POWER LAW INFLATION

HANS RINGSTRÖM

Abstract. The subject of this paper is Einstein's equations coupled to a non-linear scalar field with an exponential potential. The problem we consider is that of proving future global non-linear stability of a class of spatially locally homogeneous solutions to the equations. There are solutions on $\mathbb{R}_+ \times \mathbb{R}^n$ with accelerated expansion of power law type. We prove a result stating that if we have initial data that are close enough to those of such a solution on a ball of a certain radius, say $B_{4R_0}(p)$, then all causal geodesics starting in $B_{R_0}(p)$ are complete to the future in the maximal globally hyperbolic development of the data we started with. In other words, we only make local assumptions in space and obtain global conclusions in time. We also obtain asymptotic expansions in the region over which we have control. As a consequence of this result and the fact that one can analyze the asymptotic behaviour in most of the spatially homogeneous cases, we obtain quite a general stability statement in the spatially locally homogeneous setting.

Contents

1. Introduction

1.1. **Background and motivation.** The spacetimes currently used by physicists to model the universe are ones with accelerated expansion. However, such expansion can be achieved by many different mechanisms, and which one to choose is not completely clear. Some examples of candidates are a positive cosmological constant, quintessence and k-essence, cf. e.g. [16, 17, 18]. Due to this uncertainty, it seems reasonable to try to understand the behaviour of solutions under as general assumptions on the model as possible. One particular question of interest is that of future global non-linear stability, i.e., for the purposes of the present discussion, the following question: given initial data for the equations such that the corresponding maximal globally hyperbolic development (MGHD) is future causally geodesically complete, do small perturbations of the initial data also yield future causally geodesically complete MGHD's? It is of course also of interest to analyze the asymptotics in the causally geodesically complete direction, but that the answer to the above question be yes is a minimum requirement for stating that the MGHD of the given initial data is future stable. In [20], we built a framework for considering the question of future global non-linear stability for Einstein's equations coupled to a non-linear scalar field. The actual case considered in [20] was that of a potential with a non-degenerate positive local minimum, Einstein's vacuum equations with a positive cosmological constant being contained as a special case, and the resulting expansion being exponential. As a test of the framework of [20], and of the preconception that situations with accelerated expansion are stable, it is of interest to use it to prove stability in some other context. Here we study the behaviour in the case of an exponential potential. There are solutions of the corresponding equations on $\mathbb{R}_+ \times \mathbb{R}^n$ such that the metric is of the form

$$-dt^2 + t^{2p}\delta_{ij}dx^i \otimes dx^j, \tag{1}$$

where $p > 1$ is a real number, δ is the Kronecker delta and t and x^i are standard coordinates on $\mathbb{R}_+$ and $\mathbb{R}^n$ respectively. In other words, the expansion is of power law type, and in the limiting case, $p = 1$, it is not accelerated. One might thus expect the problem of proving stability to be harder in this setting, and, in fact, it is more difficult to analyze the behaviour of the solutions to the PDE's that result in the end. To our knowledge, the first author to study an exponential potential was Halliwell, cf. [7], who considered the spatially homogeneous and isotropic case. Later, the spatially homogeneous but non-isotropic case was studied in [10]. The question of stability in the case of $3+1$ dimensions has also been considered, see [9]. In [9], Heinzle and Rendall used the results of Michael Anderson on the stability of even dimensional de Sitter space, cf. [1], together with Kaluza Klein reduction techniques, in order to obtain stability of the metrics (1) and the corresponding scalar fields, for a discrete set of values of p converging to 1. It is of interest to note that the methods used in [1] avoid the problem of proving global existence of a system of PDE's by an intelligent and geometric choice of equations, see also [5]. In other words, the arguments used to prove the stability results of [9] are essentially geometric in flavour. In the present paper, the focus is rather on the analysis aspect, and though the perspective taken is less geometric, the results are more robust; we get stability in $n+1$ dimensions of the metrics (1) together with the corresponding scalar fields for any $p > 1$. We also formulate a result which makes local assumptions in space and yields global conclusions in time. From a conceptual point of view, this is the natural type of result to prove due to the extreme nature of the causal structure in the case of accelerated expansion. However, it is also very convenient in practice to have such a statement; combining it with the results concerning the asymptotic behaviour in the spatially homogeneous setting, we get a non-linear stability result for quite general spatially locally homogeneous solutions to the equations under consideration.

1.2. **Equations.** The subject of this paper is Einstein's equations, given by

$$G_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Sg_{\mu\nu},$$

$R_{\mu\nu}$ are the components of the Ricci tensor of a Lorentz metric g on an $n+1$-dimensional manifold M, and S is the associated scalar curvature. In this paper, we shall be interested in stress energy tensors of the form

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \left[\frac{1}{2}\nabla^\gamma\phi\nabla_\gamma\phi + V(\phi)\right]g_{\mu\nu}, \tag{3}$$

where ∇ is the Levi-Civita connection associated with the metric g, ϕ is a smooth function on M,

$$V(\phi) = V_0 e^{-\lambda\phi}, \tag{4}$$

and V_0 and λ are positive constants. We shall refer to the matter model defined by (3) as the *non-linear scalar field* model, to V as the *potential* and to ϕ as the *scalar field.* Note that in this situation, (2) is equivalent to

$$R_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi + \frac{2}{n-1}V(\phi)g_{\mu\nu}. \tag{5}$$

It should of course be coupled to a matter equation for ϕ, which is given by

$$\nabla^\mu\nabla_\mu\phi - V'(\phi) = 0. \tag{6}$$

Observe that this equation is a sufficient, but not necessary, condition for the stress energy tensor to be divergence free. We do, however, impose it. The system of equations of interest is thus (5)-(6).

1.3. **Initial value problem.** Concerning the system of equations under consideration, there is a natural initial value problem. The idea is to specify initial data that would correspond to the metric, second fundamental form, scalar field and normal derivative of the scalar field induced on a spacelike hypersurface in the Lorentz manifold one wishes to construct. However, in order for this to make sense, the initial data cannot be specified freely; they have to satisfy certain *constraint equations* that are implied by the Gauß and Codazzi equations, cf. [20] for more details.

Definition 1. *Initial data* for (5) and (6) consist of an n dimensional manifold Σ, a Riemannian metric h, a symmetric covariant 2-tensor k and two functions ϕ_a and ϕ_b on Σ, all assumed to be smooth and to satisfy

$$r - k_{ij}k^{ij} + (\mathrm{tr}_h k)^2 = \phi_b^2 + D^i\phi_a D_i\phi_a + 2V(\phi_a), \tag{7}$$

$$D^j k_{ji} - D_i(\mathrm{tr}_h k) = \phi_b D_i\phi_a, \tag{8}$$

where D is the Levi-Civita connection of h, r is the associated scalar curvature and indices are raised and lowered by h. Given initial data, the *initial value problem* is that of finding

- an $n+1$ dimensional manifold M with a Lorentz metric g and a $\phi \in C^\infty(M)$ such that (5) and (6) are satisfied, and
- an embedding $i : \Sigma \to M$

such that $i(\Sigma)$ is a Cauchy hypersurface in (M, g), $i^*g = h$, $\phi \circ i = \phi_a$, and if N is the future directed unit normal and κ is the second fundamental form of $i(\Sigma)$, then $i^*\kappa = k$ and $(N\phi) \circ i = \phi_b$. Such a triple (M, g, ϕ) is referred to as a *globally hyperbolic development* of the initial data, the existence of an embedding i being tacit.

Remark. A Cauchy hypersurface is a set in a Lorentz manifold which is intersected exactly once by every inextendible timelike curve, see [14] or [20] for more details. In the above definition, and below, we assume all Lorentz manifolds to be time oriented. One can of course define the concept of initial data and development for a lower degree of regularity. We shall, however, restrict our attention to the smooth case in this paper.

For results concerning the existence of initial data in the current setting, we refer the reader to [3] and [8].

Definition 2. Given initial data $(\Sigma, h, k, \phi_a, \phi_b)$ for (5) and (6), a *maximal globally hyperbolic development* of the data is a globally hyperbolic development (M, g, ϕ), with embedding $i : \Sigma \to M$, such that if (M', g', ϕ') is any other globally hyperbolic development of the same data, with embedding $i' : \Sigma \to M'$, then there is a map $\psi : M' \to M$ which is a diffeomorphism onto its image such that $\psi^* g = g'$, $\psi^* \phi = \phi'$ and $\psi \circ i' = i$.

Theorem 1. *Given initial data for (5) and (6), there is a maximal globally hyperbolic development of the data which is unique up to isometry.*

Remark. When we say that (M, g, ϕ) is unique up to isometry, we mean that if (M', g', ϕ') is another maximal globally hyperbolic development, then there is a diffeomorphism $\psi : M \to M'$ such that $\psi^* g' = g$, $\psi^* \phi' = \phi$ and $\psi \circ i = i'$, where i and i' are the embeddings of Σ into M and M' respectively.

The proof is as in [2]. This is an important result and will be of use to us in this paper. However, it does not yield any conclusions concerning e.g. causal geodesic completeness.

1.4. **Background solution.** The basic background solution we are interested in is (in Lemma 1 below, we shall prove that it is a solution)

$$g_0 = -dt^2 + e^{2K}(t/t_0)^{2p}\delta_{ij}dx^i \otimes dx^j, \tag{9}$$

$$\phi_0 = \frac{2}{\lambda}\ln t - \frac{1}{\lambda}c_0, \tag{10}$$

on $\mathbb{R}_+ \times \mathbb{T}^n$, where $\mathbb{R}_+ = (0, \infty)$, $t_0 > 0$, K and $p > 1$ are constants and

$$\lambda = \frac{2}{[(n-1)p]^{1/2}}, \tag{11}$$

$$c_0 = \ln\left[\frac{(n-1)(np-1)p}{2V_0}\right]. \tag{12}$$

Note that given the dimension n, there is a one to one correspondence between p and λ, and we shall prefer to specify p rather than λ. The above constructions make sense for $p > 1/n$, but in order for us to get accelerated expansion, we need to have $p > 1$.

Consider the metric (9) on $\mathbb{R}_+ \times \mathbb{R}^n$. Let h denote the Riemannian metric induced on $\{t_0\} \times \mathbb{R}^n$ by g_0 and let $\gamma : [0, T) \to \mathbb{R}_+ \times \mathbb{R}^n$ be a future directed causal curve with $\gamma(0) \in \{t_0\} \times \mathbb{R}^n$. Then, if $\gamma_\flat$ is the projection of γ to $\mathbb{R}^n$,

$$l_h[\gamma_\flat] := \int_0^T [h_{ij}\dot{\gamma}^i\dot{\gamma}^j]^{1/2} ds \leq \frac{t_0}{p-1},$$

where Latin indices run from 1 to n, a convention that will be used consistently in what follows, as well as the convention that Greek indices run from 0 to n. Furthermore, the indices used on $\mathbb{R}_+ \times \mathbb{T}^n$ and $\mathbb{R}_+ \times \mathbb{R}^n$ will be the ones associated with the standard frame $\partial_0 = \partial_t$ and ∂_i unless otherwise specified. As a consequence, if we define

$$\ell(t_0) := \frac{t_0}{p-1}, \tag{13}$$

then

$$J^+[\{t_0\} \times B_{\ell(t_0)}(\xi)] \subseteq D^+[\{t_0\} \times B_{3\ell(t_0)}(\xi)], \tag{14}$$

where $J^+(A)$ is the causal future of a set A and $D^+(A)$ is the future Cauchy development of a set A, cf. [14] or [20] for detailed definitions. This demonstrates that $\ell(t_0)$ is a fundamental length scale and, similarly to the case studied in [20], that if we want to control the behaviour of a solution to the linear wave equation (on $\mathbb{R}_+ \times \mathbb{T}^n$ with metric given by (9)) to the future of $\{t_0\} \times B_{\ell(t_0)}(\xi)$, then we only need to control the initial data on $\{t_0\} \times B_{3\ell(t_0)}(\xi)$. However, it also demonstrates that there is a difference between the case considered in the present paper and the case considered in [20]. In [20], the fundamental length scale was a *constant*, determined by the dimension and the minimum of the scalar field. In the present case, it depends on the starting time and tends to infinity with the starting time. As a consequence, the size of the ball over which it is necessary to have control in order to predict what happens along causal geodesics that start at the center tends to infinity with time. However, if we consider the above situation on $\mathbb{R}_+ \times \mathbb{T}^n$, then we see that the size of the torus grows even more rapidly if $p > 1$, so that the fraction of the volume of the torus that the ball constitutes tends to zero. Another problem that arises in the present setting is the fact that it is necessary to make a choice of t_0 given initial data $(\Sigma, \rho, \kappa, \phi_a, \phi_b)$. We shall here do so by using the relation (10), in which we shall replace ϕ_0 by the mean value of ϕ_a in the ball of interest, cf. Theorem 2 (in particular (15)) for a more precise statement.

