$ dimensional manifold M with a Lorentz metric g and a $\phi \in C^\infty(M)$ such that (12) and (13) are satisfied, and an embedding $i : \Sigma \to M$ such that $i(\Sigma)$ is a Cauchy hypersurface in (M, g), $i^*g = h$, $\phi \circ i = \phi_0$, and if N is the future directed unit normal and κ is the second fundamental form of $i(\Sigma)$, then $i^*\kappa = k$ and $(N\phi) \circ i = \phi_1$. Such a triple (M, g, ϕ) is referred to as a *globally hyperbolic development* of the initial data, the existence of an embedding i being tacit.

Remark. The concept of a Cauchy hypersurface is defined in Subsection 3.2. One can of course define the concept of initial data and development for a lower degree of regularity. We shall, however, restrict our attention to the smooth case in this paper.

For results concerning the existence of initial data in the current setting, we refer the reader to [9].

Definition 2. Given initial data for (12)-(13), a *maximal globally hyperbolic development* of the data is a globally hyperbolic development (M, g, ϕ), with embedding $i : \Sigma \to M$, such that if (M', g', ϕ') is any other globally hyperbolic development of the same data, with embedding $i' : \Sigma \to M'$, then there is a map $\psi : M' \to M$ which is a diffeomorphism onto its image such that $\psi^*g = g'$, $\psi^*\phi = \phi'$ and $\psi \circ i' = i$.

Theorem 1. *Given initial data for (12)-(13), there is a maximal globally hyperbolic development of the data which is unique up to isometry.*

Remark. When we say that (M, g, ϕ) is unique up to isometry, we mean that if (M', g', ϕ') is another maximal globally hyperbolic development, then there is a diffeomorphism $\psi : M \to M'$ such that $\psi^* g' = g$, $\psi^* \phi' = \phi$ and $\psi \circ i = i'$, where i and i' are the embeddings of Σ into M and M' respectively.

The proof is as in [8]. This is an important result and will be of use to us in this paper. However, it does not yield any conclusions concerning e.g. causal geodesic completeness.

1.4. **Results.** Recall that the constants $H > 0$ and $\chi > 0$ are determined by the potential V through the equations (10) and (11). Before we state the main result, we need to introduce some terminology. Let Σ be an n dimensional manifold. We shall be interested in coordinate systems x on open subsets U of Σ such that $x : U \to B_1(0)$ is a diffeomorphism. If S is a tensor field on Σ, we shall use the notation

$$\|S\|_{H^l(U)} = \left(\sum_{i_1,\dots,i_s=1}^{n} \sum_{j_1,\dots,j_r=1}^{n} \sum_{|\alpha|\leq l} \int_{x(U)} |\partial^\alpha S^{i_1\cdots i_s}_{j_1\cdots j_r} \circ x^{-1}|^2 dx^1 \cdots dx^n \right)^{1/2},$$

where the components of S are computed with respect to x and the derivatives are with respect to x. When we write $\|S\|_{H^l(U)}$, we shall take it to be understood that there are coordinates x as above. Below, we shall use δ to denote the Kronecker delta with respect to the x coordinates. In particular, we shall use the notation

$$\|g - a\delta\|_{H^l(U)} = \left(\sum_{i,j=1}^{n} \sum_{|\alpha|\leq l} \int_{x(U)} |\partial^\alpha (g_{ij} - a\delta_{ij}) \circ x^{-1}|^2 dx^1 \cdots dx^n \right)^{1/2}.$$

Theorem 2. *Let V be a smooth function such that $V(0) = V_0 > 0$, $V'(0) = 0$ and $V''(0) > 0$. Let $H, \chi > 0$ be defined by (10) and (11) respectively and let $n \geq 3$. There is an $\epsilon > 0$, depending on n and V, such that if $(\Sigma, h, \kappa, \phi_0, \phi_1)$ are initial data for (12) and (13) with $\dim\Sigma = n$, $x : U \to B_1(0)$ are coordinates as above and*

$$\begin{aligned} \|h - 16H^{-2}\delta\|_{H^{k_0+1}(U)} &+ \|\kappa - 16H^{-1}\delta\|_{H^{k_0}(U)} \\ &+ \|\phi_0\|_{H^{k_0+1}(U)} + \|\phi_1\|_{H^{k_0}(U)} \leq \epsilon, \end{aligned} \tag{19}$$

where k_0 is the smallest integer satisfying $k_0 > n/2 + 1$, the maximal globally hyperbolic development (M, g, ϕ) has the property that if $i : \Sigma \to M$ is the associated embedding, then all causal geodesics that start in $i\{x^{-1}[B_{1/4}(0)]\}$ are future complete. Furthermore, there is a $t_- < 0$ and a smooth map

$$\psi : (t_-, \infty) \times B_{5/8}(0) \to M, \tag{20}$$

which is a diffeomorphism onto its image such that all causal curves that start in $i\{x^{-1}[B_{1/4}(0)]\}$ remain in the image of ψ to the future and g and ϕ have expansions (21)-(31) in the solid cylinder $[0, \infty) \times B_{5/8}(0)$ when pulled back by ψ. Finally, $\psi(0, p) = i \circ x^{-1}(p)$ for $p \in B_{5/8}(0)$. In the formulas below, Latin indices refer to *the natural Euclidean coordinates on* $B_{5/8}(0)$ *and* t *is the natural time coordinate on the solid cylinder. Define* $\zeta = 4\chi/n^2$, $\lambda = n[1-(1-\zeta)^{1/2}]/2$ *for* $\zeta \in (0,1)$, $\lambda = n/2$ *for* $\zeta \geq 1$ *and* $\lambda_{\mathrm{m}} = \min\{1, \lambda\}$. *There is a smooth Riemannian metric* ρ *on* $B_{5/8}(0)$ *and constants* K_l *such that*

$$\|e^{2Ht}g^{ij}(t,\cdot) - \rho^{ij}\|_{C^l} + \|e^{-2Ht}g_{ij}(t,\cdot) - \rho_{ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}Ht}, \tag{21}$$

$$\|e^{-2Ht}\partial_t g_{ij}(t,\cdot) - 2H\rho_{ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}Ht}, \tag{22}$$

for every $l \geq 0$, *where* ρ^{ij} *are the components of the inverse. Here* C^l *denotes the* C^l *norm on* $B_{5/8}(0)$. *Concerning* g_{0m}, *there is an* $\alpha > 0$ *and constants* K_l *such that for all* $l \geq 0$,

$$\left\| g_{0m}(t,\cdot) - \frac{1}{(n-2)H}\rho^{ij}\gamma_{imj} \right\|_{C^l} + \|\partial_0 g_{0m}(t,\cdot)\|_{C^l} \leq K_l e^{-\alpha Ht}, \tag{23}$$

where γ_{imj} *are the Christoffel symbols of the metric* ρ. *Let* $k(t,\cdot)$ *be the second fundamental form induced on* $\{t\} \times B_{5/8}(0)$. *The estimates for* g_{00} *and* k_{ij} *depend on the value of* λ_{m}. *If* $\lambda_{\mathrm{m}} < 1$, *we have*

$$\|g_{00}(t,\cdot) + 1\|_{C^l} + \|\partial_0 g_{00}(t,\cdot)\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}Ht}, \tag{24}$$

$$\|e^{-2Ht}k_{ij}(t,\cdot) - H\rho_{ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}Ht}, \tag{25}$$

but for $\lambda_{\mathrm{m}} = 1$, *we have*

$$\|[\partial_0 g_{00} + 2\lambda_{\mathrm{m}}H(g_{00}+1)](t,\cdot)\|_{C^l} \leq K_l e^{-2Ht}, \tag{26}$$

$$\|g_{00}(t,\cdot) + 1\|_{C^l} \leq K_l(1+t^2)^{1/2}e^{-2Ht}, \tag{27}$$

$$\|e^{-2Ht}k_{ij}(t,\cdot) - H\rho_{ij}\|_{C^l} \leq K_l(1+t^2)^{1/2}e^{-2Ht}. \tag{28}$$

Concerning ϕ *there are three cases to consider. Let us define* $\varphi = e^{\lambda Ht}\phi$. *If* $\zeta < 1$, *then there is a smooth function* φ_0 *such that*

$$\|\varphi(t,\cdot) - \varphi_0\|_{C^l} + \|\partial_0\varphi\|_{C^l} \leq K_l e^{-\alpha Ht}. \tag{29}$$

If $\zeta = 1$, *there are smooth functions* φ_0 *and* φ_1 *such that*

$$\|\partial_0\varphi(t,\cdot) - \varphi_1\|_{C^l} + \|\varphi(t,\cdot) - \varphi_1 t - \varphi_0\|_{C^l} \leq K_l e^{-\alpha Ht}. \tag{30}$$

Finally, if $\zeta > 1$, *there is an anti symmetric matrix* A, *given by*

$$A = \begin{pmatrix} 0 & \delta H \\ -\delta H & 0 \end{pmatrix},$$

where $\delta = n(\zeta-1)^{1/2}/2$, *and smooth functions* φ_0 *and* φ_1 *such that*

$$\left\| e^{-At}\begin{pmatrix} \delta H\varphi \\ \partial_0\varphi \end{pmatrix}(t,\cdot) - \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \right\|_{C^l} \leq K_l e^{-\alpha Ht}. \tag{31}$$

Remark. Since the metric h is essentially $16H^{-2}\delta$ with respect to the x-coordinates, the ball of radius 1 with respect to the x-coordinates is approximately a ball of radius $4H^{-1}$ with respect to h. Recall the discussion concerning the linear wave equation on a background of the form (4). In order to predict what happens to geodesics that start inside a ball of radius H^{-1}, we need to control the initial data in a ball of radius $3H^{-1}$, cf. (5). In fact, we then control the behaviour in a cylinder of radius $2H^{-1}$, $t \geq t_0$. In our case, we make assumptions on a ball of radius $4H^{-1}$, the reason being that we need a margin, which we have arbitrarily chosen to be H^{-1} in the above statement. If one is interested in having a smaller margin or only interested in getting conclusions in a smaller region, one can reformulate the theorem accordingly. It would be nice to have a result with purely geometric conditions and it should be possible to reformulate the theorem in such a way using harmonic coordinates. However, that would require an unwarranted effort, since the statement is technical anyway and a geometric formulation is not needed in the applications. It is of interest to note that the estimates break down as $\chi \to 0+$; in this limit, $\zeta \to 0+$ so that $\lambda, \lambda_{\mathrm{m}}, \to 0+$. In other words, we certainly need the condition $V''(0) > 0$. The reason the condition on the initial data involves more than $n/2 + 2$ derivatives of the metric is that we need this condition in the global existence proof. Let us note that the expansions obtained here are not complete in the sense that the number of free functions appearing in the expansions is less than the number of free functions one gets to specify as initial data. For a thorough analysis of the question of asymptotic expansions in the case of a positive cosmological constant, we refer the reader to the work of Alan Rendall [28]. Note, however, that this analysis is based on a Gaussian foliation, which is not the type of foliation we obtain in the above result, cf. (23). For a discussion of asymptotics on a formal level in the case of curvature coupled scalar field models, we refer the reader to [7]. Finally, let us observe that the example (6) is such that regardless of the choice of Cauchy hypersurface Σ in (M_R, g_R) and of the choice of open set U in Σ, the above theorem is not applicable to U and the initial data induced on Σ by g_R. The reason is as follows. Assume the theorem were applicable and let γ be an inextendible causal curve in (M_R, g_R), i.e. the maximal globally hyperbolic development of Σ with the appropriate induced initial data, passing through the subset of U corresponding to $B_{1/4}(0)$ under x. Then the proof of the theorem implies that $J^-(\gamma) \cap \Sigma$ would be contained in a set diffeomorphic to a 3-ball, in contradiction with Lemma 21.

The proof of the above theorem is to be found in Section 16. In [15], no results of the above form were stated, but it should be possible to derive a result such as Theorem 2, possibly with more detailed information concerning the asymptotics, for Einstein's equations with a positive cosmological constant using the results of [15]. The reason it should be possible is that the main tool that is needed is the stability of metrics of the form (4) to the future. In fact, one needs to have a hyperbolic reduction of the equations which is such that one can prove stability even for data that do not satisfy the constraints. Since global non-linear stability from the point of view of Friedrich's conformal field equations corresponds to local stability, this is not a problem.

One consequence of the above result is that if one perturbs initial data corresponding to solutions of the form (3), for any dimension $n \geq 3$, inside the class of models we are considering, one gets a causally geodesically complete spacetime with asymptotic behaviour of the form given in the statement of Theorem 2 both to the future and to the past.

Theorem 3. *Let V be a smooth function such that $V(0) = V_0 > 0$, $V'(0) = 0$ and $V''(0) > 0$. Let $H, \chi > 0$ be defined by (10) and (11) respectively. Finally, let Σ be an $n \geq 3$ dimensional compact manifold and assume that it admits a Riemannian metric g_Σ such that $\mathrm{Ric}[g_\Sigma] = (n-1)H^2 g_\Sigma$. Given t_0 and a fixed choice of Sobolev norms $\|\cdot\|_{H^l}$ on tensors on Σ, there is an $\epsilon > 0$ such that if $(\Sigma, h, \kappa, \phi_0, \phi_1)$ are* *initial data for (12)-(13) satisfying*

$$\|h - \cosh^2(Ht_0)g_\Sigma\|_{H^{k_0+1}} + \|\phi_0\|_{H^{k_0+1}} + \|\phi_1\|_{H^{k_0}} + \|\kappa - H\sinh(Ht_0)\cosh(Ht_0)g_\Sigma\|_{H^{k_0}} \leq \epsilon$$

where k_0 is the smallest integer satisfying $k_0 > n/2 + 1$, then the maximal globally hyperbolic development associated with $(\Sigma, h, \kappa, \phi_0, \phi_1)$ is causally geodesically complete and admits expansions as stated in Theorem 2 both to the future and to the past.

Remark. The metric (3) is a solution to (12)-(13) with $\phi = 0$. Concerning the definition of Sobolev norms on tensorfields on manifolds, we refer the reader to Subsection 3.4. The above statement constitutes a generalization of some of the stability results of [15] and [1], but we are of course not able to prove any results starting at infinity. The statement that there are expansions to the future should be interpreted as saying that there is a Cauchy hypersurface S in the maximal globally hyperbolic development of $(\Sigma, h, \kappa, \phi_0, \phi_1)$ such that for every $p \in S$, there is a neighbourhood of p to which Theorem 2 applies. The statement concerning the past is similar. In particular, all the spacetimes M constructed in the above theorem have the property that late time observers in M are completely oblivious to topology.

The proof of the above theorem is to be found at the end of Section 17.

Let us consider spatially locally homogeneous solutions to Einstein's equations with a positive cosmological constant. Concerning this situation, there are results due to Wald, cf. [33], in the case of $n = 3$. Since the next theorem is partly based on the results of [33], we shall thus restrict our attention to the dimension $n = 3$ for the remainder of this subsection. Due to the analysis presented in [33], one would expect most spatially locally homogeneous solutions to Einstein's equations with a positive cosmological constant to have the property that at a "late enough" Cauchy hypersurface, Theorem 2 would be applicable in a neighourhood of every point. The possible exceptions would be spacetimes whose Cauchy hypersurfaces have universal covering spaces diffeomorphic to $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$, cf. (6) and the remark made after the statement of Theorem 2. In this paper, we shall only consider spatially locally homogeneous spacetimes that have compact spatial topology, and we shall exclude solutions whose Cauchy hypersurfaces have universal covering spaces diffeomorphic to $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. One could of course also consider the case of non-compact spatial topology, but as far as we know, there are no methods that would guarantee the existence of suitable non-trivial perturbations of homogeneous initial data that do not admit a compact quotient.

Theorem 4. *Let V be a smooth function such that $V(0) = V_0 > 0$, $V'(0) = 0$ and $V''(0) > 0$. Let $H, \chi > 0$ be defined by (10) and (11) respectively, let M be a connected and simply connected 3-dimensional manifold and let (M, g, k) be initial data for Einstein's equations with a positive cosmological constant $\Lambda = 3H^2$. Assume, furthermore, that one of the following conditions are satisfied:*

- *M is a unimodular Lie group different from $\mathrm{SU}(2)$ and g and k are left invariant under the action of this group.*
- *$M = \mathbb{H}^3$, where $\mathbb{H}^n$ is n-dimensional hyperbolic space, and the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^3$.*

- *$M = \mathbb{H}^2 \times \mathbb{R}$ and the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^2 \times \mathbb{R}$.*

Assume finally that $\mathrm{tr}_g k > 0$. Let Γ be a cocompact subgroup of M in the case that M is a unimodular Lie group and a cocompact subgroup of the isometry group otherwise. Let Σ be the compact quotient. Then (Σ, g, k) are initial data. Make a choice of Sobolev norms $\|\cdot\|_{H^l}$ on tensorfields on Σ. Then there is an $\epsilon > 0$ such that if $(\Sigma, \rho, \kappa, \phi_0, \phi_1)$ are initial data for (12)-(13) satisfying

$$\|\rho - g\|_{H^{k_0+1}} + \|k - \kappa\|_{H^{k_0}} + \|\phi_0\|_{H^{k_0+1}} + \|\phi_1\|_{H^{k_0}} \leq \epsilon,$$

where k_0 is the smallest integer satisfying $k_0 > n/2+1$, then the maximal globally hyperbolic development corresponding to $(\Sigma, \rho, \kappa, \phi_1, \phi_0)$ is future causally geodesically complete and there are expansions of the form given in the statement of Theorem 2 to the future.

Remark. If M is a 3-dimensional unimodular Lie group it contains a cocompact subgroup Γ, cf. [26]. Concerning the definition of Sobolev norms on tensorfields on manifolds, we refer the reader to Subsection 3.4. The statement that there are expansions to the future should be interpreted as saying that there is a Cauchy hypersurface S in the maximal globally hyperbolic development of $(\Sigma, \rho, \kappa, \phi_0, \phi_1)$ such that for every $p \in S$, there is a neighbourhood of p to which Theorem 2 applies. As a consequence, all the spacetimes M constructed in the above theorem have the property that late time observers in M are completely oblivious to topology.

The proof of the above theorem is to be found in Section 17.

The proof of Theorem 4 is based on Theorem 2 and illustrates the usefulness of a result in which local (in space) assumptions yield global (in time) conclusions; one can ignore the global topology of the compact spacelike hypersurfaces in the argument. Since Theorem 2 could probably have been proved in the case of a positive cosmological constant using the methods of [15], Theorem 4 could reasonably also have been proved in that context quite some time ago. The latter theorem constitutes a quite general stability result for spatially locally homogeneous solutions to Einstein's equations with a positive cosmological constant and compact Cauchy hypersurfaces. It should also be noted that the spacetimes resulting from the perturbed initial data are not only future causally geodesically complete; they also have asymptotics "similar" to those of the spacetimes around which one is perturbing, cf. the asymptotics obtained as a result of Theorem 2. Note that the corresponding result is not to be expected in the case of a vanishing cosmological constant. As a justification for this statement, let us quote the following result from [31] (based on the results of [29]). Consider the maximal globally hyperbolic development (M, g) of left invariant vacuum initial data on $\widetilde{\mathrm{Sl}}(2, \mathbb{R})$, the universal covering group of $\mathrm{Sl}(2, \mathbb{R})$ (which is a unimodular Lie group). Let Σ be a Cauchy hypersurface of homogeneity and for $p \in M$, let $t_\Sigma(p)$ be the proper time distance from Σ to p. Define

$$\eta_\Sigma = \sup\{a \geq 0 \mid \exists C < \infty : |(t_\Sigma^a R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})(p)| \leq C \ \forall p \in J^+(\Sigma)\}.$$

Then $\eta_\Sigma = 6$ if the initial data have an extra rotational symmetry and $\eta_\Sigma = 2$ if not. In particular, η_Σ does not depend on Σ. This proves that the initial data with an additional rotational symmetry lead to developments with radically different behaviour from those corresponding to initial data without this symmetry.

Let us make a brief comment concerning the topologies allowed in Theorem 4. In the case of $\mathbb{H}^2 \times \mathbb{R}$, the resulting topologies are trivial circle bundles over a higher genus surface, in the case of $\mathbb{H}^3$ one gets compact 3-dimensional hyperbolic manifolds, and in the case that the initial data are left invariant under the group action of a unimodular Lie group, we refer the reader to [26] for a discussion of the resulting topologies. The restriction to cocompact subgroups of unimodular Lie groups is in a sense artificial. Most unimodular Lie groups admit metrics with a 4-dimensional isometry group, and such metrics admit a much larger class of cocompact subgroups of the isometry group. The reason we have excluded these possibilities is mainly technical; complications arise when analyzing how isometries of the development resulting from isometries of the data relate to the specific foliation under consideration. Since these technical issues are not the main subject of this paper, we shall not treat them here, though there is no reason to believe that it could not be done. Note, however, that restricting one's attention to more symmetric metrics reduces the freedom in specifying initial data.

The above result is only intended to give an example of results that follow from Theorem 2. As another example, Alan Rendall has obtained results concerning spatially homogeneous solutions to exactly the type of model considered in this paper, cf. [27]. Combining those results with the ones obtained in this paper, it should be possible to prove stability results for spatially locally homogeneous spacetimes with compact spatial Cauchy hypersurfaces for which the scalar field is not necessarily small initially.

In Section 18 we demonstrate that there are initial data on manifolds of arbitrary compact topology that yield future causally geodesically complete maximal globally hyperbolic developments.

1.5. **Outline of the proof of Theorem 2.** As we shall explain in more detail below, the essential problem is that of proving stability of the metric (4) on $\mathbb{R}\times\mathbb{T}^n$. In order to obtain a hyperbolic problem, we shall use gauge source functions, cf. [18]. The idea is to replace $R_{\mu\nu}$ in the equation (12) with $\hat{R}_{\mu\nu}$ given by (47), where F_μ are the gauge source functions. In other words, we have the relations (49)-(50). The F_μ are allowed to depend on the spacetime coordinates and on the metric, but not on any derivatives of the metric. The modified system, obtained by replacing $R_{\mu\nu}$ with $\hat{R}_{\mu\nu}$ in (12)-(13), is then a system of hyperbolic PDE's. If the constraint equations are satisfied originally and the initial data for the modified system are set up in the right way, one can prove that $\mathcal{D}_\mu$, defined in (50), vanishes where the solution is defined so that one obtains solutions to the original equations by solving the modified system, cf. Section 4 for the details. However, for practical reasons, we shall be interested in initial data that do not satisfy the constraint equations. The most naive choice of gauge source functions would be the contracted Christoffel symbols of the background. When considering a solution such as Minkowski space, there is a natural candidate, namely $F_\mu = 0$. In the case of the metric (4), there are, however, at least two choices. For (4), $\Gamma_0 = -nH$ and $\Gamma_i = 0$ with respect to the standard coordinates on $\mathbb{R}\times\mathbb{T}^n$. As a consequence, there are two naive choices; either one fixes F^μ to be the contracted Christoffel symbols of the background with indices upstairs or one fixes F_μ to be the contracted Christoffel symbols of the background with indices downstairs. It turns out to be useful to choose the former of these possibilities; our choice $F_\mu = nHg_{0\mu}$ yields

$$\nabla_{(\mu}F_{\nu)} = \frac{1}{2}nH\partial_0 g_{\mu\nu},$$

cf. the proof of Lemma 4 (for the case under discussion here, $\omega = H$), which acts as a damping term. Unfortunately, it turns out to be insufficient to only use gauge source functions, we need to add corrections as well, cf. (53)-(54), where the $M_{\mu\nu}$ and M_ϕ vanish when $\mathcal{D}_\mu$ vanishes and contain at most first derivatives of the metric and scalar field. The reason we need to add these corrections is because we are interested in initial data that violate the constraints. The specific choices we make, cf. (51)-(52), lead to the equations (144)-(147), where $u = g_{00} + 1$, $u_i = g_{0i}$ and $h_{ij} = e^{-2Ht}g_{ij}$. In order to indicate on a heuristic level why these equations are to be preferred over the ones obtained without using the corrections, let us consider the equations that result, starting with (144)-(147), when we ignore terms that are quadratic in expressions that vanish for the background solution and when we replace $g^{\alpha\beta}\partial_\alpha\partial_\beta$ with $-\partial_t^2 + e^{-2Ht}\Delta$, where Δ is the ordinary Laplace operator on $\mathbb{T}^n$ (these equations give a rough impression of the asymptotic behaviour, but they do not give the correct decay rates). The corresponding equations for u, h_{ij} and ϕ then decouple and we obtain exponential decay for u and ϕ and convergence for h using straightforward energy estimates. Let us illustrate how one obtains decay by considering the idealized equation for u:

$$u_{tt} - e^{-2Ht}\Delta u + (n+2)H\partial_0 u + 2nH^2 u = 0. \tag{32}$$

One can find positive constants, γ, δ, ζ, η such that

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{T}^n}[u_t^2 + e^{-2Ht}|\nabla u|^2 + 2\gamma H u u_t + \delta H^2 u^2]dx,$$

where ∇ is the ordinary gradient on $\mathbb{T}^n$, satisfies

$$\partial_t\mathcal{E} \leq -\eta H\mathcal{E}, \quad \mathcal{E} \geq \zeta\int_{\mathbb{T}^n}[u_t^2 + e^{-2Ht}|\nabla u|^2 + H^2u^2]dx.$$

The argument to prove this is similar to, but simpler than, the proof of Lemma 15. As a consequence, $\mathcal{E}$ decays exponentially and by applying spatial derivatives to (32), we can use the same argument to obtain exponential decay of u in any C^l norm. The equation for u_i does not decouple, but it is possible to use the information already obtained concerning the other components of the metric to analyze its behaviour. If we had not added the corrections, we would not have obtained this structure. Considering the proof of Lemma 6, we see that if we had not added the term M_{00}, then the equation for u obtained after ignoring quadratic terms etc. would have coupled to both u_m and h_{ij}. Similarly, had we not added M_{0i}, the equation for u_i resulting after the above idealizations would have been even less decoupled than before. However, the worst aspect of not adding the corrections would be that the “damping structure” of the equations for u and u_i would be lost, where by “damping structure” we mean the second and third terms on the left hand side of (144) and (145) (note also that in order to get this damping structure, we need to have $n \geq 3$ due to (145)).

When proving future global existence of solutions to (144)-(147), the main problem is of course to find bootstrap assumptions that

- make it easy to estimate the non-linear terms that are quadratic in expressions that vanish on the background (in the end we devise a method whereby it is enough to compute the number of spatial indices of a such a term and the number of derivatives it contains in order to estimate it in the Sobolev space of interest),
- can be expressed naturally in terms of energies that, in their turn, fit together naturally with the structure of the equations (144)-(147).

Ideally, one would like the bootstrap assumptions to be such that they yield optimal control of the solutions. However, it is difficult to see how that could be achieved in the present situation due to the fact that the leading order part of the energies naturally associated with the equations is, up to some factor, equivalent to

$$E_{\mathrm{lo},l} = \sum_{|\alpha|\leq l} \int_{\mathbb{T}^n} [(\partial^\alpha \partial_t v)^2 + g^{ij}(\partial_i \partial^\alpha v)(\partial_j \partial^\alpha v)]dx, \tag{33}$$

where v should be thought of as one of u, u_i, h_{ij} and ϕ. The problem arises due to the fact that g^{ij} is of the order of magnitude e^{-2Ht}. This has as a consequence that for a given bound on $E_{\mathrm{lo},l}$, we typically obtain a much worse estimate for the highest order spatial derivatives of v than for the spatial derivatives of v of order l (in the latter case, integration of the estimate for the first part of the integrand in (33) typically produces an improved estimate). However, there is no reason to expect the spatial derivatives of order $l+1$ to have worse decay than the spatial derivatives of order l, and, in the end, this is of course not the case. The bootstrap assumptions are therefore not optimal and, in particular, they allow for an exponentially increasing g_{0i}, even though g_{0i} can be proved to converge after global existence has been proved. In fact, the main motivation for the particular bootstrap assumptions we have chosen is that they ensure that it is easy to estimate the non-linear terms, as opposed to bootstrap assumptions chosen to fit the actual behaviour.

The essence of the proof of global existence is in proving that the bootstrap assumptions can be improved if the initial data are close enough to what one is perturbing around. The means by which one achieves this goal are the equations (144)-(147). It is very important to note that, given the bootstrap assumptions, these equations have a hierarchical structure. The bootstrap assumptions can be used to estimate Δ_{00},...,Δ_ϕ appearing in (144)-(147), to estimate the commutators that appear when differentiating these equations with respect to the spatial variables and to estimate various terms arising in connection with the time differentiation of the energies. As a consequence, the equations for u, h_{ij} and ϕ can in practice be treated as if though they were decoupled, and it is possible to improve the bootstrap assumptions for the energies associated with these quantities before turning to the improvement of the energy associated with u_i. In other words, it is here crucial that we are dealing with a system of hyperbolic equations as opposed to a scalar hyperbolic equation.

Concerning the asymptotics, let us note that it is necessary to improve what one obtains in the bootstrap argument significantly after having proved global existence, cf. Section 14.

The rough idea of the proof of Theorem 2 is to take the given initial data with respect to the local coordinates assumed to exist and to interpret them as the components of initial data on a subset of $\mathbb{T}^n$ with respect to standard coordinates.

By using a cut-off function, one obtains a metric and a second fundamental form on $\mathbb{T}^n$ coinciding with the original data on an open subset, say U, and which are close to the initial data corresponding to (4). As a consequence of the construction, the constraints will typically be violated outside of U. However, the stability argument on $\mathbb{T}^n$ described above works even for initial data violating the constraints, so that one obtains a global solution to the future, and this is the main step. In the Cauchy development of U, one obtains a solution to the Einstein-scalar field equations, and this can be used as one patch in the construction of a globally hyperbolic development of the initial data given in the statement of Theorem 2. This globally hyperbolic development can then be embedded into the maximal globally hyperbolic development (MGHD) by the abstract properties of the MGHD.

Let us compare the arguments carried out here with other proofs of stability in the case of Einstein's equations. Beyond the work of Helmut Friedrich, which has already been mentioned, it is natural to mention the work of Christodoulou and Klainerman [11], the work of Andersson and Moncrief [3] and the work of Lindblad and Rodnianski [22, 23]. The methods used in the work of Friedrich, in [11] and in [3] are very different from the ones used in the present work and consequently, making a comparison is not very fruitful. In the case of [22], the situation studied is very different. In particular, [22] represents a more subtle situation due to the fact that the rate of decay is on the borderline of what can be handled; it is necessary to find additional structure in the non-linearity such as a null-condition, cf. [10, 21] or weak null condition, cf. [22, 23]. Consequently, as far as the rate of decay is concerned, the present situation is easier to deal with. Nevertheless, there are similarities between the present argument and the one presented in [22]. Let us focus on two aspects: the fact that the equations under consideration are systems, as opposed to scalar equations, and the use of the wave coordinate condition in the case of [22]. In [22], the metrics of interest are close to that of Minkowski space, and it turns out that derivatives tangential to the future Minkowski light cones have better decay than the derivatives transverse to these cones. As a consequence, it is natural to divide the equation for the components of the metric into different parts with respect to a null frame and pay special attention to the terms that correspond to two vectors transverse to the future light cone. In this process, two important observations are made in [22]. First, the solutions obtained obey not only the Einstein equations but also the wave coordinate condition; this can be used to improve the decay estimates for certain components of the metric. Second, the components can be divided into two groups, let us call them "good" and "bad". Acting with the wave operator on the good components yields terms for which there are good estimates, and acting with the wave operator on the bad components yields terms for which there are good estimates and terms that can be estimated in terms of the good components. In other words, there is a hierarchy similar to the one described above; one can improve one's knowledge concerning the good components first, and then turn to the bad components. We refer the reader to [22] for a careful discussion of these issues, cf. in particular (2.19)-(2.20) of [22]. There are also some similarities as far as the importance of the wave coordinate condition is concerned. In our case, we wish to study the equations in a setting where the constraint equations are violated, and consequently, we cannot assume that we have the condition analogous to the wave coordinates gauge. However, the corrections fill a function similar to that of using the wave coordinates condition.