1.5. **Results.** Before we state the main result, we need to introduce some terminology. Let Σ be an n dimensional manifold. We shall be interested in coordinate systems x on open subsets U of Σ such that $x : U \to B_1(0)$ is a diffeomorphism. If s is a tensor field on Σ, we shall use the notation

$$\|s\|_{H^l(U)} = \left(\sum_{i_1,\dots,i_q=1}^{n} \sum_{j_1,\dots,j_r=1}^{n} \sum_{|\alpha|\leq l} \int_{x(U)} |\partial^\alpha s^{i_1\cdots i_q}_{j_1\cdots j_r} \circ x^{-1}|^2 dx^1 \cdots dx^n \right)^{1/2},$$

where the components of s are computed with respect to x and the derivatives are with respect to x. When we write $\|s\|_{H^l(U)}$, we shall take it to be understood that there are coordinates x as above. Below, we shall use δ to denote the Kronecker delta with respect to the x coordinates. In particular, we shall use the notation

$$\|g - a\delta\|_{H^l(U)} = \left(\sum_{i,j=1}^{n} \sum_{|\alpha|\leq l} \int_{x(U)} |\partial^\alpha (g_{ij} - a\delta_{ij}) \circ x^{-1}|^2 dx^1 \cdots dx^n \right)^{1/2}.$$

Theorem 2. *Let V be given by (4), where V_0 is a positive number and λ is given by (11) in which $n \geq 3$ is an integer and $1 < p \in \mathbb{R}$. There is an $\epsilon > 0$, depending on n and p, such that if*

- *$(\Sigma, \rho, \kappa, \phi_a, \phi_b)$ are initial data for (5) and (6), with* $\dim\Sigma = n$,
- *$x : U \to B_1(0)$ is a diffeomorphism, where $U \subseteq \Sigma$,*
- *the objects $\langle\phi_a\rangle$, t_0 and K are defined by*

$$\langle\phi_a\rangle := \frac{1}{\omega_n} \int_{B_1(0)} \phi_a \circ x^{-1} dx, \ t_0 := \exp\left[\frac{1}{2}(\lambda\langle\phi_a\rangle + c_0)\right], \ K := \ln[4\ell(t_0)], \tag{15}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ with respect to the ordinary Euclidean metric, c_0 is defined in (12) and $\ell(t_0)$ is defined in (13), and
- *the inequality*

$$\begin{aligned} &\|e^{-2K}\rho-\delta\|_{H^{k_0+1}(U)}+\left\|e^{-2K}t_0\kappa-p\delta\right\|_{H^{k_0}(U)} \\ &+\|\phi_a-\phi_0(t_0)\|_{H^{k_0+1}(U)}+\|t_0\phi_b-t_0(\partial_t\phi_0)(t_0)\|_{H^{k_0}(U)} \leq \epsilon \end{aligned} \tag{16}$$

holds, where k_0 is the smallest integer satisfying $k_0 > n/2+1$,

then the maximal globally hyperbolic development (M, g, ϕ) of $(\Sigma, \rho, \kappa, \phi_a, \phi_b)$ has the property that if $i : \Sigma \to M$ is the associated embedding, then all causal geodesics that start in $i\{x^{-1}[B_{1/4}(0)]\}$ are future complete. Furthermore, there is a $t_- \in (0, t_0)$ and a smooth map,

$$\Psi : (t_-, \infty) \times B_{5/8}(0) \to M, \tag{17}$$

which is a diffeomorphism onto its image, such that all causal curves that start in $i\{x^{-1}[B_{1/4}(0)]\}$ remain in the image of Ψ to the future, and g and ϕ have expansions (18)-(23) in the solid cylinder $[0, \infty) \times B_{5/8}(0)$ when pulled back by Ψ. Finally, $\Psi(0, p) = i \circ x^{-1}(p)$ for $p \in B_{5/8}(0)$. In the formulas below, Latin indices refer to the natural Euclidean coordinates on $B_{5/8}(0)$ and t is the natural time coordinate on the solid cylinder. There is a positive constant α, a Riemannian metric χ on $B_{5/8}(0)$ and constants K_l such that if $\|\cdot\|_{C^l}$ denotes the C^l norm on $B_{5/8}(0)$, we have, for $t \geq t_0$,

$$\|\phi(t,\cdot)-\phi_0(t)\|_{C^l}+\|(t\partial_t\phi)(t,\cdot)-t\partial_t\phi_0(t)\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha} \tag{18}$$

$$\|(g_{00}+1)(t,\cdot)\|_{C^l}+\|(t\partial_t g_{00})(t,\cdot)\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha} \tag{19}$$

$$\begin{aligned} &\left\|t^{-1}g_{0i}(t,\cdot)-\frac{1}{np-2p+1}\chi^{jm}\gamma_{jim}\right\|_{C^l} \\ &+\|[t\partial_t(t^{-1}g_{0i})](t,\cdot)\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha} \end{aligned} \tag{20}$$

$$\begin{aligned} &\|(t/t_0)^{-2p}e^{-2K}g_{ij}(t,\cdot)-\chi_{ij}\|_{C^l} \\ &+\|(t/t_0)^{-2p}e^{-2K}t\partial_t g_{ij}(t,\cdot)-2p\chi_{ij}\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha}, \end{aligned} \tag{21}$$

$$\|(t/t_0)^{2p}e^{2K}g^{ij}(t,\cdot)-\chi^{ij}\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha}, \tag{22}$$

$$\|(t/t_0)^{-2p}e^{-2K}tk_{ij}(t,\cdot)-p\chi_{ij}\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha}, \tag{23}$$

where γ_{jim} are the Christoffel symbols associated with the metric χ and k is the second fundamental form of the hypersurfaces $\{t\} \times B_{5/8}(0)$.

Remark. Remarks similar to those made in connection with the analogous theorem in [20] remain valid and need not be repeated here. Let us simply point out that t_0 is chosen so that $\phi_0(t_0) = \langle\phi_a\rangle$, a choice which is essentially necessary, and that K is chosen so that the ball of radius 1 with respect to the x-coordinates roughly corresponds to a ball of radius $4\ell(t_0)$ with respect to ρ. The latter choice should be compared with (14); if we replace $3\ell(t_0)$ with $4\ell(t_0)$ on the right hand side, the inclusion still holds, but with a margin, so that the corresponding statement can be expected to hold in the MGHD's corresponding to perturbed initial data. Due to (22) and (23), we have, for $t \geq t_0$,

$$\|t(g^{ij}k_{jl})(t,\cdot)-p\delta^i_l\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha},$$

and in this sense, we have isotropization. The expansions are incomplete but with more work it should be possible to obtain more detailed information. In [9], more detailed asymptotic expansions were provided, though it should be pointed out that the foliation considered here differs from that considered in [9]. Note that as a consequence of Theorem 2 and Cauchy stability, cf. Theorem 7 of [20], we get future global non-linear stability of the solutions (9) and (10) on $\mathbb{R}_+ \times \mathbb{T}^n$ for $n \geq 3$, since we can apply Theorem 2 in a neighbourhood of every point at late enough times. The reason for this is that $[4\ell(t)]^{-2}e^{2K}(t/t_0)^{2p}$ tends to infinity, so that a ball in $\mathbb{T}^n$ of fixed positive radius $\epsilon > 0$ with respect to fixed coordinates will sooner or later contain a ball of radius $4\ell(t)$ with respect to the metric induced on $\{t\} \times \mathbb{T}^n$ by the metric g_0.

The proof of the above theorem is to be found in Section 11.

Let us consider the 4-dimensional spatially homogeneous case. In other words, let us restrict our attention to 3-dimensional initial data with a transitive isometry group. Due to the work of Kitada and Maeda, cf. [10], it is reasonable to hope that Theorem 2 will be applicable in a neighbourhood of every point on a late enough hypersurface of spatial homogeneity, with some exceptions. If Σ is $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or quotients thereof, then it is not clear that the corresponding solution needs to expand; it might recollapse. The reason for this is that $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{R}$ admit homogeneous metrics with positive scalar curvature. To simplify the statement, we shall thus exclude this possibility. Furthermore, we are only interested in the case that the isometry group admits a cocompact subgroup.

Theorem 3. *Let V be given by (4), where V_0 is a positive number and λ is given by (11) in which $n = 3$ and $p > 1$. Let M be a connected and simply connected 3-dimensional manifold and let $(M, h, k, \phi_a, \phi_b)$ be initial data for (5) and (6). Assume, furthermore, that one of the following conditions is satisfied:*

- *M is a unimodular Lie group different from $\mathrm{SU}(2)$ and the isometry group of the initial data contains the left translations.*
- *$M = \mathbb{H}^3$, where $\mathbb{H}^n$ is the n-dimensional hyperbolic space, and the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^3$.*
- *$M = \mathbb{H}^2 \times \mathbb{R}$ and the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^2 \times \mathbb{R}$.*

Assume finally that $\mathrm{tr}_h k > 0$. Let Γ be a cocompact subgroup of M in the case that M is a unimodular Lie group and a cocompact subgroup of the isometry group otherwise. Let Σ be the compact quotient. Then $(\Sigma, h, k, \phi_a, \phi_b)$ are initial data. Make a choice of Sobolev norms $\|\cdot\|_{H^l}$ on tensorfields on Σ. Then there is an $\epsilon > 0$ such that if $(\Sigma, \rho, \kappa, \varphi_a, \varphi_b)$ are initial data for (5) and (6) satisfying

$$\|\rho - h\|_{H^4} + \|\kappa - k\|_{H^3} + \|\varphi_a - \phi_a\|_{H^4} + \|\varphi_b - \phi_b\|_{H^3} \leq \epsilon,$$

then the maximal globally hyperbolic development corresponding to $(\Sigma, \rho, \kappa, \varphi_a, \varphi_b)$ is future causally geodesically complete and there are expansions of the form given in the statement of Theorem 2 to the future.

Remark. If M is a 3-dimensional unimodular Lie group it contains a cocompact subgroup Γ, cf. [15]. Concerning the definition of Sobolev norms on tensorfields on manifolds, we refer the reader to e.g. [20]. The statement that there are expansions to the future should be interpreted as saying that there is a Cauchy hypersurface Σ' in the maximal globally hyperbolic development of $(\Sigma, \rho, \kappa, \varphi_a, \varphi_b)$ such that for every $p \in \Sigma'$, there is a neighbourhood of p to which Theorem 2 applies. In [20], we made several comments that are equally relevant in the present context, but for the sake of brevity, we do not wish to repeat them here.

The proof of the above theorem is to be found in Section 12.

1.6. **Outline.** Let us start by discussing the proof of Theorem 2. Due to the nature of the causal structure, it is sufficient to study the future stability of the solutions given by (9)-(12) on $\mathbb{R}_+ \times \mathbb{T}^n$. The procedure leading to this reduction can briefly be described as follows. Given initial data and a diffeomorphism $x : U \to B_1(0)$ as described in the statement of Theorem 2, pull back the initial data to $B_1(0)$ by x^{-1}. Using a cut-off function and a suitable choice of t_0 and K, one can fit the initial data on $B_1(0)$ to the initial data on $\mathbb{T}^n$ corresponding to a $t = t_0$ slice of (9)-(12). The resulting data on $\mathbb{T}^n$ in general violate the constraints in an annular region. However, they are close to those of the $t = t_0$ slice of (9)-(12), and it is possible to demonstrate stability in the class of constraint violating data for a suitable modification of Einstein's equations, described below. Thus one obtains a solution to the modified equations which is global to the future. Furthermore, the Cauchy development of the part of $B_1(0)$ unaffected by the cut-off function yields a patch of spacetime corresponding to the original initial data. For the purposes of the present discussion, we shall refer to this patch as the *global patch*. The statements concerning future completeness of causal geodesics starting in $i\{x^{-1}[B_{1/4}(0)]\}$ and asymptotic expansions hold in the global patch. Constructing local patches corresponding to the other points of the original initial manifold, one obtains a globally hyperbolic development of the original initial data which includes the global patch. By the abstract properties of the MGHD of the initial data, this globally hyperbolic development can be embedded into the maximal globally hyperbolic one, and the statement of the theorem follows.

Due to the above observations, it is clear that the essential step of the argument is to prove future stability of the solutions defined by (9)-(12) in a situation where the constraints are violated. Such a result presupposes a hyperbolic formulation of the equations, which we provide in the beginning of Section 2. The formulation we use is based on gauge source functions, cf. [6], together with some additional modifications, cf. (24)-(25). The gauge source functions are chosen so that they coincide with the contracted Christoffel symbols of the background, the equality holding for upstairs indices, cf. (26). The main purpose of adding the modifications is that they make it possible to prove stability for data violating the constraints. However, the modifications, additionally, yield a partial decoupling at the linear level, which leads to a hierarchy we shall describe below, and they yield damping terms which are of crucial importance when proving stability. In the beginning of Section 2, we briefly discuss the hyperbolic formulation we shall use, the associated initial data and a division of the terms appearing into ones that have to be taken into account and ones that can, in the end, in practice be ignored. Readers interested in a more complete presentation are referred to [20]. After a discussion of the background solution, we then reformulate the equations. The first reformulation serves the purpose of expressing the equations in terms of quantities concerning which we have definite expectations; we subtract the background scalar field ϕ_0

from the scalar field ϕ and consider $\psi = \phi - \phi_0$, $u = g_{00} + 1$, $u_i = g_{0i}$ and $h_{ij} = (t/t_0)^{-2p} g_{ij}$. We expect ψ and u to converge to zero and h_{ij} to converge. Concerning u_i, it seems reasonable to expect that if we rescale it by a factor of t^{-p} (the logic being that every downstairs spatial index corresponds to a factor t^p), then the resulting object remains small or converges to zero. Thus, it might seem natural to carry out such a rescaling. However, in the case of u_i, we shall do this rescaling at the level of the energies, cf. Section 5. The resulting equations, (45)-(48), have a certain structure; considering the linear terms, it is clear that the terms involving zeroth order derivatives have a factor in front of them of the form of a constant divided by t^2, and the terms involving first order derivatives have a factor in front of them of the form of a constant divided by t. Consequently, it seems natural to multiply the equations with t^2 and to change the time coordinate so that $t\partial_t = \partial_\tau$ for some new time coordinate τ. This is the purpose of the second reformulation, which leads to the equations (61)-(64) with which we shall be working.