If we were to solve the equations on $\mathbb{T}^n$ and were to assume that the constraints were satisfied, it would not be necessary to add the corrections; we could simply use the fact that the gauge source functions in that case would equal the contracted Christoffel symbols. It would be of interest to know if one could add corrections to the equations in [22], similar to the ones used in the present paper, in such a way that one could avoid using the wave coordinates condition and prove a stability result for initial data violating the constraints. Such a result might be of interest to numerical analysts. Finally, let us note that in recent numerical work concerning Einstein's equations, methods similar to the use of gauge source functions have been employed, cf. [25]. However, there is one significant difference; in the present paper, as well as in [22], the gauge source functions are given explicitly in terms of the metric whereas in [25], the gauge source functions obey certain equations, so that one obtains a coupled system for the metric components, the scalar field and the gauge source functions.

1.6. **Further applications.** In this paper we discuss the case of a non-linear scalar field with a potential satisfying $V(0) > 0$, $V'(0) = 0$ and $V''(0) > 0$. The methods developed here can, however, be used as a basis for proving similar stability results in the case that the potential is of the form $V(\phi) = V_0 \exp(-\lambda\phi)$ where $\lambda < \sqrt{2}$ for $n = 3$, as we shall demonstrate in a future paper. These types of scalar fields have been used by physicists as a mechanism for generating accelerated expansion.

2. Linear algebra

The proof that certain causal geodesics are future complete is based on a PDE argument in which the basic unknowns are the scalar field and the components of the metric. It will be natural to divide the latter part of the unknowns into three different types according to their asymptotic behaviour, and in the present section, we wish to specify the notation and make some observations that will be of use in making this division.

Let g be a symmetric $(n+1)\times(n+1)$-dimensional real valued matrix with components $g_{\mu\nu}$, $\mu,\nu = 0,...,n$. We shall denote the $n\times n$ matrix with components g_{ij}, $i,j = 1,...,n$, by $g_\flat$ and if g is invertible, we shall denote the components of the inverse by $g^{\mu\nu}$, $\mu,\nu = 0,...,n$ and the $n\times n$ matrix with components g^{ij}, $i,j = 1,...,n$, by $g^\sharp$. We shall use $v[g]$ to denote the n-vector with components g_{0i}, $i = 1,...,n$ and for any symmetric and positive definite $n\times n$-matrix ξ and any n-vector v, we shall write

$$|v|_\xi = \left(\sum_{i,j=1}^{n} \xi_{ij} v^i v^j\right)^{1/2}.$$

We shall also use the notation $|v| = |v|_\delta$, where δ_{ij} is the Kronecker delta. Furthermore, if A is an $n\times n$ real valued matrix (not necessarily symmetric), we shall denote the $(n+1)\times(n+1)$-dimensional matrix with 00-component 1, $0i$ and $i0$-components 0 and ij components given by A_{ij} by M_A. Finally, if ρ is a symmetric, real valued $(n+1)\times(n+1)$-matrix with one negative eigenvalue and n positive ones, we shall say that it is a *Lorentz matrix.*

Lemma 1. *Let ρ be a symmetric $(n+1)\times(n+1)$ real valued matrix. Assume that $\rho_{00} < 0$ and that $\rho_\flat$ is positive definite. Then ρ is a Lorentz matrix.*

Proof. Let A be an orthogonal $n \times n$-matrix diagonalizing $\rho_\flat$ and $h = M_A^t \rho M_A$. Then $h_\flat = A^t \rho_\flat A$ is diagonal, with diagonal elements $\lambda_i > 0$, $i = 1, ..., n$. Furthermore $h_{00} = \rho_{00} < 0$ and the eigenvalues of ρ and h coincide. If we compute the determinant of $h - \lambda \mathrm{Id}$, we obtain

$$p(\lambda) = \left(\rho_{00} - \lambda - \frac{h_{01}^2}{\lambda_1 - \lambda} - ... - \frac{h_{0n}^2}{\lambda_n - \lambda}\right)(\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

Let us define

$$f(\lambda) = \rho_{00} - \lambda - \frac{h_{01}^2}{\lambda_1 - \lambda} - ... - \frac{h_{0n}^2}{\lambda_n - \lambda}.$$

If we differentiate this function, we obtain

$$f'(\lambda) = -1 - \frac{h_{01}^2}{(\lambda_1 - \lambda)^2} - ... - \frac{h_{0n}^2}{(\lambda_n - \lambda)^2}.$$

Note that $\lambda_1, ..., \lambda_n$ are all positive. Let us denote the smallest of the λ_i by $\lambda_{\min}$. For λ belonging to the interval $(-\infty, \lambda_{\min})$, we obtain the conclusion that $f'(\lambda) < 0$. Furthermore, $f(-\infty) = \infty$ and $f(0) < 0$. Thus there is a unique negative value of λ, say λ_0, for which $f(\lambda) = 0$. This is clearly an eigenvalue of h. Since it is easy to see that $p'(\lambda_0) \neq 0$, we see that λ_0 is a root with multiplicity one to the polynomial equation $p(\lambda) = 0$. There is in other words only one eigenvalue in the interval $(-\infty, \lambda_{\min})$. Since h is a symmetric matrix, it only has real eigenvalues, so the remaining n eigenvalues have to be positive. $\square$

Lemma 2. *Let g be a symmetric $(n+1) \times (n+1)$ real valued matrix. Assume $g_{00} < 0$ and that $g_\flat$ is positive definite. Then g is a Lorentz matrix,*

$$g^{00} = \frac{1}{g_{00} - d^2}, \tag{34}$$

where $d = |v[g]|_{g_\flat^{-1}}$, $g^\sharp$ is positive definite, with

$$\frac{g_{00}}{g_{00} - d^2} |w|^2_{g_\flat^{-1}} \leq |w|^2_{g^\sharp} \leq |w|^2_{g_\flat^{-1}} \tag{35}$$

for any $w \in \mathbb{R}^n$ and

$$v[g^{-1}] = \frac{1}{d^2 - g_{00}} g_\flat^{-1} v[g]. \tag{36}$$

Note that g^{00} is negative, since g_{00} is negative, and that there is an upper bound on this quantity depending only on g_{00} and d.

Proof. The first statement of the lemma is a direct consequence of Lemma 1. To prove the remaining statements, let A be the square root of $g_\flat^{-1}$, i.e. the positive definite, symmetric matrix with the property that $A^2 = g_\flat^{-1}$. Then $A^t g_\flat A = \mathrm{Id}$. Consider $h = M_A^t g M_A$. Then $h_{00} = g_{00}$, $h_\flat = \mathrm{Id}$ and $v[h] = A^t v[g]$. Let B be an orthogonal matrix such that

$$B^t A^t v[g] = |A^t v[g]| e_1, \tag{37}$$

where $e_1 = (1, 0, ..., 0)^t$. Note that

$$|A^t v[g]| = |v[g]|_{g_\flat^{-1}} = d. \tag{38}$$

Consider $\rho = M_B^t M_A^t g M_A M_B$. Then $\rho_{00} = g_{00}$, $\rho_\flat = \mathrm{Id}$ and $v[\rho] = de_1$. Note that the inverse of the 2×2-matrix with components $\rho_{\mu\nu}$, $\mu, \nu = 0, 1$ is given by

$$\frac{1}{g_{00} - d^2} \begin{pmatrix} 1 & -d \\ -d & g_{00} \end{pmatrix}.$$

Since

$$g^{-1} = M_A M_B \rho^{-1} M_B^t M_A^t,$$

and the matrices M_A and M_B preserve the 00-component of a matrix, we obtain (34). Furthermore

$$g^\sharp = AB\rho^\sharp B^t A^t.$$

We are interested in the supremum and infimum, for $w \neq 0$, of

$$\frac{|w|_{g^\sharp}^2}{|w|_{g_\flat^{-1}}^2} = \frac{(g^\sharp w, w)}{(Aw, Aw)} = \frac{(\rho^\sharp B^t A^t w, B^t A^t w)}{(B^t A^t w, B^t A^t w)},$$

where we have used the fact that B is orthogonal and A is symmetric. Since $\rho^\sharp$ is diagonal with 11-component equal to $g_{00}/(g_{00} - d^2) \leq 1$ and the ii-components equal 1 for $i > 1$, we obtain (35). Since

$$v[\rho^{-1}] = -\frac{d}{g_{00} - d^2} e_1, \quad dBe_1 = A^t v[g],$$

where we have used (37), (38) and the fact that B is orthogonal to obtain the second equality, we obtain

$$v[g^{-1}] = ABv[\rho^{-1}] = -\frac{1}{g_{00} - d^2} A^2 v[g] = -\frac{1}{g_{00} - d^2} g_\flat^{-1} v[g],$$

which implies (36). Note that one could also have obtained this equality by applying $g_\flat^{-1}$ to

$$g^{0i} g_{ij} + g^{00} g_{0j} = 0$$

and using the fact that (34) holds. □

3. Background material

In this section we state the background material we shall be needing. For most statements, we shall not provide any proofs, but we wish to make precise statements. Let us start by local existence of non-linear wave equations.

3.1. **Standard local existence.**

Let g be a smooth function from $\mathbb{R}^{nN+2N+n+1}$ to the set of symmetric real valued $(n+1) \times (n+1)$-matrices. We shall denote the components $g_{\mu\nu}$ and assume that for every multiindex $\alpha = (\alpha_1, ..., \alpha_{nN+2N+n+1})$ and compact interval $I = [T_1, T_2]$, there is a continuous, increasing function $h_{I,\alpha} : \mathbb{R} \to \mathbb{R}$ such that

$$|(\partial^\alpha g_{\mu\nu})(t, x, \xi)| \leq h_{I,\alpha}(|\xi|) \tag{39}$$

for all $\mu, \nu = 0, ..., n$, $t \in I$, $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{nN+2N}$. Assume that there are constants $a_1, a_2, a_3 > 0$ such that $g_{00} \leq -a_1$, $g_\flat \geq a_2$ and $|g_{\mu\nu}| \leq a_3$, with notation as in Section 2. Then g is a Lorentz matrix valued function due to Lemma 1 and we shall denote the components of the inverse by $g^{\mu\nu}$. As a consequence of our assumptions, the components of the inverse are bounded, and there are constants

$b_1, b_2 > 0$ such that $g^{00} \leq -b_1$ and $g^\sharp \geq b_2$. Given a C^1 function $u : \Omega \to \mathbb{R}^N$ for some $\Omega \subseteq \mathbb{R}^{n+1}$, we define $g[u]$ to be the function on Ω given by

$$g[u](t,x) = g\{t, x, u(t,x), \partial_0 u(t,x), ..., \partial_n u(t,x)\}.$$

Let f be a smooth function from $\mathbb{R}^{nN+2N+n+1}$ to $\mathbb{R}^N$ satisfying an estimate of the form (39). We shall use similar conventions concerning f as we do concerning g, in particular we shall write $f[u]$, the meaning being analogous to the case of g. Sometimes it will be convenient to view f as the sum of two functions, $f = f_a + f_b$, where f_b only depends on t and x and f_a has the property that $f_a(t, x, 0, ..., 0) = 0$. Given the above division, we shall assume that f_b is of locally x-compact support; a function $h : \mathbb{R}^{n+1} \to \mathbb{R}^m$ is said to be of *locally x-compact support* if for any compact interval $[T_1, T_2]$ there is a compact set K such that $h(t,x) = 0$ if $t \in [T_1, T_2]$ and $x \notin K$. Note that a smooth function $u : \mathbb{R}^{n+1} \to \mathbb{R}^m$ of locally x-compact support can be viewed as an element of $C^l[\mathbb{R}, H^k(\mathbb{R}^n, \mathbb{R}^m)]$ for any l, k. This is no longer true if we consider smooth functions with the property that for every fixed t, $u(t, \cdot)$ has compact support. A simple counterexample is obtained by taking $\phi \in C_0^\infty(\mathbb{R}^n)$ which is not identically zero and defining $u(t,x) = \phi(x^1 - 1/t, x^2, ..., x^n)$ for $t > 0$ and $u(t,x) = 0$ for $t \leq 0$.

Consider the initial value problem

$$g^{\mu\nu}\partial_\mu \partial_\nu u = f, \tag{40}$$

$$u(0, \cdot) = U_0 \tag{41}$$

$$\partial_t u(0, \cdot) = U_1, \tag{42}$$

where we write g instead of $g[u]$ and f instead of $f[u]$.

Theorem 5. *Let $U_0, U_1 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$. Then there are $T_- < 0 < T_+$ and a unique solution $u \in C^\infty[(T_-, T_+) \times \mathbb{R}^n, \mathbb{R}^N]$ to (40)-(42). The solution is of locally x-compact support and T_+ can be chosen so that either $T_+ = \infty$ or*

$$\lim_{\tau \to T_+ -} \sup_{0 \leq t \leq \tau} \sum_{|\alpha| + j \leq 2} \sup_{x \in \mathbb{R}^n} |\partial^\alpha \partial_t^j u(t,x)| = \infty,$$

where the α are spatial multiindices $\alpha = (\alpha_1, ..., \alpha_n)$. The statement concerning T_- is similar.

This can be proved e.g. along the lines of [20], Theorem 6.4.11.

3.2. **Causality.** Let us remind the reader of the basic causality concepts in Lorentz geometry, cf. [24], Chapter 14. If (M, g) is a time oriented Lorentz manifold, we shall use the notation that $p < q$ if there is a future pointing causal curve from p to q and $p \leq q$ if $p < q$ or $p = q$. Given a subset A of M, define

$$J^+(A) = \{p \in M \mid \exists\, q \in A : q \leq p\},$$
$$J^-(A) = \{p \in M \mid \exists\, q \in A : p \leq q\}.$$

The sets $J^+(A)$ and $J^-(A)$ are called the *causal future* and *past* of A respectively. The *strong causality condition* is said to hold at $p \in M$ provided that given any neighbourhood U of p there is a neighbourhood $V \subseteq U$ of p such that every causal curve segment with endpoints in V lies entirely in U. A Lorentz manifold is said to be *globally hyperbolic* if the strong causality condition holds at each of its points and if for each pair $p < q$, the set $J(p,q) = J^+(p) \cap J^-(q)$ is compact. The assumption that a Lorentz manifold M be globally hyperbolic has many consequences, e.g. if $p, q \in M$ and $p \leq q$, then there is a causal geodesic from p to q such that no causal curve from p to q can have greater length, cf. Proposition 19, p. 411 of [24]. Furthermore, the causality relation $\leq$ is closed on M, i.e. if $p_n \to p$, $q_n \to q$ and $p_n \leq q_n$, then $p \leq q$, cf. Lemma 22, p. 412 of [24].

A subset A of a Lorentz manifold (M, g) is said to be *achronal* if there is no pair of points $p, q \in A$ that can be connected by a timelike curve and it is said to be *acausal* if no pair of points in A can be connected by a causal curve. Given an achronal subset A of M, the *future Cauchy development* of A is the set $D^+(A)$ of all points p of M such that every past inextendible causal curve through p meets A. The past Cauchy development $D^-(A)$ is defined analogously and we write $D(A) = D^+(A) \cup D^-(A)$. A *Cauchy hypersurface* in M is a subset S that is met exactly once by every inextendible timelike curve in M. Then $D(S) = M$ due to Lemma 29, p. 415 of [24]. One can prove that a Lorentz manifold is globally hyperbolic if and only if it admits a Cauchy hypersurface, cf. Corollary 39, p. 422 of [24] and [19]. Furthermore, a globally hyperbolic Lorentz manifold admits a smooth spacelike Cauchy hypersurface, cf. [4]-[6].

In the end, we shall need the following, somewhat more technical, statements.

Lemma 3. *Let (M, g) be a Lorentz manifold and assume it admits a smooth spacelike Cauchy hypersurface S. Then, for $p \in J^+(S)$, $J^-(p) \cap J^+(S)$ is compact. If S is compact and $\Omega \subseteq S$ is open, with respect to the topology induced on S, then $D(\Omega)$ is open. If $U \subseteq M$ is open, $q \in J^+(S)$ and $J^-(q) \cap J^+(S) \subseteq U$, then if $q_i \in J^+(S)$ are such that $q_i \to q$, we have $J^-(q_i) \cap J^+(S) \subseteq U$ for i large enough. If $\Omega \subseteq S$ is closed, then $D(\Omega)$ is closed. If $q_i \leq q$, $q \in I^+(S)$ and $q_i \to q$, then the closure of the union of the $J^-(q_i) \cap J^+(S)$ is $J^-(q) \cap J^+(S)$.*

Proof. The first statement follows from Lemma 40 p. 423 of [24]. Assume $\Omega \subseteq S$ is open with respect to the topology induced on S and that $D(\Omega)$ is not open. Then there is a $q \in D(\Omega)$ and $q_i \to q$ such that $q_i \notin D(\Omega)$. We conclude that there are $r_i \in S - \Omega$ such that $r_i \leq q_i$ or vice versa. Assume, without loss of generality, that $r_i \leq q_i$ for all i. Since $S - \Omega$ is compact, we can assume $r_i \to r \in S - \Omega$. Since the relation $\leq$ is closed on a globally hyperbolic manifold, $r \leq q$, contradicting the fact that $q \in D(\Omega)$. In order to prove the third statement, let p be such that there is a future directed timelike curve from q to p. Then $J^-(p) \cap J^+(S)$ is compact and $J^-(p)$ contains q in its interior, cf. Lemma 3, p. 403 of [24]. Since $q_i \to q$, $J^-(q_i) \subseteq J^-(p)$ for i large enough. Assuming the desired statement is not true, there is a subsequence q_{i_k} and points $r_{i_k} \in J^-(q_{i_k}) \cap J^+(S)$ such that $r_{i_k} \notin U$. Since the r_{i_k} are in the compact set $J^-(p) \cap J^+(S) - U$ for k large enough, we can assume that they converge to a point r. Then $r \in J^-(q) \cap J^+(S) - U$, a contradiction. To prove the fourth statement, assume, in order to obtain a contradiction, that $q_i \in D(\Omega)$ and $q_i \to q \notin D(\Omega)$. Assume, without loss of generality, that $q, q_i \in J^+(S)$ and let $p \in S - \Omega$ be such that $p \leq q$. Then, due to the time reversal of the third statement, there is an r in the timelike past of p such that $J^+(r)$ does not intersect Ω. Thus q is in the timelike future of r, cf. Corollary 1, p. 402 of [24], so that $q_i \in J^+(r)$ for i large enough, so that $q_i \notin D(\Omega)$, contradicting the assumptions. To prove the last statement, let $p \in J^-(q)$. If $p \in I^+(S)$, let $p_k \to p$ be such that $p_k \in J^+(S)$ is in the timelike past of p. Then q is in the timelike future of p_k. Thus there is an i_k such that q_{i_k} is in the timelike future of p_k. Thus all the p_k are in the union of the $J^-(q_i)$, so that p is in the closure of the union of $J^-(q_i) \cap J^+(S)$. Assume $p \in S \cap J^-(q)$. Let $p_k \ll p$ be such that $p_k \to p$, let i_k be such that q_{i_k} is in the timelike future of p_k and let γ_k be a timelike curve from p_k to q_{i_k}. Denote the point of intersection between γ_k and S by p'_k. Since $p'_k \in J^-(q) \cap J^+(S)$, which is compact, we can choose a subsequence so that it converges to, say, r. Since $p_k \leq p'_k$ and p_k converges to p, we conclude that $p \leq r$. Since $p \in S$ and S is a spacelike Cauchy hypersurface, we have to have $p = r$. The conclusion follows. □

3.3. **Uniqueness.** We shall need the following uniqueness result.

Theorem 6. *Let (M, g) be a globally hyperbolic $n+1$-dimensional Lorentz manifold and let S be a smooth spacelike Cauchy hypersurface. Let $\Omega \subseteq S$ and assume that U is an open set containing $\overline{D^+(\Omega)}$. Assume $u : U \to \mathbb{R}^l$ is a smooth solution to the equation*

$$\nabla^\alpha \nabla_\alpha u + Xu + cu = 0,$$

where X is an $l \times l$ matrix of smooth vectorfields on U and c is a smooth $l \times l$ matrix valued function on U. Assume furthermore that u and grad$u$ *vanish on Ω. Then u and* grad$u$ *vanish on $D^+(\Omega)$.*

Remark. The equation need only be satisfied in $D^+(\Omega)$. There is a similar statement concerning $D^-(\Omega)$.

3.4. **Stability.** In order to prove Theorems 3 and 4, we need to have a Cauchy stability result. Let us start by specifying the topology we shall be using.

Definition 3. Let M be a compact n dimensional manifold, and assume ϕ_i, $i = 1, ..., l$ is a finite partition of unity such that supp$\phi_i \subset U_i$ for open sets U_i. Assume furthermore that (x_i, U_i) are coordinates. Given $T \in \mathcal{T}^r_s(M)$, define

$$\|T\|_{H^k} = \left(\sum_{i=1}^{l} \sum_{j_1,...,j_r=1}^{n} \sum_{i_1,...,i_s=1}^{n} \sum_{|\alpha|\leq k} \int_{U_i} \phi_i |\partial^\alpha T^{j_1\cdots j_r}_{i_1\cdots i_s}|^2 dx^1_i \cdots dx^n_i \right)^{1/2} \tag{43}$$

where $T^{j_1\cdots j_r}_{i_1\cdots i_s}$ are the components of T relative to the coordinates x_i and ∂^α signifies differentiation with respect to x_i.

Remark. In order not to get too cumbersome notation we abuse notation by not clearly indicating with respect to which coordinates we compute components of tensors etc.

Note that (43) defines a norm on the space of smooth tensorfields $\mathcal{T}^r_s(M)$. If one uses a different partition of unity one clearly gets a different norm, but they are all equivalent. Consequently, they define the same topology, and it makes sense to say that $T_j \to T$ with respect to H^k without any reference to a partition of unity.

In order to make a precise statement concerning stability, we need to be specific concerning the requirements of the background solution.

Definition 4. Let M be a compact n dimensional manifold. Let g be a smooth Lorentz metric on $I \times M$ where $I = (T_-, T_+)$. Assume that ∂_t is timelike and that the hypersurfaces $\{\tau\} \times M$ are spacelike with respect to g for $\tau \in I$. Finally, assume that there is a $\phi \in C^\infty(I \times M)$ such that ϕ and g satisfy (12)-(13). Then we shall call $(I \times M, g, \phi)$ a *background solution.*

Definition 5. Let g be a Lorentz metric on $I \times M$, with $I = (T_-, T_+)$, let $\phi \in C^\infty(I \times M)$ and let $\tau \in I$. Assume $\{\tau\} \times M$ is spacelike with respect to g and let $i : M \to I \times M$ be defined by $i(p) = (\tau, p)$. Let h be the Riemannian metric on M obtained by using i to pull back the Riemannian metric induced on $\{\tau\} \times M$ by g, let k be the covariant 2-tensor obtained by using i to pull back the second fundamental form induced on $\{\tau\} \times M$ by g, let $\phi_0 = \phi \circ i$ and let $\phi_1 = (N\phi) \circ i$, where N is the future directed unit normal to $\{\tau\} \times M$ with respect to g. Then we shall refer to (h, k, ϕ_0, ϕ_1) as the *initial data induced on* $\{\tau\} \times M$ by (g, ϕ), or simply the initial data induced on $\{\tau\} \times M$ if the solution is understood from the context.

Theorem 7. *Let $(I \times M, g, \phi)$ be a background solution. Let $(\rho, \kappa, \phi_0, \phi_1)$ be the initial data induced on $\{T_0\} \times M$ by (g, ϕ). Assume ρ_j is a sequence of Riemannian metrics on M, κ_j a sequence of covariant 2-tensors and $\psi_{0,j}$ and $\psi_{1,j}$ are a sequence of smooth functions such that ρ_j and $\psi_{0,j}$ converge to ρ and ϕ_0 respectively in H^{l+1} and κ_j and $\psi_{1,j}$ converge to κ and ϕ_1 respectively in H^l, where $l > n/2+1$. Assume furthermore that $(\rho_j, \kappa_j, \psi_{0,j}, \psi_{1,j})$ satisfy the constraint equations (17)-(18) with (h, k, ϕ_0, ϕ_1) replaced by $(\rho_j, \kappa_j, \psi_{0,j}, \psi_{1,j})$. Then there are $T_{j,-} < T_0 < T_{j,+}$, a Lorentz metric h_j on $\bar{M}_j = (T_{j,-}, T_{j,+}) \times M$ and a smooth function ψ_j on $\bar{M}_j$ such that (h_j, ψ_j) satisfy (12)-(13) on $\bar{M}_j$. Furthermore, the initial data induced on $\{T_0\} \times M$ by (h_j, ψ_j) are $(\rho_j, \kappa_j, \psi_{0,j}, \psi_{1,j})$, ∂_t is timelike with respect to h_j and $\{\tau\} \times M$ is a spacelike Cauchy hypersurface with respect to h_j for all $\tau \in (T_{j,-}, T_{j,+})$. If $T \in I$, then $T \in (T_{j,-}, T_{j,+})$ for j large enough and the initial data induced on $\{T\} \times M$ by (h_j, ψ_j) converge to the corresponding initial data of (g, ϕ).*

Remark. The topology we have in mind when we speak of convergence of the initial data induced on $\{T\} \times M$ is the same as we used for the data induced on $\{T_0\} \times M$. In other words, H^{l+1} for the induced metric and scalar field and H^l for the second fundamental form and time derivative of the scalar field.

4. Equations on $\mathbb{T}^n$

In the introduction, we formulated the initial value problem for a general Σ. In practice, due to the causal structure of the type of spacetimes we are interested in, the global topology of Σ will turn out to be irrelevant. For convenience we shall thus assume $\Sigma = \mathbb{T}^n$ and consider the equations on $\mathbb{R} \times \mathbb{T}^n$. On this manifold, we have coordinates $x = (x^0, ..., x^n)$; x^μ giving the μ:th coordinate. Strictly speaking, these coordinates are of course not globally well defined, but in the end, we are only interested in $\partial_{x^\mu} = \partial_\mu$, and these vectorfields are globally well defined. In what follows, we shall take for granted that everything is computed with respect to these coordinates, or, to be more precise, with respect to the frame given by ∂_μ.

Let us start by describing the type of metric around which we wish to perturb. The model metric is given by

$$g = -dt^2 + e^{2\Omega}\delta_{ij}dx^i \otimes dx^j, \tag{44}$$

where Ω is a smooth function of t. For the purposes of the present paper, it is enough to think of Ω as being Ht where H is a constant, but in a later paper, we shall apply the same methods to the case $\Omega = p\ln t$, where $p > 1$ is constant. We shall also use the notation

$$\omega = \dot{\Omega}. \tag{45}$$

A metric of the form (44) has the property that $\Gamma^0 = n\omega$ and $\Gamma^i = 0$, where, as always, Latin indices range from 1 to n, Greek indices range from 0 to n and where

$$\Gamma^\mu = \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}).$$

Let us define

$$F_\mu = n\omega g_{0\mu} \tag{46}$$

and, following the ideas of [18], let

$$\begin{aligned} \hat{R}_{\mu\nu} &= -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \nabla_{(\mu} F_{\nu)} \\ &\quad + g^{\alpha\beta} g^{\gamma\delta}[\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}], \end{aligned} \tag{47}$$

where

$$\begin{aligned} \Gamma_{\alpha\gamma\beta} &= \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}), \\ \nabla_\mu F_\nu &= \partial_\mu F_\nu - \Gamma^\alpha_{\mu\nu} F_\alpha \end{aligned} \tag{48}$$

and a parenthesis denotes symmetrization, i.e.

$$\nabla_{(\mu} F_{\nu)} = \frac{1}{2}(\nabla_\mu F_\nu + \nabla_\nu F_\mu).$$

In other words,

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{(\mu}\mathcal{D}_{\nu)}, \tag{49}$$

where

$$\mathcal{D}_\mu = F_\mu - \Gamma_\mu. \tag{50}$$

Observe that the notation (48) is somewhat questionable since F_μ will not in general be the components of a covector. We shall, however, use it. Note that $\hat{R}_{\mu\nu}$, considered as a differential operator acting on the metric, is hyperbolic.

Let us define

$$M_{00} = -2\omega g^{0\mu}\mathcal{D}_\mu, \quad M_{0i} = 2\omega\mathcal{D}_i, \tag{51}$$

$$M_{ij} = 0, \quad M_\phi = -g^{\mu\nu}\mathcal{D}_\mu\partial_\nu\phi, \tag{52}$$

and consider the equations

$$\hat{R}_{\mu\nu} - \nabla_\mu\phi\nabla_\nu\phi - \frac{2}{n-1}V(\phi)g_{\mu\nu} + M_{\mu\nu} = 0 \tag{53}$$

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi - \Gamma^\mu\partial_\mu\phi - V'(\phi) + M_\phi = 0. \tag{54}$$

Note that the system (53)-(54) is a quasi-linear system of hyperbolic PDE's for the metric and the scalar field. In other words, if we specify $g_{\mu\nu}$, $\partial_0 g_{\mu\nu}$, ϕ and $\partial_0\phi$ at $t = 0$, we obtain a unique local solution to the system (at least if we assume $g_{00} < 0$ and g_{ij} to be the components of a positive definite matrix initially). Let us assume we have solutions to (53)-(54). Due to (53), we have

$$G_{\mu\nu} - T_{\mu\nu} = -\nabla_{(\mu}\mathcal{D}_{\nu)} + \frac{1}{2}(\nabla^\gamma\mathcal{D}_\gamma)g_{\mu\nu} - M_{\mu\nu} + \frac{1}{2}(g^{\alpha\beta}M_{\alpha\beta})g_{\mu\nu}. \tag{55}$$

Furthermore, $G_{\mu\nu}$ is divergence free due to the Bianchi identities and $T_{\mu\nu}$ satisfies

$$\nabla^\mu T_{\mu\nu} = -M_\phi \nabla_\nu \phi$$

due to (54). Consequently, taking the divergence of (55), we obtain

$$\nabla_\mu \nabla^\mu \mathcal{D}_\nu + R_\nu{}^\mu \mathcal{D}_\mu = -2M_\phi \nabla_\nu \phi - 2\nabla^\mu M_{\mu\nu} + g^{\alpha\beta}\nabla_\nu M_{\alpha\beta}. \tag{56}$$

Assuming that there are smooth solutions to (53)-(54) on some set $(T_-, T_+) \times \mathbb{R}^n$ with $T_- < 0 < T_+$, we see that there are smooth functions $A_{\alpha\beta\gamma}$ and $B_{\alpha\beta}$ such that $\mathcal{D}$ satisfies

$$g^{\alpha\beta}\partial_\alpha\partial_\beta \mathcal{D}_\mu + A_\mu{}^{\alpha\beta}\partial_\alpha \mathcal{D}_\beta + B_\mu{}^\alpha \mathcal{D}_\alpha = 0. \tag{57}$$

If it is possible to set up initial data for (53) and (54) in such a way that $\mathcal{D}_\mu$ and $\partial_0 \mathcal{D}_\mu$ are zero for $t = 0$, we are thus allowed to conclude that $\mathcal{D}_\mu$ is zero where the solution is defined. Consequently, $M_{\mu\nu}$ and M_ϕ are also zero, and we get a solution to (12)-(13).

4.1. **Initial data.** In practice, we shall be interested in initial data that do not satisfy the constraint equations on the entire initial manifold. We shall thus assume that we are given $(h, \kappa, \phi_0, \phi_1)$ on $\mathbb{T}^n$, where h is a Riemannian metric, κ is a covariant 2-tensor and ϕ_0, ϕ_1 are smooth functions on $\mathbb{T}^n$. Furthermore, we shall assume that (17)-(18) are satisfied on $S \subseteq \mathbb{T}^n$. Starting with these initial data, let us construct initial data for (53)-(54). The spatial part of the metric, g_{ij} is determined by h:

$$g_{ij}|_{t=0} = h(\partial_i, \partial_j), \tag{58}$$

for $i, j = 1, ..., n$. However, g_{00} and g_{0i} are not specified by the initial data. Let us choose them to satisfy

$$g_{00}|_{t=0} = -1, \quad g_{0i}|_{t=0} = 0. \tag{59}$$

Due to this choice, the future directed unit normal to the hypersurface $t = 0$ is ∂_t, so that if we had a metric g whose second fundamental form were κ, we would have

$$\kappa_{ij} = \frac{1}{2}\partial_0 g_{ij}.$$

It is thus natural to require that

$$\partial_0 g_{ij}|_{t=0} = 2\kappa(\partial_i, \partial_j). \tag{60}$$

Concerning ϕ, we require

$$\phi|_{t=0} = \phi_0, \quad (\partial_t \phi)|_{t=0} = \phi_1, \tag{61}$$

since ∂_t is the future directed unit normal to $\{0\} \times \mathbb{T}^n$. The only objects that remain to be determined are $\partial_0 g_{00}$ and $\partial_0 g_{0i}$. We shall let the condition $\mathcal{D}_\mu|_{t=0} = 0$ determine these quantities. Assuming we had a metric g, we would obtain, for $t = 0$,

$$\Gamma_0 = -\frac{1}{2}\partial_0 g_{00} - \text{tr}\kappa.$$

where we have used (59) and (60). We thus require

$$\partial_0 g_{00}|_{t=0} = -2F_0|_{t=0} - 2\text{tr}\kappa. \tag{62}$$

Note that since F_0 only depends on the coordinates and on the metric, the right hand side has already been defined for $t = 0$. We also have, assuming we had a metric g,

$$\Gamma_l = -\partial_0 g_{0l} + \frac{1}{2} g^{ij}(2\partial_i g_{jl} - \partial_l g_{ij}).$$

Consequently we require

$$\partial_0 g_{0l}|_{t=0} = \left[-F_l + \frac{1}{2} g^{ij}(2\partial_i g_{jl} - \partial_l g_{ij}) \right] \Big|_{t=0}. \tag{63}$$

4.2. **Development of the data.** Due to (62) and (63), we know that $\mathcal{D}_\mu = 0$ for $t = 0$. However, in order to be allowed to conclude that $\mathcal{D}_\mu$ is zero, we need to know that $\partial_0 \mathcal{D}_\mu$ is zero for $t = 0$. On the other hand, we have no more freedom left in specifying initial data. However, it will turn out that the last condition is a consequence of the constraint equations. We thus obtain the following result.

Proposition 1. *Let $(h, \kappa, \phi_0, \phi_1)$ be given on $\mathbb{T}^n$, where h is a Riemannian metric, κ is a covariant 2-tensor and ϕ_0, ϕ_1 are functions. Assume*

$$\begin{aligned} (h, \kappa) &\in H^{k+1}[\mathbb{T}^n, M_n(\mathbb{R})] \times H^k[\mathbb{T}^n, M_n(\mathbb{R})], \\ (\phi_0, \phi_1) &\in H^{k+1}(\mathbb{T}^n) \times H^k(\mathbb{T}^n), \end{aligned}$$

where $k > n/2 + 1$. Define $g_{\mu\nu}|_{t=0}$ by (58)-(59), $(\partial_t g_{\mu\nu})|_{t=0}$ by (60) and (62)-(63) and define $\phi|_{t=0}$, $(\partial_t \phi)|_{t=0}$ by (61). Then there are $T_- < 0 < T_+$ and a unique solution

$$g \in C^2[I \times \mathbb{T}^n, M_{n+1}(\mathbb{R})], \quad \phi \in C^2[I \times \mathbb{T}^n, \mathbb{R}] \tag{64}$$

to (53)-(54), where $I = (T_-, T_+)$, such that $g_{00} < 0$ and g_{ij} are the components of a positive definite matrix. Furthermore

$$g \in L^\infty\{I, H^{k+1}[\mathbb{T}^n, M_{n+1}(\mathbb{R})]\}, \ \phi \in L^\infty[I, H^{k+1}(\mathbb{T}^n)] \tag{65}$$

$$\partial_t g \in L^\infty\{I, H^k[\mathbb{T}^n, M_{n+1}(\mathbb{R})]\}, \ \partial_t \phi \in L^\infty[I, H^k(\mathbb{T}^n)]. \tag{66}$$

Let $T_{\max}$ be the supremum of the times $T_+ > 0$ such that there is a solution (g, ϕ) on $[0, T_+)$ satisfying the above conditions. If $T_{\max} < \infty$ one of the following two statements have to be true. 1. There is a sequence $(t_l, x_l) \in [0, T_{\max}) \times \mathbb{T}^n$ such that either $g_{00}(t_l, x_l) \to 0$ or the smallest eigenvalue of $\{g_{ij}(t_l, x_l)\}$ tends to zero as l tends to infinity. 2. We have the following limit:

$$\lim_{t \to T_{\max}-} \sup_{0 \le \tau \le t} \sum_{|\alpha| + j \le 2} \sup_{x \in \mathbb{T}^n} [\|\partial^\alpha \partial_t^j g(\tau, x)\| + |\partial^\alpha \partial_t^j \phi(\tau, x)|] = \infty.$$

There is an analogous statement concerning $T_{\min}$ which is defined analogously to $T_{\max}$. In particular, $T_{\max}$ and $T_{\min}$ are independent of k. If we assume the initial data to be smooth, we get a unique smooth solution (g, ϕ) to (53)-(54) on $I_{\max} = (T_{\min}, T_{\max})$ such that $g_{00} < 0$ and g_{ij} are the components of a positive definite matrix. Then g is a smooth Lorentz metric on $M = I_{\max} \times \mathbb{T}^n$ and $\{t\} \times \mathbb{T}^n$ are Cauchy hypersurfaces in the Lorentz manifold (M, g) for $t \in I_{\max}$. If we furthermore assume that (17)-(18) are satisfied on an open subset $S \subseteq \mathbb{T}^n$, then (g, ϕ) satisfy (12)-(13) on $D(S)$, where $D(S)$ is defined with respect to the metric g.

Remark. Here $M_n(\mathbb{R})$ denotes the set of $n \times n$ matrices and we think of h and k as such matrices whose elements are given by the corresponding components with respect to the standard basis $\{\partial_i\}$ of the tangent space of $\mathbb{T}^n$. Furthermore, we think of g as being $M_{n+1}(\mathbb{R})$ valued, the elements of the matrix being the the components of g with respect to $\{\partial_\mu\}$. The regularity condition in the existence statement is of course far from optimal. However, with the methods we use to close the bootstrap, we need this degree of regularity. When we write $D(S)$, we, strictly speaking, mean $D(\{0\} \times S)$, cf. the notation of Subsection 3.2.

Proof. The existence result and continuation criterion can, up to small modifications, be found in standard textbooks on non-linear hyperbolic PDE's, so we shall take this for granted. Since $g_{00} < 0$ and g_{ij} are the components of a positive definite metric, a linear algebra argument suffices to conclude that $g_{\mu\nu}$ are the components of a Lorentz metric and that $g^{00} < 0$, cf. Lemma 2. This means that the gradient of the function $t : M \to \mathbb{R}$ defined by $t(x^0, ..., x^n) = x^0$ is past directed timelike (here we use the convention that ∂_t is future directed). Consequently, if $\gamma : (s_-, s_+) \to M$ is a future directed causal curve, $t \circ \gamma$ is a strictly monotonically increasing function. Thus a causal curve can intersect the hypersurfaces $\{t\} \times \mathbb{T}^n$ at most once. If the image of γ is contained in the past of, say, $\{t\} \times \mathbb{T}^n$, then $\gamma([s_0, s_+))$ is contained in a compact subset of M for $s_0 \in (s_-, s_+)$. Using this fact, the causality of γ, the fact that $g_{00} < 0$ and the fact that g_{ij} are the components of a positive definite metric, one can conclude that γ is extendible to the future. We conclude that all the hypersurfaces $\{t\} \times \mathbb{T}^n$ are Cauchy hypersurfaces.

In order to prove that (g, ϕ) satisfy (12)-(13) on $D(S)$, all we need to prove is that $\mathcal{D}_\mu$ and $\partial_0 \mathcal{D}_\mu$ equal zero on S. The reason for this is that on M, we have the equation (57). Given that the initial data for $\mathcal{D}_\mu$ are zero on S, standard uniqueness results for linear equations on globally hyperbolic Lorentz manifolds yield the desired conclusion, cf. Theorem 6. We already know that $\mathcal{D}_\mu = 0$ initially due to our choice of initial data, but we need to prove that $\partial_0 \mathcal{D}_\mu = 0$ initially.

The solution we obtain solves (55). Note that in this equation, $M_{\alpha\beta} = 0$ initially, since $\mathcal{D}_\mu = 0$ initially. Let us contract (55) with $N^\mu X^\nu$ for $t = 0$, where X is orthogonal to N. Then, on S, the left hand side is zero since the constraints are fulfilled and the right hand side is

$$-\frac{1}{2} N^\mu X^\nu (\partial_\mu \mathcal{D}_\nu + \partial_\nu \mathcal{D}_\mu).$$

Note that the part of the covariant derivative involving Christoffel symbols vanishes due to the fact that $\mathcal{D}_\mu = 0$ initially. Since $X^\nu \partial_\nu \mathcal{D}_\mu = 0$ for $t = 0$, we obtain

$$\partial_0 \mathcal{D}_i = 0$$

on S for $t = 0$, $i = 1, ..., n$. If we contract (55) with $N^\mu N^\nu$, we obtain

$$\partial_0 \mathcal{D}_0 = 0$$

on S by a similar argument. The proposition follows. □

4.3. **The equations.** To sum up, we shall in the end restrict our attention to the equations

$$\hat{R}_{00} + 2\omega\Gamma^0 - 2n\omega^2 - (\partial_t\phi)^2 - \frac{2}{n-1}V(\phi)g_{00} = 0 \quad (67)$$

$$\hat{R}_{0i} - 2\omega(\Gamma_i - n\omega g_{0i}) - \partial_t\phi\partial_i\phi - \frac{2}{n-1}V(\phi)g_{0i} = 0 \quad (68)$$

$$\hat{R}_{ij} - \partial_i\phi\partial_j\phi - \frac{2}{n-1}V(\phi)g_{ij} = 0 \quad (69)$$

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi - n\omega\partial_0\phi - V'(\phi) = 0, \quad (70)$$

where the indices i, j run from 1 to n and $\hat{R}_{\mu\nu}$ is given by (47). Furthermore, we shall only consider these equations on $\mathbb{R} \times \mathbb{T}^n$ and one should not think of any of the objects appearing as tensors but rather as the components with respect to the standard basis for the tangent space of $\mathbb{R} \times \mathbb{T}^n$. The advantage of this system is that it behaves well even when the initial data do not satisfy the constraints, cf. the comments made in the introduction.

5. The modified Ricci tensor

Lemma 4. *Let $\hat{R}_{\mu\nu}$ be given by (47), where F_μ is defined in (46). Then*

$$\hat{R}_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + ng_{0(\mu}\partial_{\nu)}\omega + \frac{1}{2}n\omega\partial_0 g_{\mu\nu} + A_{\mu\nu},$$

where

$$A_{\mu\nu} = g^{\alpha\beta}g^{\gamma\delta}[\partial_\alpha g_{\nu\gamma}\partial_\beta g_{\mu\delta} - \Gamma_{\alpha\nu\gamma}\Gamma_{\beta\mu\delta}].$$

Proof. Consider

$$\nabla_\mu F_\nu = \partial_\mu F_\nu - \Gamma^\alpha_{\mu\nu}F_\alpha.$$

Note that

$$\Gamma^\alpha_{\mu\nu}F_\alpha = n\omega g_{0\alpha}\Gamma^\alpha_{\mu\nu} = n\omega\Gamma_{\mu 0\nu} = \frac{1}{2}n\omega(\partial_\mu g_{0\nu} + \partial_\nu g_{0\mu} - \partial_0 g_{\mu\nu}).$$

Since

$$\partial_\mu F_\nu = n\omega(\partial_\mu\omega)g_{0\nu} + n\omega\partial_\mu g_{0\nu},$$

we obtain

$$\nabla_{(\mu}F_{\nu)} = ng_{0(\nu}\partial_{\mu)}\omega + \frac{1}{2}n\omega\partial_0 g_{\mu\nu}.$$

Let us turn to the squares of the Christoffel symbols. The expression of interest is

$$A_{\mu\nu} = g^{\alpha\beta}g^{\gamma\delta}[\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}].$$

Since

$$\Gamma_{\beta\delta\nu} + \Gamma_{\beta\nu\delta} = \partial_\beta g_{\delta\nu},$$

we have

$$\begin{aligned} A_{\mu\nu} &= g^{\alpha\beta}g^{\gamma\delta}[\Gamma_{\alpha\gamma\mu}\partial_\beta g_{\delta\nu} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}] \\ &= g^{\alpha\beta}g^{\gamma\delta}[\Gamma_{\beta\delta\mu}\partial_\alpha g_{\gamma\nu} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}] \\ &= g^{\alpha\beta}g^{\gamma\delta}[\partial_\beta g_{\mu\delta}\partial_\alpha g_{\gamma\nu} + (\Gamma_{\alpha\gamma\nu} - \partial_\alpha g_{\gamma\nu})\Gamma_{\beta\mu\delta}] \\ &= g^{\alpha\beta}g^{\gamma\delta}[\partial_\beta g_{\mu\delta}\partial_\alpha g_{\gamma\nu} - \Gamma_{\alpha\nu\gamma}\Gamma_{\beta\mu\delta}], \end{aligned}$$

where the second step only involves a renaming of indices. The lemma follows. □

Note that we can write

$$A_{\mu\nu} = \mathrm{I}_{\mu\nu} + \mathrm{II}_{\mu\nu} + \mathrm{III}_{\mu\nu} + \mathrm{IV}_{\mu\nu} + \mathrm{V}_{\mu\nu} + \mathrm{VI}_{\mu\nu},$$

where

$$\mathrm{I}_{\mu\nu} = g^{00}g^{00}(\partial_0 g_{0\mu}\partial_0 g_{0\nu} - \Gamma_{0\mu 0}\Gamma_{0\nu 0}) \tag{71}$$

$$\mathrm{II}_{\mu\nu} = g^{00}g^{0p}[\partial_0 g_{0\mu}(\partial_0 g_{p\nu} + \partial_p g_{0\nu}) + \partial_0 g_{0\nu}(\partial_0 g_{p\mu} + \partial_p g_{0\mu}) - 2\Gamma_{0\mu 0}\Gamma_{0\nu p} - 2\Gamma_{0\nu 0}\Gamma_{0\mu p}] \tag{72}$$

$$\mathrm{III}_{\mu\nu} = g^{00}g^{pl}(\partial_0 g_{p\mu}\partial_0 g_{l\nu} + \partial_p g_{0\mu}\partial_l g_{0\nu} - 2\Gamma_{0\mu p}\Gamma_{0\nu l}) \tag{73}$$

$$\mathrm{IV}_{\mu\nu} = g^{0j}g^{0p}[\partial_0 g_{0\mu}\partial_j g_{p\nu} + \partial_0 g_{p\mu}\partial_j g_{0\nu} + \partial_j g_{0\mu}\partial_0 g_{p\nu} + \partial_j g_{p\mu}\partial_0 g_{0\nu} - \Gamma_{0\mu 0}\Gamma_{j\nu p} - 2\Gamma_{0\mu p}\Gamma_{0\nu j} - \Gamma_{j\mu p}\Gamma_{0\nu 0}] \tag{74}$$

$$\mathrm{V}_{\mu\nu} = g^{pl}g^{0j}(\partial_0 g_{p\mu}\partial_j g_{l\nu} + \partial_j g_{p\mu}\partial_0 g_{l\nu} + \partial_l g_{0\mu}\partial_p g_{j\nu} + \partial_l g_{j\mu}\partial_p g_{0\nu} - 2\Gamma_{0\mu p}\Gamma_{j\nu l} - 2\Gamma_{j\mu p}\Gamma_{0\nu l}) \tag{75}$$

$$\mathrm{VI}_{\mu\nu} = g^{pl}g^{ij}(\partial_i g_{p\mu}\partial_j g_{l\nu} - \Gamma_{i\mu p}\Gamma_{j\nu l}). \tag{76}$$

Before we start separating the relevant parts from the irrelevant ones in these terms, let us comment on what can be considered to be small and what has to be taken into account. In the end, we shall be perturbing around a metric of the form (44). For the metric (44), $\partial_t g_{ij} = 2\omega g_{ij}$. Consequently, from a perturbation point of view, all terms that involve spatial derivatives can be considered to be small, as well as $g_{00}+1$, $g^{00}+1$, g_{0i}, g^{0i}, $\partial_0 g_{00}$, $\partial_0 g_{0i}$ and $\partial_0 g_{ij} - 2\omega g_{ij}$. The relevant part of $A_{\mu\nu}$ is the one which involves terms consisting of at most one small factor. From this point of view, we see that $\mathrm{I}_{\mu\nu}$, $\mathrm{II}_{\mu\nu}$ and $\mathrm{IV}_{\mu\nu}$ do not contain any relevant terms.

Lemma 5. *Given the definitions (71)-(76), we have*

$$\mathrm{III}_{0m} = 2\omega g^{00}\left(\partial_0 g_{0m} - \frac{1}{2}\partial_m g_{00}\right) + \Delta_{\mathrm{III},0m}, \tag{77}$$

$$\mathrm{III}_{ij} = 2\omega g^{00}\partial_0 g_{ij} - 2\omega^2 g^{00} g_{ij} + \Delta_{\mathrm{III},ij}, \tag{78}$$

$$\mathrm{V}_{0m} = -2\omega^2 g^{00} g_{0m} + \Delta_{\mathrm{V},0m}, \tag{79}$$

$$\mathrm{VI}_{00} = n\omega^2 - \omega g^{ij}\partial_0 g_{ij} + 2\omega g^{ij}\partial_i g_{j0} + \Delta_{\mathrm{VI},00}, \tag{80}$$

$$\mathrm{VI}_{0m} = \omega g^{ij}\Gamma_{imj} + \Delta_{\mathrm{VI},0m}, \tag{81}$$

where $\Delta_{\mathrm{III},0m}$, $\Delta_{\mathrm{III},ij}$, $\Delta_{\mathrm{V},0m}$ $\Delta_{\mathrm{VI},00}$ and $\Delta_{\mathrm{VI},0m}$ are given by (82)-(86) respectively.

Remark. In the end, we shall not need to know much concerning the structure of the non-linear terms in order to be able to estimate them in H^k. In fact, it will be enough to count that the number of factors that are "small", cf. the discussion prior to the statement of the lemma, is two or greater, something which is automatically true due to our definition of error terms, and to count the number of downstairs spatial indices minus the number of upstairs spatial indices, something which is also clear from the context, in order to be able to write down the estimate in H^k immediately.

Proof. The result is obtained by straightforward computations. The irrelevant terms are given by

$$\Delta_{\mathrm{III},0m} = g^{00}(g^{pl}\partial_0 g_{lm} - 2\omega\delta^p_m)\left(\partial_0 g_{p0} - \frac{1}{2}\partial_p g_{00}\right) + \frac{1}{2}g^{00}g^{pl}\partial_p g_{00}(\partial_l g_{0m} + \partial_m g_{0l}), \tag{82}$$

for III_{0m},

$$\begin{aligned}\Delta_{\mathrm{III},ij} = {} & g^{00}g^{pl}\left[\partial_p g_{0i}\partial_l g_{0j} - \frac{1}{2}(\partial_p g_{0i} - \partial_i g_{0p})(\partial_l g_{0j} - \partial_j g_{0l})\right] \\ & -\frac{1}{2}g^{00}[(g^{pl}\partial_0 g_{ip} - 2\omega\delta^l_i)(\partial_l g_{0j} - \partial_j g_{0l}) \\ & +(g^{pl}\partial_0 g_{jl} - 2\omega\delta^p_j)(\partial_p g_{0i} - \partial_i g_{0p})] \\ & +\omega g^{00}(g_{jl}g^{pl} - \delta^p_j)\partial_0 g_{ip} \\ & +\frac{1}{2}g^{00}(g^{pl}\partial_0 g_{ip} - 2\omega\delta^l_i)(\partial_0 g_{jl} - 2\omega g_{jl}),\end{aligned} \tag{83}$$

for III_{ij} (note that $g_{jl}g^{pl} - \delta^p_j = -g_{j0}g^{p0}$),

$$\begin{aligned}\Delta_{\mathrm{V},0m} = {} & g^{pl}g^{0j}(\partial_0 g_{p0}\partial_j g_{lm} + \partial_j g_{p0}\partial_0 g_{lm} + \partial_l g_{00}\partial_p g_{jm} \\ & +\partial_l g_{j0}\partial_p g_{0m} - 2\Gamma_{00p}\Gamma_{jml}) - g^{pl}g^{0j}[(\partial_j g_{p0} + \partial_p g_{j0})\Gamma_{0ml} \\ & -\frac{1}{2}\partial_0 g_{jp}(\partial_l g_{0m} - \partial_m g_{0l})] + \omega g^{0l}(\partial_0 g_{ml} - 2\omega g_{ml}) \\ & +\frac{1}{2}g^{0j}(g^{pl}\partial_0 g_{jp} - 2\omega\delta^l_j)\partial_0 g_{ml}\end{aligned} \tag{84}$$

for V_{0m},

$$\begin{aligned}\Delta_{\mathrm{VI},00} = {} & g^{ij}g^{pl}\partial_i g_{p0}\partial_j g_{l0} + \frac{1}{2}g^{pl}(g^{ij}\partial_0 g_{ip} - 2\omega\delta^j_p)(\partial_j g_{l0} + \partial_l g_{j0}) \\ & -\frac{1}{4}g^{ij}g^{pl}(\partial_i g_{p0} + \partial_p g_{i0})(\partial_j g_{l0} + \partial_l g_{j0}) \\ & -\frac{1}{4}(g^{ij}\partial_0 g_{ip} - 2\omega\delta^j_p)(g^{pl}\partial_0 g_{jl} - 2\omega\delta^p_j)\end{aligned} \tag{85}$$

for VI_{00} and

$$\Delta_{\mathrm{VI},0m} = g^{pl}g^{ij}\left[\partial_i g_{p0}\partial_j g_{lm} - \frac{1}{2}(\partial_i g_{p0} + \partial_p g_{i0})\Gamma_{jml}\right] + \frac{1}{2}g^{ij}(g^{pl}\partial_0 g_{pi} - 2\omega\delta^l_i)\Gamma_{jml} \tag{86}$$

for VI_{0m}. □

Let

$$\begin{aligned}\Delta_{\mathrm{I},\mu\nu} &= \mathrm{I}_{\mu\nu}, \quad \Delta_{\mathrm{II},\mu\nu} = \mathrm{II}_{\mu\nu}, \quad \Delta_{\mathrm{IV},\mu\nu} = \mathrm{IV}_{\mu\nu}, \quad \Delta_{\mathrm{III},00} = \mathrm{III}_{00}, \\ \Delta_{\mathrm{V},00} &= \mathrm{V}_{00}, \quad \Delta_{\mathrm{V},ij} = \mathrm{V}_{ij}, \quad \Delta_{\mathrm{VI},ij} = \mathrm{VI}_{ij},\end{aligned}$$

and

$$\Delta_{A,\mu\nu} = \Delta_{\mathrm{I},\mu\nu} + \ldots + \Delta_{\mathrm{VI},\mu\nu}. \tag{87}$$

Then

$$\begin{aligned} A_{00} &= n\omega^2 - \omega g^{ij}\partial_0 g_{ij} + 2\omega g^{ij}\partial_i g_{j0} + \Delta_{A,00} \\ A_{0m} &= 2\omega g^{00}\partial_0 g_{0m} - 2\omega^2 g^{00} g_{0m} - \omega g^{00}\partial_m g_{00} + \omega g^{ij}\Gamma_{imj} + \Delta_{A,0m} \\ A_{ij} &= 2\omega g^{00}\partial_0 g_{ij} - 2\omega^2 g^{00} g_{ij} + \Delta_{A,ij}. \end{aligned} \tag{88}$$

Let us turn our attention to the correction terms introduced.

Lemma 6. *We have*

$$\begin{aligned} A_{00} + 2\omega\Gamma^0 - 2n\omega^2 &= \omega\partial_0 g_{00} + n\omega^2(g_{00}+1) + n\omega^2 g_{00} \\ &\quad +\Delta_{A,00} + \Delta_{C,00} \end{aligned} \tag{89}$$

$$\begin{aligned} A_{0m} - 2\omega(\Gamma_m - n\omega g_{0m}) &= 2(n-1)\omega^2 g_{0m} - \omega g^{ij}\Gamma_{imj} \\ &\quad +\Delta_{A,0m} + \Delta_{C,0m}, \end{aligned} \tag{90}$$

where $\Delta_{C,00}$ *and* $\Delta_{C,0m}$ *are given by (92) and (93) respectively.*

Proof. Note that

$$(g^{00}+1)g_{00} = g^{00}g_{00} + g^{0i}g_{0i} - g^{0i}g_{0i} + g_{00} = g_{00} + 1 - g^{0i}g_{0i},$$

so that

$$g^{00} + 1 = \frac{1}{g_{00}}(g_{00} + 1 - g^{0i}g_{0i}). \tag{91}$$

Using this observation, one can compute that

$$2\omega\Gamma^0 = \omega\partial_0 g_{00} + \omega g^{ij}\partial_0 g_{ij} + 2n\omega^2(g_{00}+1) - 2\omega g^{ij}\partial_j g_{i0} + \Delta_{C,00},$$

where

$$\begin{aligned} \Delta_{C,00} &= -\frac{2n\omega^2}{g_{00}}[(1+g_{00})^2 - g^{0i}g_{0i}] \\ &\quad -\omega(g^{00}+1)(g^{ij}\partial_0 g_{ij} - 2n\omega) + 2\omega(g^{00}+1)g^{ij}\partial_i g_{j0} \\ &\quad +\omega(g^{00}g^{00}-1)\partial_0 g_{00} + 2\omega g^{00}g^{0i}(\Gamma_{0i0} + 2\Gamma_{00i}) \\ &\quad +4\omega g^{0i}g^{0j}\Gamma_{0ji} + 2\omega g^{ij}g^{0p}\Gamma_{ipj}. \end{aligned} \tag{92}$$

Thus we obtain (89). Let us compute

$$\begin{aligned} -2\omega(\Gamma_m - n\omega g_{0m}) &= -2\omega g^{00}\partial_0 g_{0m} + \omega g^{00}\partial_m g_{00} + 2\omega^2 g^{00} g_{0m} \\ &\quad -2\omega g^{ij}\Gamma_{imj} + 2(n-1)\omega^2 g_{0m} + \Delta_{C,0m}, \end{aligned}$$

where

$$\Delta_{C,0m} = 2\omega^2(g^{00}+1)g_{0m} - 2\omega g^{0i}(\partial_0 g_{mi} - 2\omega g_{mi} + \partial_i g_{m0} - \partial_m g_{i0}). \tag{93}$$

Consequently, (90) holds. □

6. Rough control

The precise form of the equations depends on the particular potential, and therefore we wish to postpone writing down the equations for as long as possible. Some of the bootstrap assumptions we shall make in the end do, however, have consequences independent of the potential, and so we wish to begin with them.

Assume we have a solution to (67)-(70) on some time interval $[t_0, T)$. On this interval we make the following **bootstrap assumptions**. Assume that there are constants K and $c_1 > 1$ such that, using the notation $u[g] = g_{00}+1$ and the notation of Section 2,

$$c_1^{-1}|w|^2 \leq e^{-2\Omega-2K}|w|^2_{g_\flat} \leq c_1|w|^2, \tag{94}$$

$$|u[g]| \leq \eta, \tag{95}$$

$$|v[g]|^2 \leq \eta c_1^{-1} e^{2\Omega-2r+2K}, \tag{96}$$

for all $w \in \mathbb{R}^n$, and all $(t,x) \in [t_0,T)\times\mathbb{T}^n$. Here Ω and r are non-negative functions of t in $[t_0,\infty)$ and $\eta \in (0,1)$ is a constant. At a later stage, we shall impose more strict conditions on these quantities. Due to the bootstrap assumptions (94)-(95) and Lemma 1, we conclude that $g_{\mu\nu}$ are the components of a Lorentz metric. Thus we can speak of the inverse of g and we denote the components of the inverse by $g^{\mu\nu}$.

Lemma 7. *Let $g_{\mu\nu}$ be the components of a matrix valued function on $[t_0,T)\times\mathbb{T}^n$ satisfying the conditions (94)-(96) where $\Omega, r \geq 0$ for $t \geq t_0$. Then g is a Lorentz metric and there is a numerical constant $\eta_0 > 0$ such that if we assume $\eta \leq \eta_0$ in (95)-(96), we have*

$$|v[g^{-1}]| \leq 2c_1 e^{-2\Omega-2K}|v[g]| \tag{97}$$

$$|(v[g], v[g^{-1}])| \leq 2c_1 e^{-2\Omega-2K}|v[g]|^2 \tag{98}$$

$$|u[g^{-1}]| \leq 4\eta, \tag{99}$$

$$\frac{2}{3c_1}|w|^2 \leq e^{2\Omega+2K}|w|^2_{g^\sharp} \leq \frac{3c_1}{2}|w|^2 \tag{100}$$

for all $w \in \mathbb{R}^n$, $t \in [t_0,T)$ and $x \in \mathbb{T}^n$. Here we use the notation (ξ,ζ) for the ordinary scalar product of $\xi,\zeta \in \mathbb{R}^n$.

Proof. Let A be the square root of $g_\flat^{-1}$. Using (94), we get

$$|v[g^{-1}]|^2 \leq c_1 e^{-2\Omega-2K}|v[g^{-1}]|^2_{g_\flat}.$$

However, multiplying (36) with A^{-1} and taking absolute values, we get

$$|v[g^{-1}]|^2_{g_\flat} = \frac{1}{(d^2-g_{00})^2}|Av[g]|^2 \leq \frac{c_1 e^{-2\Omega-2K}}{(d^2-g_{00})^2}|v[g]|^2,$$

where we have also used (94). Combining these two inequalities, we get

$$|v[g^{-1}]|^2 \leq \frac{c_1^2 e^{-4\Omega-4K}}{(d^2-g_{00})^2}|v[g]|^2.$$

Since $1/(d^2-g_{00})^2 \leq 1/g_{00}^2$, which can be assumed to be arbitrarily close to 1 by imposing conditions on η_0, we get (97). The estimate (98) is then an immediate consequence of this.

In order to prove (99), let us first note that

$$d^2 = |v[g]|^2_{g_\flat^{-1}} = |Av[g]|^2 \leq c_1 e^{-2\Omega-2K}|v[g]|^2 \leq \eta, \tag{101}$$

where we have used (94), (96) and the fact that $r \geq 0$. Using (34) we get

$$|u[g^{-1}]| \leq \frac{|u[g]|+d^2}{d^2-g_{00}}.$$

Combining this estimate with (95) and (101), we see that for η_0 small enough, (99) holds.