Starting with (61)-(64), one can generate a model problem by dropping the terms given by $\Delta_{\mu\nu}$ and Δ_ψ, and by replacing the wave operator $\hat{\Box}_g$ by the wave operator associated with the background. Considering (61)-(64) with these simplifications in mind, one sees that some of the equations partly decouple; the equations for u and ψ, (61) and (64), do not involve the remaining unknowns, and the equation for h_{ij}, (63), does not involve u_i. In other words, there is a hierarchy in the model problem. One can start by analyzing the model equations for u and ψ, then turn to the model equations for h_{ij}, and finally consider the equation for u_i. Even though this hierarchy does not persist in the non-linear case, some aspects of it remain and are of central importance in the proof of future global non-linear stability; given suitable bootstrap assumptions, the hierarchy does, for all practical purposes, persist. Given the structure of the hierarchy, it is natural to start by considering the model equations for u and ψ. Such an analysis is the subject of Section 3. It turns out that one can construct an energy which decays exponentially. For this to hold, one does, however, need to require that $n \geq 3$ and $p > 1$; for $p = 1$, there are constant, non-zero, solutions to the model equations. In Section 4, we write down the energies, not only for u and ψ, but also for h_{ij} and u_i, with which we shall be working in the non-linear setting, the construction in part being based on that of the model problem. We also derive the estimates for the time derivatives of the energies on which the bootstrap argument will be based.

In Section 5, we specify the bootstrap assumptions. There are two levels of assumptions. The first level consists of assumptions ensuring that g remains a Lorentz metric, with quantitative bounds, cf. Subsection 5.1. Thanks to this assumption, it is, among other things, possible to define the energies. The second level assumption consists of an upper bound for the energy, cf. Subsection 5.4.

The main tool for proving future global existence is the system of differential inequalities derived in Lemma 16 of Section 7. Corollaries 1 and 2 of Section 4 and the equations (61)-(64) constitute the starting point for the derivation. However, it is necessary to estimate the terms that are of higher order in the expressions that vanish on the background, cf. Lemma 11, to estimate the commutator terms that arise when applying spatial derivatives to the equations, cf. Lemma 13, and to estimate the remainder terms that appear in the estimates for the time derivatives of the energies in Corollaries 1 and 2, cf. Lemma 15. Section 6 is devoted to deriving the necessary estimates. All the estimates are of course based on the bootstrap assumptions, and deriving them requires an effort. However, applying general techniques developed in [20] leads to a significant reduction of the amount of work. Using these estimates, we then derive the system of differential inequalities in Section 7. The hierarchy mentioned above is apparent in this system. Disregarding the terms involving ϵ in (139)-(141) (the corresponding terms can be estimated using the bootstrap assumptions), it is clear that only $\hat{H}_{\mathrm{lp},k}$ appears on the right hand side of the differential inequality for $\hat{H}_{\mathrm{lp},k}$, cf. (139), so that one can improve the bootstrap assumptions for this quantity first. Considering (141), the second and third terms on the right hand side may appear hard to control. However, since it is possible to improve the bootstrap for $\hat{H}_{\mathrm{lp},k}$ to say that, not only is it small but it decays exponentially, the second and third terms on the right hand side of (141) do not constitute a problem. Finally, turning to (140), the second term on the right hand side can be controlled using the information already obtained concerning $\hat{H}_{\mathrm{lp},k}$ and $\hat{H}_{\mathrm{m},k}$. To conclude, it is of crucial importance to derive a *system* of differential inequalities; combining (139)-(141) into one differential inequality yields an estimate which does not appear to be very useful. In Section 8, we then prove future global existence of solutions corresponding to initial data on $\mathbb{T}^n$ close to those of a model solution. Note, however, that given initial data on $\mathbb{T}^n$, it is necessary to determine an initial time, since some of the unknowns, i.e. ψ and h_{ij}, depend on it. We carry out a discussion concerning how to achieve this in the beginning of Section 8. After the proof of global existence, we derive some basic conclusions; in the case of hyperbolic PDE's, it is natural to make smallness assumptions for a finite degree of regularity and then to draw conclusions for any degree of regularity, and a first step in this direction is taken in Theorem 5, following the proof of future global existence. In Section 9, we then carry out a rough analysis of the causal structure. This analysis yields information concerning the future Cauchy development of subsets of the initial data, which is of crucial importance when carrying out the arguments described at the beginning of the present subsection. Furthermore, we prove future causal geodesic completeness. In Section 10, we derive asymptotic expansions for the solution and in Section 11 we prove the main theorem along the lines described above. The spatially homogeneous solutions of interest were already analyzed in [10], but the perspective taken here is somewhat different. Furthermore we need somewhat more detailed knowledge concerning the asymptotics, and consequently, we discuss the spatially homogeneous solutions in detail in Section 12. Note, however, that the results of [10] cover a much more general situation than we discuss in the present paper. At the end of Section 12 we then prove Theorem 3.

Let us comment on the differences and the similarities between the situation studied in [20] and the one studied in the present paper. The main purpose of [20] was to build a framework for proving future global non-linear stability in the Einstein-non-linear scalar field setting. In particular, specific choices of gauge source functions and corrections to the equation were made that work equally well for the case studied in [20] as for the case studied here. Furthermore, in [20], we wrote down bootstrap assumptions as well as a partial division of the terms appearing in the equations, separating out the ones of higher order. Finally, and perhaps most importantly, we constructed an algorithm yielding estimates for the non-linear terms given that the bootstrap assumptions hold, the advantage of the algorithm being that in order to estimate a term in H^k, it is enough to make simple computations such as counting the number of downstairs spatial indices minus the number of upstairs spatial indices. All of these constructions carry over, and will be very useful in the present situation. On the other hand, the actual PDE problems that result are quite different in the different cases. In the case of a potential with a positive non-degenerate minimum, the background scalar field is zero, but in the case of an exponential potential, the scalar field tends to infinity as $t \to \infty$. As described above, it is thus, in the case of an exponential potential, necessary to subtract the background solution. The process of doing so introduces couplings between the equations for the scalar field and the different components of the metric, even on the linear level, and this makes the resulting equations harder to analyze. Above, we discussed the equations for u and ψ that result after having dropped the terms that are quadratic in the quantities that vanish on the background and after having changed the coefficients of the highest order derivatives to those corresponding to the background. In particular, we noted that these equations are coupled, and it turns out that finding an energy that decays exponentially does require an effort. If one were to consider the corresponding equations for u and ψ in the case studied in [20], one would see that the equations for u and ψ decouple, and that one easily obtains exponential decay for both of them separately. To sum up, there are several aspects concerning the general set up of the equations and the general methods for estimating the non-linearity that are common to the analysis carried out in [20] and the analysis carried out here. However, the actual PDE problems that one has to deal with in the end are quite different, the present one being the more difficult.

Finally, let us note that in the outline of the proof of the theorem in [20] corresponding to Theorem 2 in the present paper, we motivated the choice of gauge source functions, the choice of corrections, and we made comparisons between our method and the methods used by Lindblad and Rodnianski to prove the stability of Minkowski space in [12] and [13] (simplifying the original proof by Christodoulou and Klainerman [4], though not obtaining as detailed asymptotics). As a consequence, we shall not do so here.

2. Reformulation of the equations on $\mathbb{T}^n$

As we pointed out in the outline, the central problem in the proof of Theorem 2 is that of proving future global non-linear stability of the solutions (9)-(12) on $\mathbb{R}_+ \times \mathbb{T}^n$. In [20], we considered (5) and (6) in the context of perturbations around metrics of the form

$$-dt^2 + e^{2\Omega}\delta_{ij}dx^i \otimes dx^j$$

on $\mathbb{R}_+ \times \mathbb{T}^n$. Thus the problem we are interested in here fits exactly into the general framework developed in [20], provided we choose $\Omega = p\ln t + K - p\ln t_0$ (below, we shall, for various reasons, make a somewhat different choice). As in [20], we shall use the notation $\omega = \dot{\Omega}$, so that $\omega = p/t$. The choice of equations, the relevant estimates for the non-linearity etc. then follow from [20]. Consequently, we shall consider the equations

$$\hat{R}_{\mu\nu} - \nabla_\mu\phi\nabla_\nu\phi - \frac{2}{n-1}V(\phi)g_{\mu\nu} + M_{\mu\nu} = 0, \tag{24}$$

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi - \Gamma^\mu\partial_\mu\phi - V'(\phi) + M_\phi = 0, \tag{25}$$

cf. (53) and (54) of [20], where all the indices are with respect to the standard vectorfields on $\mathbb{R}_+ \times \mathbb{T}^n$, i.e. $\partial_0 = \partial_t$, $\partial_i = \partial_{x^i}$ for $i = 1, ..., n$, if x^i are the standard "coordinates" on $\mathbb{T}^n$. Here

$$\mathcal{D}_\mu = F_\mu - \Gamma_\mu, \quad \hat{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{(\mu}\mathcal{D}_{\nu)}, \quad \nabla_{(\mu}\mathcal{D}_{\nu)} = \frac{1}{2}(\partial_\mu \mathcal{D}_\nu + \partial_\nu \mathcal{D}_\mu) - \Gamma^\alpha_{\mu\nu}\mathcal{D}_\alpha,$$

and

$$F_\mu = n\omega g_{0\mu}, \quad M_{00} = 2\omega g^{0\mu}(\Gamma_\mu - F_\mu), \quad M_{0i} = -2\omega(\Gamma_i - n\omega g_{0i}), \tag{26}$$

$$M_{ij} = 0, \quad M_\phi = g^{\mu\nu}(\Gamma_\mu - F_\mu)\partial_\nu \phi. \tag{27}$$

The equations (24) and (25) imply a homogeneous wave equation for $\mathcal{D}_\mu$, cf. (56) and (57) of [20]. If the initial data satisfy the constraints and one sets up the initial for the equations (24) and (25) in the correct way, the initial data for $\mathcal{D}_\mu$ vanish. This leads to the conclusion that $\mathcal{D}_\mu = 0$ where the solution is defined. As a consequence, we obtain a solution to (5) and (6). For more details on this argument, the reader is referred to [20], cf., in part, Proposition 1.

2.1. **Initial data.** The initial data for (24) and (25) are not completely determined by initial data for (5) and (6). However, part of the corresponding freedom has to be used to ensure that $\mathcal{D}_\mu = 0$ initially. In practice, we shall be interested in initial data that do not satisfy the constraint equations on the entire initial manifold. We shall thus assume that we are given $(\varrho, \varsigma, \Phi_a, \Phi_b)$ on $\mathbb{T}^n$, where ϱ is a Riemannian metric, ς is a symmetric covariant 2-tensor and Φ_a, Φ_b are smooth functions on $\mathbb{T}^n$. Furthermore, we shall assume that (7) and (8) are satisfied on $S \subseteq \mathbb{T}^n$ (with (h, k, ϕ_a, ϕ_b) replaced by $(\varrho, \varsigma, \Phi_a, \Phi_b)$). Starting with these initial data, we construct initial data for (24) and (25) as in [20]:

$$g_{ij}(t_0, \cdot) = \varrho(\partial_i, \partial_j), \tag{28}$$

$$g_{00}(t_0, \cdot) = -1, \quad g_{0i}(t_0, \cdot) = 0, \tag{29}$$

for $i, j = 1, ..., n$, cf. (58) and (59) of [20]. Due to this choice, the future directed unit normal to the hypersurface $t = t_0$ is ∂_t. Note, furthermore, that this fixes $F_\mu(t_0, \cdot)$, cf. (26). Concerning the first time derivatives, we choose

$$\partial_0 g_{ij}(t_0, \cdot) = 2\varsigma(\partial_i, \partial_j), \tag{30}$$

$$\partial_0 g_{00}(t_0, \cdot) = -2F_0(t_0, \cdot) - 2\mathrm{tr}\varsigma, \tag{31}$$

$$\partial_0 g_{0l}(t_0, \cdot) = \left[-F_l + \frac{1}{2}g^{ij}(2\partial_i g_{jl} - \partial_l g_{ij})\right](t_0, \cdot), \tag{32}$$

cf. (60), (62) and (63) of [20] respectively. Due to these choices, $\mathcal{D}_\mu(t_0, \cdot) = 0$. Concerning ϕ, we require

$$\phi(t_0, \cdot) = \Phi_a, \quad (\partial_t \phi)(t_0, \cdot) = \Phi_b, \tag{33}$$

cf. (61) of [20], since ∂_t is the future directed unit normal to $\{t_0\} \times \mathbb{T}^n$.

With these initial data, we get a local existence and uniqueness result. Furthermore, we get a continuation criterion and the conclusion that (5) and (6) are satisfied in $D(\{t_0\} \times S)$, where D signifies the Cauchy development (for a definition of Cauchy development, see [14] or [20]). For an exact statement, cf. Proposition 1 of [20].

2.2. **Equations.** To conclude, we consider the equations

$$\hat{R}_{00} + 2\omega\Gamma^0 - 2n\omega^2 - \nabla_0\phi\nabla_0\phi - \frac{2}{n-1}V(\phi)g_{00} = 0, \tag{34}$$

$$\hat{R}_{0i} - 2\omega(\Gamma_i - n\omega g_{0i}) - \nabla_0\phi\nabla_i\phi - \frac{2}{n-1}V(\phi)g_{0i} = 0, \tag{35}$$

$$\hat{R}_{ij} - \nabla_i\phi\nabla_j\phi - \frac{2}{n-1}V(\phi)g_{ij} = 0, \tag{36}$$

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi - n\omega\partial_0\phi - V'(\phi) = 0. \tag{37}$$

In order to analyze what terms are relevant and what terms are irrelevant in the expressions for $\hat{R}_{\mu\nu} + M_{\mu\nu}$, one can use the results of [20]. Combining Lemma 4, Lemma 6 and (88) of [20], we obtain

$$\begin{aligned}
\hat{R}_{00} + 2\omega\Gamma^0 - 2n\omega^2 &= -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{00} + \frac{1}{2}(n+2)\omega\partial_0 g_{00} \\
&\quad + n(\dot{\omega} + \omega^2)g_{00} + n\omega^2(g_{00}+1) \\
&\quad + \Delta_{A,00} + \Delta_{C,00},
\end{aligned} \tag{38}$$

$$\begin{aligned}
\hat{R}_{0m} - 2\omega(\Gamma_m - n\omega g_{0m}) &= -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{0m} + \frac{1}{2}n\omega\partial_0 g_{0m} \\
&\quad + \left[2(n-1)\omega^2 + \frac{1}{2}n\dot{\omega}\right]g_{0m} - \omega g^{ij}\Gamma_{imj} \\
&\quad + \Delta_{A,0m} + \Delta_{C,0m},
\end{aligned} \tag{39}$$

$$\begin{aligned}
\hat{R}_{ij} &= -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{ij} + \frac{1}{2}n\omega\partial_0 g_{ij} + 2\omega g^{00}\partial_0 g_{ij} \\
&\quad - 2\omega^2 g^{00}g_{ij} + \Delta_{A,ij},
\end{aligned} \tag{40}$$

where the higher order terms $\Delta_{A,\mu\nu}$, $\Delta_{C,\mu\nu}$ are defined in (87), (92) and (93) of [20]. The point of these expressions is that $\Delta_{A,\mu\nu}$ and $\Delta_{C,\mu\nu}$ are sums of terms that are quadratic in factors that vanish for the background solution.

2.3. **Background solution, revisited.** Before we proceed, let us prove that the basic solution around which we are perturbing actually is a solution.

Lemma 1. *Let $n \geq 3$, $p > 1$, $V_0 > 0$ and define λ, c_0 and V by (11), (12) and (4) respectively. Then the metric g_0, given by (9), and the function ϕ_0, given by (10), on $\mathbb{R}_+ \times \mathbb{T}^n$ satisfy (5) and (6). In particular, ϕ_0 satisfies the equation*

$$\ddot{\phi}_0 + n\omega\dot{\phi}_0 + V'(\phi_0) = 0. \tag{41}$$

Proof. One can compute that for g_0 given in (9), we have $\Gamma^0 = n\omega$, where $\omega = p/t$, and $\Gamma^i = 0$. In other words, F_μ defined in (26) coincides with Γ_μ, so that for g_0, $\hat{R}_{\mu\nu} = R_{\mu\nu}$ and the modifications $M_{\mu\nu}$ and M_ϕ vanish. Note also that $\Delta_{A,\mu\nu} = 0$ for the metric under consideration, and that $\Delta_{C,00} = \Delta_{C,0m} = 0$, cf. [20] (note that this is clear due to the idea behind the definition of these quantities). The 00 component of (5) is thus, due to (34) and (38), equivalent to

$$-n\dot{\omega} - n\omega^2 - \dot{\phi}_0^2 + \frac{2V(\phi_0)}{n-1} = 0. \tag{42}$$

Since ϕ_0 only depends on t, the $0m$ equations are automatically satisfied and the ij equations are equivalent to

$$\dot{\omega} + n\omega^2 - \frac{2V(\phi_0)}{n-1} = 0. \tag{43}$$

With ϕ_0 as in (10), equation (43) is equivalent to

$$-p + np^2 - \frac{2V_0 e^{c_0}}{n-1} = 0,$$

which is equivalent to

$$e^{c_0} = \frac{(n-1)(np-1)p}{2V_0},$$

which holds due to (12). In particular,

$$\frac{2V(\phi_0)}{n-1} = \frac{p(np-1)}{t^2}. \tag{44}$$

Using this information, (42) is equivalent to

$$np - np^2 - \frac{4}{\lambda^2} + p(np-1) = 0.$$

In other words, (11) implies (42). Thus (5) is satisfied. To check that ϕ_0 satisfies the last equation, which in the current situation is equivalent to (37), is simply a computation. Since (37) is equivalent to (6) for the metric under consideration, the lemma follows. □

2.4. **Linear algebra.** Before reformulating the equations, let us introduce some terminology concerning Lorentz matrices. Let g be a real valued $(n+1)\times(n+1)$-matrix with components $g_{\mu\nu}$. We shall denote the matrix with components g_{ij}, $i,j = 1,...,n$ by $g_\flat$, denote the vector with components g_{0i} by $v[g]$ and denote $g_{00}+1$ by $u[g]$. If g is symmetric and has one negative and n positive eigenvalues, we shall say that g is a *Lorentz matrix*. In case g is an invertible $(n+1)\times(n+1)$ matrix, we shall let $g^{\mu\nu}$ denote the components of the inverse and we shall let $g^\sharp$ denote the matrix with components g^{ij}, $i,j = 1,...,n$. It is of interest to note the following, cf. Lemma 1 and 2 of [20].

Lemma 2. *Let h be a symmetric $(n+1)\times(n+1)$ real valued matrix. Assume that $u[h] < 1$ and that $h_\flat$ is positive definite. Then h is a Lorentz matrix, $h^\sharp$ is positive definite and $u[h^{-1}] < 1$.*

Remark. Below, we shall sometimes use the notation $h_\flat > 0$ to indicate that $h_\flat$ is positive definite.

Definition 3. A *canonical Lorentz matrix* is a symmetric $(n+1)\times(n+1)$-dimensional real valued matrix g such that $u[g] < 1$ and $g_\flat > 0$. Let $\mathcal{C}_n$ denote the set of $(n+1)\times(n+1)$-dimensional canonical Lorentz matrices.

Note that, due to Lemma 2, the inverse of an element of $\mathcal{C}_n$ is in $\mathcal{C}_n$.

2.5. **First reformulation of the equations.** Since the background scalar field ϕ_0 tends to infinity as $t \to \infty$, it seems natural to reformulate the equations in terms of $\psi = \phi - \phi_0$. Furthermore, since the 00- and ij-components of the background metric are -1 and $e^{2K}(t/t_0)^{2p}\delta_{ij}$ respectively, it seems natural to consider $u = g_{00} + 1$ and $h_{ij} = (t/t_0)^{-2p} g_{ij}$. Isolating terms that involve, at worst (in terms of number of derivatives) first order derivatives of the unknowns and are quadratic in quantities that vanish on the background, we obtain the following reformulation.

Lemma 3. *Let $V_0 > 0$, $p > 1$ and let $n \geq 3$ be an integer. Define λ by (11), V by (4) and let ϕ_0 be given by the right hand side of (10), where c_0 is given by (12). Finally, fix $0 < t_0 \in \mathbb{R}$ and let U be an open subset of $\mathbb{R}_+ \times \mathbb{T}^n$. Then the following statements are equivalent:*

- *the functions g and ϕ, with values in $\mathcal{C}_n$ and $\mathbb{R}$ respectively, are C^∞ and satisfy (34)-(37) on U,*
- *the functions $\psi = \phi - \phi_0$, $u = g_{00} + 1$, $u_i = g_{0i}$, $h_{ij} = (t/t_0)^{-2p} g_{ij}$ $(i, j = 1, ..., n)$ are C^∞, where $u < 1$ and h_{ij} are the components of a positive definite metric, and satisfy*

$$-g^{\mu\nu}\partial_\mu\partial_\nu u + (n+2)\omega\partial_0 u + \frac{\beta_1}{t^2}u - \frac{8}{\lambda t}\partial_0\psi - \frac{2\lambda p(np-1)}{t^2}\psi + \tilde{\Delta}_{00} = 0, \tag{45}$$

$$-g^{\mu\nu}\partial_\mu\partial_\nu u_i + n\omega\partial_0 u_i + \frac{\beta_2}{t^2}u_i - 2\omega g^{lm}\Gamma_{lim} - \frac{4}{\lambda t}\partial_i\psi + \tilde{\Delta}_{0i} = 0, \tag{46}$$

$$-g^{\mu\nu}\partial_\mu\partial_\nu h_{ij} + n\omega\partial_0 h_{ij} + \left[-\frac{2p}{t^2}u + \frac{2\lambda p(np-1)}{t^2}\psi\right] h_{ij} + \tilde{\Delta}_{ij} = 0, \tag{47}$$

$$-g^{\mu\nu}\partial_\mu\partial_\nu\psi + n\omega\partial_0\psi + \frac{2(np-1)}{t^2}\psi - \frac{2}{\lambda t^2}u + \tilde{\Delta}_\psi = 0 \tag{48}$$

on U, where $\omega = p/t$, $\beta_1 = 2p[n(p-1)+1]$ and $\beta_2 = p(n-2)(2p-1)$. Furthermore, $\tilde{\Delta}_{00}$, $\tilde{\Delta}_{0i}$, $\tilde{\Delta}_{ij}$ and $\tilde{\Delta}_\psi$ are defined by (51), (52), (55) and (56) respectively.

Remark. Given u, u_i and h_{ij}, one can construct $g_{\mu\nu}$ and thereby $g^{\mu\nu}$. Note that the equivalence presupposes that t_0 has been fixed. Recall that $\mathcal{C}_n$ was defined in Definition 3. It is of interest to note that (45)-(48) are independent of V_0; an expression of the form $V_0 e^{-\lambda\phi_0}$ appears in $\Delta_{E,\phi}$, cf. (49), but this expression is independent of V_0 due to (44). On the other hand, it is necessary to know V_0 in order to be able to reconstruct ϕ from ψ.

Proof. Note that

$$e^{-\lambda\phi} = e^{-\lambda\phi_0}(e^{-\lambda\psi} - 1 + \lambda\psi) + e^{-\lambda\phi_0}(1 - \lambda\psi).$$

Since the first term is quadratic in ψ, which vanishes on the background, we define

$$\Delta_{E,\phi} = V_0 e^{-\lambda\phi_0}(e^{-\lambda\psi} - 1 + \lambda\psi). \tag{49}$$

With this notation, we can write

$$\frac{2V(\phi)}{n-1} = \frac{p(np-1)}{t^2}(1 - \lambda\psi) + \frac{2\Delta_{E,\phi}}{n-1},$$

cf. (44). We thus have (in this proof, we shall use the notation $\dot{f} = \partial_t f$)

$$\begin{aligned}\dot{\phi}^2 + \frac{2V(\phi)}{n-1}g_{00} &= \frac{4}{\lambda^2 t^2} - \frac{p(np-1)}{t^2} \\ &+\frac{4}{\lambda t}\dot{\psi} + \frac{p(np-1)}{t^2}(g_{00}+1) + \frac{\lambda p(np-1)}{t^2}\psi + \Delta_{\phi,00},\end{aligned}$$

where

$$\Delta_{\phi,00} = \dot{\psi}^2 - \frac{\lambda p(np-1)}{t^2}\psi(g_{00}+1) + \frac{2\Delta_{E,\phi}}{n-1}g_{00}. \tag{50}$$

Before we reformulate (34), let us note that, due to (38),

$$\begin{aligned}\hat{R}_{00} + 2\omega\Gamma^0 - 2n\omega^2 &= -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{00} + \frac{1}{2}(n+2)\omega\partial_0 g_{00} + n(\dot{\omega}+2\omega^2)(g_{00}+1) \\ &-n(\dot{\omega}+\omega^2) + \Delta_{A,00} + \Delta_{C,00}.\end{aligned}$$

Since

$$-n(\dot{\omega}+\omega^2) - \frac{4}{\lambda^2 t^2} + \frac{p(np-1)}{t^2} = 0,$$

cf. (42) and (44), we get

$$\begin{aligned}\hat{R}_{00} + 2\omega\Gamma^0 - 2n\omega^2 - \dot{\phi}^2 - \frac{2V(\phi)}{n-1}g_{00} &= -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{00} + \frac{1}{2}(n+2)\omega\partial_0 g_{00} \\ &+\left[n(\dot{\omega}+2\omega^2) - \frac{p(np-1)}{t^2}\right](g_{00}+1) \\ &-\frac{4}{\lambda t}\dot{\psi} - \frac{\lambda p(np-1)}{t^2}\psi + \frac{1}{2}\tilde{\Delta}_{00},\end{aligned}$$

where

$$\tilde{\Delta}_{00} = 2\Delta_{A,00} + 2\Delta_{C,00} - 2\Delta_{\phi,00}. \tag{51}$$

Thus (34) is equivalent to (45). By similar arguments, using (39), (35) is equivalent to (46), where

$$\tilde{\Delta}_{0i} = 2\Delta_{A,0i} + 2\Delta_{C,0i} - 2\partial_t\psi\partial_i\psi + \frac{2p(np-1)}{t^2}\lambda\psi g_{0i} - \frac{4\Delta_{E,\phi}}{n-1}g_{0i}. \tag{52}$$

Using (40), equation (36) can be reformulated to

$$\begin{aligned}&-g^{\mu\nu}\partial_\mu\partial_\nu g_{ij} + (n+4g^{00})\omega\partial_0 g_{ij} \\ &\quad -2\left[2\omega^2 g^{00} + \frac{p(np-1)}{t^2} - \frac{\lambda p(np-1)}{t^2}\psi\right]g_{ij} + \hat{\Delta}_{ij} = 0,\end{aligned}$$

where

$$\hat{\Delta}_{ij} = 2\Delta_{A,ij} - \frac{4\Delta_{E,\phi}}{n-1}g_{ij} - 2\partial_i\psi\partial_j\psi. \tag{53}$$