Finally, note that due to (35) and (94), we have

$$|w|^2_{g^\sharp} \leq |w|^2_{g_\flat^{-1}} = |Aw|^2 \leq c_1 e^{-2\Omega-2K}|w|^2.$$

Similarly,

$$|w|^2_{g^\sharp} \geq \frac{g_{00}}{g_{00}-d^2}|w|^2_{g_\flat^{-1}} \geq \frac{g_{00}}{g_{00}-d^2}c_1^{-1}e^{-2\Omega-2K}|w|^2.$$

Since $g_{00}/(d^2-g_{00})$ can be assumed to be arbitrarily close to 1 by demanding that η_0 be small enough, the lemma follows. □

7. Energies

The exact form of the energies will in the end depend on the particular potential, but in the cases we are interested in, they will be equivalent to objects we now define. In some cases, the background scalar field converges to zero, but in others it tends to infinity. Consequently, it is sometimes necessary to subtract the background solution.

Definition 6. Assume that the scalar field corresponding to the background solution around which we are perturbing is ϕ_0. Then we let

$$\psi = \phi - \phi_0. \tag{102}$$

Furthermore, we define

$$u = g_{00}+1, \quad u_i = g_{0i}, \quad h_{ij} = e^{-2\Omega}g_{ij}. \tag{103}$$

Let us also introduce the notation

$$\|f\|_{H^k} = \left(\sum_{|\alpha|\leq k}\int_{\mathbb{T}^n}(\partial^\alpha f)^2 dx\right)^{1/2},$$

where

$$\partial^\alpha f = \frac{\partial^{|\alpha|}f}{\partial(x^1)^{\alpha_1}\cdots\partial(x^n)^{\alpha_n}},$$

and α is a multiindex, $\alpha = (\alpha_1,...,\alpha_n)$ for non-negative integers α_i. Even when $f \in C^\infty(I\times\mathbb{T}^n)$ for some interval I, we shall take it to be understood that $\partial^\alpha f$ only means differentiation with respect to the last n variables. When we write $\partial_\mu f$ on the other hand, we take it to be understood that μ is a number from 0 to n.

We shall express the estimates in terms of the following quantities

$$\begin{aligned}
E_{\mathrm{lp},k} &= \frac{1}{2}\sum_{|\alpha|\leq k}\int_{\mathbb{T}^n}\{(\partial^\alpha\partial_t u)^2 + (\partial^\alpha\partial_t\psi)^2 \\
&\quad + g^{ij}(\partial^\alpha\partial_i u\partial^\alpha\partial_j u + \partial^\alpha\partial_i\psi\partial^\alpha\partial_j\psi) + \omega^2[(\partial^\alpha u)^2 + (\partial^\alpha\psi)^2]\}dx \\
E_{\mathrm{s},k} &= \frac{1}{2}\sum_{|\alpha|\leq k}\sum_i\int_{\mathbb{T}^n}[(\partial^\alpha\partial_t u_i)^2 + g^{lm}\partial^\alpha\partial_l u_i\partial^\alpha\partial_m u_i + \omega^2(\partial^\alpha u_i)^2]dx \\
E_{\mathrm{m},k} &= \frac{1}{2}\sum_{|\alpha|\leq k}\sum_{i,j}\int_{\mathbb{T}^n}[(\partial^\alpha\partial_t h_{ij})^2 + g^{lm}\partial^\alpha\partial_l h_{ij}\partial^\alpha\partial_m h_{ij} \\
&\quad + a_\alpha\omega^2 e^{-2r}(\partial^\alpha h_{ij})^2]dx,
\end{aligned}$$

where $a_\alpha = 0$ for $\alpha = 0$ and $a_\alpha = 1$ otherwise. The reason we have to introduce a_α is that we want all of these quantities to be zero for the solution around which we are perturbing. The reason we include the factor e^{-2r} is that we want $E_{\rm m}$ to decay, which is also not to be expected without this added factor. It will be more natural to express the bootstrap assumptions in terms of the following quantities:

$$\begin{aligned}\hat{E}_{{\rm lp},k} &= \omega^{-2}e^{2r}E_{{\rm lp},k}, \quad \hat{E}_{{\rm s},k} = \omega^{-2}e^{-2\Omega+2r-2K}E_{{\rm s},k},\\ \hat{E}_{{\rm m},k} &= \omega^{-2}e^{2r-4K}E_{{\rm m},k}.\end{aligned} \tag{104}$$

We shall also use the notation

$$\hat{E}_k = \hat{E}_{{\rm lp},k} + \hat{E}_{{\rm s},k} + \hat{E}_{{\rm m},k}.$$

The main **bootstrap assumption** we shall be making in the end is that

$$\hat{E}_{k_0}^{1/2}(t) \leq \epsilon \tag{105}$$

for some $k_0 > n/2+1$, $\epsilon \leq 1$ and for all t in some time interval $[t_0, T)$.

Lemma 8. *Assuming (94)-(96) hold on $[t_0,T)\times\mathbb{T}^n$ where $\eta \leq \eta_0$, $n \geq 3$ and $\Omega, r \geq 0$ for $t \geq t_0$, we have, on $[t_0,T)\times\mathbb{T}^n$:*

$$e^r[\|\psi\|_{H^k} + \omega^{-1}\|\partial_t\psi\|_{H^k} + e^{-\Omega-K}\omega^{-1}\|\partial_i\psi\|_{H^k}] \leq C\hat{E}_{{\rm lp},k}^{1/2} \tag{106}$$

$$e^r[\|u\|_{H^k} + \omega^{-1}\|\partial_t u\|_{H^k} + e^{-\Omega-K}\omega^{-1}\|\partial_i u\|_{H^k}] \leq C\hat{E}_{{\rm lp},k}^{1/2} \tag{107}$$

$$\begin{aligned}e^{-\Omega+r-K}[\|u_m\|_{H^k} + \omega^{-1}\|\partial_t u_m\|_{H^k}&\\ +e^{-\Omega-K}\omega^{-1}\|\partial_i u_m\|_{H^k}] &\leq C\hat{E}_{{\rm s},k}^{1/2}\end{aligned} \tag{108}$$

$$\begin{aligned}e^{-2\Omega+r-2K}\left[\omega^{-1}\|\partial_t g_{ij} - 2\omega g_{ij}\|_{H^k}\right.&\\ \left.+e^{-\Omega-K}\omega^{-1}\|\partial_l g_{ij}\|_{H^k}\right] &\leq C\hat{E}_{{\rm m},k}^{1/2}\end{aligned} \tag{109}$$

$$e^{-2\Omega-2K}\|\partial^\alpha g_{ij}\|_2 \leq C\hat{E}_{{\rm m},k}^{1/2} \tag{110}$$

where the last estimate is valid for $0 < |\alpha| \leq k$ and the constants depend on c_1.

Remark. The constant η_0 is the one appearing in the statement of Lemma 7. There is structure in the above estimates which is worth keeping in mind. The number of $-\Omega - K$:s appearing in the exponent corresponds to the number of spatial indices downstairs inside the H^k norm, including spatial derivatives. Note also that there is a gain of e^r in all the estimates except one, namely (110). All the norms appearing on the left hand side are on a fixed time slice. In other words, when we write $\|\psi\|_{H^k}$, we strictly speaking mean $\|\psi(t,\cdot)\|_{H^k}$. Below, we shall quite consistently abuse notation in this fashion.

Proof. Using (100), the estimates are immediate consequences of the definitions. Note for instance that

$$\partial_t h_{ij} = e^{-2\Omega}(\partial_t g_{ij} - 2\omega g_{ij}).$$

The lemma follows. □

8. Sobolev estimates for the inverse

Let us turn to estimating the derivatives of the components of the inverse of the metric. We shall use the following standard result, where $\|\cdot\|_p$ signifies the L^p norm of a measurable function defined on $\mathbb{T}^n$.

Theorem 8. *Assume $f_1, ..., f_l \in H^m(\mathbb{T}^n) \cap L^\infty(\mathbb{T}^n)$. Then there is a constant C depending on n, m and l such that if $\beta_1, ..., \beta_l$ are multiindices with $|\beta_1| + ... + |\beta_l| = m$, then*

$$\|\partial^{\beta_1} f_1 \cdots \partial^{\beta_l} f_l\|_2 \le C \sum_{i=1}^{l} \prod_{j \neq i} \|f_j\|_\infty \sum_{|\beta|=m} \|\partial^\beta f_i\|_2.$$

Due to this theorem and the bootstrap assumptions, we obtain estimates for the inverse of the metric.

Lemma 9. *Let $g_{\mu\nu}$ be the components of a matrix valued function on $[t_0, T) \times \mathbb{T}^n$ satisfying the conditions (94)-(96), where $\eta \le \eta_0$ and $\Omega, r \ge 0$ for $t \ge t_0$. Then g is a Lorentz metric, and if we denote the components of the inverse by $g^{\mu\nu}$, we have, for $0 < |\alpha| \le k$,*

$$e^r \|\partial^\alpha g^{00}\|_2 \le C\hat{E}_k^{1/2}, \tag{111}$$

$$e^{2\Omega+2K} \|\partial^\alpha g^{lm}\|_2 \le C\hat{E}_k^{1/2}, \tag{112}$$

$$e^{\Omega+r+K} \|g^{0l}\|_{H^k} \le C\hat{E}_k^{1/2}. \tag{113}$$

Proof. Note that

$$\partial_i g^{\lambda\sigma} = -g^{\mu\lambda} g^{\sigma\nu} \partial_i g_{\mu\nu}.$$

In general, $\partial^\alpha g^{\lambda\sigma}$ is, up to numerical factors, a sum of terms of the form

$$g^{\mu_1\lambda} g^{\mu_2\nu_1} \cdots g^{\mu_l\nu_{l-1}} g^{\sigma\nu_l} \partial^{\alpha_1} g_{\mu_1\nu_1} \cdots \partial^{\alpha_l} g_{\mu_l\nu_l}, \tag{114}$$

where $\alpha_1 + ... + \alpha_l = \alpha$, and $\alpha_i \neq 0$ (note that here $\alpha_1, ..., \alpha_l$ are multiindices, not the components of α). Let us introduce the notation that the number of g^{00} factors in (114) is l_a, the number of g^{0i} factors is l_b and the number of g^{ij} factors is l_c. Furthermore, let us denote the number of factors of the form $\partial^\beta g_{00}$ for some multiindex β by l_1, the number of factors of the form $\partial^\beta g_{0i}$ by l_2 and the number of factors of the form $\partial^\beta g_{ij}$ by l_3. It is of interest to analyze how these numbers change when we apply ∂_i to (114). If ∂_i hits a factor of the form $\partial^\beta g_{\mu\nu}$, then the numbers do not change. If ∂_i hits a factor of the form g^{00}, there are three possibilities for how the numbers can change

$$(l_a, l_1) \mapsto (l_a + 1, l_1 + 1) \tag{115}$$

$$(l_b, l_2) \mapsto (l_b + 1, l_2 + 1) \tag{116}$$

$$(l_a, l_b, l_3) \mapsto (l_a - 1, l_b + 2, l_3 + 1). \tag{117}$$

By this notation we mean that only the numbers that appear on the left hand side are changed. Thus in the first case, l_2, l_3, l_b, l_c remain unchanged. Consider the case when ∂_i hits a g^{0l} factor. There are four cases to consider:

$$(l_a, l_1) \mapsto (l_a + 1, l_1 + 1) \tag{118}$$

$$(l_b, l_2) \mapsto (l_b + 1, l_2 + 1) \tag{119}$$

$$(l_c, l_3) \mapsto (l_c + 1, l_3 + 1) \tag{120}$$

$$(l_a, l_b, l_c, l_2) \mapsto (l_a + 1, l_b - 1, l_c + 1, l_2 + 1). \tag{121}$$

Finally, let us assume ∂_i hits a g^{lm} factor. We have

$$(l_b, l_c, l_1) \mapsto (l_b + 2, l_c - 1, l_1 + 1) \tag{122}$$
$$(l_b, l_2) \mapsto (l_b + 1, l_2 + 1) \tag{123}$$
$$(l_c, l_3) \mapsto (l_c + 1, l_3 + 1). \tag{124}$$

Note that l_1, l_2, l_3 are monotonically increasing, but that l_a, l_b, l_c are not.

Let us estimate an expression of the form (114) using Theorem 8. Due to (96), (97), (99) and (100), we have the estimates

$$\|g^{00}\|_\infty \leq 5, \quad \|g^{0i}\|_\infty \leq Ce^{-\Omega-r-K}, \quad \|g^{ij}\|_\infty \leq Ce^{-2\Omega-2K},$$

where C is a constant depending on c_1. For this reason, we rewrite the factors of the form $g^{\mu\nu}$ in the following way:

$$g^{0i} = e^{-\Omega-r-K}(e^{\Omega+r+K}g^{0i}), \quad g^{ij} = e^{-2\Omega-2K}(e^{2\Omega+2K}g^{ij}).$$

Since we have the estimates

$$|g_{00}| \leq 2, \quad |g_{0i}| \leq Ce^{\Omega-r+K}, \quad |g_{ij}| \leq Ce^{2\Omega+2K},$$

due to (94)-(96), we shall also rewrite the factors of the form $\partial^\beta g_{\mu\nu}$ according to

$$\partial^\beta g_{0i} = e^{\Omega-r+K}(e^{-\Omega+r-K}\partial^\beta g_{0i}), \ \partial^\beta g_{ij} = e^{2\Omega+2K}(e^{-2\Omega-2K}\partial^\beta g_{ij}).$$

We get the estimate

$$\begin{aligned}
&\|g^{\lambda\mu_1}g^{\mu_2\nu_1}\cdots g^{\mu_l\nu_{l-1}}g^{\nu_l\sigma}\partial^{\alpha_1}g_{\mu_1\nu_1}\cdots\partial^{\alpha_l}g_{\mu_l\nu_l}\|_2 \\
&\leq \ C\exp\{-(\Omega+K)[2(l_c-l_3)+l_b-l_2]-r(l_b+l_2)\} \\
&\quad \sum_{|\beta|=|\alpha|}\left(l_1\|\partial^\beta g_{00}\|_2 + l_2 e^{-\Omega+r-K}\sum_l \|\partial^\beta g_{0l}\|_2\right. \\
&\quad \left. + l_3 e^{-2\Omega-2K}\sum_{l,m}\|\partial^\beta g_{lm}\|_2\right),
\end{aligned}$$

where $\alpha = \alpha_1 + ... + \alpha_l$ (note that before applying Theorem 8, we take out all the $g^{\mu\nu}$ factors in the L^∞-norm). The reason we have included the factors l_i is that we wish to distinguish between the cases $l_i = 0$ and $l_i > 0$. The point is to consider how the expressions

$$l_H = l_b - l_2 + 2(l_c - l_3), \quad l_D = l_b + l_2 \tag{125}$$

change when applying ∂_i to (114). Due to (115)-(124), we conclude that l_H is conserved, but that l_D is monotonically increasing. That l_H is conserved is not so surprising since it simply counts the number of upstairs spatial indices minus the number of downstairs spatial indices (disregarding derivatives). Note that the operations that increase l_D are (116), (117), (119), (122) and (123).

Let us estimate $\partial^\alpha g^{00}$ for some $\alpha \neq 0$. Note that

$$\partial_i g^{00} = -g^{00}g^{00}\partial_i g_{00} - 2g^{00}g^{0l}\partial_i g_{0l} - g^{0l}g^{0m}\partial_i g_{lm}. \tag{126}$$

Let us start with $g^{00}g^{00}\partial_i g_{00}$. In this case, $l_H = l_D = 0$. However, if, when differentiating, we are supposed to obtain a term with a factor of the form $\partial_j g_{0l}$

or $\partial_j g_{lm}$, then we have to go through one of the processes (116) or (117) both of which increase l_D by 2. Consequently, we get an estimate

$$\begin{aligned}\|\partial^\alpha(g^{00}g^{00}\partial_i g_{00})\|_2 \quad &\leq \quad C \sum_{|\beta|=|\alpha|+1}\left(\|\partial^\beta g_{00}\|_2 + e^{-\Omega-r-K}\sum_l \|\partial^\beta g_{0l}\|_2\right.\\ &\left. + e^{-2\Omega-2r-2K}\sum_{l,m}\|\partial^\beta g_{lm}\|_2\right).\end{aligned}$$

For the other two terms in the right hand side of (126), $l_H = 0$ initially and $l_D = 2$. The estimates for the last two terms are thus better than the estimate for the first term. Adding up, we obtain

$$e^r\|\partial^\alpha g^{00}\|_2 \leq CS_k,$$

where

$$\begin{aligned}S_k \quad &= \quad \sum_{0<|\beta|\leq k}\left(e^r\|\partial^\beta g_{00}\|_2 + e^{-\Omega+r-K}\sum_l \|\partial^\beta g_{0l}\|_2\right.\\ &\left. + e^{-2\Omega-2K}\sum_{l,m}\|\partial^\beta g_{lm}\|_2\right).\end{aligned}$$

In fact, we obtain a somewhat better result as far as the factors of e^r in front of different terms are concerned, but we shall have no reason to use this improvement. Using (107), (108) and (110), this proves (111). Let us turn to $\partial^\alpha g^{0l}$. We have

$$\partial_i g^{0l} = -g^{00}g^{0l}\partial_i g_{00} - g^{0m}g^{0l}\partial_i g_{0m} - g^{00}g^{lm}\partial_i g_{0m} - g^{0j}g^{lm}\partial_i g_{jm}.$$

For all the terms in the right hand side except for the second one, we have $l_H = 1$ and $l_D = 1$. For the second term, we have $l_H = 1$ and $l_D = 3$, but this improvement will not be of any use to us. We thus obtain

$$e^{\Omega+K+r}\|\partial^\alpha g^{0l}\|_2 \leq CS_k,$$

which, in combination with (97) and (108) (note that these are necessary to deal with the case $\alpha = 0$), implies (113). Finally,

$$\partial_i g^{lm} = -g^{l0}g^{m0}\partial_i g_{00} - g^{lp}g^{m0}\partial_i g_{p0} - g^{l0}g^{mp}\partial_i g_{p0} - g^{lp}g^{mq}\partial_i g_{pq}.$$

We obtain (112) by arguments similar to ones presented above. □

9. Estimates for the non-linearity

In this section we shall be making the bootstrap assumptions that (105) holds as well as (94)-(96), where $\eta \leq \eta_0$ and $\Omega, r \geq 0$ for $t \geq t_0$. Before we write down the consequences of these assumptions, let us observe that due to Theorem 8, we have an estimate of the form

$$(127)\quad \|f_1\cdots f_l\|_{H^k} \leq C\left(\sum_{0<|\alpha|\leq k}\sum_{i=1}^{l-1}\|\partial^\alpha f_i\|_2\prod_{j\neq i}\|f_j\|_\infty + \|f_1\|_\infty\cdots\|f_{l-1}\|_\infty\|f_l\|_{H^k}\right).$$

The point of this estimate is that the only function we estimate in L^2 is f_l, cf. (111) and (112), which are only valid for $|\alpha| > 0$. It will be of interest to estimate e.g. $1/g_{00}$ in H^k and in order to be able to do that, we need the following result.

Lemma 10. *Let $F \in C^\infty(I)$ for some open interval I and let $f \in H^k(\mathbb{T}^n) \cap L^\infty(\mathbb{T}^n)$, where $k > 0$. Let $J = [a, b]$, where a is the essential infimum and b is the essential supremum of f and assume that $J \subset I$. Then there is a constant, depending on k, the supremum of F and its derivatives up to order k on J and on $\|f\|_\infty$ such that for any α with $|\alpha| = k$,*

$$\|\partial^\alpha F \circ f\|_2 \leq C \sum_{|\beta|=k} \|\partial^\beta f\|_2.$$

Remark. The case $\alpha = 0$ is special. If $F(0) = 0$, it can be dealt with similarly, but not otherwise.

Proof. The result follows from the fact that $\partial^\alpha F \circ f$ is, up to numerical factors, a sum of terms of the form

$$(\partial^l F) \circ f \partial^{\alpha_1} f \cdots \partial^{\alpha_l} f,$$

where $\alpha_1 + ... + \alpha_l = \alpha$. □

Lemma 11. *Let $g_{\mu\nu}$ be the components of a matrix valued function satisfying (105) and (94)-(96) where $\eta \leq \eta_0$ and $\Omega, r \geq 0$ for $t \geq t_0$. Then $g_{\mu\nu}$ are the components of a Lorentz metric and we have the following estimates:*

$$e^{-2\Omega-2K}\omega^{-1}\|\partial_t g_{lm}\|_{C^1} \leq C, \tag{128}$$

$$e^{-2\Omega-2K}\omega^{-1}\|\partial^\alpha\partial_t g_{lm}\|_2 \leq C\hat{E}^{1/2}_{\mathrm{m},k}, \tag{129}$$

the last inequality being valid for $0 < |\alpha| \leq k$. Furthermore,

$$e^r\omega^{-1}\|g^{ij}\partial_t g_{jm} - 2\omega\delta^i_m\|_\infty \leq C\epsilon, \tag{130}$$

$$e^r\omega^{-1}\|g^{ij}\partial_t g_{jm} - 2\omega\delta^i_m\|_{H^k} \leq C\hat{E}^{1/2}_k, \tag{131}$$

$$e^r\|g^{00} + 1\|_{H^k} \leq C\hat{E}^{1/2}_k, \tag{132}$$

$$e^r\|g^{00} + 1\|_\infty \leq C\epsilon, \tag{133}$$

$$e^r\left\|\partial^\alpha\left(\frac{1}{g_{00}}\right)\right\|_2 \leq C\hat{E}^{1/2}_{\mathrm{lp},k}, \tag{134}$$

for $0 < |\alpha| \leq k$. Note that the constants in the estimates are allowed to depend on c_1. Finally,

$$e^{r-2K}[\omega^{-1}\|\partial_t h_{ij}\|_{H^k} + e^{-\Omega-K}\omega^{-1}\|\partial_m h_{ij}\|_{H^k}] \leq C\hat{E}^{1/2}_{\mathrm{m},k}, \tag{135}$$

$$e^{r-2K}[\omega^{-1}\|\partial_t\partial_l h_{ij}\|_\infty + e^{-\Omega-K}\omega^{-1}\|\partial_m\partial_l h_{ij}\|_\infty] \leq C\epsilon. \tag{136}$$

Proof. The estimate (128) follows from (94), (105), (109), (110) and Sobolev embedding (note that $k_0 > n/2 + 1$). The inequality (129) follows from (110) and (109). Note that

$$g^{ij}\partial_t g_{jm} - 2\omega\delta^i_m = g^{ij}(\partial_t g_{jm} - 2\omega g_{jm}) - 2\omega g^{i0}g_{m0}.$$

Using (100), (105), (108), (109), (112), (113) and (127), we obtain (130) and (131). If we apply Lemma 10 with $F(f) = 1/f$ and $f = g_{00}$, keeping (95) in mind, we obtain (134) for $0 < |\alpha| \leq k$. Due to (91), we conclude that (132) and (133) hold.

Finally, (135)-(136) are immediate consequences of the definition of the energies and (105). □

9.1. **Algorithm for estimating the non-linear terms.** Let us write down a general algorithm for dealing with the non-linear terms, assuming the bootstrap assumptions hold as stated in the beginning of the present section. A general term will consist of factors of the form $F(g_{00})$, $G(g^{00})$ and $R(\psi)$, where F, G and R are smooth in the intervals g_{00}, g^{00} and ψ belong to. Furthermore, there will be l_a, l_b, l_c, l_d, l_e and l_f factors of the form g^{ij}, g^{0i}, $1+g_{00}$, $1+g^{00}$, g_{0i} and g_{ij} respectively. Finally, let us denote the number of $\partial_t g_{ij}$, $\partial_t g_{00}$, $\partial_i g_{00}$, $\partial_t g_{0i}$, $\partial_i g_{0j}$, $\partial_i g_{jl}$, $g^{ij}\partial_t g_{jl} - 2\omega\delta^i_l$, $\partial_t g_{jm} - 2\omega g_{jm}$, ψ, $\partial_t\psi$ and $\partial_i\psi$ factors by $l_1, ..., l_{11}$ respectively.

Step 1. Rescale all the factors. The relevant factor to take out is

$$\begin{aligned}
&(e^{-2\Omega-2K})^{l_a}(e^{-\Omega-r-K})^{l_b}(e^{-r})^{l_c}(e^{-r})^{l_d}(e^{\Omega-r+K})^{l_e}(e^{2\Omega+2K})^{l_f}\\
&\quad(\omega e^{2\Omega+2K})^{l_1}(\omega e^{-r})^{l_2}(\omega e^{\Omega-r+K})^{l_3}(\omega e^{\Omega-r+K})^{l_4}(\omega e^{2\Omega-r+2K})^{l_5}\\
&\quad(\omega e^{3\Omega-r+3K})^{l_6}(\omega e^{-r})^{l_7}(\omega e^{2\Omega-r+2K})^{l_8}(e^{-r})^{l_9}(\omega e^{-r})^{l_{10}}(\omega e^{\Omega-r+K})^{l_{11}}.
\end{aligned}$$

Note that $F(g_{00})$, $G(g^{00})$, $R(\psi)$, $e^{2\Omega+2K}g^{ij}$, $e^{-2\Omega-2K}g_{ij}$ as well as $e^{-2\Omega-2K}\omega^{-1}\partial_t g_{ij}$ are bounded in L^∞. All the remaining factors are bounded by $C\epsilon$ in L^∞ and by $C\hat{E}_k^{1/2}$ in H^k after rescaling. Let us define

$$\begin{aligned}
l_\epsilon &= l_b + l_c + l_d + l_e + l_2 + ... + l_{11}\\
l_h &= -2l_a - l_b + l_e + 2l_f + 2l_1 + l_3 + l_4 + 2l_5 + 3l_6 + 2l_8 + l_{11},\\
l_\partial &= l_1 + ... + l_8 + l_{10} + l_{11}.
\end{aligned}$$

Note that l_h coincides with the number of downstairs spatial indices minus the number of upstairs spatial indices, including derivatives, and that l_∂ is the number of factors that are derivatives (note that we e.g. regard terms of the form $\partial_t g_{ij} - 2\omega g_{ij}$ as derivatives). With this notation, the factor we have taken out is

$$\omega^{l_\partial} e^{l_h(\Omega+K) - l_\epsilon r}.$$

Step 2. Assume $l_\epsilon \geq 1$ and $\epsilon \leq 1$. Then the rescaled quantity is bounded in the H^k-norm by CR_k, where

$$R_k = \hat{E}_k^{1/2}\epsilon^{l_\epsilon - 1}. \tag{137}$$

In order to prove this statement, let us apply (127) to the rescaled quantity with f_l chosen to be one of the factors that contribute to l_ϵ. In other words, if we have to estimate one of

$$F(g_{00}),\ G(g^{00}),\ R(\psi),\ e^{2\Omega+2K}g^{ij},\ e^{-2\Omega-2K}g_{ij},\ e^{-2\Omega-2K}\omega^{-1}\partial_t g_{ij}$$

in anything but L^∞, there will always be a derivative hitting these factors. If the derivatives hit one of

$$F(g_{00}),\ G(g^{00}),\ R(\psi),$$

we can estimate the result by $Ce^{-r}\epsilon R_k$. If the derivatives hit one of

$$e^{2\Omega+2K}g^{ij},\ e^{-2\Omega-2K}g_{ij},\ e^{-2\Omega-2K}\omega^{-1}\partial_t g_{ij},$$

we get an estimate $C\epsilon R_k$. The remaining terms are bounded by CR_k. Since $\epsilon \leq 1$ and $r \geq 0$, we obtain the desired conclusion.

Step 3. The estimate we obtain in the end is

$$C\omega^{l_\partial}e^{l_h(\Omega+K)-l_\epsilon r}\hat{E}_k^{1/2}\epsilon^{l_\epsilon-1}. \tag{138}$$

Assuming $l_\epsilon \geq 2$ and $\epsilon \leq 1$, we obtain

$$C\epsilon\omega^{l_\partial}e^{l_h(\Omega+K)-l_\epsilon r}\hat{E}_k^{1/2}. \tag{139}$$

Algorithm. Given a term of the above type, compute that $l_\epsilon \geq 1$. Note that one obtains l_ϵ simply by adding all the factors that are assumed small in the perturbation argument. After that, compute l_h, i.e. the number of downstairs spatial indices minus the number of upstairs spatial indices, including derivatives. Finally, compute l_∂, the number of derivatives. The estimate for the corresponding term is then of the form (138) if $l_\epsilon = 1$ and of the form (139) if $l_\epsilon \geq 2$.

9.2. **Estimates for the non-linearity.** The algorithm we have developed makes it trivial to estimate $\Delta_{A,\mu\nu}$, $\Delta_{C,00}$ and $\Delta_{C,0m}$.

Lemma 12. *Let $g_{\mu\nu}$ be the components of a smooth matrix valued function on $[t_0,T)\times\mathbb{T}^n$ satisfying (105) and (94)-(96) where $\eta \leq \eta_0$ and $\Omega, r \geq 0$ for $t \geq t_0$. Then*

$$\begin{aligned}
\|\Delta_{A,00}\|_{H^k} + \|\Delta_{C,00}\|_{H^k} &\leq C\epsilon\omega^2 e^{-2r}\hat{E}_k^{1/2}\\
\|\Delta_{A,0m}\|_{H^k} + \|\Delta_{C,0m}\|_{H^k} &\leq C\epsilon\omega^2 e^{\Omega+K-2r}\hat{E}_k^{1/2}\\
\|\Delta_{A,ij}\|_{H^k} &\leq C\epsilon\omega^2 e^{2\Omega+2K-2r}\hat{E}_k^{1/2},
\end{aligned}$$

where $\Delta_{A,\mu\nu}$, $\Delta_{C,00}$ and $\Delta_{C,0m}$ are given by (87), (92) and (93) respectively.

Proof. By construction, for any term appearing in $\Delta_{A,\mu\nu}$ or $\Delta_{C,\mu\nu}$, l_h is simply the number of spatial indices in the set $\{\mu,\nu\}$. That $l_\epsilon \geq 2$ is again true by construction. The reason for the factor ω^2 on the right hand side is that whenever a derivative is missing it is compensated for by a factor of ω. □

One object one has to estimate is the commutator between $\hat{\Box}_g = -g^{\mu\nu}\partial_\mu\partial_\nu$ and ∂^α, acting on some suitable function, say v. In order to be able to do so one needs to know something about $\hat{\Box}_g v$. However, since we do not wish to write down the equations, we shall make assumptions on $\hat{\Box}_g v$ in the statement of the lemma.

Lemma 13. *Let $g_{\mu\nu}$ be the components of a smooth matrix valued function on $[t_0,T)\times\mathbb{T}^n$ satisfying (105) and (94)-(96) where $\eta \leq \eta_0$ and $\Omega, r \geq 0$ for $t \geq t_0$. Let v be a smooth function on $[t_0,T)\times\mathbb{T}^n$ such that*

$$\omega^{-1}\|\partial_t v\|_{H^k} + e^{-\Omega-K}\omega^{-1}\|\partial_i v\|_{H^k} + \omega^{-2}\|\hat{\Box}_g v\|_{H^k} \leq Ce^{l_h(\Omega+K)-r}\hat{E}_k^{1/2} \tag{140}$$

for some $k > n/2+1$. Then, for $0 < |\alpha| \leq k$,

$$\|[\hat{\Box}_g, \partial^\alpha]v\|_2 \leq C\epsilon\omega^2 e^{l_h(\Omega+K)-2r}\hat{E}_k^{1/2}, \tag{141}$$

where the constant depends on

$$\sup_{t\in[t_0,T)} \omega^{-1}e^{-\Omega-K+r}, \tag{142}$$

which we assume to be finite.