We wish to reformulate this equation in terms of $h_{ij} = (t/t_0)^{-2p}g_{ij}$. Note that

$$(t/t_0)^{-2p}\partial_0 g_{ij} = \partial_0 h_{ij} + 2\omega h_{ij},\ (t/t_0)^{-2p}\partial_0^2 g_{ij} = \partial_0^2 h_{ij} + 4\omega\partial_0 h_{ij} + \frac{2p(2p-1)}{t^2}h_{ij}.$$

Using

$$g^{00} + 1 = -(g_{00}+1) + \frac{1}{g_{00}}[(g_{00}+1)^2 - g^{0i}g_{0i}], \tag{54}$$

we conclude that (36) is equivalent to (47) where

$$\tilde{\Delta}_{ij} = -4g^{0l}\omega\partial_l h_{ij} + \frac{2p}{t^2}\frac{1}{g_{00}}[(g_{00}+1)^2 - g^{0l}g_{0l}]h_{ij} + (t/t_0)^{-2p}\hat{\Delta}_{ij} \tag{55}$$

and $\hat{\Delta}_{ij}$ is given by (53). Finally, let us turn to (37). Note that

$$V'(\phi) = -\lambda V_0 e^{-\lambda\phi_0} + \lambda^2 V_0 e^{-\lambda\phi_0}\psi - \lambda\Delta_{E,\phi},$$

so that (37) is equivalent to

$$-g^{\mu\nu}\partial_\mu\partial_\nu\psi + n\omega\partial_0\psi + \frac{2(np-1)}{t^2}\psi - (g^{00}+1)\partial_0^2\phi_0 - \lambda\Delta_{E,\phi} = 0,$$

where we have used the fact that ϕ_0 satisfies (41). Due to (54), (37) is equivalent to (48), where

$$\tilde{\Delta}_\psi = -\frac{1}{g_{00}}[(g_{00}+1)^2 - g^{0i}g_{0i}]\partial_0^2\phi_0 - \lambda\Delta_{E,\phi}. \tag{56}$$

The lemma follows. □

2.6. **Second reformulation of the equations.** Consider (45). All the terms on the left hand side but the first and the last have a certain structure: terms involving $\partial_t u$ and $\partial_t\psi$ are multiplied by a factor in the form of a constant divided by t (recall that $\omega = p/t$) and terms involving u and ψ are multiplied by a factor in the form of a constant divided by t^2. Similar comments can be made concerning the remaining equations (46)-(48). Consequently, in order to minimize the number of time dependent coefficients, it seems natural to multiply the equations with t^2 and to change the time coordinate to τ, where τ is such that $\partial_\tau = t\partial_t$.

Lemma 4. *Let $V_0 > 0$, $p > 1$ and let $n \geq 3$ be an integer. Define λ by (11), V by (4) and let ϕ_0 be given by the right hand side of (10), where c_0 is given by (12). Fix $0 < t_0 \in \mathbb{R}$, let the time coordinate τ be defined by $\tau = \ln(t/t_0)$, τ_0 be defined by $\tau_0 = \ln t_0$, and let U be an open subset of $\mathbb{R}_+ \times \mathbb{T}^n$. Finally, let $\Upsilon : \mathbb{R}_+ \times \mathbb{T}^n \to \mathbb{R} \times \mathbb{T}^n$ be defined by $\Upsilon(t,x) = [\ln(t/t_0), x]$. Then the following statements are equivalent:*

- *the functions g and ϕ, with values in $\mathcal{C}_n$ and $\mathbb{R}$ respectively, are C^∞ and satisfy (34)-(37) on U,*
- *the functions h_{ij}, u_i, u and ψ ($i,j = 1,...,n$) defined by*

$$\begin{aligned}
h_{ij}(\tau,x) &= e^{-2p\tau}g_{ij}(e^{\tau+\tau_0},x), && (57)\\
u_i(\tau,x) &= g_{0i}(e^{\tau+\tau_0},x), && (58)\\
u(\tau,x) &= g_{00}(e^{\tau+\tau_0},x)+1, && (59)\\
\psi(\tau,x) &= \phi(e^{\tau+\tau_0},x) - \phi_0(e^{\tau+\tau_0}) && (60)
\end{aligned}$$

are C^∞, where $u < 1$ and h_{ij} are the components of a positive definite metric, and satisfy

$$\hat{\Box}_g u + \alpha_1 \partial_\tau u + \beta_1 u - \frac{8}{\lambda}\partial_\tau \psi - 2\lambda p(np-1)\psi + \Delta_{00} = 0, \tag{61}$$

$$\hat{\Box}_g u_i + \alpha_2 \partial_\tau u_i + \beta_2 u_i - 2pe^{\tau+\tau_0} g^{lm}\Gamma_{lim} - \frac{4e^{\tau+\tau_0}}{\lambda}\partial_i \psi + \Delta_{0i} = 0, \tag{62}$$

$$\hat{\Box}_g h_{ij} + (np-1)\partial_\tau h_{ij} + [-2pu + 2\lambda p(np-1)\psi] h_{ij} + \Delta_{ij} = 0, \tag{63}$$

$$\hat{\Box}_g \psi + (np-1)\partial_\tau \psi + 2(np-1)\psi - \frac{2}{\lambda} u + \Delta_\psi = 0 \tag{64}$$

on $\Upsilon(U)$, where

$$\hat{\Box}_g = -g^{00}\partial_\tau^2 - 2e^{\tau+\tau_0} g^{0i}\partial_\tau \partial_i - e^{2(\tau+\tau_0)} g^{ij}\partial_i \partial_j, \tag{65}$$

$\alpha_1 = (n+2)p - 1$, $\beta_1 = 2p[n(p-1)+1]$, $\alpha_2 = np - 1$, $\beta_2 = p(n-2)(2p-1)$ *and Δ_{00}, Δ_{0i}, Δ_{ij} and Δ_ψ are given by (66)-(69).*

Remark. From time to time, we shall abuse notation by writing $g_{ij}(\tau, x)$ when $g_{ij}(e^{\tau+\tau_0}, x)$ would be the correct expression etc. Note that the functions h_{ij} etc. are different from the ones of the previous lemma, the difference amounting to a change of time coordinate.

Proof. Note that

$$t^2 \partial_0^2 f = -\partial_\tau f + \partial_\tau^2 f.$$

The conclusions follow by straightforward computations, and we have

$$\Delta_{00} = (g^{00}+1)\partial_\tau u + e^{2(\tau+\tau_0)}\tilde{\Delta}_{00}, \tag{66}$$

$$\Delta_{0i} = (g^{00}+1)\partial_\tau u_i + e^{2(\tau+\tau_0)}\tilde{\Delta}_{0i}, \tag{67}$$

$$\Delta_{ij} = (g^{00}+1)\partial_\tau h_{ij} + e^{2(\tau+\tau_0)}\tilde{\Delta}_{ij}, \tag{68}$$

$$\Delta_\psi = (g^{00}+1)\partial_\tau \psi + e^{2(\tau+\tau_0)}\tilde{\Delta}_\psi, \tag{69}$$

where $\tilde{\Delta}_{00}$, $\tilde{\Delta}_{0i}$, $\tilde{\Delta}_{ij}$ and $\tilde{\Delta}_\psi$ are defined by (51), (52), (55) and (56) respectively. The lemma follows. □

3. Model problem

As was discussed in the introduction, the system of equations (61)-(64) has, in a certain sense, a hierarchical structure; dropping the $\Delta_{\mu\nu}$ and Δ_ψ terms and changing the coefficients of the highest order derivatives to those of the background, the equations for u and ψ involve neither u_i nor h_{ij} and the equations for h_{ij} do not involve u_i. As has been mentioned, this structure will be of essential importance in the bootstrap argument used to prove future global existence. As a consequence of the structure of the hierarchy, a natural problem to consider is that of proving decay of solutions to the resulting model equations for u and ψ, cf. (70) and (71) below. In order for the analysis to be of use in the non-linear setting, it is preferable to prove decay by constructing a decaying energy; arguments based on energies tend to be more robust. The purpose of the present section is to construct such an energy.

3.1. **Model equations.** If we consider (61) and (64), ignore the higher order terms and replace the metric with the background metric, i.e. if we assume $g^{00} = -1$, $g^{0i} = 0$ and $g^{ij} = (t/t_0)^{-2p}\delta^{ij}$ and assume, for the sake of simplicity, $t_0 = 1$, we obtain the equations

$$u_{\tau\tau} - e^{-2H\tau}\Delta u + \alpha_1 u_\tau + \beta_1 u + \gamma_1 \psi_\tau + \delta_1 \psi = 0, \tag{70}$$

$$\psi_{\tau\tau} - e^{-2H\tau}\Delta \psi + \beta_3 u + \gamma_3 \psi_\tau + \delta_3 \psi = 0. \tag{71}$$

Here $H = p - 1$ and

$$\alpha_1 = (n+2)p - 1, \quad \beta_1 = 2p[n(p-1)+1], \quad \gamma_1 = -\frac{8}{\lambda}, \tag{72}$$

$$\delta_1 = -2\lambda p(np-1), \quad \beta_3 = -\frac{2}{\lambda}, \quad \gamma_3 = np - 1, \quad \delta_3 = 2(np-1), \tag{73}$$

where $n \geq 3$, $p > 1$ and λ is given by (11). Let us define

$$A = \begin{pmatrix} \beta_1 & \delta_1 \\ \beta_3 & \delta_3 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha_1 & \gamma_1 \\ 0 & \gamma_3 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ \psi \end{pmatrix}. \tag{74}$$

Then (70) and (71) can be written

$$\mathbf{u}_{\tau\tau} - e^{-2H\tau}\Delta\mathbf{u} + C\mathbf{u}_\tau + A\mathbf{u} = 0. \tag{75}$$

Let T be an invertible 2×2 matrix and apply T^{-1} to (75). We obtain

$$\hat{\mathbf{u}}_{\tau\tau} - e^{-2H\tau}\Delta\hat{\mathbf{u}} + T^{-1}CT\hat{\mathbf{u}}_\tau + T^{-1}AT\hat{\mathbf{u}} = 0,$$

where $\hat{\mathbf{u}} := T^{-1}\mathbf{u}$.

3.2. **Positive definiteness of the coefficient matrices.** Let us try to find a matrix T so that $T^{-1}AT$ is diagonal. The eigenvalues of A are given by

$$\lambda_\pm := \frac{\beta_1+\delta_3}{2} \pm \left[\frac{(\beta_1+\delta_3)^2}{4} - \beta_1\delta_3 + \delta_1\beta_3\right]^{1/2}.$$

Note that

$$\frac{(\beta_1+\delta_3)^2}{4} - \beta_1\delta_3 + \delta_1\beta_3 = \frac{1}{4}[(\beta_1-\delta_3)^2 + 4\delta_1\beta_3] > 0,$$

since $\delta_1\beta_3 > 0$ for $n \geq 3$ and $p > 1$. The eigenvalues are thus real and different. Note also that

$$\beta_1\delta_3 - \delta_1\beta_3 = 4np(np-1)(p-1) > 0$$

for the range of n and p we are interested in. This computation shows that $\lambda_- = 0$ when $p = 1$. Since $\beta_1 + \delta_3 > 0$, we conclude that both eigenvalues are positive. Let

$$T := \begin{pmatrix} \lambda_- - \delta_3 & \lambda_+ - \delta_3 \\ \beta_3 & \beta_3 \end{pmatrix}. \tag{76}$$

Then $\det T > 0$ and

$$\hat{A} := T^{-1}AT = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \tag{77}$$

where the first equality is a definition. Let $\hat{C} := T^{-1}CT$. The main question is then whether $\hat{C} + \hat{C}^t$ is positive definite or not.

Lemma 5. *With definitions as above, $\hat{C} + \hat{C}^t$ is positive definite.*

Proof. Define

$$R := \begin{pmatrix} \beta_3 & \delta_3 - \lambda_+ \\ -\beta_3 & \lambda_- - \delta_3 \end{pmatrix}.$$

Note that $T^{-1} = R/\det T$. In other words, R coincides with T^{-1} up to a positive factor. The question is then if RCT plus its transpose is positive definite. Let us define a, b, c and d by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = RCT.$$

In order to prove that RCT plus its transpose is positive definite, all we need to prove is that

$$a + d > 0, \quad (a+d)^2 - (a-d)^2 - (b+c)^2 > 0. \tag{78}$$

One can compute that

$$a + d = -\beta_3(\lambda_+ - \lambda_-)(\alpha_1 + \gamma_3).$$

Since $\beta_3 < 0$, $\alpha_1 + \gamma_3 > 0$ and $\lambda_+ - \lambda_- > 0$, we conclude that $a + d > 0$. One can also compute that

$$\begin{aligned} b + c &= -\beta_3(\lambda_+ - \lambda_-)(\gamma_3 - \alpha_1) \\ a - d &= \beta_3[(\alpha_1 - \gamma_3)(\beta_1 - \delta_3) + 2\beta_3\gamma_1]. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left(\frac{a+d}{\beta_3}\right)^2 - \left(\frac{a-d}{\beta_3}\right)^2 - \left(\frac{b+c}{\beta_3}\right)^2 \\ &= 4\alpha_1\gamma_3[(\beta_1 - \delta_3)^2 + 4\beta_3\delta_1] - (\alpha_1 - \gamma_3)^2(\beta_1 - \delta_3)^2 \\ &\quad -4(\alpha_1 - \gamma_3)(\beta_1 - \delta_3)\beta_3\gamma_1 - 4\beta_3^2\gamma_1^2. \end{aligned}$$

After inserting the values for the different constants, we obtain

$$\begin{aligned} &\left(\frac{a+d}{2\beta_3}\right)^2 - \left(\frac{a-d}{2\beta_3}\right)^2 - \left(\frac{b+c}{2\beta_3}\right)^2 \\ &= [(np-1)^2 + 2p(np-1)](\beta_1 - \delta_3)^2 + 16p(np-1)^2(np-1+2p) \\ &\quad -p^2(\beta_1 - \delta_3)^2 - 8(n-1)p^2(\beta_1 - \delta_3) - 16(n-1)^2p^2. \end{aligned}$$