Proof. Note that

$$[\partial^{\alpha}, g^{\mu\nu}\partial_{\mu}\partial_{\nu}]v$$

is, up to constant factors, a sum of terms of the form

$$\partial^{\alpha_1}\partial_i g^{\mu\nu}\partial^{\alpha_2}\partial_{\mu}\partial_{\nu}v,$$

where $|\alpha_1| + |\alpha_2| = |\alpha| - 1$. It is natural to divide these terms into two different categories. Either $\mu = \nu = 0$ or one of $\mu,\nu \neq 0$. Let us consider the second case first. Assuming $|\alpha| \leq k$, we have the estimate

$$\begin{aligned} &\|\partial^{\alpha_1}\partial_i g^{j\nu}\partial^{\alpha_2}\partial_j\partial_{\nu}v\|_2 \\ \leq\ & C\sum_j [\|\partial_i g^{j\nu}\|_{\infty}\|\partial_{\nu}v\|_{H^k} + \|\partial_j\partial_{\nu}v\|_{\infty}\|\partial_i g^{j\nu}\|_{H^{k-1}}], \end{aligned} \tag{143}$$

where we take it to be understood that we sum over ν and over j in the left hand side. Due to (112) and (113), we obtain

$$\begin{aligned} &\|\partial^{\alpha_1}\partial_i g^{j\nu}\partial^{\alpha_2}\partial_j\partial_{\nu}v\|_2 \\ \leq\ & Ce^{-\Omega-K}\sum_{j,l}[\epsilon(e^{-r}\|\partial_t v\|_{H^k} + e^{-\Omega-K}\|\partial_l v\|_{H^k}) \\ & +(e^{-r}\|\partial_j\partial_t v\|_{\infty} + e^{-\Omega-K}\|\partial_j\partial_l v\|_{\infty})\hat{E}_k^{1/2}] \end{aligned}$$

where we have used the bootstrap assumptions and Sobolev embedding, in view of the fact that $k_0 > n/2 + 1$. Due to (140), the fact that $r \geq 0$ and the fact that (142) is bounded, we obtain an estimate of the form (141).

In order to deal with the case $\mu = \nu = 0$, we rewrite the corresponding term

$$\partial^{\alpha_1}\partial_i g^{00}\partial^{\alpha_2}\partial_t^2 v = -\partial^{\alpha_1}\partial_i g^{00}\partial^{\alpha_2}\left[\frac{1}{g^{00}}(2g^{0j}\partial_j\partial_t v + g^{jl}\partial_j\partial_l v + F)\right],$$

where $F = \hat{\Box}_g v$ and $|\alpha_1 + \alpha_2| = |\alpha| - 1 \leq k - 1$. Let us consider the term

$$\begin{aligned} \partial^{\alpha_1}\partial_i g^{00}\partial^{\alpha_2}\left[\frac{1}{g^{00}}g^{0j}\partial_j\partial_t v\right] &= \omega e^{(l_h-1)(\Omega+K)-3r} \\ & \partial^{\alpha_1}(e^r\partial_i g^{00})\partial^{\alpha_2}\left[\frac{1}{g^{00}}(e^{\Omega+r+K}g^{0j})(\omega^{-1}e^{-l_h(\Omega+K)+r}\partial_j\partial_t v)\right]. \end{aligned}$$

We can estimate this expression in L^2 by

$$\omega e^{(l_h-1)(\Omega+K)-3r}\epsilon^2\hat{E}_k^{1/2}.$$

Using the fact that $\epsilon \leq 1$, that $r \geq 0$ and that we allow the constants to depend on an upper bound of (142), we get the desired estimate. The argument to deal with

$$\partial^{\alpha_1}\partial_i g^{00}\partial^{\alpha_2}\left[\frac{1}{g^{00}}g^{jl}\partial_j\partial_l v\right]$$

is similar, though the estimate is somewhat worse. Finally, using (127), (111), (140) and the bootstrap assumptions, we can estimate

$$\left\|\partial^{\alpha_1}\partial_i g^{00}\partial^{\alpha_2}\left[\frac{1}{g^{00}}F\right]\right\|_2$$

as desired. □

10. Equations

From now on, we shall restrict our attention to potentials of the form described in connection with (10)-(11). In the general setup we have been considering up till now, the background metric is given by (44) with $\Omega = Ht$, where $H > 0$ is defined by (10). Consequently $\omega = H$ is constant. We shall choose $r = aHt$ for some constant $a > 0$, which is to be determined. From now on, we shall also let $t_0 = 0$, so that the conditions that $r, \Omega \geq 0$ will be satisfied automatically. The background ϕ_0 around which we are perturbing is 0, and we shall use the variables defined in (102) and (103).

Lemma 14. *Let $V \in C^\infty(\mathbb{R})$ be such that $V(0) > 0$, $V'(0) = 0$ and $V''(0) > 0$ and define $H > 0$ and χ by (10) and (11). Then (67)-(70) are equivalent to*

$$\hat{\Box}_g u + (n+2)H\partial_0 u + 2nH^2 u + \Delta_{00} = 0, \tag{144}$$

$$\hat{\Box}_g u_m + nH\partial_0 u_m + 2(n-2)H^2 u_m - 2Hg^{ij}\Gamma_{imj} + \Delta_{0m} = 0, \tag{145}$$

$$\hat{\Box}_g h_{ij} + nH\partial_0 h_{ij} + \Delta_{ij} = 0, \tag{146}$$

$$\hat{\Box}_g \phi + nH\partial_0\phi + H^2\chi\phi + \Delta_\phi = 0, \tag{147}$$

where $\Delta_{00}, ..., \Delta_\phi$ are given by (151)-(154) and we use the notation

$$\hat{\Box}_g = -g^{\alpha\beta}\partial_\alpha\partial_\beta.$$

Proof. Let us define

$$\Delta_{\phi,\mu\nu} = -\partial_\mu\phi\partial_\nu\phi - \frac{2}{n-1}[V(\phi) - V_0]g_{\mu\nu} \tag{148}$$

$$\hat{\Delta}_\phi = V'(\phi) - H^2\chi\phi, \tag{149}$$

where χ was defined in (11). Letting $\Delta_{C,ij} = 0$,

$$\hat{\Delta}_{\mu\nu} = \Delta_{A,\mu\nu} + \Delta_{C,\mu\nu} + \Delta_{\phi,\mu\nu}, \tag{150}$$

and using Lemma 4, Lemma 6 and (88), the equations (67)-(70) take the form

$$-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{00} + \frac{1}{2}(n+2)H\partial_0 g_{00} + nH^2(g_{00}+1) + \hat{\Delta}_{00} = 0$$

$$-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{0m} + \frac{1}{2}nH\partial_0 g_{0m} + (n-2)H^2 g_{0m} - Hg^{ij}\Gamma_{imj} + \hat{\Delta}_{0m} = 0$$

$$-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{ij} + \frac{1}{2}nH\partial_0 g_{ij} + 2Hg^{00}\partial_0 g_{ij} - 2H^2 g^{00} g_{ij} - nH^2 g_{ij} + \hat{\Delta}_{ij} = 0$$

$$-g^{\alpha\beta}\partial_\alpha\partial_\beta\phi + nH\partial_0\phi + H^2\chi\phi + \hat{\Delta}_\phi = 0.$$

We obtain (144)-(147), where Δ_{00},..., Δ_ϕ are given by

$$\Delta_{00} = 2\hat{\Delta}_{00} \tag{151}$$

$$\Delta_{0m} = 2\hat{\Delta}_{0m} \tag{152}$$

$$\Delta_{ij} = -4Hg^{0p}\partial_p h_{ij} + 2e^{-2Ht}\hat{\Delta}_{ij} \tag{153}$$

$$\Delta_\phi = \hat{\Delta}_\phi. \tag{154}$$

Here $\hat{\Delta}_{\mu\nu}$ is defined in (150), in which $\Delta_{\phi,\mu\nu}$ is defined by (148), and $\hat{\Delta}_{\phi}$ is defined by (149). □

11. Energy estimates

Lemma 15. *Consider a solution to the equation*

$$\hat{\Box}_g v + \alpha H \partial_0 v + \beta H^2 v = F, \tag{155}$$

where $\alpha > 0$ and $\beta \geq 0$. Then there are constants $\eta_{\mathrm{c}}, \zeta > 0$ and $\gamma, \delta \geq 0$, depending on α and β, such that if

$$|g^{00} + 1| \leq \eta_{\mathrm{c}}, \tag{156}$$

and

$$\mathcal{E}_{\gamma,\delta}[v] = \frac{1}{2}\int_{\mathbb{T}^n}[-g^{00}(\partial_0 v)^2 + g^{ij}\partial_i v \partial_j v - 2\gamma H g^{00} v \partial_0 v + \delta H^2 v^2]dx, \tag{157}$$

then

$$\mathcal{E}_{\gamma,\delta}[v] \geq \zeta \int_{\mathbb{T}^n}[(\partial_0 v)^2 + g^{ij}\partial_i v \partial_j v + \iota_\beta H^2 v^2]dx, \tag{158}$$

where $\iota_\beta = 0$ if $\beta = 0$ and $\iota_\beta = 1$ if $\beta > 0$. Furthermore

$$\frac{d\mathcal{E}_{\gamma,\delta}}{dt} \leq -\eta_{\mathrm{c}} H \mathcal{E}_{\gamma,\delta} + \int_{\mathbb{T}^n}\{(\partial_0 v + \gamma H v)F + \Delta_{E,\gamma,\delta}[v]\}dx,$$

where $\Delta_{E,\gamma,\delta}[v]$ is given by (159).

Remark. If $\beta = 0$, then $\gamma = \delta = 0$.

Proof. If $\beta > 0$, choose $\gamma = \alpha/2$ and $\delta = \beta + \alpha^2/2$. Then $\gamma^2 < \delta$, and it is clear that there is a constant $\zeta > 0$ such that (158) holds, assuming g^{00} is close enough to 1. If $\beta = 0$, we simply let $\gamma = \delta = 0$, and the existence of a $\zeta > 0$ such that (158) holds again follows from the assumption that g^{00} is close enough to 1. Compute

$$\begin{aligned}\frac{d\mathcal{E}_{\gamma,\delta}}{dt} &= \int_{\mathbb{T}^n}\{-(\alpha-\gamma)H(\partial_0 v)^2 + (\delta - \beta - \gamma\alpha)H^2 v \partial_0 v - \beta\gamma H^3 v^2 \\ &\quad -(1+\gamma)H g^{ij}\partial_i v \partial_j v + (\partial_0 v + \gamma H v)F + \Delta_{E,\gamma,\delta}[v]\}dx,\end{aligned}$$

where

$$\begin{aligned}\Delta_{E,\gamma,\delta}[v] &= -\gamma H(\partial_i g^{ij})v\partial_j v - 2\gamma H(\partial_i g^{0i})v\partial_0 v - 2\gamma H g^{0i}\partial_i v \partial_0 v \\ &\quad -(\partial_i g^{0i})(\partial_0 v)^2 - (\partial_j g^{ij})\partial_i v \partial_0 v - \frac{1}{2}(\partial_0 g^{00})(\partial_0 v)^2 \\ &\quad + \left(\frac{1}{2}\partial_0 g^{ij} + H g^{ij}\right)\partial_i v \partial_j v - \gamma H \partial_0 g^{00} v \partial_0 v \\ &\quad -\gamma H(g^{00}+1)(\partial_0 v)^2.\end{aligned} \tag{159}$$

Due to our choices, we have, assuming $\beta > 0$,

$$\begin{aligned}\frac{d\mathcal{E}_{\gamma,\delta}}{dt} &= -\frac{1}{2}H\int_{\mathbb{T}^n}[\alpha(\partial_0 v)^2 + (\alpha+2)g^{ij}\partial_i v \partial_j v + \alpha\beta H^2 v^2]dx \\ &\quad + \int_{\mathbb{T}^n}\{(\partial_0 v + \gamma H v)F + \Delta_{E,\gamma,\delta}[v]\}dx.\end{aligned}$$

Since the opposite inequality to (158) also holds, provided we replace ζ by ζ^{-1} for ζ small enough, we obtain the conclusion of the lemma for $\beta > 0$. The conclusion in the case $\beta = 0$ follows for similar reasons. □

Corollary 1. *Under the assumptions of Lemma 15, let*

$$\mathfrak{E}_k = \sum_{|\alpha|\leq k} \mathcal{E}_{\gamma,\delta}[\partial^\alpha v].$$

Then

$$\frac{d\mathfrak{E}_k}{dt} \leq -\eta_c H \mathfrak{E}_k + \sum_{|\alpha|\leq k} \int_{\mathbb{T}^n} \{(\partial_0\partial^\alpha v + \gamma H \partial^\alpha v)(\partial^\alpha F + [\hat{\Box}_g, \partial^\alpha]v) + \Delta_{E,\gamma,\delta}[\partial^\alpha v]\} dx.$$

Proof. Given that v satisfies (155), $\partial^\alpha v$ satisfies

$$\hat{\Box}_g(\partial^\alpha v) + \alpha H \partial_0(\partial^\alpha v) + \beta H(\partial^\alpha v) = \partial^\alpha F + [\hat{\Box}_g, \partial^\alpha]v.$$

The statement follows from Lemma 15. □

We are now in a position to define the energies with which we shall be working. Note that all the equations (144)-(147) are of the form considered in Lemma 15. In the case of (144), (145) and (147), we simply identify the first three terms on the left hand side with the terms on the left hand side of (155) and identify the remaining terms with $-F$. In the case of (146), we proceed similarly, but in this case, $\beta = 0$. With u, u_m, h_{ij} and ϕ, we can thus associate constants $(\alpha_{\mathrm{l}}, \beta_{\mathrm{l}})$, $(\alpha_{\mathrm{s}}, \beta_{\mathrm{s}})$, $(\alpha_{\mathrm{m}}, \beta_{\mathrm{m}})$ and $(\alpha_{\mathrm{sf}}, \beta_{\mathrm{sf}})$ respectively. Due to Lemma 15 we get γ, δ, η_c and ζ with corresponding indices (we replace the index c with the corresponding index as well). Note that all these constants only depend on n and χ. From now on we shall assume $\eta \leq \eta_{\min}$ in (95)-(96), where

$$\eta_{\min} := \min\{1, \eta_0, \eta_{\mathrm{l}}, \eta_{\mathrm{s}}, \eta_{\mathrm{m}}, \eta_{\mathrm{sf}}\}/4. \tag{160}$$

Note that if (94)-(96) are satisfied with $\eta \leq \eta_{\min}$, then (156) is satisfied with η_c replaced by η_{l},...,η_{sf} due to (99). Note also that $\eta_{\min}$ only depends on n and χ. Let us define

$$H_{\mathrm{l},k} = \sum_{|\alpha|\leq k} \mathcal{E}_{\gamma_{\mathrm{l}},\delta_{\mathrm{l}}}[\partial^\alpha u], \quad H_{\mathrm{s},k} = \sum_i \sum_{|\alpha|\leq k} \mathcal{E}_{\gamma_{\mathrm{s}},\delta_{\mathrm{s}}}[\partial^\alpha u_i],$$
$$H_{\mathrm{sf},k} = \sum_{|\alpha|\leq k} \mathcal{E}_{\gamma_{\mathrm{sf}},\delta_{\mathrm{sf}}}[\partial^\alpha \phi].$$

Since there is no advantage in separating u and ϕ, let us introduce

$$H_{\mathrm{lp},k} = H_{\mathrm{l},k} + H_{\mathrm{sf},k}$$

and $\eta_{\mathrm{lp}} = \min\{\eta_{\mathrm{l}}, \eta_{\mathrm{sf}}\}$. Finally, let us introduce

$$H_{\mathrm{m},k} = \sum_{i,j} \sum_{|\alpha|\leq k} \left[\mathcal{E}_{\gamma_{\mathrm{m}},\delta_{\mathrm{m}}}[\partial^\alpha h_{ij}] + \frac{1}{2} \int_{\mathbb{T}^n} a_\alpha H^2 e^{-2aHt} (\partial^\alpha h_{ij})^2 dx \right], \tag{161}$$

where $a_\alpha = 0$ if $\alpha = 0$ and $a_\alpha = 1$ otherwise, $\gamma_{\mathrm{m}} = \delta_{\mathrm{m}} = 0$, cf. the proof of Lemma 15, and a is given by

$$a := \frac{2\eta_{\min}}{3}. \tag{162}$$

Note that, since $\eta_{\min} \leq 1/4$, we have $a \leq 1/6$. Note also that $E_{\mathrm{lp},k}$ and $H_{\mathrm{lp},k}$ are equivalent in the sense that there is a constant $c_E > 1$ depending on n and χ such that

$$c_E^{-1} E_{\mathrm{lp},k} \leq H_{\mathrm{lp},k} \leq c_E E_{\mathrm{lp},k}$$

assuming (94)-(96) are satisfied with $\eta \leq \eta_{\min}$. Similarly, $E_{\mathrm{s},k}$ and $H_{\mathrm{s},k}$ are equivalent and $E_{\mathrm{m},k}$ and $H_{\mathrm{m},k}$ are equivalent. Let us rescale similarly to (104). Since $\omega = H$, $r = aHt$ and $\Omega = Ht$, we define

$$\begin{aligned} \hat{H}_{\mathrm{lp},k} &= H^{-2} e^{2aHt} H_{\mathrm{lp},k}, \quad \hat{H}_{\mathrm{s},k} = H^{-2} e^{-2Ht+2aHt-2K} H_{\mathrm{s},k}, \\ \hat{H}_{\mathrm{m},k} &= H^{-2} e^{2aHt-4K} H_{\mathrm{m},k}. \end{aligned}$$

Finally, we let

$$\hat{H}_k = \hat{H}_{\mathrm{lp},k} + \hat{H}_{\mathrm{s},k} + \hat{H}_{\mathrm{m},k}. \tag{163}$$

Note that in the current context, $\hat{E}_k$ and $\hat{H}_k$ are equivalent.

12. Differential inequalities

The purpose of this section is to prove the differential inequalities that will be the essential tools for proving future global existence. When we say that the **bootstrap assumptions** hold, we here mean that we have smooth solutions $g_{\mu\nu}$ and ϕ to (144)-(147) such that (94)-(96) hold on some time interval $[0, T)$, where $\eta \leq \eta_{\min}$, cf. (160), and that

$$\hat{H}_{k_0}^{1/2}(t) \leq \epsilon \tag{164}$$

on the same time interval for some ϵ and k_0 such that $\epsilon \leq 1$ and $k_0 > n/2+1$. Note that under these assumptions, $\hat{H}_k$ and $\hat{E}_k$ are equivalent, the relevant constants only depending on n and χ, so that we, for all practical purposes, can assume that (105) holds. Recall that $r, \Omega \geq 0$ for $t \geq 0$ by construction in the current setting. Let us write down the estimates that will be of relevance.

Lemma 16. *Assume that the bootstrap assumptions hold. Then*

$$\begin{aligned} \|\Delta_{00}\|_{H^k} &\leq CH^2 \epsilon e^{-2aHt} \hat{H}_k^{1/2}, \\ \|\Delta_{0m}\|_{H^k} &\leq CH^2 \epsilon e^{Ht+K-2aHt} \hat{H}_k^{1/2}, \\ \|\Delta_{ij}\|_{H^k} &\leq CH^2 \epsilon e^{2K-2aHt} \hat{H}_k^{1/2}, \\ \|\Delta_{\phi}\|_{H^k} &\leq CH^2 \epsilon e^{-2aHt} \hat{H}_k^{1/2}, \end{aligned}$$

where the constants depend on n, k, χ, c_1 and the C^{k+3} norm of V/V_0 in a neighbourhood of 0.

Remark. The bootstrap assumptions, via Sobolev embedding, imply a bound for ϕ only depending on n and χ (recall that $\epsilon \leq 1$). This bound corresponds to the neighbourhood mentioned in the lemma.

Proof. Let us first consider $\hat{\Delta}_{\mu\nu}$. The terms arising from $\Delta_{A,\mu\nu}$ and $\Delta_{C,\mu\nu}$ are already under control due to Lemma 12. Concerning $\Delta_{\phi,\mu\nu}$, note that

$$V(\phi) - V_0 = H^2 \phi^2 R(\phi),$$

where

$$R(\phi) = \frac{n(n-1)}{2V_0} \int_0^1 \int_0^s V''(\tau\phi) d\tau ds.$$

Since R is a smooth function, we can use the algorithm for estimating $\Delta_{\phi,\mu\nu}$. The argument to deal with Δ_ϕ is similar. This yields all the estimates except the one for Δ_{ij}. The first term on the right hand side of (153) is the only term we have not yet estimated, but it can be dealt with using the algorithm. □

Lemma 17. *Assume that the bootstrap assumptions hold. Then*

$$\begin{aligned}
\|[\hat{\Box}_g, \partial^\alpha]u\|_2 &\leq CH^2\epsilon e^{-2aHt}\hat{H}_k^{1/2}, \\
\|[\hat{\Box}_g, \partial^\alpha]u_m\|_2 &\leq CH^2\epsilon e^{Ht+K-2aHt}\hat{H}_k^{1/2}, \\
\|[\hat{\Box}_g, \partial^\alpha]h_{ij}\|_2 &\leq CH^2\epsilon e^{2K-2aHt}\hat{H}_k^{1/2}, \\
\|[\hat{\Box}_g, \partial^\alpha]\phi\|_2 &\leq CH^2\epsilon e^{-2aHt}\hat{H}_k^{1/2},
\end{aligned}$$

for all $|\alpha| \leq k$, where the constants depend on n, k, χ, c_1, the C^{k+3} norm of V/V_0 in a neighbourhood of 0 *and on an upper bound on $H^{-1}e^{-K}$.*

Proof. This follows from Lemma 13, Lemma 16, (144)-(147), (106)-(110) and the algorithm (138) (note that we have used $a \leq 1$). Strictly speaking, the estimate for $[\hat{\Box}_g, \partial^\alpha]h_{ij}$ is obtained by applying Lemma 13 to $e^{-2K}h_{ij}$. □

In preparation for the final estimate, let us note that the following estimates hold.

Lemma 18. *Assume that the bootstrap assumptions hold. Then*

$$\begin{aligned}
\left\|\frac{1}{2}\partial_0 g^{ij} + Hg^{ij}\right\|_\infty &\leq CH\epsilon e^{-2Ht-2K}e^{-aHt}, \\
\|\partial_0 g^{00}\|_\infty &\leq CH\epsilon e^{-aHt}.
\end{aligned}$$

Proof. The estimates follow in a straightforward way from estimates we have already written down. □

Finally, we need the following estimates.

Lemma 19. *Assume that the bootstrap assumptions hold. Then*

$$\|\Delta_{E,\gamma_\mathrm{l},\delta_\mathrm{l}}[\partial^\alpha u]\|_1 \leq CH\epsilon e^{-aHt}H_{\mathrm{l},k}, \tag{165}$$

$$\|\Delta_{E,\gamma_\mathrm{s},\delta_\mathrm{s}}[\partial^\alpha u_m]\|_1 \leq CH\epsilon e^{-aHt}H_{\mathrm{s},k}, \tag{166}$$

$$\|\Delta_{E,\gamma_\mathrm{m},\delta_\mathrm{m}}[\partial^\alpha h_{ij}]\|_1 \leq CH\epsilon e^{-aHt}H_{\mathrm{m},k}, \tag{167}$$

$$\|\Delta_{E,\gamma_\mathrm{sf},\delta_\mathrm{sf}}[\partial^\alpha \phi]\|_1 \leq CH\epsilon e^{-aHt}H_{\mathrm{sf},k}, \tag{168}$$

for $|\alpha| \leq k$, where $\Delta_{E,\gamma,\delta}$ is defined in (159) and the constants depend on n, χ and an upper bound for $H^{-1}e^{-K}$.

Proof. Let $\mathcal{E}_{\gamma,\delta}$ be defined as in Lemma 15, $\Delta_{E,\gamma,\delta}$ be defined as in (159) and recall that $\gamma_\mathrm{m} = \delta_\mathrm{m} = 0$. If the bootstrap assumptions hold, we see that

$$\|\Delta_{E,\gamma,\delta}[v]\|_1 \leq CH\epsilon e^{-aHt}\mathcal{E}_{\gamma,\delta}[v].$$

This proves the lemma. □

Lemma 20. *Assume that the bootstrap assumptions hold. Then*

$$\frac{d\hat{H}_{\mathrm{lp},k}}{dt} \leq -4aH\hat{H}_{\mathrm{lp},k} + CH\epsilon e^{-aHt}\hat{H}_k^{1/2}\hat{H}_{\mathrm{lp},k}^{1/2}, \tag{169}$$

$$\begin{aligned}\frac{d\hat{H}_{\mathrm{s},k}}{dt} &\leq -4aH\hat{H}_{\mathrm{s},k} + CH\hat{H}_{\mathrm{m},k}^{1/2}\hat{H}_{\mathrm{s},k}^{1/2} \\ &\quad + CH\epsilon e^{-aHt}\hat{H}_k^{1/2}\hat{H}_{\mathrm{s},k}^{1/2},\end{aligned} \tag{170}$$

$$\frac{d\hat{H}_{\mathrm{m},k}}{dt} \leq He^{-aHt}\hat{H}_{\mathrm{m},k} + CH\epsilon e^{-aHt}\hat{H}_k^{1/2}\hat{H}_{\mathrm{m},k}^{1/2}, \tag{171}$$

where the constants depend an upper bound on $H^{-1}e^{-K}$, n, k, χ, c_1, and the C^{k+3} norm of V/V_0 in a neighbourhood of 0.

Proof. Recall that a is defined by (162), so that e.g. $\eta_l \geq 6a$, a fact we shall use. The inequalities (169) and (171) follow from the estimates we have written down so far. In the derivation of (171), recall that $\gamma_{\mathrm{m}} = \delta_{\mathrm{m}} = 0$ and note that when ∂_t hits the last factor in the last term of (161), the estimate

$$\begin{aligned}&a_\alpha H^2 e^{-2aHt}\partial^\alpha h_{ij}\partial_t\partial^\alpha h_{ij} \\ \leq\ & \frac{1}{2}He^{-aHt}a_\alpha[H^2e^{-2aHt}(\partial^\alpha h_{ij})^2 - g^{00}(\partial_t\partial^\alpha h_{ij})^2 \\ &+ (g^{00}+1)(\partial_t\partial^\alpha h_{ij})^2]\end{aligned}$$

is of use. If the second to last term on the left hand side of (145) did not exist, we would get (170) without the middle term on the right hand side. What remains to be estimated is a constant times

$$\begin{aligned}&H^{-2}e^{-2Ht+2aHt-2K}H_{\mathrm{s},k}^{1/2}\|Hg^{ij}\Gamma_{imj}\|_{H^k} \\ =\ & e^{-Ht+aHt-K}\hat{H}_{\mathrm{s},k}^{1/2}\|g^{ij}\Gamma_{imj}\|_{H^k}.\end{aligned}$$

Note that

$$\|g^{ij}\Gamma_{imj}\|_{H^k} \leq C\left[\|g^{ij}\|_\infty\|\Gamma_{imj}\|_{H^k} + \sum_{0<|\alpha|\leq k}\|\partial^\alpha g^{ij}\|_2\|\Gamma_{imj}\|_\infty\right].$$

In order to estimate the right hand side, let us use (110), (109) and (112). We use (110), the bootstrap assumptions and the fact that $k_0 > n/2+1$ to estimate $\|\Gamma_{imj}\|_\infty$. We then obtain

$$\|g^{ij}\Gamma_{imj}\|_{H^k} \leq Ce^{(1-a)Ht+K}H\hat{H}_{\mathrm{m},k}^{1/2} + C\epsilon\hat{H}_k^{1/2}.$$

The lemma follows. □

13. Global existence

We are now in a position to prove global existence of solutions to (144)-(147), given that the initial energy is small enough.

Theorem 9. *Let V be a smooth function such that $V(0) = V_0 > 0$, $V'(0) = 0$ and $V''(0) > 0$. Let $H, \chi > 0$ be defined by (10) and (11) respectively. Let $(\rho, \kappa, \phi_0, \phi_1)$ be given on $\mathbb{T}^n$, where ρ is a smooth Riemannian metric, κ is a smooth covariant 2-tensor and ϕ_0, ϕ_1 are smooth functions. Define $g_{\mu\nu}|_{t=0}$ by (58)-(59) (with h replaced by ρ), $(\partial_t g_{\mu\nu})|_{t=0}$ by (60) and (62)-(63) (with F_μ given by (46) where $\omega = H$), and* *define $\phi|_{t=0}$, $(\partial_t\phi)|_{t=0}$ by (61). This defines initial data for (144)-(147). Assume that there are constants $c_1 > 2$ and K such that*

$$\frac{2}{c_1}|v|^2 \leq e^{-2K} g_{ij}(0,x)v^i v^j \leq \frac{c_1}{2}|v|^2, \tag{172}$$

for all $v \in \mathbb{R}^n$ and $x \in \mathbb{T}^n$. Let $k_0 > n/2+1$ and define $\hat{H}_{k_0}$ by (163). There are $\epsilon_0, c_0 \in (0,1)$, where ϵ_0 and c_0 should be small enough, depending on an upper bound on $H^{-1}e^{-K}$, on χ, n, k_0, c_1 and the sup norm of $V^{(i)}/V_0$, $i = 1, ..., k_0+3$ in a neighbourhood of zero, such that if

$$\hat{H}^{1/2}_{k_0}(0) \leq c_0\epsilon, \tag{173}$$

for some $\epsilon \leq \epsilon_0$, then the solution to (144)-(147) with initial data specified as above exists for all future times, (94)-(96) are satisfied with $\eta = \eta_{\min}$ for all $t \geq 0$ and

$$\hat{H}^{1/2}_{k_0}(t) \leq \epsilon \tag{174}$$

for all $t \geq 0$.

Remark. The size of the neighbourhood on which we need to estimate V/V_0 is determined by n and χ, cf. the remark following Lemma 16.

Proof. Let $\mathcal{A}$ denote the set of $s \in [0,\infty)$ such that

- there exists a smooth solution to (144)-(147) on $[0,s)$,
- (94)-(96) are satisfied on $[0,s)$ with $\eta = \eta_{\min}$,
- (174) is satisfied on $[0,s)$.

Note that if $s \in \mathcal{A}$, then the conditions necessary for deriving the different inequalities above are satisfied on $[0,s)$. In particular, (169)-(171) hold. Note also that (144)-(147) are equivalent to (67)-(70), which, in their turn, are equivalent to (53)-(54), given the choices (46) and (51)-(52). Thus, due to Proposition 1, we have a unique smooth solution to (144)-(147) on some interval $(T_{\min}, T_{\max})$. Assume $c_0 \leq 1/2$. Then (174) is satisfied with a margin for $t = 0$ and so it will be satisfied on an open interval containing 0. Since (172) holds, as well as $g_{00} = -1$ and $g_{0i} = 0$ for $t = 0$, (94)-(96) are satisfied on an open interval containing 0. We conclude that there is a $T > 0$ such that $T \in \mathcal{A}$. That $\mathcal{A}$ is closed and connected follows from the definition. What remains to be proved is that it is open.