One can see that the terms involving $(\beta_1 - \delta_3)^2$ add up to something non-negative. Consider the second term on the right hand side. If we write the last factor in this term as $np - 1 + p + p$, take the term that arises from one of the p's and add it to the last two terms, we obtain

$$\begin{aligned} &16p^2[(np-1)^2 - (n-1)^2] - 8(n-1)p^2(\beta_1 - \delta_3) \\ &\quad = 16np^2(p-1)[n(p+1) - 2] - 8(n-1)p^2(\beta_1 - \delta_3). \end{aligned}$$

However,

$$\beta_1 - \delta_3 = 2np(p-1) + 2p - 2np + 2,$$

so that

$$-8(n-1)p^2(\beta_1 - \delta_3) = -16np^2(p-1)(n-1)p + 16(n-1)p^2(np - p - 1).$$

We conclude that

$$\begin{aligned} &16p^2[(np-1)^2 - (n-1)^2] - 8(n-1)p^2(\beta_1 - \delta_3) \\ &\quad = 16np^2(p-1)(n+p-2) + 16(n-1)p^2(np-p-1). \end{aligned}$$

Thus (78) holds and $\hat{C}+\hat{C}^t$ is positive definite. $\square$

3.3. **Model energy.** Let us consider a solution to (70) and (71), where $\tau\in\mathbb{R}$ and $x\in\mathbb{T}^n$. Let us use the notation

$$\begin{pmatrix}\hat{u}\\ \hat{\psi}\end{pmatrix}=\hat{\mathbf{u}}=T^{-1}\mathbf{u},$$

where T is given by (76). Then

$$\hat{\mathbf{u}}_{\tau\tau}-e^{-2H\tau}\Delta\hat{\mathbf{u}}+\hat{C}\hat{\mathbf{u}}_\tau+\hat{A}\hat{\mathbf{u}}=0.$$

Note that $\hat{A}$ is given by (77) and that $\hat{C}+\hat{C}^t$ is positive definite. We shall denote the components of $\hat{C}$ by $\hat{C}_{ij}$. Let us define an energy

$$E=\frac{1}{2}\int_{\mathbb{T}^n}[|\hat{\mathbf{u}}_\tau|^2+e^{-2H\tau}(|\nabla\hat{u}|^2+|\nabla\hat{\psi}|^2)+2c\hat{\mathbf{u}}^t\hat{\mathbf{u}}_\tau+b_1\hat{u}^2+b_2\hat{\psi}^2]dx,$$

where the constants c and b_i are to be determined. To start with, the only condition we impose is that $c^2<b_i$ for $i=1,2$. Note that this implies that there is an $\eta>0$, depending on c, b_1 and b_2, such that

$$\frac{1}{2}\int_{\mathbb{T}^n}[|\hat{\mathbf{u}}_\tau|^2+e^{-2H\tau}(|\nabla\hat{u}|^2+|\nabla\hat{\psi}|^2)+|\hat{\mathbf{u}}|^2]dx\le\eta E.$$

Let us compute

$$\begin{aligned}\frac{dE}{d\tau} &= \int_{\mathbb{T}^n}\Big[-\frac{1}{2}\hat{\mathbf{u}}_\tau^t(\hat{C}+\hat{C}^t)\hat{\mathbf{u}}_\tau+c|\hat{\mathbf{u}}_\tau|^2-(H+c)e^{-2H\tau}(|\nabla\hat{u}|^2+|\nabla\hat{\psi}|^2)\\ &\quad -c\lambda_-\hat{u}^2-c\lambda_+\hat{\psi}^2+(b_1-\lambda_--c\hat{C}_{11})\hat{u}\hat{u}_\tau+(b_2-\lambda_+-c\hat{C}_{22})\hat{\psi}\hat{\psi}_\tau\\ &\quad -c\hat{C}_{12}\hat{u}\hat{\psi}_\tau-c\hat{C}_{21}\hat{\psi}\hat{u}_\tau]dx.\end{aligned}$$

Let us choose

$$b_1=\lambda_-+c\hat{C}_{11},\quad b_2=\lambda_++c\hat{C}_{22}. \tag{79}$$

Since the $\lambda_\pm$ are positive, we obtain $c^2<b_i$ by choosing c small enough. Note that

$$|c\hat{C}_{12}\hat{u}\hat{\psi}_\tau|\le c^{3/2}\hat{u}^2+\frac{1}{4}c^{1/2}\hat{C}_{12}^2\hat{\psi}_\tau^2,$$

and similarly for $c\hat{C}_{21}\hat{\psi}\hat{u}_\tau$. Choosing b_i as in (79), we obtain

$$\begin{aligned}\frac{dE}{d\tau} &\le \int_{\mathbb{T}^n}\Big[-\frac{1}{2}\hat{\mathbf{u}}_\tau^t(\hat{C}+\hat{C}^t)\hat{\mathbf{u}}_\tau+c|\hat{\mathbf{u}}_\tau|^2+\frac{1}{4}c^{1/2}(\hat{C}_{12}^2\hat{\psi}_\tau^2+\hat{C}_{21}^2\hat{u}_\tau^2)\\ &\quad -(H+c)e^{-2H\tau}(|\nabla\hat{u}|^2+|\nabla\hat{\psi}|^2)\\ &\quad -c(\lambda_--c^{1/2})\hat{u}^2-c(\lambda_+-c^{1/2})\hat{\psi}^2]dx.\end{aligned}$$

Due to Lemma 5, $\hat{C}^t+\hat{C}$ is positive definite, so that by choosing c small enough, there is a constant $a_1>0$, depending on n and p, such that

$$\frac{dE}{d\tau}\le -a_1\int_{\mathbb{T}^n}[\hat{u}_\tau^2+\hat{\psi}_\tau^2+e^{-2H\tau}(|\nabla\hat{u}|^2+|\nabla\hat{\psi}|^2)+\hat{u}^2+\hat{\psi}^2]dx.$$

This of course implies the existence of a $\kappa>0$, depending on n and p, such that

$$\frac{dE}{d\tau}\le -2\kappa E.$$

4. Energy estimates

Let us turn back to the actual equations. The purpose of the present section is to construct the energies on which the bootstrap argument will be based. Let us start by constructing the energy associated with (61) and (64). Note that we can write (61) and (64) as

$$\hat{\Box}_g \mathbf{u} + C\mathbf{u}_\tau + A\mathbf{u} + \mathbf{\Delta} = 0,$$

where A, C and $\mathbf{u}$ are defined in (74), $\hat{\Box}_g$ is defined in (65) and

$$\mathbf{\Delta} = \begin{pmatrix} \Delta_{00} \\ \Delta_\psi \end{pmatrix}.$$

Letting T be defined by (76), $\hat{\mathbf{u}} = T^{-1}\mathbf{u}$, $\hat{\mathbf{\Delta}} = T^{-1}\mathbf{\Delta}$, $\hat{A} = T^{-1}AT$ and $\hat{C} = T^{-1}CT$, we obtain

$$\hat{\Box}_g \hat{\mathbf{u}} + \hat{C}\hat{\mathbf{u}}_\tau + \hat{A}\hat{\mathbf{u}} + \hat{\mathbf{\Delta}} = 0. \tag{80}$$

We shall also use the terminology

$$\begin{pmatrix} \hat{u} \\ \hat{\psi} \end{pmatrix} := \hat{\mathbf{u}}.$$

Lemma 6. *Let $p > 1$ and τ_0 be real numbers and $n \geq 3$ be an integer. Let $g : I \times \mathbb{T}^n \to \mathcal{C}_n$, where I is an interval, and denote the components of g by $g_{\mu\nu}$. Consider a solution $\hat{\mathbf{u}}$ to the equation*

$$\hat{\Box}_g \hat{\mathbf{u}} + \hat{C}\hat{\mathbf{u}}_\tau + \hat{A}\hat{\mathbf{u}} = \mathbf{F} \tag{81}$$

on $I \times \mathbb{T}^n$, where $\hat{\Box}_g$ is defined in (65), $\mathbf{F}$ is a given function and $\hat{A}$ and $\hat{C}$ are defined above. Given constants c_{lp} and b_i, $i = 1, 2$, we define

$$\mathcal{E}[\hat{\mathbf{u}}] = \frac{1}{2}\int_{\mathbb{T}^n} \{-g^{00}\partial_\tau \hat{\mathbf{u}}^t \partial_\tau \hat{\mathbf{u}} + \hat{g}^{ij}\partial_i \hat{\mathbf{u}}^t \partial_j \hat{\mathbf{u}} - 2c_{\text{lp}} g^{00}\hat{\mathbf{u}}^t \partial_\tau \hat{\mathbf{u}} + b_1 \hat{u}^2 + b_2 \hat{\psi}^2\} dx \tag{82}$$

on I, where we use the notation $\hat{g}^{ij} = e^{2(\tau+\tau_0)} g^{ij}$. Below we shall also use the notation $\hat{g}^{0i} = e^{\tau+\tau_0} g^{0i}$ and $H = p - 1$. There are constants $\eta_{\text{lp}}, \zeta_{\text{lp}}, b_i, c_{\text{lp}} > 0$, depending on n and p, such that if $\mathcal{E}$ is defined by (82) with this choice of b_i and c_{lp} and

$$|g^{00} + 1| \leq \eta_{\text{lp}}, \tag{83}$$

then

$$\mathcal{E} \geq \zeta_{\text{lp}} \int_{\mathbb{T}^n} \{\partial_\tau \hat{\mathbf{u}}^t \partial_\tau \hat{\mathbf{u}} + \hat{g}^{ij}\partial_i \hat{\mathbf{u}}^t \partial_j \hat{\mathbf{u}} + \hat{\mathbf{u}}^t \hat{\mathbf{u}}\} dx \tag{84}$$

and

$$\frac{d\mathcal{E}}{d\tau} \leq -2\eta_{\text{lp}}\mathcal{E} + \int_{\mathbb{T}^n} \{(\partial_\tau \hat{\mathbf{u}}^t + c_{\text{lp}}\hat{\mathbf{u}}^t)\mathbf{F} + \Delta_E[\hat{\mathbf{u}}]\} dx$$

where $\Delta_E[\hat{\mathbf{u}}]$ is given in (85).

Remark. Note that since g is a map into $\mathcal{C}_n$, Lemma 2 implies that g^{ij} are the components of a positive definite matrix and that $g^{00} < 0$.

Proof. Let us compute

$$\begin{aligned}\frac{d\mathcal{E}}{d\tau} &= \int_{\mathbb{T}^n}\{-\frac{1}{2}\partial_\tau\hat{\mathbf{u}}^t(\hat{C}+\hat{C}^t)\partial_\tau\hat{\mathbf{u}} - \partial_\tau\hat{\mathbf{u}}^t\hat{A}\hat{\mathbf{u}} + \partial_\tau\hat{\mathbf{u}}^t\mathbf{F} - (H+c_{\mathrm{lp}})\hat{g}^{ij}\partial_i\hat{\mathbf{u}}^t\partial_j\hat{\mathbf{u}} \\ &\quad + c_{\mathrm{lp}}|\partial_\tau\hat{\mathbf{u}}|^2 - c_{\mathrm{lp}}\hat{\mathbf{u}}^t\hat{C}\partial_\tau\hat{\mathbf{u}} - c_{\mathrm{lp}}\hat{\mathbf{u}}^t\hat{A}\hat{\mathbf{u}} + c_{\mathrm{lp}}\hat{\mathbf{u}}^t\mathbf{F} + b_1\hat{u}\partial_\tau\hat{u} + b_2\hat{\psi}\partial_\tau\hat{\psi} \\ &\quad + \Delta_E[\hat{\mathbf{u}}]\}dx,\end{aligned}$$

where

$$\begin{aligned}\Delta_E[\hat{\mathbf{u}}] &= -c_{\mathrm{lp}}(g^{00}+1)\partial_\tau\hat{\mathbf{u}}^t\partial_\tau\hat{\mathbf{u}} - 2c_{\mathrm{lp}}\hat{g}^{0i}\partial_i\hat{\mathbf{u}}^t\partial_\tau\hat{\mathbf{u}} - 2c_{\mathrm{lp}}(\partial_i\hat{g}^{0i})\hat{\mathbf{u}}^t\partial_\tau\hat{\mathbf{u}} \\ &\quad - c_{\mathrm{lp}}(\partial_j\hat{g}^{ij})\partial_i\hat{\mathbf{u}}^t\hat{\mathbf{u}} - \frac{1}{2}(\partial_\tau g^{00})\partial_\tau\hat{\mathbf{u}}^t\partial_\tau\hat{\mathbf{u}} + \left[\frac{1}{2}\partial_\tau\hat{g}^{ij} + H\hat{g}^{ij}\right]\partial_i\hat{\mathbf{u}}^t\partial_j\hat{\mathbf{u}} \\ &\quad - (\partial_i\hat{g}^{0i})\partial_\tau\hat{\mathbf{u}}^t\partial_\tau\hat{\mathbf{u}} - (\partial_j\hat{g}^{ij})\partial_\tau\hat{\mathbf{u}}^t\partial_i\hat{\mathbf{u}} - c_{\mathrm{lp}}(\partial_\tau g^{00})\hat{\mathbf{u}}^t\partial_\tau\hat{\mathbf{u}}.\end{aligned} \tag{85}$$

Choosing c_{lp} and b_i similarly to how we chose them in Subsection 3.3, we get the desired conclusion, assuming g^{00} to be close enough to -1. □

Corollary 1. *With assumptions as in Lemma 6, let $\mathcal{E}$ be defined by (82) with constants chosen as in the statement of Lemma 6. Let*

$$\mathfrak{E}_k = \sum_{|\alpha|\leq k}\mathcal{E}[\partial^\alpha\hat{\mathbf{u}}].$$

Then, assuming (83) holds,

$$\frac{d\mathfrak{E}_k}{d\tau} \leq -2\eta_{\mathrm{lp}}\mathfrak{E}_k + \sum_{|\alpha|\leq k}\int_{\mathbb{T}^n}\{(\partial^\alpha\partial_\tau\hat{\mathbf{u}}^t + c_{\mathrm{lp}}\partial^\alpha\hat{\mathbf{u}}^t)(\partial^\alpha\mathbf{F} + [\hat{\Box}_g,\partial^\alpha]\hat{\mathbf{u}}) + \Delta_E[\partial^\alpha\hat{\mathbf{u}}]\}dx.$$

Remark. When we write ∂^α, we shall always take for granted that the Greek index used *upstairs* is a multiindex, $\alpha = (l_1, ..., l_n)$, where the l_i are non-negative integers so that

$$\partial^\alpha = \partial_1^{l_1}\cdots\partial_n^{l_n},$$

where ∂_i is the standard differential operator with respect to the i:th "coordinate" on $\mathbb{T}^n$. Note in particular that ∂^α never contains any derivatives with respect to the time coordinate. Note also that in an expression ∂_α, the Greek index *downstairs* means a number from 0 to n.