Assume $0 < T < \infty$ is such that $T \in \mathcal{A}$. Note that the bootstrap assumptions together with the equations ensure that the C^2 norms of g and ϕ do not blow up and that g_{00} and the smallest eigenvalue of $\{g_{ij}\}$ stay bounded well away from zero on $[0,T)$. Consequently, $T < T_{\max}$ due to Proposition 1. We thus have a smooth solution beyond T, and we conclude that (94)-(96), with $\eta = \eta_{\min}$, and (174) hold on $[0,T]$. In order to go beyond T, let us first prove that an improvement of (94)-(96) holds in $[0,T]$. Due to (174) and Sobolev embedding, we obtain

$$H^{-1}e^{aHt-2K}\|\partial_0 h_{ij}\|_\infty \leq C\epsilon.$$

Consequently,

$$\|e^{-2Ht-2K}g_{ij}(t,\cdot) - e^{-2K}g_{ij}(0,\cdot)\|_\infty \leq Ca^{-1}\epsilon \tag{175}$$

for all $t \in [0,T]$. By assuming ϵ to be small enough, we obtain (94) with c_1 replaced by $2c_1/3$. By assuming ϵ to be small enough, we also obtain (95) and (96) with $\eta_{\min}$ replaced by $\eta_{\min}/2$, due to the definition of the energies and Sobolev embedding.

Thus (94)-(96) hold in an open neighbourhood of T. In the interval $[0,T]$, we have, due to (169) and (174),

$$\frac{d\hat{H}_{\mathrm{lp},k_0}}{dt} \leq CH\epsilon^3 e^{-aHt}.$$

Integrating this inequality and assuming ϵ to be small enough, we get,

$$\hat{H}^{1/2}_{\mathrm{lp},k_0}(t) \leq 2c_0\epsilon \tag{176}$$

for all $t \in [0,T]$. Note that the bound on ϵ depends on c_0 which we have not specified. We shall, however, specify c_0 in the end and it will only depend on the mentioned constants, so that this is not a problem. In order to get an estimate for $\hat{H}_{\mathrm{m},k_0}$, let us define

$$f = \exp\left[\frac{1}{a}(e^{-aHt}-1)\right].$$

Note that $\exp(-1/a) \leq f \leq 1$ for all $t \in [0,T]$. Defining $\tilde{H}_{\mathrm{m},k_0} = f\hat{H}_{\mathrm{m},k_0}$, we get, using (171) and (174),

$$\frac{d\tilde{H}_{\mathrm{m},k_0}}{dt} \leq CH\epsilon^3 e^{-aHt},$$

so that

$$\hat{H}_{\mathrm{m},k_0}(t) \leq e^{1/a}\hat{H}_{\mathrm{m},k_0}(0) + e^{1/a}Ca^{-1}\epsilon^3.$$

Assuming ϵ to be small enough, depending on the quantities mentioned and c_0, we obtain

$$\hat{H}^{1/2}_{\mathrm{m},k_0}(t) \leq C_{\mathrm{m}}c_0\epsilon, \tag{177}$$

where $C_{\mathrm{m}} = 2\exp[1/(2a)]$. Consider (170). In the last term, there is one part which can be written

$$CH\epsilon e^{-aHt}\hat{H}_{\mathrm{s},k_0}.$$

By assuming ϵ to be small enough, we can absorb this term in the first one, at the price of reducing the constant. All that remains of the last two terms in (170) can be estimated by

$$C_{\mathrm{s}}Hc_0\epsilon\hat{H}^{1/2}_{\mathrm{s},k_0}$$

due to (176) and (177). We get

$$\frac{d\hat{H}_{\mathrm{s},k_0}}{dt} \leq -3aH\hat{H}_{\mathrm{s},k_0} + C_{\mathrm{s}}Hc_0\epsilon\hat{H}^{1/2}_{\mathrm{s},k_0}.$$

Thus $\hat{H}_{\mathrm{s},k_0}$ decays as soon as $3a\hat{H}^{1/2}_{\mathrm{s},k_0}(t) \geq C_{\mathrm{s}}c_0\epsilon$. Assuming c_0 to be small enough, only depending on a, C_{s} and C_{m} and ϵ to be small enough only depending on the stated quantities, we obtain (174) with ϵ in the right hand side replaced by $\epsilon/2$. Thus $\mathcal{A}$ is open. □

The conclusions of Theorem 9 are global existence and that we have estimates of the form (174). It is, however, of interest to obtain estimates for the higher derivatives.

Theorem 10. *Consider a solution to (144)-(147) corresponding to smooth initial data satisfying the conditions of Theorem 9. Then for every k, there is a constant C_k such that*

$$\hat{H}^{1/2}_k(t) \leq C_k \tag{178}$$

for all $t \geq 0$.

Proof. Since we have (174) and (94)-(96), with $\eta = \eta_{\min}$, for all $t \geq 0$, we have (169)-(171) for all k and all $t \geq 0$. Let us define

$$\tilde{H}_{\mathrm{s},k} = e^{-aHt}\hat{H}_{\mathrm{s},k}.$$

Then

$$\frac{d\tilde{H}_{\mathrm{s},k}}{dt} \leq -5aH\tilde{H}_{\mathrm{s},k} + CHe^{-aHt/2}\hat{H}^{1/2}_{\mathrm{m},k}\tilde{H}^{1/2}_{\mathrm{s},k} + CH\epsilon e^{-3aHt/2}\hat{H}^{1/2}_{k}\tilde{H}^{1/2}_{\mathrm{s},k}.$$

Due to this inequality, (169) and (171), we obtain

$$\frac{d\mathcal{H}_k}{dt} \leq CHe^{-aHt/2}\mathcal{H}_k,$$

where

$$\mathcal{H}_k = \hat{H}_{\mathrm{l},k} + \tilde{H}_{\mathrm{s},k} + \hat{H}_{\mathrm{m},k} + \hat{H}_{\mathrm{sf},k}.$$

Consequently $\mathcal{H}_k$ is bounded. This leads to the conclusion that $\hat{H}_{\mathrm{lp},k}$ and $\hat{H}_{\mathrm{m},k}$ are both bounded. If we insert this information into (170), we get

$$\frac{d\hat{H}_{\mathrm{s},k}}{dt} \leq -4aH\hat{H}_{\mathrm{s},k} + CHe^{-aHt}\hat{H}_{\mathrm{s},k} + CH\hat{H}^{1/2}_{\mathrm{s},k}.$$

By assuming t to be great enough, the second term on the right hand side can be absorbed in the first. The inequality that results immediately implies that $\hat{H}_{\mathrm{s},k}$ is bounded, since it implies that $\hat{H}_{\mathrm{s},k}$ decays as soon as it exceeds a certain value. The theorem follows. □

14. Asymptotics

The estimates we have obtained so far, i.e. (178), are what naturally comes out of the bootstrap assumptions, and they are far from optimal. Let us try to improve them.

Proposition 2. *Consider a solution to (144)-(147) corresponding to smooth initial data satisfying the conditions of Theorem 9. Let us define* $\zeta = 4\chi/n^2$ *and*

$$\lambda = \frac{n}{2}[1 - (1-\zeta)^{1/2}]$$

for $\zeta \in (0,1)$ *and* $\lambda = n/2$ *for* $\zeta \geq 1$. *We shall also need the notation* $\lambda_{\mathrm{m}} = \min\{1, \lambda\}$. *There is a smooth Riemannian metric* ρ *on* $\mathbb{T}^n$ *and constants* K_l *such that*

$$\begin{aligned} &\|e^{2Ht}g^{ij}(t,\cdot) - \rho^{ij}\|_{C^l} \\ &+\|e^{-2Ht}g_{ij}(t,\cdot) - \rho_{ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}Ht}, \quad (179)\\ &\|e^{-2Ht}\partial_t g_{ij}(t,\cdot) - 2H\rho_{ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}Ht}, \quad (180) \end{aligned}$$

for every $l \geq 0$ *and* $t \geq 0$, *where* ρ^{ij} *are the components of the inverse. Here and below, we shall, for the sake of brevity, write* C^l *instead of* $C^l(\mathbb{T}^n)$. *Concerning* g_{0m}, *there is an* $\alpha > 0$ *and constants* K_l *such that for all* $l \geq 0$ *and* $t \geq 0$,

$$\left\|g_{0m}(t,\cdot) - \frac{1}{(n-2)H}\rho^{ij}\gamma_{imj}\right\|_{C^l} + \|\partial_0 g_{0m}(t,\cdot)\|_{C^l} \leq K_l e^{-\alpha Ht}, \quad (181)$$

where γ_{imj} *are the Christoffel symbols of the metric* ρ. *The estimates for* g_{00} *and* k_{ij}, *the components of the second fundamental form induced on the hypersurfaces*

$t = \text{const.}$ *with respect to the standard coordinates on* $\mathbb{T}^n$*, depend on the value of* $\lambda_{\rm m}$*. If* $\lambda_{\rm m} < 1$*, there are constants* K_l *such that for every* $l \geq 0$ *and* $t \geq 0$*,*

$$\begin{aligned} \|g_{00}(t,\cdot)+1\|_{C^l} + \|\partial_0 g_{00}(t,\cdot)\|_{C^l} &\leq K_l e^{-2\lambda_{\rm m} Ht}, \\ \|e^{-2Ht}k_{ij}(t,\cdot) - H\rho_{ij}\|_{C^l} &\leq K_l e^{-2\lambda_{\rm m} Ht} \end{aligned}$$

and if $\lambda_{\rm m} = 1$*, there are constants* K_l *such that for every* $l \geq 0$ *and* $t \geq 1$*,*

$$\begin{aligned} \|[\partial_0 g_{00} + 2\lambda_{\rm m} H(g_{00}+1)](t,\cdot)\|_{C^l} &\leq K_l e^{-2Ht}, \\ \|g_{00}(t,\cdot)+1\|_{C^l} &\leq K_l t e^{-2Ht}, \\ \|e^{-2Ht}k_{ij}(t,\cdot) - H\rho_{ij}\|_{C^l} &\leq K_l t e^{-2Ht}. \end{aligned}$$

Concerning ϕ *there are three cases to consider. Let us define* $\varphi = e^{\lambda Ht}\phi$*. If* $\zeta < 1$*, then there is a smooth function* φ_0 *and constants* $K_l, \alpha > 0$ *such that for all* $l \geq 0$ *and* $t \geq 0$*,*

$$\|\varphi(t,\cdot) - \varphi_0\|_{C^l} + \|\partial_0\varphi\|_{C^l} \leq K_l e^{-\alpha Ht}. \tag{182}$$

If $\zeta = 1$*, there are smooth functions* φ_0 *and* φ_1 *and constants* $K_l, \alpha > 0$ *such that for all* $l \geq 0$ *and* $t \geq 0$*,*

$$\|\partial_0\varphi(t,\cdot) - \varphi_1\|_{C^l} + \|\varphi(t,\cdot) - \varphi_1 t - \varphi_0\|_{C^l} \leq K_l e^{-\alpha t}. \tag{183}$$

Finally, if $\zeta > 1$*, there is an anti symmetric matrix* A*, given in (193), where* $\delta = n(\zeta-1)^{1/2}/2$*, smooth functions* φ_0 *and* φ_1 *and constants* $K_l, \alpha > 0$ *such that for all* $l \geq 0$ *and* $t \geq 0$*,*

$$\left\| e^{-At} \begin{pmatrix} \delta H\varphi \\ \partial_0\varphi \end{pmatrix} (t,\cdot) - \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \right\|_{C^l} \leq K_l e^{-\alpha t}. \tag{184}$$

Remark. In the above proposition all constants are allowed to depend on H. The statement is certainly not a complete result concerning the asymptotics; it is possible to get more information. However, we shall be content with the above estimates.

Proof. Note that due to (178), we have

$$e^{2Ht}\|g^{ij}\|_{C^l} + e^{-2Ht}\|g_{ij}\|_{C^l} + e^{Ht+aHt}\|g^{0i}\|_{C^l} + e^{-Ht+aHt}\|g_{0i}\|_{C^l} \leq K_l, \tag{185}$$

and similarly for other quantities. Note in particular that we do not lose any decay by taking derivatives; in order to bound $\partial_i g_{0m}$ in H^l, we use the fact that $\hat{H}_{l+1}$ is bounded. Our first goal is to prove that $H_{{\rm s},l}$ is bounded. Let us estimate Δ_{0m}. We wish to prove that

$$\|\Delta_{0m}\|_{H^l} \leq Ce^{-bHt}(1 + H_{{\rm s},l}^{1/2}), \tag{186}$$

for some $b > 0$. Since ϕ, $\partial_i\phi$ and $\partial_t\phi$ are decaying exponentially in any C^l norm, we have this sort of estimate for $\Delta_{\phi,0m}$ (in this case we can in fact choose $b = 2a$). Using the fact that

$$g^{0i} = -\frac{1}{g_{00}} g^{ij} g_{0j}, \tag{187}$$

we obtain a similar estimate for $\Delta_{C,0m}$. Before we turn to estimating $\Delta_{A,0m}$, let us consider the case that we have the type of term dealt with by the algorithm for estimating the non-linearity with $l_\epsilon \geq 2$ and $l_h = 1$, and let us assume that the term includes a factor g^{0i}. By rewriting g^{0i} according to (187) and considering what remains after taking away the factor g_{0j}, we get a term such as the ones estimated by the algorithm with $l_\epsilon \geq 1$ and $l_h = 0$. In other words, the term we started with is g_{0j} times something which decays exponentially in any C^l norm, cf. (138). This allows us to estimate II_{0m}, IV_{0m} and $\Delta_{\mathrm{V},0m}$ by the right hand side of (186). That I_{0m}, $\Delta_{\mathrm{III},0m}$ and $\Delta_{\mathrm{VI},0m}$ satisfy the same sort of estimate follows from (178). Consequently (186) holds. We conclude that

$$\|\hat{\Box}_g u_m\|_{H^l} \leq C + CH_{\mathrm{s},l}^{1/2}.$$

By arguments similar to, but simpler than, the proof of Lemma 13, we conclude that for $|\alpha| \leq l$,

$$\|[\hat{\Box}_g, \partial^\alpha]u_m\|_2 \leq Ce^{-bHt}(1 + H_{\mathrm{s},l}^{1/2}).$$

Finally note that we have (166) and that

$$\|F_{0m}\|_{H^l} \leq C + Ce^{-bHt}H_{\mathrm{s},l}^{1/2},$$

where $-F_{0m}$ is given by the last two terms on the left hand side of (145), cf. (185). Combining these observations with Corollary 1, we conclude that

$$\frac{dH_{\mathrm{s},l}}{dt} \leq -\eta_{\mathrm{s}} H H_{\mathrm{s},l} + CH_{\mathrm{s},l}^{1/2} + Ce^{-bHt}H_{\mathrm{s},l},$$

which proves that $H_{\mathrm{s},l}$ remains bounded to the future. Note that as a consequence, $e^{2Ht}\|g^{0i}\|_{C^l}$ is bounded.

Let us turn to ϕ. Note that if we introduce $\varphi = e^{\lambda Ht}\phi$, (147) can be rewritten

$$-g^{00}\partial_0^2\varphi + (n - 2\lambda)H\partial_0\varphi + (\lambda^2 - n\lambda + \chi)H^2\varphi = R, \tag{188}$$

where

$$\begin{aligned} R \;=\;& (g^{00} + 1)(-2\lambda H\partial_0\varphi + \lambda^2 H^2\varphi) - e^{\lambda Ht}\Delta_\phi \\ & +2g^{0i}\partial_i(\partial_0\varphi - \lambda H\varphi) + g^{ij}\partial_i\partial_j\varphi. \end{aligned}$$

Let us introduce the quantity

$$L_l = \frac{1}{2}\sum_{|\alpha|\leq l}[(\partial^\alpha\partial_0\varphi)^2 + e^{-2bHt}(\partial^\alpha\varphi)^2],$$

where b is a positive constant such that for every $m \geq 0$, there are constants C_m such that

$$\|g_{00}(t,\cdot) + 1\|_{C^m} + \|g^{00}(t,\cdot) + 1\|_{C^m} + \|\phi(t,\cdot) + 1\|_{C^m} \leq C_m e^{-2bHt}$$

for all $t \geq 0$. Assume furthermore that there are constants C_m, c such that

$$\|\varphi\|_{C^m} + \|\partial_0\varphi\|_{C^m} \leq C_m e^{(c+2)Ht} \tag{189}$$

for all m and $t \geq 0$. Then, for all $|\alpha| \leq l$,

$$|\partial^\alpha R| \leq Ce^{-bHt}L_l^{1/2} + Ce^{cHt}. \tag{190}$$

Note that, due to (188), for $|\alpha| \leq l$,

$$|\partial^\alpha\partial_0^2\varphi| \leq Ce^{bHt}L_l^{1/2} + Ce^{cHt}.$$

Thus $\partial^\alpha[(g^{00} + 1)\partial_0^2\varphi]$ can be estimated by the right hand side of (190) for $|\alpha| \leq l$ and (188) turns into

$$\partial_0^2\varphi + (n - 2\lambda)H\partial_0\varphi + (\lambda^2 - n\lambda + \chi)H^2\varphi = \hat{R}, \tag{191}$$

where we have modified R in an obvious way in order to obtain $\hat{R}$ and $\hat{R}$ satisfies an estimate of the form (190). Let us introduce ζ by $n^2\zeta/4 = \chi$. Then the solutions to the equation $\lambda^2 - n\lambda + \chi = 0$ are given by

$$\lambda_{\pm} = \frac{n}{2}[1 \pm (1-\zeta)^{1/2}].$$

If we consider the ODE that results by putting the right hand side of (191) to zero, the behaviour is quite different depending on whether $\zeta \in (0,1)$, $\zeta = 1$ or $\zeta > 1$. Let λ be defined as in the statement of the proposition and let us consider the first case. Letting $\delta = n(1-\zeta)^{1/2}$, (191) turns into

$$\partial_0^2\varphi + \delta H \partial_0 \varphi = \hat{R}.$$

Consequently

$$\frac{dL_l}{dt} \leq -2\min\{\delta, b\} H L_l + Ce^{-bHt} L_l + Ce^{cHt} L_l^{1/2}. \tag{192}$$

Since δ and b are positive, the second term on the right hand side can be absorbed by the first. We conclude that for $c > 0$, L_l can be estimated by Ce^{2cHt}, and for $c < 0$, L_l is exponentially decaying. If $c > 0$, $\partial_0\varphi$ can be estimated by Ce^{cHt} in any C^l norm. By integrating this estimate, we get the same conclusion for φ. In other words, if $c > 0$ and we have the estimate (189), we can improve this estimate and replace c by $c - 2$. By carrying out this argument a finite number of times, we get the conclusion that L_l decays exponentially. Thus $\partial_0\varphi$ decays exponentially in any C^l norm and there is a smooth function φ_0 such that (182) holds. In the case $\zeta = 1$, we still have (192), but in that case, $\delta = 0$. All the same, for $c > 0$, we get the conclusion that L_l can be estimated by Ce^{2cHt}. Thus we can improve the estimate (189) until c becomes negative. Since $\delta = 0$, (192) only yields the conclusion that L_l is bounded for $c < 0$. Consequently $e^{\lambda Ht}\phi$ may grow linearly. This is, however, not a great surprise, since t is a solution of the ODE resulting by putting the right hand side of (191) to zero. On the other hand, by inserting the fact that L_l is bounded into the equation, we get the conclusion that $\partial_0^2\varphi$ converges to zero exponentially in any C^l norm, so that there are smooth φ_0 and φ_1 such that (183) holds. Let us turn to the case $\zeta > 1$. Letting $\delta = n(\zeta-1)^{1/2}/2$, (191) turns into

$$\partial_0^2\varphi + \delta^2 H^2 \varphi = \hat{R}.$$

Defining $u_0 = \delta H \varphi$ and $u_1 = \partial_0\varphi$, we obtain

$$\partial_t \mathbf{u} = A\mathbf{u} + \mathbf{R},$$

where

$$A = \begin{pmatrix} 0 & \delta H \\ -\delta H & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 \\ \hat{R} \end{pmatrix}. \tag{193}$$

Letting $\hat{\mathbf{u}} = e^{-At}\mathbf{u}$, we obtain

$$\partial_t \hat{\mathbf{u}} = e^{-At}\mathbf{R}. \tag{194}$$

Note that e^{-At} is an orthogonal matrix and define

$$M_l = \frac{1}{2}\sum_{|\alpha|\leq l} (\partial^\alpha \hat{\mathbf{u}})^t (\partial^\alpha \hat{\mathbf{u}}).$$

Assuming that (189) holds, we obtain

$$\partial_t M_l \leq Ce^{-bHt}M_l + Ce^{cHt}M_l^{1/2},$$

by arguments similar to ones given above. After using this a finite number of times, we conclude that we can assume c in (189) to be negative. After a finite number of iterates, we thus get the conclusion that M_l is bounded. Consequently (189) holds with $c = -2$. Inserting this information into (194), we obtain the existence of two smooth functions φ_0 and φ_1 such that (184) holds.

Let us improve our control of u_m. Note that since $\partial_0 h_{ij}$ converges to zero exponentially in any C^l norm, there are smooth functions ρ_{ij} such that

$$\|e^{-2Ht}g_{ij}(t,\cdot) - \rho_{ij}\|_{C^l} \leq K_l e^{-\alpha t}. \tag{195}$$

Note that for a given x, $\rho_{ij}(x)$ are necessarily the components of a symmetric positive semi definite matrix. One can also check that $\partial_0(e^{2Ht}g^{ij})$ converges to zero exponentially in any C^l norm. Consequently, we have an estimate similar to (195), and we shall use the notation ρ^{ij} for the limit of $e^{2Ht}g^{ij}$. Since $g^{ij}g_{jk} + g^{i0}g_{0k} = \delta^i_k$ and $g^{i0}g_{0k}$ converges to zero exponentially in any C^l norm, we conclude that $\rho^{ij}\rho_{jk} = \delta^i_k$. In other words, ρ_{ij} must be the components of a positive definite matrix and ρ^{ij} are the components of its inverse. If we let

$$\gamma_{ijk} = \frac{1}{2}(\partial_i\rho_{kj} + \partial_k\rho_{ij} - \partial_j\rho_{ik}),$$

we thus get

$$\|(g^{ij}\Gamma_{imj})(t,\cdot) - \rho^{ij}\gamma_{imj}\|_{C^l} \leq K_l e^{-\alpha t}.$$

Note that due to (186) and the fact that $H_{\mathrm{s},l}$ is bounded for any l, Δ_{0m} decays exponentially in any C^l norm. By (145), we conclude that $\partial_0^2 u_m$ is bounded in any C^l norm, which leads to the conclusion that $(g^{00}+1)\partial_0^2 u_m$ is exponentially decaying in any C^l norm. The same is true of $g^{ij}\partial_i\partial_j u_m$ and $g^{0i}\partial_0\partial_i u_m$. We conclude from (145) that

$$\partial_0^2\hat{u}_m + nH\partial_0\hat{u}_m + 2(n-2)H^2\hat{u}_m = R,$$

where R decays exponentially in any C^l norm and

$$\hat{u}_m = u_m - \frac{1}{(n-2)H}\rho^{ij}\gamma_{imj}.$$

Consequently $\hat{u}_m$ and $\partial_0\hat{u}_m$ converge to zero exponentially, so that (181) holds.

Let us study the behaviour of h_{ij} in greater detail. The contribution of the scalar field to Δ_{ij} is $2e^{-2Ht}\Delta_{\phi,ij}$. If we let $\lambda_{\mathrm{m}} = \min\{1,\lambda\}$, we get

$$e^{-2Ht}\|\Delta_{\phi,ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}Ht}$$

for any l. Since $\Delta_{C,ij} = 0$, let us turn to $\Delta_{A,ij}$. It is clear that VI_{ij} is bounded in any C^l norm, but there is no better bound. It is easy to see that I_{ij}, II_{ij}, IV_{ij} and V_{ij} are bounded in any C^l norm as well. What remains is $\Delta_{\mathrm{III},ij}$, given by (83). The only term we do not already know to be bounded in any C^l norm is the last one. However, this term can be written as a factor times $\partial_0 h_{ij}$ where the factor can be bounded by $e^{2Ht-bHt}$ in any C^l norm. We conclude that if we define

$$N_l = \frac{1}{2}\sum_{|\alpha|\leq l}\sum_{i,j}(\partial^\alpha\partial_0 h_{ij})^2,$$

we get, for $|\alpha| \leq l$,

$$|\partial^\alpha \Delta_{ij}| \leq Ce^{-2\lambda_{\rm m}Ht} + Ce^{-bHt}N_l^{1/2}, \tag{196}$$

since the first term on the right hand side of (153) is, up to constants, bounded by e^{-2Ht} with respect to any C^l norm. We conclude from (146) that

$$-g^{00}\partial_0^2 h_{ij} + nH\partial_0 h_{ij} = \mathcal{R}_{ij},$$

where $\mathcal{R}_{ij}$ satisfies an estimate of the form (196). From this we conclude that $(g^{00}+1)\partial_0^2 h_{ij}$ satisfies the same sort of estimate so that

$$\partial_0^2 h_{ij} + nH\partial_0 h_{ij} = \hat{\mathcal{R}}_{ij},$$

where $\hat{\mathcal{R}}_{ij}$ satisfies the same sort of estimate as $\mathcal{R}_{ij}$. Consequently,

$$\partial_t N_l \leq -2nHN_l + Ce^{-2\lambda_{\rm m}Ht}N_l^{1/2} + Ce^{-bHt}N_l.$$

As a consequence,

$$N_l \leq Ce^{-4\lambda_{\rm m}Ht}.$$

In particular, (179) and (180) hold.

Let us turn to g_{00}. Letting

$$P_l = \sum_{|\alpha|\leq l}[(\partial^\alpha\partial_0 u)^2 + H^2(\partial^\alpha u)^2],$$

one can, by arguments similar to ones given above, prove that

$$\partial_0^2 u + (n+2)H\partial_0 u + 2nH^2 u = R_0,$$

where, for $|\alpha| \leq l$,

$$|\partial^\alpha R_0| \leq Ce^{-2\lambda_{\rm m}Ht} + Ce^{-bHt}P_l^{1/2}.$$

Changing variables to $v = e^{\lambda_{\rm m}Ht}u$, we obtain

$$\partial_0^2 v + (n+2-2\lambda_{\rm m})H\partial_0 v + [\lambda_{\rm m}^2 - (n+2)\lambda_{\rm m} + 2n]H^2 v = e^{\lambda_{\rm m}Ht}R_0. \tag{197}$$

Note that if we consider the factor in front of $H^2 v$ as a polynomial in $\lambda_{\rm m}$, it has zeros at 2 and at n. Below and above it is positive and in between it is negative. Since $\lambda_{\rm m} \leq 1$, the factor in front of $H^2 v$ is thus positive. Consequently, there are γ and δ such that

$$\hat{P}_l = \frac{1}{2}\sum_{|\alpha|\leq l}[(\partial^\alpha\partial_0 v)^2 + 2\gamma H\partial^\alpha v\partial^\alpha\partial_0 v + \delta H^2(\partial^\alpha v)^2]$$

is equivalent to $e^{2\lambda_{\rm m}Ht}P_l$ and

$$\partial_t\hat{P}_l \leq -\eta H\hat{P}_l + Ce^{-\lambda_{\rm m}Ht}\hat{P}_l^{1/2} + Ce^{-bHt}\hat{P}_l$$

for some $\eta > 0$. We conclude that $\hat{P}_l$ is bounded (in fact we're allowed to conclude that it decays to zero exponentially). This leads to the improved estimate

$$\|R_0\|_{C^l} \leq Ce^{-2\lambda_{\rm m}Ht}.$$

Changing variables again to $\hat{v} = e^{2\lambda_{\rm m}Ht}u$, we obtain (197) with $\lambda_{\rm m}$ replaced by $2\lambda_{\rm m}$. Since $\lambda_{\rm m} \leq 1$, the factor in front of $H^2 v$ is still non-negative, but if $\lambda_{\rm m} = 1$ it is zero. The factor in front of $H\partial_0 v$ is, however, always positive, assuming $n \geq 3$. Regardless of whether $\lambda_{\rm m} = 1$ or not, we get the conclusion that

$$\|\partial_0(e^{2\lambda_{\rm m}Ht}u)\|_{C^l} \leq K_l.$$

In the case that $\lambda_{\rm m} < 1$, we get the additional conclusion that

$$\|e^{2\lambda_{\rm m}Ht}u\|_{C^l} \leq K_l.$$

Finally, let us turn to the second fundamental form. Note that the future directed unit normal is given by

$$N = -(-g^{00})^{-1/2}g^{0\mu}\partial_\mu.$$

Thus

$$k_{ij} = \langle \nabla_{\partial_i} N, \partial_j\rangle = -\partial_i[(-g^{00})^{-1/2}g^{0\mu}]g_{\mu j} - (-g^{00})^{-1/2}g^{0\mu}\Gamma_{ij\mu},$$

so that

$$\left\| \left[k_{ij} - \frac{1}{2}(-g^{00})^{1/2}\partial_0 g_{ij}\right](t,\cdot)\right\|_{C^l} \leq K_l.$$

The proposition follows from the inequalities already derived. □

15. Causal structure

Let us first prove the statements made in the introduction concerning the metric (6).

Lemma 21. *Let $\Lambda > 0$, $H = \Lambda^{1/2}$ and let g_R be the metric given by (6) and defined on $M_R = \mathbb{R}\times\mathbb{S}^1\times\mathbb{S}^2$. Then (M_R, g_R) is causally geodesically complete and satisfies Einstein's vacuum equations with a cosmological constant Λ. Furthermore, if Σ is an arbitrary Cauchy hypersurface in (M_R, g_R) and γ is an arbitrary inextendible causal curve, then the intersection of the causal past of γ with Σ is not contained in a subset of Σ homeomorphic to a 3-ball.*

Proof. That (M_R, g_R) solves Einstein's vacuum equations with a positive cosmological constant Λ follows by a computation. Furthermore, for every $t \in \mathbb{R}$, $S_t = \{t\}\times\mathbb{S}^1\times\mathbb{S}^2$ is a Cauchy hypersurface in (M_R, g_R). Let γ be a future directed causal geodesic and let s_0 be such that $\gamma^0(s_0) = 0$. The zeroth component of the geodesic equation then implies that $\ddot{\gamma}^0(s) \leq 0$ when $s \geq s_0$ and $\ddot{\gamma}^0(s) \geq 0$ when $s \leq s_0$. Thus $0 < \dot{\gamma}^0(s) \leq \dot{\gamma}^0(s_0)$, i.e. $\dot{\gamma}^0$ is bounded. This implies that γ is complete, since γ has to intersect every S_t. Let Σ and γ be as in the statement of the lemma. Then Σ is homeomorphic to $\mathbb{S}^1\times\mathbb{S}^2$ due to Corollary 32, p. 417 of [24]. In particular, Σ is compact, so that there is a τ such that S_τ is strictly to the future of Σ. Considering the metric (6), it is clear that there is $0 < T \in \mathbb{R}$ such that if $(t,p,q) \in M_R$, then $\{t-T\}\times\{p\}\times\mathbb{S}^2 \subseteq J^-(\{(t,p,q)\})$. Thus, since γ and $S_{\tau+T}$ intersect, $J^-(\gamma)$ contains $\{\tau\}\times\{p\}\times\mathbb{S}^2$ for some $p\in\mathbb{S}^1$. Assume $J^-(\gamma)\cap\Sigma$ is contained in a set B, homeomorphic to a 3-ball. Let $f_1 : \mathbb{S}^2 \to \{\tau\}\times\{p\}\times\mathbb{S}^2$ be defined by $f_1(q) = (\tau,p,q)$. Let $f_2 : \{\tau\}\times\{p\}\times\mathbb{S}^2 \to B$ be defined by following the flow lines of ∂_t; note that following the flow lines of ∂_t to the past from $\{\tau\}\times\{p\}\times\mathbb{S}^2$ to Σ implies that we end up in $\Sigma\cap J^-(\gamma)$, i.e. in a subset of B. Let $f_3 : B \to \mathbb{S}^2$ be defined by projection onto the $\mathbb{S}^2$ factor in M_R. Due to Proposition 31, p. 417 of O'Neill, f_2 is continuous. Furthermore $f_3\circ f_2\circ f_1$ is the identity on $\mathbb{S}^2$, so that our assumptions lead to the conclusion that we have factored the identity map from $\mathbb{S}^2$ to itself through the 3-ball. Since the second homology group of $\mathbb{S}^2$ is $\mathbb{Z}$ and the second homology group of B is $\{0\}$, we obtain a contradiction, and the lemma follows. □

Let us turn to the causal structure of the metrics constructed in Theorem 9.