Proof. Differentiating (81), we obtain

$$\hat{\Box}_g\partial^\alpha\hat{\mathbf{u}} + \hat{C}\partial_\tau\partial^\alpha\hat{\mathbf{u}} + \hat{A}\partial^\alpha\hat{\mathbf{u}} = \partial^\alpha\mathbf{F} + [\hat{\Box}_g,\partial^\alpha]\hat{\mathbf{u}},$$

so that we only need to apply Lemma 6 in order to get the desired conclusion. □

The energies we shall construct for u_i and h_{ij} will be based on the following lemma.

Lemma 7. *Let τ_0 be a real number and $n \geq 3$ be an integer. Let $g : I \times \mathbb{T}^n \to \mathcal{C}_n$, where I is an interval, and denote the components of g by $g_{\mu\nu}$. Consider a solution to the equation*

$$\hat{\Box}_g v + \alpha\partial_\tau v + \beta v = F, \tag{86}$$

on $I \times \mathbb{T}^n$, where $\hat{\Box}_g$ is defined in (65), F is a given function and $\alpha > 0$ and $\beta \geq 0$ are constants. Then there are constants $\eta_{\mathrm{c}}, \zeta > 0$ and $\gamma, \delta \geq 0$, depending on α and β, such that if

$$|g^{00}+1| \leq \eta_{\mathrm{c}} \tag{87}$$

and

$$\mathcal{E}_{\gamma,\delta}[v] = \frac{1}{2}\int_{\mathbb{T}^n}[-g^{00}(\partial_\tau v)^2 + \hat{g}^{ij}\partial_i v\partial_j v - 2\gamma g^{00}v\partial_\tau v + \delta v^2]dx,$$

then

$$\mathcal{E}_{\gamma,\delta} \geq \zeta\int_{\mathbb{T}^n}[(\partial_\tau v)^2 + \hat{g}^{ij}\partial_i v\partial_j v + \iota_\beta v^2]dx, \tag{88}$$

where $\iota_\beta = 0$ *if* $\beta = 0$ *and* $\iota_\beta = 1$ *otherwise, and*

$$\frac{d\mathcal{E}_{\gamma,\delta}}{d\tau} \leq -2\eta_c\mathcal{E}_{\gamma,\delta} + \int_{\mathbb{T}^n}\{(\partial_\tau v + \gamma v)F + \Delta_{E,\gamma,\delta}[v]\}dx,$$

where $\Delta_{E,\gamma,\delta}[v]$ *is given by (89). If* $\beta = 0$*, then* $\gamma = \delta = 0$*.*

Proof. If $\beta > 0$, choose $\gamma = \alpha/2$ and $\delta = \beta + \alpha^2/2$. Then $\gamma^2 < \delta$, and it is clear that there is a constant $\zeta > 0$ such that (88) holds, assuming g^{00} is close enough to -1. If $\beta = 0$, we simply let $\gamma = \delta = 0$, and the existence of a $\zeta > 0$ such that (88) holds again follows from the assumption that g^{00} is close enough to -1. Compute

$$\begin{aligned}\frac{d\mathcal{E}_{\gamma,\delta}}{d\tau} &= \int_{\mathbb{T}^n}\{-(\alpha-\gamma)(\partial_\tau v)^2 + (\delta - \beta - \gamma\alpha)v\partial_\tau v - \beta\gamma v^2 \\ &\quad -(H+\gamma)\hat{g}^{ij}\partial_i v\partial_j v + (\partial_\tau v + \gamma v)F + \Delta_{E,\gamma,\delta}[v]\}dx,\end{aligned}$$

where

$$\begin{aligned}\Delta_{E,\gamma,\delta}[v] &= -\gamma(\partial_i\hat{g}^{ij})v\partial_j v - 2\gamma(\partial_i\hat{g}^{0i})v\partial_\tau v - 2\gamma\hat{g}^{0i}\partial_i v\partial_\tau v - (\partial_i\hat{g}^{0i})(\partial_\tau v)^2 \\ &\quad -(\partial_j\hat{g}^{ij})\partial_i v\partial_\tau v - \frac{1}{2}(\partial_\tau g^{00})(\partial_\tau v)^2 + \left(\frac{1}{2}\partial_\tau\hat{g}^{ij} + H\hat{g}^{ij}\right)\partial_i v\partial_j v \\ &\quad -\gamma\partial_\tau g^{00}v\partial_\tau v - \gamma(g^{00}+1)(\partial_\tau v)^2.\end{aligned} \tag{89}$$

Due to our choices, we have, assuming $\beta > 0$,

$$\begin{aligned}\frac{d\mathcal{E}_{\gamma,\delta}}{d\tau} &= -\frac{1}{2}\int_{\mathbb{T}^n}[\alpha(\partial_\tau v)^2 + (\alpha + 2H)\hat{g}^{ij}\partial_i v\partial_j v + \alpha\beta v^2]dx \\ &\quad + \int_{\mathbb{T}^n}\{(\partial_\tau v + \gamma v)F + \Delta_{E,\gamma,\delta}[v]\}dx.\end{aligned}$$

Since the opposite inequality to (88) also holds, provided we replace ζ by ζ^{-1} for ζ small enough, we obtain the conclusion of the lemma for $\beta > 0$. The conclusion in the case $\beta = 0$ follows for similar reasons. □

Corollary 2. *Under the assumptions of Lemma 7, let*

$$\mathfrak{E}_k = \sum_{|\alpha|\leq k}\mathcal{E}_{\gamma,\delta}[\partial^\alpha v].$$

Then, assuming (87) holds,

$$\frac{d\mathfrak{E}_k}{d\tau} \leq -2\eta_c\mathfrak{E}_k + \sum_{|\alpha|\leq k}\int_{\mathbb{T}^n}\{(\partial_\tau\partial^\alpha v + \gamma\partial^\alpha v)(\partial^\alpha F + [\hat{\Box}_g, \partial^\alpha]v) + \Delta_{E,\gamma,\delta}[\partial^\alpha v]\}dx.$$

Proof. Given that v satisfies (86), $\partial^\alpha v$ satisfies

$$\hat{\Box}_g(\partial^\alpha v) + \alpha\partial_\tau(\partial^\alpha v) + \beta(\partial^\alpha v) = \partial^\alpha F + [\hat{\Box}_g, \partial^\alpha]v.$$

The statement follows from Lemma 7. □

5. Bootstrap assumptions

Before we write down the basic bootstrap assumptions, let us introduce some terminology. If A is a symmetric positive definite $n \times n$ matrix with components A_{ij} and $w \in \mathbb{R}^n$, we shall use the notation

$$|w|_A = \left(\sum_{i,j=1}^{n} A_{ij} w^i w^j \right)^{1/2}.$$

If Id is the identity matrix, we define $|w| := |w|_{\mathrm{Id}}$. We shall also use the notation introduced in Subsection 2.4.

5.1. Primary bootstrap assumptions.

The purpose of the primary bootstrap assumptions is to ensure that the metric remains Lorentzian, with quantitative bounds.

Definition 4. Let $p > 1$, $a > 0$, $c_1 > 1$, $\eta \in (0, 1)$, K_0 and τ_0 be real numbers and $n \geq 3$ be an integer. We shall say that a function $g : I \times \mathbb{T}^n \to \mathcal{C}_n$, where I is an interval, satisfies the *primary bootstrap assumptions* on I (the relevant constants being understood from the context) if

$$c_1^{-1}|w|^2 \leq e^{-2\Omega-2K}|w|^2_{g_\flat} \leq c_1|w|^2, \tag{90}$$

$$|u[g]| \leq \eta, \tag{91}$$

$$|v[g]|^2 \leq \eta c_1^{-1} e^{2\Omega-2r+2K}, \tag{92}$$

for all $w \in \mathbb{R}^n$ and all $(\tau, x) \in I \times \mathbb{T}^n$, where $\Omega = p\tau$, $r = a\tau$ and $K = \tau_0 + K_0$.

Remark. We shall specify a and η in (101) and (100) below. In the end we shall apply the above conditions to a situation in which K_0 only depends on p, so that factors of e^{-K_0} and e^{K_0} can be considered to be constants of which one need not keep track. In fact, the natural choice to make for e^K is a numerical multiple of the basic length scale $\ell(t_0)$. Furthermore, the constants η and a we shall use only depend on n and p, and c_1 will, in our applications, be a numerical constant. In other words, the only quantity that in practice needs to be specified (beyond n and p) is τ_0.

Lemma 7 of [20] gives the following conclusions of the bootstrap assumptions.

Lemma 8. *Let $p > 1$, $a > 0$, $c_1 > 1$, $\eta \in (0, 1)$, K_0 and τ_0 be real numbers and $n \geq 3$ be an integer. Assume that $g : I \times \mathbb{T}^n \to \mathcal{C}_n$ satisfies the primary bootstrap assumptions on I, where I is an interval. There is a numerical constant $\eta_0 \in (0, 1/4)$ such that if we assume $\eta \leq \eta_0$ in (91) and (92), then*

$$|v[g^{-1}]| \leq 2c_1 e^{-2\Omega-2K}|v[g]| \tag{93}$$

$$|(v[g], v[g^{-1}])| \leq 2c_1 e^{-2\Omega-2K}|v[g]|^2 \tag{94}$$

$$|u[g^{-1}]| \leq 4\eta, \tag{95}$$

$$\frac{2}{3c_1}|w|^2 \leq e^{2\Omega+2K}|w|^2_{g^\sharp} \leq \frac{3c_1}{2}|w|^2 \tag{96}$$

for all $w \in \mathbb{R}^n$ and $(\tau, x) \in I \times \mathbb{T}^n$. Here we use the notation (ξ, ζ) for the ordinary scalar product of $\xi, \zeta \in \mathbb{R}^n$.

Remark. The lemma holds irrespective of the value of a.

5.2. **Energies.** Let $p > 1$, $a > 0$, $c_1 > 1$, $\eta \in (0, \min\{\eta_0, \eta_{\rm lp}/4\}]$, K_0 and τ_0 be real numbers and $n \geq 3$ be an integer. Assume that $g : I \times \mathbb{T}^n \to \mathcal{C}_n$ satisfies the primary bootstrap assumptions on I, where I is an interval. Then (83) is satisfied due to (95). In order to define the energy associated with u and ψ, let us note that (61) and (64) can be combined into (80). Using the notation introduced in connection with (80), let

$$H_{{\rm lp},k} = \sum_{|\alpha|\leq k} \mathcal{E}[\partial^\alpha \hat{\mathbf{u}}], \tag{97}$$

where $\mathcal{E}$ is defined in (82) with the constants that are obtained as a result of Lemma 6.

Consider (62). If we take all the terms on the left hand side except for the first three to the right hand side, we get an equation of the type discussed in Lemma 7 with α replaced by α_2 and β replaced by β_2. Since $\alpha_2, \beta_2 > 0$, Lemma 7 yields positive constants $\gamma_{\rm s}$, $\delta_{\rm s}$, $\eta_{\rm s}$ and $\zeta_{\rm s}$ such that the conclusions of that lemma holds, and we define

$$H_{{\rm s},k} = \sum_i \sum_{|\alpha|\leq k} \mathcal{E}_{\gamma_{\rm s},\delta_{\rm s}}[\partial^\alpha u_i], \tag{98}$$

where $\mathcal{E}_{\gamma_{\rm s},\delta_{\rm s}}$ is defined in Lemma 7.

Consider (63). Taking all but the first two terms on the left hand side to the right hand side, we obtain an equation of the type considered in Lemma 7 with α replaced by $np-1 > 0$ and β replaced by 0. We thus get $\gamma_{\rm m} = \delta_{\rm m} = 0$ and $\eta_{\rm m}, \zeta_{\rm m} > 0$ such that the conclusions of Lemma 7 hold. We define the energy associated with h_{ij} to be

$$H_{{\rm m},k} = \frac{1}{2}\sum_{i,j} \sum_{|\alpha|\leq k} \left(\mathcal{E}_{\gamma_{\rm m},\delta_{\rm m}}[\partial^\alpha h_{ij}] + \int_{\mathbb{T}^n} e^{-2a\tau} a_\alpha (\partial^\alpha h_{ij})^2 dx\right), \tag{99}$$

where $a > 0$ is given by (101) and $a_\alpha = 1$ for $|\alpha| > 0$, $a_\alpha = 0$ for $\alpha = 0$. From now on, we shall assume that g satisfies the primary bootstrap assumption on an interval I where η is defined by

$$\eta := \min\{\eta_0, \eta_{\rm lp}/4, \eta_{\rm s}/4, \eta_{\rm m}/4\}. \tag{100}$$

Note that as a consequence, the conclusions of Lemma 6 and 7 hold for the energies of interest, cf. (95). Furthermore, we define

$$a := \frac{1}{4}\min\{p-1, \eta_{\rm lp}, \eta_{\rm s}, \eta_{\rm m}\}. \tag{101}$$

Note that a and η only depend on n and p.