Proposition 3. *Consider a future directed causal curve γ with domain $[s_0, s_{\max})$ in the Lorentz manifold constructed in Theorem 9 such that $\gamma^0(s_0) = 0$. Let γ^μ denote the coordinates of this curve in the universal covering space of the spacetime, i.e. $[0,\infty)\times\mathbb{R}^n$. Assuming ϵ to be small enough (independent of K, H and γ), $\dot\gamma^0 > 0$ and the length of the spatial part of the curve with respect to the metric at $t = 0$ satisfies*

$$\int_{s_0}^{s_{\max}} [g_{ij}(0,\gamma_{\mathrm{r}})\dot\gamma^i\dot\gamma^j]^{1/2} ds \le d(\epsilon)H^{-1}, \tag{198}$$

where $d(\epsilon)\to 1$ as $\epsilon\to 0$ and $\gamma_{\mathrm{r}} = \pi\circ\gamma$ where $\pi : [0,\infty)\times\mathbb{T}^n\to\mathbb{T}^n$ is given by $\pi(t,x) = x$. Finally, if γ is future inextendible, $\gamma^0(s)\to\infty$ as $s\to s_{\max}$.

Remark. The timelike vectorfield ∂_t is defined to be future directed.

Proof. Due to causality, we have

$$g_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu \le 0, \tag{199}$$

and the condition that the curve be future directed is equivalent to

$$g_{00}\dot\gamma^0 + g_{0i}\dot\gamma^i < 0. \tag{200}$$

Let us work out the consequences of this. Due to (96), we have

$$\begin{aligned} |2g_{0i}\dot\gamma^0\dot\gamma^i| &\le \eta^{1/2}|\dot\gamma^0|^2 + \eta^{-1/2}|g_{0i}\dot\gamma^i|^2 \\ &\le \eta^{1/2}|\dot\gamma^0|^2 + \eta^{1/2}c_1^{-1}e^{2Ht+2K-2aHt}\delta_{ij}\dot\gamma^i\dot\gamma^j. \end{aligned}$$

Note that when we write t in this equation, we of course mean γ^0. Since the last term can be bounded by $\eta^{1/2}g_{ij}\dot\gamma^i\dot\gamma^j$, due to (94), we obtain

$$g_{ij}\dot\gamma^i\dot\gamma^j \le c(\eta)\dot\gamma^0\dot\gamma^0, \tag{201}$$

where $c(\eta)\to 1$ as $\eta\to 0$ and we have used (95) and (199). Due to (94), we conclude that

$$\delta_{ij}\dot\gamma^i\dot\gamma^j \le c_1c(\eta)e^{-2Ht-2K}\dot\gamma^0\dot\gamma^0. \tag{202}$$

Combining (175) and (202), we obtain

$$|g_{ij}(0,\gamma_{\mathrm{r}})\dot\gamma^i\dot\gamma^j - e^{-2Ht}g_{ij}\dot\gamma^i\dot\gamma^j| \le Ca^{-1}\epsilon c_1c(\eta)e^{-2Ht}\dot\gamma^0\dot\gamma^0.$$

This observation, together with (201), yields

$$g_{ij}(0,\gamma_{\mathrm{r}})\dot\gamma^i\dot\gamma^j \le d^2(\epsilon)e^{-2Ht}\dot\gamma^0\dot\gamma^0, \tag{203}$$

where $d(\epsilon)\to 1$ as $\epsilon\to 0$ (note that $\eta\to 0$ as $\epsilon\to 0$). Consider (200). Note that

$$|g_{0i}\dot\gamma^i| \le [e^{-2Ht-2K}\delta^{ij}g_{0i}g_{0j}]^{1/2}[e^{2Ht+2K}\delta_{ij}\dot\gamma^i\dot\gamma^j]^{1/2} \le \xi(\epsilon)|\dot\gamma^0|,$$

where $\xi(\epsilon)\to 0$ as $\epsilon\to 0$, due to (96) and (202). Assuming ϵ to be small enough we conclude that $\dot\gamma^0 > 0$, which yields the first conclusion of the proposition. Combining this observation with (203), we obtain (198). Finally, let γ be future inextendible and assume γ^0 does not tend to ∞. Since $\dot\gamma^0 > 0$, γ^0 has to converge to a finite number and since we have (202), the same holds for γ^i. We have a contradiction. □

Proposition 4. *Consider a spacetime of the type constructed in Theorem 9. Assuming ϵ to be small enough (independent of K and H), this spacetime is future causally geodesically complete.*

Proof. Let γ be a future directed causal geodesic (i.e. a map γ from an open interval into the spacetime satisfying $\gamma'' = 0$) and assume that the maximal existence interval is given by $(s_{\min}, s_{\max})$. We shall use the notation $t = \gamma^0(s)$. Due to the equation for a geodesic, we have

$$\ddot{\gamma}^0 + \Gamma^0_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu = 0. \tag{204}$$

Due to (174),

$$\begin{aligned} |\Gamma^0_{00}| &\leq C\epsilon H e^{-aHt}, \quad |\Gamma^0_{ij} - Hg_{ij}| \leq C\epsilon H e^{2Ht+2K-aHt}, \\ |\Gamma^0_{0i}| &\leq C\epsilon H e^{Ht+K-aHt}. \end{aligned}$$

Consequently, $\Gamma^0_{ij}\dot{\gamma}^i\dot{\gamma}^j \geq 0$ for t large enough. Due to (202), we conclude that

$$|\Gamma^0_{00}\dot{\gamma}^0\dot{\gamma}^0| + 2|\Gamma^0_{0i}\dot{\gamma}^0\dot{\gamma}^i| \leq C\epsilon H e^{-aHt}|\dot{\gamma}^0|^2.$$

Combining these pieces of information with (204), we obtain

$$\ddot{\gamma}^0 \leq C\epsilon H e^{-aHt}\dot{\gamma}^0\dot{\gamma}^0$$

for $s \geq s_1$. Since $\dot{\gamma}^0 > 0$ assuming ϵ is small enough (independent of γ), we can divide by $\dot{\gamma}^0$ in this equation and integrate in order to obtain, for $s \geq s_1$,

$$\begin{aligned} \ln\frac{\dot{\gamma}^0(s)}{\dot{\gamma}^0(s_1)} &= \int_{s_1}^s \frac{\ddot{\gamma}^0(\sigma)}{\dot{\gamma}^0(\sigma)}d\sigma \leq C\epsilon H\int_{s_1}^s e^{-aH\gamma^0(\sigma)}\dot{\gamma}^0(\sigma)d\sigma \\ &= C\epsilon H\int_{\gamma^0(s_1)}^{\gamma^0(s)} e^{-aH\tau}d\tau \leq C\epsilon a^{-1}\exp[\gamma^0(s_1)] \end{aligned}$$

(recall that $t = \gamma^0(s)$) so that $\dot{\gamma}^0$ is bounded for $s \geq s_1$. Thus

$$\gamma^0(s) - \gamma^0(s_1) = \int_{s_1}^s \dot{\gamma}^0(\sigma)d\sigma \leq C|s - s_1|.$$

Since $\gamma^0(s) \to \infty$ as $s \to s_{\max}$, we conclude that $s_{\max} = \infty$. □

16. Proof of the main theorem

Proof of Theorem 2. Consider $\mathbb{T}^n$ to be $[-\pi, \pi]^n$ with the ends identified.

Construction of a global in time patch. Let us start by constructing a patch of spacetime which is essentially the development of the piece of the data over which we have some control. Let $f_c \in C_0^\infty[B_1(0)]$ be such that $f_c(p) = 1$ for $|p| \leq 15/16$ and $0 \leq f_c \leq 1$. Define initial data for a Lorentz metric $\bar{g}$ and a function Φ on $\{0\} \times \mathbb{T}^n$ by

$$\begin{aligned} \bar{g}_{00}(0,\cdot) &= -1 \\ \bar{g}_{0i}(0,\cdot) &= 0 \\ \bar{g}_{ij}(0,\cdot) &= f_c h_{ij} \circ x^{-1} + 16H^{-2}(1 - f_c)\delta_{ij} \\ \partial_t\bar{g}_{ij}(0,\cdot) &= 2f_c\kappa_{ij} \circ x^{-1} + 32H^{-1}(1 - f_c)\delta_{ij} \\ \Phi(0,\cdot) &= f_c\phi_0 \circ x^{-1} \\ \partial_t\Phi(0,\cdot) &= f_c\phi_1 \circ x^{-1}, \end{aligned}$$

where the indices on the right hand side refer to the coordinates x assumed to exist in the statement of the theorem, δ_{ij} are the components of the Kronecker delta and the indices on the left hand side refer to the standard coordinates on $\mathbb{T}^n$. Define, furthermore,

$$\begin{aligned} \partial_0 \bar{g}_{00}(0,\cdot) &= [-2nH\bar{g}_{00} - \bar{g}^{ij}\partial_t \bar{g}_{ij}](0,\cdot) \\ \partial_0 \bar{g}_{0l}(0,\cdot) &= \left[-nH\bar{g}_{0l} + \frac{1}{2}\bar{g}^{ij}(2\partial_i \bar{g}_{jl} - \partial_l \bar{g}_{ij})\right](0,\cdot). \end{aligned}$$

Note that the last two equations are simply (62) and (63) given that we define $\bar{k}_{ij} = \partial_t \bar{g}_{ij}/2$. For ϵ small enough, Theorem 9 applies to these initial data and we get solutions to (144)-(147) on $(t_-,\infty)\times\mathbb{T}^n$ for some $t_- < 0$. Let us justify this statement and check that the bound only depends on n and V. By the assumptions, $\bar{g}_{ij} - 16H^{-2}\delta_{ij}$ is small in $H^{k_0+1}(\mathbb{T}^n)$. Assuming ϵ to be small enough, we get (172) with $c_1 = 4$ and $e^{-2K} = H^2/16$. In our case, k_0 is determined by n, so that the constants c_0 and ϵ_0 appearing in the statement of Theorem 9 only depend on n and V. If we can prove that $\hat{H}_{k_0}(0) \leq C\epsilon$ for some C depending only on n and V, we are thus done. However, for $t = 0$, $\hat{H}_{k_0}$ is equivalent to the sum of H_{lp,k_0}, H_{s,k_0} and H_{m,k_0}, with the constant only depending on H. On the other hand, by the arguments given in Section 11, this sum is equivalent to the sum of E_{lp,k_0}, E_{s,k_0} and E_{m,k_0} (recall that in the expressions for these quantities, $r = aHt$, $\omega = H$ and $\psi = \phi$). However, for $t = 0$, one sees that this sum is bounded by $C\epsilon$, where the constant only depends on n and V. The statement follows. Note that we also get asymptotics as in the statement of Proposition 2. Furthermore, on $B_{15/16}(0)$, the constraint equations are satisfied, and we have chosen $\partial_0\bar{g}_{00}$ and $\partial_0\bar{g}_{0i}$ in such a way that $\mathcal{D}_\mu|_{t=0} = 0$. Due to Proposition 1, we conclude that in $D[\{0\}\times B_{15/16}(0)]$, $(\bar{g},\Phi)$ satisfy (12)-(13). If ϵ is small enough, Proposition 3 implies that

$$(t_-,\infty)\times B_{5/8}(0) \subseteq D[\{0\}\times B_{29/32}(0)], \tag{205}$$

where we increase t_- if necessary. The reason for this is that, first of all, the assumptions concerning h and Sobolev embedding yield

$$16H^{-2}|v|^2 \leq d_1^2(\epsilon)\bar{g}_{ij}(0,\cdot)v^i v^j$$

for all $v \in \mathbb{R}^n$, where $d_1(\epsilon) \to 1$ as $\epsilon \to 0$. Due to (198), we then obtain

$$4H^{-1}\int_{s_0}^{s_{\max}} [\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j]^{1/2} ds \leq d(\epsilon)d_1(\epsilon)H^{-1}$$

For ϵ small enough we thus get

$$\int_{s_0}^{s_{\max}} [\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j]^{1/2} ds \leq \frac{9}{32},$$

which implies (205). Note that due to Lemma 3,

$$\begin{aligned} U_{0,\mathrm{exc}} &= D[\{0\}\times B_{15/16}(0)], \quad U_{1,\mathrm{exc}} = D[\{0\}\times B_{29/32}(0)], \\ U_{2,\mathrm{exc}} &= D[\{0\}\times \bar{B}_{29/32}(0)] \end{aligned}$$

are open, open and closed subsets of $\mathbb{R}\times x(U)$ respectively. Consequently, $W_{i,\mathrm{exc}} = (\mathrm{Id}\times x^{-1})(U_{i,\mathrm{exc}})$ for $i = 0,1,2$ are also open, open and closed respectively.

Construction of a reference metric. In order to prove that the patches that we construct fit together to form a globally hyperbolic development, it is convenient to construct a reference metric. Let

$$\tilde{g} = (1 - f_c\circ x)(-dt^2 + h) + (f_c\circ x)(\mathrm{Id}\times x)^*\bar{g}.$$

Here h is the Riemannian metric on Σ given by the initial data. Note that ∂_t is timelike with respect to $\bar{g}$ so that ∂_t is timelike with respect to $\tilde{g}$. The hypersurfaces $\{\tau\} \times \Sigma$ are spacelike with respect to $-dt^2 + h$ and with respect to $(\mathrm{Id} \times x)^* \bar{g}$ for $\tau \in (t_-, \infty)$, so that they are spacelike with respect to $\tilde{g}$. As a consequence, $\tilde{g}$ is a Lorentz metric on $(t_-, \infty) \times \Sigma$, cf. Lemma 1.

Construction of local patches. In order to construct a globally hyperbolic development, we need to have patches starting with open subsets of the initial data for which we have no control beyond the fact that the constraints are satisfied. Let $p \in \Sigma$. Let $O \ni p$ be an open subset of Σ such that we have coordinates $y^1, ..., y^n$ on O and define coordinates $y^0, ..., y^n$ on $\mathbb{R} \times O$ by $y^0 = t$. Consider the equations

$$\hat{R}_{\mu\nu} - \nabla_\mu \phi \nabla_\nu \phi - \frac{2}{n-1} V(\phi) g_{\mu\nu} = 0, \tag{206}$$

$$\nabla^\mu \nabla_\mu \phi - V'(\phi) = 0, \tag{207}$$

where

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{(\mu} \mathcal{D}_{\nu)}, \quad \mathcal{D}_\mu = F_\mu - \Gamma_\mu, \quad F_\mu = g_{\mu\nu} g^{\alpha\beta} \tilde{\Gamma}^\nu_{\alpha\beta}, \tag{208}$$

$\tilde{\Gamma}^\nu_{\alpha\beta}$ are the Christoffel symbols of the background metric $\tilde{g}$, the curvature is computed for the unknown metric g, all indices are raised and lowered by g, etc. We would like to apply the local existence result given in Theorem 5, but this result does not immediately apply to the present situation due to the global restrictions on g made and the fact that a Lorentz metric on $\mathbb{R}^{n+1}$ can never have compact support. Let $Q \ni p$ be an open set such that its closure is compact and contained in O. Let $A_{\mu\nu}$ be the components of a Lorentz matrix valued function depending smoothly on the components $g_{\alpha\beta}$ of g with respect to the coordinates y. Let $A_{00} = g_{00}$ for all $g_{00} \in [-3/2, -1/2]$ and have the property that the range of A_{00} is contained in $[-2, -1/4]$. Let $A_{0i} = g_{0i}$ for $g_{0i} \in [-1, 1]$ and have the property that the range of A_{0i} is contained in $[-2, 2]$. Let $\mathcal{U}$ be an open subset of the set of symmetric $n \times n$ matrices such that the matrices with components $h_{ij}(q)$ for $q \in \bar{Q}$ are contained in $\mathcal{U}$ and that the closure of $\mathcal{U}$ in the set of all $n \times n$ matrices is compact and contained in the set of positive definite ones. Let A_{ij} be such that $A_{ij} = g_{ij}$ for $\{g_{ij}\} \in \mathcal{U}$ and A_{ij} is everywhere positive definite with a positive lower bound and an upper bound. Finally, assume that $A_{\mu\nu}$ is constant outside of a compact set. Note that A satisfies the conditions described in Subsection 3.1 (with g replaced by A). In particular, the derivative estimates follow easily from the fact that the derivatives of A with respect to the metric coefficients have compact support. Let $\theta_1 \in C_0^\infty[(-1, 1) \times O]$ be such that $\theta_1(q) = 1$ for $q \in [-1/2, 1/2] \times \bar{Q}$. In considering (206)-(207), we replace $g^{\mu\nu}$, wherever it appears, with $A^{\mu\nu}$, the components of the inverse of A and we replace $\tilde{\Gamma}^\alpha_{\mu\nu}$ by $\theta_1 \tilde{\Gamma}^\alpha_{\mu\nu}$. With these modifications, the resulting f, using the terminology of Subsection 3.1, has the properties required for applying Theorem 5. The reason is that f is a sum of terms that are smooth functions of ξ times functions of t and x that have compact support. Since $V'(0) = 0$, $f_b = 0$ using the terminology of Subsection 3.1. As initial data we would ideally like to prescribe that (58)-(63) hold. However, that does not lead to an equation of the type considered in Theorem 5. Let $\theta_0 \in C_0^\infty(O)$ be such that $\theta_0(q) = 1$ for all $q \in \bar{Q}$. Modify all the initial data by multiplying them with θ_0. Let u be the vector which collects ϕ and $g_{\mu\nu}$ for $\mu, \nu = 0, ..., n$. We can consider the resulting equation as an equation on $\mathbb{R}^{n+1}$. Furthermore, it is of such a form that Theorem

5 is applicable. We thus get a smooth local solution. Due to the smoothness of the solution, there is an open neighbourhood W of p in $\mathbb{R}\times\Sigma$ with the property that $\theta_1 = 1$ and $g_{\alpha\beta}$ are such that $A_{\mu\nu} = g_{\mu\nu}$ in W. Furthermore, we can assume that $\Sigma_p := W \cap \{0\}\times\Sigma \subseteq \{0\}\times Q$ and that every inextendible causal curve in W intersects Σ_p. Thus, (W, g) is globally hyperbolic with a Cauchy hypersurface Σ_p (note that since g^{00} is negative, grad$t$ is timelike on W and the time coordinate is strictly monotonically increasing along any causal curve so that causal curves intersect Σ_p at most once). Consequently, $J^-(q) \cap J^+(\Sigma_p)$ is compact and contained in W for every $q \in W$ with positive t-coordinate and similarly for points of W with negative t-coordinate, cf. Lemma 3. If we let $\mathcal{D}_\mu = F_\mu - \Gamma_\mu$, then $\mathcal{D}_\mu = \partial_0\mathcal{D}_\mu = 0$ on Σ_p by an argument similar to the one presented at the end of the proof of Proposition 1 (in the present setting $M_{\mu\nu} = M_\phi = 0$, which only simplifies the argument). Furthermore, $\mathcal{D}_\mu$ satisfies (56) with $M_{\mu\nu}$ and M_ϕ set to zero. Applying Theorem 6 on (W, g), which is globally hyperbolic, we conclude that $\mathcal{D}_\mu = 0$ in all of W. Let W_p be an open neighbourhood of p with the same properties as W and whose closure is compact and contained in W.

Patching together. We would like to define the manifold M to be the union of all the W_p and $W_{1,\text{exc}}$. The first problem we are confronted with is that of constructing a metric on M. In other words, proving that the metrics we have constructed on the different patches coincide in the intersection. Let us consider the intersection of W_p and W_q and comment on the changes one has to make if one replaces W_p by $W_{1,\text{exc}}$ as we go along. Say that $W_p \cap W_q \neq \emptyset$. The closures of W_p and W_q are compact and contained in open sets W_1, W_2, with properties as above, on which we have coordinates $z = (z^0, ..., z^n)$ and $y = (y^0, ..., y^n)$ respectively, where $z^0 = y^0 = t$. In the exceptional case, note that $W_{1,\text{exc}}$ is contained in $W_{2,\text{exc}}$, which is closed. Consequently, we shall in the exceptional case replace $\overline{W}_p$ with $W_{2,\text{exc}}$. Furthermore, $W_{2,\text{exc}} \subseteq W_{0,\text{exc}}$ and the latter set is open, so that in the exceptional case, we replace W_1 with $W_{0,\text{exc}}$. On W_1 and W_2, we have metrics g_1 and g_2 and smooth functions ϕ_a and ϕ_b respectively, both satisfying (206)-(207) when expressed with respect to the coordinates z and y respectively. Let us express both g_1 and g_2 with respect to the coordinates z in $W_1 \cap W_2$ and refer to the components as $g_{1\mu\nu}$ and $g_{2\mu\nu}$ respectively. Let us also use the notation $\Sigma_i = W_i \cap \{0\}\times\Sigma$.

Both are solutions. Note that the equations (206)-(207) are geometric, i.e. coordinate invariant. The reason is the following. Let $\tilde{\nabla}$ be the Levi-Civita connection associated with the reference metric $\tilde{g}$. Define A by

$$A(X, Y, \eta) = \eta(\nabla_X Y - \tilde{\nabla}_X Y), \tag{209}$$

for vectorfields X, Y and a 1-form field η. We see that A is multilinear over the functions, so that it is a tensor field. Writing it out in components, we get

$$A^\mu_{\alpha\beta} = A(\partial_\alpha, \partial_\beta, dx^\mu) = \Gamma^\mu_{\alpha\beta} - \tilde{\Gamma}^\mu_{\alpha\beta},$$

where

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2}\tilde{g}^{\mu\nu}(\partial_\alpha\tilde{g}_{\beta\nu} + \partial_\beta\tilde{g}_{\alpha\nu} - \partial_\nu\tilde{g}_{\alpha\beta}).$$

Compute

$$g_{\mu\nu}g^{\alpha\beta}A^\mu_{\alpha\beta} = \Gamma_\nu - g_{\mu\nu}g^{\alpha\beta}\tilde{\Gamma}^\mu_{\alpha\beta} = \Gamma_\nu - F_\nu = -\mathcal{D}_\nu.$$

The left hand side is clearly the components of a covector, so that $\mathcal{D}_\nu$ are the components of a covector as well. Due to (208), we conclude that the left hand sides of (206)-(207) transform as tensors under a change of coordinates. We conclude that both $g_{1\mu\nu}, \phi_a$ and $g_{2\mu\nu}, \phi_b$ satisfy (206)-(207). Furthermore, since $\mathcal{D}_{i\mu}$ are the components of a covector and vanishes with respect to one of the coordinate systems, it vanishes with respect to the other coordinate system.

The initial data coincide. By the construction and the specific form of the coordinate systems, it is clear that $g_{1ij} = g_{2ij}$, $g_{100} = g_{200}$ and that $g_{20i} = g_{10i}$ for $t = 0$. Since $\mathcal{D}_{i\mu} = 0$ and the metrics coincide for $t = 0$, the contracted Christoffel symbols for g_1 and g_2 with respect to the z-coordinates have to coincide. Since $k_{ij} = \partial_0 g_{ij}/2$, and the coordinates have the above special form, we conclude that $\partial_t g_{1\mu\nu} = \partial_t g_{2\mu\nu}$ for $t = 0$. Finally, it is clear that $\phi_a = \phi_b$ and $\partial_t\phi_a = \partial_t\phi_b$ for $t = 0$.

The solutions coincide. We wish to prove that the solutions coincide in $\overline{W}_p \cap \overline{W}_q$. For $t \geq 0$, let

$$S_t = [0, t] \times \Sigma \cap \overline{W}_p \cap \overline{W}_q.$$

Note that S_t is compact, and this is still the case if we replace $\overline{W}_p$ by $W_{2,\text{exc}}$. Let $\mathcal{A}$ be the set of $t \in [0, \infty)$ such that $g_1 = g_2$ and $\phi_a = \phi_b$ in S_t and that for $r \in S_t$,

$$J_1^-(r) \cap J_1^+(\Sigma_1) = J_2^-(r) \cap J_2^+(\Sigma_2), \tag{210}$$

where $J_1^-(r)$ is the causal past of r with respect to the metric g_1 in W_1 etc. Note that $0 \in \mathcal{A}$, so that $\mathcal{A}$ is non-empty. Assume $t \in \mathcal{A}$ and $r \in S_t$ with $r = (t, \xi)$. Note that $J_i^-(r) \cap J_i^+(\Sigma_i) \subseteq W_1 \cap W_2$. If $\tau > t$ is close enough to t, the same is true with r replaced by (τ, ξ) due to Lemma 3. Taking the difference of (206)-(207) for the two solutions, keeping in mind that $sg_1 + (1 - s)g_2$ is a Lorentz metric for $s \in [0, 1]$ due to the fact that $g_{i00} < 0$ and $g_{i\flat}$ is positive definite for $i = 1, 2$, we conclude that Theorem 6 is applicable with two choices for the coefficients of the highest order derivatives; either $g_1^{\mu\nu}$ or $g_2^{\mu\nu}$. We conclude that $g_1 = g_2$ and $\phi_a = \phi_b$ in

$$\{J_1^-[(\tau, \xi)] \cap J_1^+(\Sigma_1)\} \cup \{J_2^-[(\tau, \xi)] \cap J_2^+(\Sigma_2)\}.$$

Consequently (210) holds with r replaced by (τ, ξ). This proves that $\mathcal{A}$ is open, due to the following argument. Assume there is no $\epsilon > 0$ such that $[t, t + \epsilon] \subseteq \mathcal{A}$. Then there is a sequence $r_i = (t_i, p_i)$ such that $t_i \to t+$ and either $g_1(r_i) \neq g_2(r_i)$, $\phi_a(r_i) \neq \phi_b(r_i)$ or (210) does not hold for $r = r_i$. Due to compactness, we can assume p_i to converge to, say, p. Applying the above argument with $\xi = p$, i.e. $r = (t, p)$, we arrive at a contradiction for i large enough. We conclude that $[t, t + \epsilon] \subseteq \mathcal{A}$ for $\epsilon > 0$ small enough. The closedness is less complicated to prove, though some care is required in the proof of (210). However, (210) follows from Lemma 3. Since $\mathcal{A}$ is connected, we conclude that $\mathcal{A} = [0, \infty)$ so that $g_1 = g_2$ and $\phi_a = \phi_b$ in $\overline{W}_p \cap \overline{W}_q$ for $t \geq 0$. Due to the same argument in the opposite time direction, we conclude that we have a solution to (12)-(13) on M, defined to be the union of all the W_p and $W_{1,\text{exc}}$. The embedding $i : \Sigma \to M$ is simply the inclusion $i(p) = (0, p)$. By construction, it is clear that if K is the induced second fundamental form, $i^*g = h$, $i^*K = k$, $\phi \circ i = \phi_0$ and $(N\phi) \circ i = \phi_1$. Let γ be an inextendible causal curve in M. Then the image of γ has to intersect some W_p and $\gamma|_{\gamma^{-1}(W_p)}$ is an inextendible causal curve in W_p which by construction has to intersect Σ. Since grad$t$ is timelike by construction, the t-coordinate of γ is strictly monotone, so that γ intersects Σ exactly once.

Embedding into the maximal globally hyperbolic development. Above, we have constructed a globally hyperbolic development of the initial data, say (M, g, ϕ). Furthermore, all causal geodesics that start in $\{0\} \times B_{1/4}(0)$ are future complete in M due to Proposition 4 and there is an embedding ψ of the form (20) due to the inclusion (205). Finally, this embedding has the properties stated in the theorem. To get the desired conclusion, we need only observe that by the definition of a maximal globally hyperbolic development, there is an embedding of (M, g, ϕ) into the maximal globally hyperbolic development $(\bar{M}, \bar{g}, \bar{\phi})$. □

17. Stability of locally spatially homogeneous spacetimes

Proof of Theorem 4. Given the initial data, let us start by constructing a development.

Construction of a development. Let us first consider the case in which the background initial data are (G, g, k), where G is a simply connected unimodular Lie group and the isometry group of the initial data contains the left translations in G. The arguments presented below are based on a formulation of Einstein's equations, in the context of interest, that was introduced by Ellis and MacCallum, cf. [12]. Our presentation, however, follows the presentation given in the appendix of [30] quite closely. Let e'_i be an orthonormal basis of the Lie algebra and define the structure constants γ^i_{jk} by the relation

$$[e'_j, e'_k] = \gamma^i_{jk} e'_i.$$

The fact that G is unimodular is equivalent to $\gamma^i_{ji} = 0$ which is equivalent to the statement that there is a symmetric matrix ν such that $\gamma^i_{jk} = \epsilon_{jkl}\nu^{li}$, where $\epsilon_{123} = 1$ and ϵ_{ijk} is antisymmetric in all its indices. In fact, one can compute ν^{ij} by the formula

$$\nu^{ij} = \frac{1}{2}\gamma^{(i}_{kl}\epsilon^{j)kl}, \tag{211}$$

where the parenthesis signifies symmetrization. According to Lemma 21.1 of [30], one can apply an orthogonal matrix to the basis e'_i, so that ν^{ij} with respect to this new basis is diagonal. Let us denote this new basis by e'_i as well and let $k_{ij} = k(e'_i, e'_j)$. The content of the momentum constraint (18) is that $\{k_{ij}\}$ and $\{\nu_{ij}\}$ commute:

$$k_i{}^l \nu_{lj} - \nu_i{}^l k_{lj} = 0 \tag{212}$$

(note that $\phi_0 = \phi_1 = 0$ when we apply (18) here). In the above equation, and below, we raise and lower indices with δ_{ij}. In other words, there is no difference between upstairs and downstairs indices, and the only reason for making a distinction is aesthetical. As a consequence, we can assume e'_i to be such that k_{ij} are the components of a diagonal matrix as well. Define $n(0) = \nu$, $\theta(0) = \mathrm{tr}_g k$ and $\sigma_{ij}(0) = k_{ij} - \theta(0)\delta_{ij}/3$. Define n, θ, σ to be the solution to

$$\dot{n}_{ij} = 2n_{k(i}\sigma_{j)}{}^k - \frac{1}{3}\theta n_{ij} \tag{213}$$

$$\dot{\sigma}_{ij} = -\theta\sigma_{ij} - s_{ij} \tag{214}$$

$$\dot{\theta} = -\sigma_{ij}\sigma^{ij} - \frac{1}{3}\theta^2 + \Lambda. \tag{215}$$

In these equations $s_{ij} = b_{ij} - \mathrm{tr}(b)\delta_{ij}/3$, where $b_{ij} = 2n_i{}^l n_{lj} - \mathrm{tr}(n)n_{ij}$. Let (t_-, t_+) be the maximal existence interval. Note that, since $V(0) = \Lambda$, (17) is equivalent to

$$\sigma_{ij}\sigma^{ij} + \left(n_{ij}n^{ij} - \frac{1}{2}(\mathrm{tr}n)^2\right) + 2\Lambda = \frac{2}{3}\theta^2 \tag{216}$$

at $t = 0$. Due to (213)-(215) and the fact that (216) holds at $t = 0$, (216) is satisfied at all times. The reason is that if you move all the terms in (216) to the left hand side and denote the resulting expression by f, then (213)-(215) imply $\dot{f} = -2\theta f/3$. Let v be a vector collecting all the off-diagonal components of n and σ. Using (213) and (214) one can derive an equation of the form $\dot{v} = Cv$ for some matrix C depending on the unknowns. Since $v(0) = 0$, we conclude that $v(t) = 0$ for all $t \in (t_-, t_+)$. In other words n and σ remain diagonal. As a consequence, (212) holds for all $t \in (t_-, t_+)$ if we replace k with σ and ν with n.

Let us define f_i by the condition that $f_i(0) = 1$ and $\dot{f}_i = (2\sigma_i - \theta/3)f_i$, where σ_i denotes the diagonal components of σ. Define

$$a_i = (\Pi_{j \neq i} f_j)^{-1/2},$$

define $e_i = a_i^{-1} e_i'$ (no summation on i) and $e_0 = \partial_t$. The point of this definition is that the matrix $\tilde{n}$ obtained from the basis e_i using the right hand side of (211), where γ^i_{kl} are the structure constants associated with the basis e_i, coincides with n. Let $M = (t_-, t_+) \times G$ and define a metric on M by requiring that e_α be an orthonormal basis with e_0 timelike and e_i spacelike. In other words,

$$\bar{g} = -dt^2 + \sum_{i=1}^{3} a_i^2(t)\xi^i \otimes \xi^i, \tag{217}$$

where the ξ^i are the duals of the e_i'. Let ∇ be the associated Levi-Civita connection and compute $\langle \nabla_{e_0} e_i, e_j \rangle = 0$. If

$$\tilde{\theta}(X, Y) = \langle \nabla_X e_0, Y \rangle, \quad \tilde{\theta}_{\mu\nu} = \tilde{\theta}(e_\mu, e_\nu),$$

then $\tilde{\theta}_{00} = \tilde{\theta}_{0i} = \tilde{\theta}_{i0} = 0$. Furthermore,

$$a_j e_0(a_j^{-1})\delta_{ij} = -\tilde{\theta}_{ij}$$

(no summation over j) so that $\tilde{\theta}_{ij}$ is diagonal and $\mathrm{tr}\tilde{\theta} = \theta$. Finally,

$$-\tilde{\sigma}_{ii} = -\tilde{\theta}_{ii} + \frac{1}{3}\theta = -\sigma_i.$$

Let us now check that $(M, \bar{g})$ is a globally hyperbolic development of the initial data we started with. That the metric and second fundamental form induced on $\{0\} \times G$ correspond to the initial data is clear from the construction. That $(M, \bar{g})$ is globally hyperbolic and that all the hypersurfaces $\{t\} \times G$ are Cauchy hypersurfaces follows by an argument which is identical to the proof of Lemma 21.4 of [30]. What remains to be checked is that the equations,

$$\mathrm{Ric}[\bar{g}] = \Lambda \bar{g},$$

are satisfied. However, (212), with k_{ij} replaced by θ_{ij} and ν_{ij} replaced by n_{ij}, is equivalent to the $0i$ components of Einstein's equations, (215) is the 00 component of the equations, (214) is the traceless part of the ij components of the equations and the trace part of the ij equations satisfy the correct equation due to (215)-(216). We conclude that the constructed metric satisfies Einstein's equations with a positive cosmological constant.

Let us consider the case that the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^3$. Let q be a symmetric covariant 2-tensor field on $\mathbb{H}^3$ with such invariance properties and assume that at $p = 0 \in B^3$,

$$q_p = b_{ij} dx^i|_p \otimes dx^j|_p,$$

where x^i are the standard coordinates in the ball model. Since, for each $A \in \mathrm{O}(3)$, there is an isometry of the standard metric of $\mathbb{H}^3$ that fixes p and maps $\partial_i|_p$ to $A_i{}^j \partial_j|_p$, we conclude that $b = AbA^t$ for all $A \in \mathrm{O}(3)$. As a consequence, b has to be a multiple of the identity (since b is symmetric there is an orthogonal matrix diagonalizing it, so that b has to be diagonal, and the fact that the diagonal components have to coincide then follows by applying permutation matrices) . Using the full isometry group, we see that there must be a number β such that

$$q = \beta g_{\mathbb{H}^3}, \quad g_{\mathbb{H}^3} = 4\frac{dx^2 + dy^2 + dz^2}{(1 - x^2 - y^2 - z^2)^2}.$$

We conclude that the initial data are given by $g = \alpha^2 g_{\mathbb{H}^3}$ and $k = \beta\alpha g_{\mathbb{H}^3}$ where α, β are constants such that $\alpha, \beta > 0$ since $\mathrm{tr}_g k > 0$. The Hamiltonian constraint (17) is equivalent to

$$-6\frac{1}{\alpha^2} + 6\left(\frac{\beta}{\alpha}\right)^2 = 2\Lambda$$

and the momentum constraint (18) is automatically satisfied. Let a satisfy

$$\begin{aligned} \ddot{a} &= \frac{1}{3}\Lambda a \\ a(0) &= \alpha \\ \dot{a}(0) &= \beta. \end{aligned} \tag{218}$$

Let $I = (t_-, t_+)$ be the interval on which $a > 0$ and let

$$f = \left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} - \frac{1}{3}\Lambda.$$

Then $\dot{f} = -2\dot{a}f/a$, so that $f = 0$ on I, since it is zero initially. As a consequence, a satisfies

$$2\left(\frac{\dot{a}}{a}\right)^2 - \frac{2}{a^2} + \frac{\ddot{a}}{a} = \Lambda. \tag{219}$$

Using (218), (219) and the formulas (1)-(3) on p. 211 of [24], we conclude that

$$-dt^2 + a^2(t) g_{\mathbb{H}^3} \tag{220}$$

is a solution to Einstein's vacuum equations with a cosmological constant Λ. Furthermore, the induced metric and second fundamental form on the $t = 0$ hypersurface give the initial data when pulled back to $\mathbb{H}^3$ using the standard embedding. Note that

$$a(t) = \alpha \cosh(Ht) + \beta H^{-1} \sinh(Ht),$$

where $H = (\Lambda/3)^{1/2}$ and that I contains $[0,\infty)$. Consequently

$$\lim_{t\to\infty} e^{-Ht}a(t) = (\alpha + \beta H^{-1})/2 > 0, \quad \lim_{t\to\infty} \frac{\dot{a}(t)}{a(t)} = H.$$

Let us consider the case that the initial data are invariant under the full isometry group of $\mathbb{H}^2 \times \mathbb{R}$. Let q be a symmetric covariant 2-tensor field on $\mathbb{H}^2 \times \mathbb{R}$ with such invariance properties and assume that at $p = 0 \in B^2 \times \mathbb{R}$,

$$q_p = a_{ij}dx^i|_p \otimes dx^j|_p + b_i(dx^i|_p \otimes dz|_p + dz|_p \otimes dx^i|_p) + c dz|_p \otimes dz|_p$$

where x^1, x^2 are the standard coordinates on the open unit disc and z is the standard coordinate on $\mathbb{R}$. Due to the invariance properties of q, we see that $b_i = 0$ and that a_{ij} must be the components of a multiple of the identity matrix. Using the full isometry group, we conclude that

$$q = c_{\mathbb{H}^2} g_{\mathbb{H}^2} + c_{\mathbb{R}} dz^2, \quad g_{\mathbb{H}^2} = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

for some constants $c_{\mathbb{H}^2}$ and $c_{\mathbb{R}}$. As a consequence of the above observation, we can assume that the initial data are given by

$$g = a_0^2 g_{\mathbb{H}^2} + b_0^2 dz^2, \quad k = a_1 a_0 g_{\mathbb{H}^2} + b_1 b_0 dz^2,$$

where $a_0, b_0 > 0$. That the initial data satisfy the Hamiltonian constraint is equivalent to

$$\left(\frac{a_1}{a_0}\right)^2 + 2\frac{a_1 b_1}{a_0 b_0} - \frac{1}{a_0^2} = \Lambda.$$

Let the functions a and b be determined by

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = \Lambda \tag{221}$$

$$\frac{\ddot{b}}{b} + \frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} = \Lambda \tag{222}$$

$$(a(0), b(0)) = (a_0, b_0) \tag{223}$$

$$\left(\dot{a}(0), \dot{b}(0)\right) = (a_1, b_1) \tag{224}$$

and let I be the intersection of the maximal existence interval and the maximal interval containing 0 on which a and b are both positive. Let

$$f = \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}\dot{b}}{ab} - \frac{1}{a^2} - \Lambda, \quad \theta = \frac{\dot{b}}{b} + 2\frac{\dot{a}}{a}.$$

Then $\dot{f} = -\theta f$ so that $f(t) = 0$ $\forall t \in I$, since $f(0) = 0$. Define the metric $\bar{g}$ on $I \times \mathbb{H}^2 \times \mathbb{R}$ by

$$\bar{g} = -dt^2 + a^2(t) g_{\mathbb{H}^2} + b^2(t) dz^2. \tag{225}$$

Then $\bar{g}$ satisfies Einstein's equations with a cosmological constant Λ and the metric and second fundamental form induced on the hypersurface $t = 0$ yield the initial data when pulled back to $\mathbb{H}^2 \times \mathbb{R}$ by the standard embedding. The fact that $f = 0$ can be reformulated to

$$\frac{1}{3}\theta^2 = \Lambda + \frac{1}{a^2} + \frac{1}{3}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)^2. \tag{226}$$

Note that θ is the trace of the second fundamental form of the hypersurfaces of constant t and that as a consequence of (226) and the assumption that $\theta(0) > 0$, we have $\theta > 3H$, where $H = (\Lambda/3)^{1/2}$. Furthermore, as long as θ remains finite, the solution to (221)-(224) cannot blow up, so that the only obstruction to global existence to the future is finite in time blow up of θ. The reason is as follows. Assume θ is bounded on $[0, T) \subset I$ for some $T < \infty$. Then $\dot{a}/a$ and $\dot{b}/b$ are bounded on $[0, T)$ due to (226). As a consequence, a and b are bounded and thus $\dot{a}$ and $\dot{b}$ are bounded. That a is bounded away from 0 is clear from (226) and that b cannot converge to zero as $t \to T-$ follows from the fact that there is a uniform bound on $\dot{b}/b$ on $[0, T)$; the assumption that b does converge to zero would lead to the conclusion that $b = 0$ in all of $[0, T)$. We conclude that the solution can be extended beyond T. Combining (221) with the fact that $f = 0$ yields $\ddot{a} = \dot{a}\dot{b}/b$. Combining this with (222) yields $\ddot{b}/b + 2\ddot{a}/a = \Lambda$, which implies

$$\dot{\theta} = \Lambda - \frac{1}{3}\theta^2 - \frac{2}{3}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)^2. \tag{227}$$

Since $\theta^2/3 > \Lambda$, we conclude that θ is strictly monotonically decreasing. Consequently, it is bounded to the future, so that we have future global existence. By an argument which is identical to one given below in the Bianchi class A case (unimodular Lie groups), cf. (230) and the two equations following it, we can use (226) and (227) to conclude that $\theta - 3H$ converges to zero exponentially. As a consequence of (226), we conclude that $\dot{a}/a - \dot{b}/b$ converges to zero exponentially, whence

$$\lim_{t\to\infty} \frac{\dot{a}(t)}{a(t)} = \lim_{t\to\infty} \frac{\dot{b}(t)}{b(t)} = H, \quad \lim_{t\to\infty} e^{-Ht}a(t) = \alpha_0, \quad \lim_{t\to\infty} e^{-Ht}b(t) = \beta_0$$

for some constants $\alpha_0, \beta_0 > 0$.

Note that the n dimensional hyperbolic space can be viewed as a Lie group. Let $\mathcal{S}_n$ be the set of $n \times n$ matrices of the following form: the first row is any $x \in \mathbb{R}^n$ such that the first element of x is positive and the remaining rows are the second to n:th rows of the $n \times n$ identity matrix. Then $\mathcal{S}_n$ is a group under matrix multiplication and we can identify it with the upper half plane. If x and y are elements of the n dimensional upper half plane, so that the first components x_1, y_1 are positive, then the corresponding product xy is given by first multiplying y by x_1 and then translating the last $n-1$ components of the result by the last $n-1$ components of x. Thus the hyperbolic metric is a left invariant metric on the Lie group $\mathcal{S}_n$. As a consequence, (225) and (220) can be considered to be of the form (217), where the a_i:s satisfy (231). As in the unimodular case, we can then use an argument which is identical to the proof of Lemma 21.4 of [30] in order to prove that the metrics (220) and (225) yield globally hyperbolic spacetimes.

Analyzing the asymptotics. The analysis of the asymptotics of solutions to (213)-(216) follows as in Wald [33]. Note first of all that $n_{ij}n^{ij} - (\text{tr}n)^2/2 \geq 0$ unless all the n_i (the diagonal components of n_{ij}) are non-zero and have the same sign. However, all the n_i being non-zero and having the same sign corresponds to a universal covering group of SU(2), which we have excluded. Since we assume that $\theta(0) > 0$, (216) then implies that

$$\theta(t) \geq (3\Lambda)^{1/2} =: \alpha \tag{228}$$

for all t. Combining this with (215), we get the conclusion that

$$\dot{\theta} \leq -\frac{1}{3}\theta^2 + \Lambda \leq 0. \tag{229}$$

Due to this equation, θ is bounded to the future. Combining this fact with (216) and the fact that the expression involving the n_{ij} is non-negative, we conclude that $\sigma_{ij}(t)$ is bounded to the future. Thus n_{ij} cannot blow up in finite time to the future due to (213). Since none of θ, σ_{ij} and n_{ij} can blow up in a finite time to the future, we conclude that $t_+ = \infty$. Concerning θ, we have two possibilities. Either $\theta(t) > \alpha$ for all $t \in (t_-, t_+)$, or there is a $t_0 \in (t_-, t_+)$ such that $\theta(t_0) = \alpha$. Let us consider the second case first. Then, due to (228) and (229), we conclude that $\theta(t) = \alpha$ for all $t \in [t_0, t_+)$. Combining this fact with (216) and the fact that the expression involving the n_{ij} is non-negative, we conclude that $\sigma_{ij}(t) = 0$ for $t \geq t_0$. In the case that $\theta(t) > \alpha$ for all t, we can proceed as in [33]. Due to (229), we have

$$\frac{\dot{\theta}}{\theta^2 - \alpha^2} \leq -\frac{1}{3}. \tag{230}$$

Integrating this inequality, we get

$$\frac{\theta - \alpha}{\theta + \alpha} \leq \psi, \quad \psi = \exp\left[-\frac{2\alpha}{3}t + C\right],$$

where C is an integration constant. For t large enough, $\psi < 1$, and then we get

$$\theta \leq \alpha\frac{1+\psi}{1-\psi}, \quad 0 < \theta - \alpha \leq \alpha\frac{2\psi}{1-\psi}.$$

As a consequence, $\theta \to \alpha$ and the error is exponentially small. Combining this observation with (216), we conclude that σ_{ij} converges to zero exponentially. Going through the definitions above, one then sees that $a_i(t) = \alpha_i \exp[\alpha t/3 + \rho_i(t)]$, for some functions ρ_i that converge to zero exponentially and that $\dot{a}_i/a_i \to H$. Note that this statement also holds if $\theta(t_0) = \alpha$ for some $t_0 \in (t_-, t_+)$.

Stability. Let us assume we have a metric of the form

$$\bar{g} = -dt^2 + \sum_{i=1}^{3} a_i^2(t)\xi^i \otimes \xi^i,$$

on $I \times G$, where G is a 3-dimensional Lie group, I is an open interval containing (t_0, ∞) for t_0 large enough and ξ^i are the duals of a basis $\{e_i\}$ for the Lie algebra. Assume furthermore that

$$\lim_{t\to\infty} e^{-Ht}a_i(t) = \alpha_i, \quad \lim_{t\to\infty} \frac{\dot{a}_i}{a_i} = H, \tag{231}$$

where H is as in the statement of the theorem, and some $\alpha_i > 0$. Assume finally that there is a group of diffeomorphisms Γ acting freely and properly discontinuously on G such that $\mathrm{Id} \times \Gamma$ is a group of isometries of $\bar{g}$ and such that the quotient of G under Γ is compact (it is clear that the groups under consideration in the theorem are of this type in the unimodular case, due to our assumptions, and in the remaining cases due to the fact that the metrics are either of the form (225) or of the form (220)). Let Σ denote the quotient and let $\pi : G \to \Sigma$ be the covering projection. Let us define a reference metric

$$h = \sum_{i=1}^{3} \alpha_i^2 \xi^i \otimes \xi^i$$

on G. Note that since

$$\hat{h} = e^{-2Ht} \sum_{i=1}^{3} a_i^2(t) \xi^i \otimes \xi^i$$

converges to the metric h as $t \to \infty$ and Γ is a group of isometries of $\hat{h}$, Γ is a group of isometries of h. Consequently, h induces a metric on Σ. In what follows it will be useful to compare ∂_{y^i} for some coordinates y with the basis e_i. Unfortunately, we cannot assume that the e_i are well defined on Σ, since the group Γ may contain diffeomorphisms that do not map e_i to itself. On the other hand, there is an $\epsilon_0 > 0$ such that if $\epsilon \le \epsilon_0$ and $p \in \Sigma$, then $B_\epsilon(p)$ (measured with respect to the metric h) is such that $\pi^{-1}[B_\epsilon(p)]$ consists of a disjoint collection of open sets such that π, restricted to any connected member of the disjoint union, is an isometry onto $B_\epsilon(p)$. One can use one of these isometries to push the basis e_i (and thus ξ^i) forward to $B_\epsilon(p)$. However, the result will in general depend on the choice of connected member of $\pi^{-1}[B_\epsilon(p)]$; below we shall speak of a choice of ξ^i on $B_\epsilon(p)$. We now wish to prove that there is an $\epsilon > 0$ and a $K > 0$ such that for every $p \in \Sigma$, there are normal coordinates y^i on $B_\epsilon(p)$ with respect to the metric h, and a choice of ξ^i such that if $\zeta^i_j = \xi^i(\partial_{y^j})$, then all the derivatives of ζ^i_j with respect to y^l up to order $k_0 + 1$ are bounded by K in the sup norm on $B_\epsilon(p)$. In order to obtain a contradiction, let us assume that the statement is not true. Given any $n > 0$ such that $1/n \le \epsilon_0$ is smaller than the injectivity radius of (Σ, h), there is thus a p_n such that regardless of the choice of ξ^i and normal coordinates on $B_{1/n}(p_n)$, there is a multiindex α with $|\alpha| \le k_0 + 1$ such that $|\partial^\alpha \zeta^i_j|$ exceeds n on $B_{1/n}(p_n)$. Since Σ is compact, there is a subsequence of the p_n, which we shall also denote by p_n, converging to a point $p \in \Sigma$. There is an $\epsilon > 0$ such that $\epsilon \le \epsilon_0$ and $\bar{B}_\epsilon(p)$ is contained in a convex neighbourhood U of p. Let v_i be an orthonormal basis of the tangent space on U. For n large enough, $\bar{B}_{1/n}(p_n) \subset U$. We can define normal coordinates x^i_n on U by letting $x^i_n(q)$ be the coefficients of $\exp^{-1}_{p_n}(q)$ with respect to $v_i|_{p_n}$. We define normal coordinates x^i on U similarly by replacing p_n by p. Since $\exp^{-1}_p(q)$ is a smooth function in both coordinates on a convex set, cf. Lemma 9, p. 131 of [24], we conclude that x^i_n, considered as smooth functions on $B_\epsilon(p)$, converge to x^i with respect to any C^k norm and coordinates that contain the closure of $B_\epsilon(p)$ in their domain of definition. For any choice of ξ^i on $B_\epsilon(p)$, $\xi^i(\partial_{x^j})$ is bounded in the C^{k_0+1} norm with respect to the coordinates x^i on $B_{\epsilon/2}(p)$. Fix a choice of ξ^i. For n large enough, this also corresponds to a choice of ξ^i on $B_{1/n}(p_n)$, and by the above observation concerning the relation between the coordinate systems x^i and x^i_n, we conclude that $\xi^i(\partial_{x^j_n})$ is bounded with respect to the C^{k_0+1} norm in the x^i_n coordinates. This contradicts the assumption.

Let $\epsilon > 0$ and $K > 0$ be as above and $p \in \Sigma$. Let y^i be normal coordinates on $B_\epsilon(p)$ with respect to the metric h, and make a choice of ξ^i such that if $\zeta^i_j = \xi^i(\partial_{y^j})$, then all the derivatives of ζ^i_j with respect to y^l up to order $k_0 + 1$ are bounded by K in the sup norm on $B_\epsilon(p)$. The initial data induced on the hypersurface $\{t\} \times G$ is given by

$$g = \sum_{i=1}^{3} a_i^2(t) \xi^i \otimes \xi^i, \quad k = \sum_{i=1}^{3} \dot{a}_i(t) a_i(t) \xi^i \otimes \xi^i.$$

Let us introduce coordinates $x^i = He^{Ht}y^i/4$. For t large enough, the range of x^i contain the ball of radius 1. Note that

$$g_{ij} = g(\partial_{x^i}, \partial_{x^j}) = 16H^{-2}\sum_{l=1}^{3} e^{-2Ht}a_l^2(t)(\xi^l \otimes \xi^l)(\partial_{y^i}, \partial_{y^j}).$$

Since $e^{-Ht}a_i(t) \to \alpha_i$ as $t \to \infty$, $h(\partial_{y^i}, \partial_{y^j}) = \delta_{ij}$ at p, the derivatives of $\xi^l(\partial_{y^i})$ with respect to y^j are bounded by K on $B_\epsilon(p)$ and the ball of radius 1 with respect to the x^i coordinates corresponds to a ball of an arbitrarily small radius with respect to the y^i coordinates for t large enough, we conclude that for t large enough (the bound being independent of p), $g_{ij} - 16H^{-2}\delta_{ij}$ is arbitrarily small in the ball of radius 1 with respect to the x^i coordinates. Since

$$\frac{\partial}{\partial x^i} = 4H^{-1}e^{-Ht}\frac{\partial}{\partial y^i},$$

and $\xi^i(\partial_{y^j})$ is bounded in C^{k_0+1}, the spatial derivatives of g_{ij} with respect to x^l are arbitrarily small for t large enough (independent of p). Similarly,

$$k_{ij} = k(\partial_{x^i}, \partial_{x^j}) = 16H^{-2}\sum_{l=1}^{3} e^{-2Ht}\dot{a}_l(t)a_l(t)(\xi^l \otimes \xi^l)(\partial_{y^i}, \partial_{y^j}).$$

Since, in addition to the above observations, $e^{-2Ht}\dot{a}_i(t) \to H\alpha_i$, we conclude that $k_{ij} - 16H^{-1}\delta_{ij}$ is arbitrarily small in a ball of radius 1 with respect to the x^i-coordinates. Furthermore, the derivatives of k_{ij} with respect to ∂_{x^l} are arbitrarily small. To conclude, there is a t_0 such that $(g, k, 0, 0)$ for $t = t_0$ satisfy (19) with ϵ replaced by $\epsilon/2$, where the coordinates are of the form described above (regardless of the point p). Using Theorem 7, we get the desired stability statement. □

Proof of Theorem 3. The proof is similar to the end of the proof of Theorem 4, but easier. Let Σ and g_Σ be as in the statement of the theorem. The metric we wish to consider is of the form (3). Similarly to the above proof, one can prove that there is an $\epsilon > 0$ and a $K > 0$ such that for every $p \in \Sigma$, there are normal coordinates y^i on $B_\epsilon(p)$ with the property that all derivatives up to order $k_0 + 1$ of $g_\Sigma(\partial_{y^i}, \partial_{y^j})$ with respect to the y-coordinates are bounded by K. Given this observation, the end of the proof is essentially the same as the end of the above proof. □

18. Appropriate initial data on an arbitrary manifold

Let (M, g) be a closed n dimensional Riemannian manifold such that g has constant scalar curvature. Let $g_\alpha = e^\alpha g$ for $\alpha \in \mathbb{R}$. Then, if r is the scalar curvature of g, $r_\alpha = e^{-\alpha}r$ is the scalar curvature of g_α. Let $k_\beta = \beta g_\alpha$. Then, assuming all indices are raised and lowered with g_α, we have

$$r_\alpha - k_{\beta ij}k_\beta^{ij} + (\mathrm{tr}k_\beta)^2 - 2V(0) = e^{-\alpha}r - n\beta^2 + n^2\beta^2 - n(n-1)H^2.$$

Choose β to be the positive solution to

$$\beta^2 = H^2 - \frac{1}{n(n-1)}e^{-\alpha}r,$$

which exists, assuming α to be big enough. Then $(g_\alpha, k_\beta, 0, 0)$ satisfy (17)-(18). Furthermore, for α large enough, the data will be such that Theorem 2 is applicable in a neighbourhood of each $p \in M$, the argument being similar to the end of the proof of Theorem 4. Thus they yield future causally geodesically complete spacetimes and we have the expansions stated in Theorem 2 to the future.

Assuming g is a Riemannian metric on a closed n dimensional manifold M with associated scalar curvature r (which is not necessarily constant), let g_α and k_β be as above. Let

$$\epsilon_\alpha = \sup_{p\in M} |r_\alpha(p)| + e^{-\alpha}.$$

Define

$$\phi_{1,\alpha} = (\epsilon_\alpha + r_\alpha)^{1/2}, \quad \phi_{0,\alpha} = 0, \quad \beta = \left[H^2 + \frac{\epsilon_\alpha}{n(n-1)}\right]^{1/2}.$$

Then $(g_\alpha, k_\beta, \phi_{0,\alpha}, \phi_{1,\alpha})$ satisfy (17)-(18), and if α is large enough, there is a neighbourhood of each $p \in M$ such that Theorem 2 applies to that neighbourhood, the argument being similar to the end of the proof of Theorem 4. In particular, the resulting spacetimes are future causally geodesically complete and we obtain expansions to the future as stated in Theorem 2.

Acknowledgments

Part of this work was carried out while the author was enjoying the hospitality of the Isaac Newton Institute for Mathematical Sciences and the Max Planck Institute for Gravitational Physics. Thanks are due to an anonymous referee for providing the important examples (6) and (7). The research was supported by the Swedish Research Council and the Göran Gustafsson Foundation. The author is a Royal Swedish Academy of Sciences Research Fellow supported by a grant from the Knut and Alice Wallenberg Foundation.

References

[1] Anderson, M. T.: Existence and Stability of even-dimensional asymptotically de Sitter spaces. Ann. Henri Poincaré **6**, 801–820 (2005)

[2] Anderson, M. T.: Asymptotic behaviour of future-complete cosmological spacetimes. A spacetime safari: essays in honour of Vincent Moncrief. Class. Quantum Grav. **21** no. 3, S11–S27 (2004)

[3] Andersson, L., Moncrief, V.: Future complete vacuum spacetimes. In: The Einstein equations and the large scale behaviour of gravitational fields. Birkhäuser, Basel (2004)

[4] Bernal, A. N., Sánchez, M.: On smooth Cauchy surfaces and Geroch's splitting theorem. Commun. Math. Phys. **243**, 461–470 (2003)

[5] Bernal, A. N., Sánchez, M.: Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. Commun. Math. Phys. **257**, 43–50 (2005)

[6] Bernal, A. N., Sánchez, M.: Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions. Lett. Math. Phys. **77**, 183–197 (2006)

[7] Bieli, R.: Algebraic expansions for curvature coupled scalar field models. Class.Quant.Grav. **22**, 4363–4376 (2005)

[8] Choquet-Bruhat, Y., Geroch R.: Global aspects of the Cauchy problem in General Relativity. Commun. Math. Phys. **14**, 329–335 (1969)

[9] Choquet-Bruhat, Y., Isenberg, J., Pollack, D.: The constraint equations for the Einstein-scalar field system on compact manifolds. Class. Quantum Grav. **24**, 809–828 (2007)

[10] Christodoulou, D.: Global solutions of nonlinear hyperbolic equations for small initial data. Commun. Pure Appl. Math. **39**, 267–282 (1986)

[11] Christodoulou, D., Klainerman, S.: The global non-linear stability of the Minkowski space. Princeton University Press, Princeton (1993)

[12] Ellis, G., MacCallum, M.: A class of homogeneous cosmological models. Comm. Math. Phys. **12**, 108–141 (1969)

[13] Fefferman, C., Graham, C.R.: Conformal invariants, in Elie Cartan et les Mathématiques d'Ajourd'hui. Astérisque, numero hors série, Soc. Math. France, Paris, 95–116 (1985)
[14] Fischer, A. E., Moncrief, V.: The reduced Einstein equations and the conformal volume collapse of 3-manifolds. Class. Quantum Grav. **18**, no. 21, 4493–4515 (2001)
[15] Friedrich, H.: On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. Commun. Math. Phys. **107**, 587–609 (1986)
[16] Friedrich, H.: Conformal Einstein evolution. In: The Conformal Structure of Space-Time, J. Frauendiener and H. Friedrich, Eds., Lecture Notes in Physics **604**, Springer Verlag, Berlin, 1–50 (2002)
[17] Friedrich, H.: On the global existence and the asymptotic behavior of solutions to the Einstein-Maxwell-Yang-Mills equations. J. Differential Geom. **34**, no. 2, 275–345 (1991)
[18] Friedrich, H., Rendall, A. D.: The Cauchy problem for the Einstein equations. In: Einstein's field equations and their physical implications. Lecture Notes in Phys. **540**, Springer, Berlin (2000)
[19] Geroch, R.: Domain of dependence. J. Math. Phys. **11**, 437–439 (1970)
[20] Hörmander, L.: Lectures on Nonlinear Hyperbolic Differential Equations. Springer-Verlag, Berlin Heidelberg New York (1997)
[21] Klainerman, S.: The null condition and global existence to nonlinear wave equations. Lect. Appl. Math. **23**, 293–326 (1986)
[22] Lindblad, H., Rodnianski, I.: Global existence for the Einstein vacuum equations in wave coordinates. Commun. Math. Phys. **256**, 43–110 (2005)
[23] Lindblad, H., Rodnianski, I.: The global stability of Minkowski space-time in harmonic gauge. Ann. of Math., accepted
[24] O'Neill, B.: Semi Riemannian Geometry. Academic Press, Orlando (1983)
[25] Pretorius, F.: Numerical relativity using a generalized harmonic decomposition. Class. Quantum Grav. **22**, 425–451 (2005)
[26] Raymond, F., Vasquez, T.: 3-manifolds whose universal coverings are Lie groups. Topology and its applications **12**, 161–179 (1981)
[27] Rendall, A. D.: Accelerated cosmological expansion due to a scalar field whose potential has a positive lower bound. Class.Quant.Grav. **21**, 2445–2454 (2004)
[28] Rendall, A. D.: Asymptotics of solutions of the Einstein equations with positive cosmological constant. Ann. Henri Poincaré **5**, 1041–1064 (2004)
[29] Ringström, H.: Future asymptotic expansions of Bianchi VIII vacuum metrics. Class. Quant. Grav. **20**, 1943–1990 (2003)
[30] Ringström, H.: The Bianchi IX attractor. Annales Henri Poincaré **2**, 405–500 (2001)
[31] Ringström, H.: On curvature decay in expanding cosmological models. Commun. Math. Phys. **264**, 613–630 (2006)
[32] Wald, R.: General Relativity. University of Chicago Press, Chicago (1984)
[33] Wald, R.: Asymptotic behaviour of homogeneous cosmological models in the presence of a positive cosmological constant. Phys. Rev. D **28**, 2118–2120 (1983)

Department of Mathematics, KTH, 100 44 Stockholm, Sweden

E-mail address: `hansr@kth.se`