5.3. **Basic estimates.** Let us use the notation

$$\|f\|_{H^k} = \left(\sum_{|\alpha|\leq k} \int_{\mathbb{T}^n} (\partial^\alpha f)^2 dx\right)^{1/2}$$

for the Sobolev norms (note that we shall use this notation even when f depends on t, and then the derivatives will still only be with respect to the spatial coordinates). We wish to express the Sobolev norms of the quantities of interest in terms of the geometrically defined energies $H_{\text{lp},k}$ etc. In the end it will turn out to be convenient to use the following energies instead:

$$\hat{H}_{\text{lp},k} = e^{2a\tau} H_{\text{lp},k},\ \hat{H}_{\text{s},k} = e^{-2p\tau+2a\tau-2K} H_{\text{s},k},\ \hat{H}_{\text{m},k} = e^{2a\tau-4K} H_{\text{m},k},$$

where $a > 0$ is given by (101). We shall also use the notation

$$\hat{H}_k = \hat{H}_{\text{lp},k} + \hat{H}_{\text{s},k} + \hat{H}_{\text{m},k}. \tag{102}$$

Note that, using the notation of Section 7 in [20], $\hat{H}_{\text{lp},k}$, $\hat{H}_{\text{s},k}$ and $\hat{H}_{\text{m},k}$ are equivalent to $\hat{E}_{\text{lp},k}$, $\hat{E}_{\text{s},k}$ and $\hat{E}_{\text{m},k}$ respectively; in the formulas for $\hat{E}$, the quantity r should be replaced by $a\tau$ and it is convenient to note that

$$\omega^{-1}\partial_t = p^{-1}\partial_\tau, \quad \omega^{-1} g^{0i} = p^{-1}\hat{g}^{0i}, \quad \omega^{-2} g^{ij} = p^{-2}\hat{g}^{ij}. \tag{103}$$

In particular, $\hat{H}_k$ is equivalent to $\hat{E}_k$. Furthermore, we have the following lemma.

Lemma 9. *Let $p > 1$, $c_1 > 1$, K_0 and τ_0 be real numbers and $n \geq 3$ be an integer. Let η and a be defined by (100) and (101) respectively and assume that $g : I \times \mathbb{T}^n \to \mathcal{C}_n$ satisfies the primary bootstrap assumptions on an interval I. Then*

$$e^{a\tau}[\|\psi\|_{H^k} + \|\partial_\tau\psi\|_{H^k} + e^{-H\tau-K_0}\|\partial_i\psi\|_{H^k}] \leq C\hat{H}^{1/2}_{\text{lp},k}, \tag{104}$$

$$e^{a\tau}[\|u\|_{H^k} + \|\partial_\tau u\|_{H^k} + e^{-H\tau-K_0}\|\partial_i u\|_{H^k}] \leq C\hat{H}^{1/2}_{\text{lp},k}, \tag{105}$$

$$e^{-p\tau+a\tau-K}[\|u_m\|_{H^k} + \|\partial_\tau u_m\|_{H^k} + e^{-H\tau-K_0}\|\partial_i u_m\|_{H^k}] \leq C\hat{H}^{1/2}_{\text{s},k}, \tag{106}$$

$$e^{-2p\tau+a\tau-2K}\left[\|\partial_\tau g_{ij} - 2pg_{ij}\|_{H^k} + e^{-H\tau-K_0}\|\partial_l g_{ij}\|_{H^k}\right] \leq C\hat{H}^{1/2}_{\text{m},k}, \tag{107}$$

$$e^{-2p\tau-2K}\|\partial^\alpha g_{ij}\|_2 \leq C\hat{H}^{1/2}_{\text{m},k} \tag{108}$$

hold on I, where $K = \tau_0 + K_0$, the last estimate is valid for $0 < |\alpha| \leq k$ and the constants depend on c_1, n and p.

Proof. The lemma follows from Lemma 8 of [20] given the above mentioned equivalence of the energies (though it is not difficult to prove the statement directly). Note, however, that this is based on observations such as (103) and

$$\omega^{-1}e^{-p\tau-K} = p^{-1}e^{-H\tau-K_0}$$

and the fact that 1 is as good a constant as p^{-1}. □

We shall need estimates for the components of the inverse of the metric. Such estimates follow from the results of [20].

Lemma 10. *Let $p > 1$, $c_1 > 1$, K_0 and τ_0 be real numbers and $n \geq 3$ be an integer. Let η and a be defined by (100) and (101) respectively and assume that $g : I \times \mathbb{T}^n \to \mathcal{C}_n$ satisfies the primary bootstrap assumptions on an interval I. Then, for $0 < |\alpha| \leq k$,*

$$e^{a\tau}\|\partial^\alpha g^{00}\|_2 \leq C\hat{H}^{1/2}_k, \tag{109}$$

$$e^{2p\tau+2K}\|\partial^\alpha g^{lm}\|_2 \leq C\hat{H}^{1/2}_k, \tag{110}$$

$$e^{p\tau+a\tau+K}\|g^{0l}\|_{H^k} \leq C\hat{H}^{1/2}_k \tag{111}$$

hold on I, where $K = \tau_0 + K_0$, $\hat{H}_k$ is defined in (102) and the constants depend on n, p, k and c_1.

Proof. See Lemma 9 of [20]. □

5.4. **The main bootstrap assumption.** Using the primary bootstrap assumptions, it is possible to define the energy $\hat{H}_k$ in terms of which the main bootstrap assumption is phrased.

Definition 5. Let $p > 1$, $c_1 > 1$, K_0, $0 < \epsilon \leq 1$ and τ_0 be real numbers and $n \geq 3$ and $k_0 > n/2 + 1$ be integers. We shall then say that (g, ψ) satisfy the *main bootstrap assumption* on I (the relevant constants being understood from the context), where I is an interval, if

- $g : I \times \mathbb{T}^n \to \mathcal{C}_n$ and $\psi : I \times \mathbb{T}^n \to \mathbb{R}$ are C^∞,
- g satisfies the primary bootstrap assumptions on I, where η and a are defined by (100) and (101) respectively,
- g and ψ satisfy

$$\hat{H}_{k_0}^{1/2}(\tau) \leq \epsilon \tag{112}$$

for all $\tau \in I$, where $K = K_0 + \tau_0$.

Remark. Note that these bootstrap assumptions correspond exactly to the bootstrap assumptions made in [20], given the specific form of Ω and r, cf. (105) of [20].

6. Estimates for the non-linearity

In the proof of future global existence of solutions, the main tool is the system of differential inequalities given in Section 7. The first step in the derivation of these inequalities has already been taken, cf. Corollary 1 and 2. However, in order to obtain (139)-(141), it is necessary to estimate $\Delta_{\mu\nu}$, Δ_ψ, $[\hat{\Box}_g, \partial^\alpha]\mathbf{\hat{u}}$, $\Delta_E[\partial^\alpha \mathbf{\hat{u}}]$ etc. in H^k, cf. Corollary 1 and 2. The present section is devoted to a derivation of such estimates.

In Subsection 9.1 of [20], we described an algorithm for estimating the higher order terms. The current context is only a special case of what was considered there. However, a few things should be kept in mind when making the comparison. First of all, in the estimates in [20], time derivatives were computed with respect to the original time t and not with respect to τ. Furthermore, $\Omega = p\tau$, $K = \tau_0 + K_0$, $\omega = p/t$ and $r = a\tau$. The relationship between t and τ is of course given by $\tau = \ln t - \tau_0$. When using the algorithm described in [20], it is convenient to note that (103) holds. In particular, changing ∂_t to ∂_τ corresponds to multiplication with ω^{-1} as far as estimates are concerned.

6.1. Estimates for the quadratic terms.

Lemma 11. *Let $p > 1$, $c_1 > 1$, K_0, $0 < \epsilon \leq 1$ and τ_0 be real numbers and $n \geq 3$ and $k_0 > n/2+1$ be integers. Assume that (g, ψ) satisfy the main bootstrap assumption on an interval I. Then*

$$\|\Delta_{00}\|_{H^k} \leq C\epsilon e^{-2a\tau}\hat{H}_k^{1/2}, \tag{113}$$

$$\|\Delta_{0l}\|_{H^k} \leq C\epsilon e^{p\tau-2a\tau+K}\hat{H}_k^{1/2}, \tag{114}$$

$$\|\Delta_{ij}\|_{H^k} \leq C\epsilon e^{-2a\tau+2K}\hat{H}_k^{1/2}, \tag{115}$$

$$\|\Delta_{\psi}\|_{H^k} \leq C\epsilon e^{-2a\tau}\hat{H}_k^{1/2} \tag{116}$$

on I, where Δ_{00}, Δ_{0i}, Δ_{ij} and Δ_ψ are given by (66)-(69), $K = K_0 + \tau_0$, $\hat{H}_k$ is defined in (102) and the constants depend on n, p, k and c_1.

Remark. The bootstrap assumptions only constitute control of $k_0 + 1$ derivatives, but the conclusions of the present lemma, as well as several lemmas to follow, hold for any non-negative integer k.

Proof. Consider

$$\Delta_{00} = (g^{00} + 1)\partial_\tau u + e^{2(\tau+\tau_0)}\tilde{\Delta}_{00}.$$

To estimate the first term using the algorithm, cf. Subsection 9.1 of [20], note that it can be rewritten

$$(g^{00} + 1)p\omega^{-1}\partial_t u. \tag{117}$$

The expression

$$(g^{00} + 1)\partial_t u, \tag{118}$$

is of the type dealt with by the algorithm, and, in the terminology of [20], we compute that $l_\epsilon = 2$, $l_h = 0$ and $l_\partial = 1$. Here l_ϵ gives the number of terms that are "small" (for a precise definition, see [20]), l_h gives the number of downstairs spatial indices minus the number of upstairs spatial indices, including spatial derivatives, and l_∂ is the number of derivatives occurring. Due to the algorithm, the expression (118) can thus be estimated by

$$C\epsilon\omega^{l_\partial} e^{l_h(\Omega+K)-l_\epsilon r}\hat{E}_k^{1/2} = C\epsilon\omega e^{-2a\tau}\hat{E}_k^{1/2},$$

which yields the desired estimate for (117) in view of the fact that $\hat{E}_k$ and $\hat{H}_k$ are equivalent. What remains to be considered is thus

$$e^{2(\tau+\tau_0)}\tilde{\Delta}_{00} = 2e^{2(\tau+\tau_0)}\Delta_{A,00} + 2e^{2(\tau+\tau_0)}\Delta_{C,00} - 2e^{2(\tau+\tau_0)}\Delta_{\phi,00}, \tag{119}$$

cf. (51). Due to Lemma 12 of [20], we have the estimate

$$\|\Delta_{A,00}\|_{H^k} + \|\Delta_{C,00}\|_{H^k} \le C\epsilon\omega^2 e^{-2r}\hat{E}_k^{1/2}.$$

Noting that $\omega^{-2} = p^{-2}e^{2(\tau+\tau_0)}$, this estimate implies

$$e^{2(\tau+\tau_0)}\|\Delta_{A,00}\|_{H^k} + e^{2(\tau+\tau_0)}\|\Delta_{C,00}\|_{H^k} \le C\epsilon e^{-2a\tau}\hat{H}_k^{1/2},$$

which yields the desired estimate for the first two terms on the right hand side of (119). Let us turn to $e^{2(\tau+\tau_0)}\Delta_{\phi,00}$, where $\Delta_{\phi,00}$ is given by (50). An estimate for the first two terms in (50), after multiplication by $e^{2(\tau+\tau_0)}$, follows by estimating $\omega^{-2}\dot{\psi}^2$ and $u\psi$. These objects can be estimated by the algorithm; in both cases $l_\epsilon = 2$ and $l_h = 0$ and in the first case $l_\partial = 2$ whereas $l_\partial = 0$ in the last case. Finally, we need to estimate

$$e^{2(\tau+\tau_0)}g_{00}\Delta_{E,\phi}. \tag{120}$$

Note that $\Delta_{E,\phi}$ is given by (49) and that $V_0 e^{-\lambda\phi_0} = p(np-1)(n-1)/(2t^2)$, cf. (44), so that estimating (120) is the same as estimating

$$g_{00}(e^{-\lambda\psi} - 1 + \lambda\psi) = R(\psi)g_{00}\psi^2$$

for some smooth function R, cf. the proof of Lemma 16 in [20]. This is an object which can be estimated by the algorithm; $l_\epsilon = 2$ and $l_h = l_\partial = 0$. The arguments to derive (114)-(116) are similar. □

6.2. **Estimates for the commutators.** We shall need estimates for the H^k-norm of

$$\hat{F}_0 := \hat{\Box}_g u = -\alpha_1 \partial_\tau u - \beta_1 u + \frac{8}{\lambda}\partial_\tau \psi + 2\lambda p(np-1)\psi - \Delta_{00}, \tag{121}$$

$$\hat{F}_i := \hat{\Box}_g u_i = -\alpha_2 \partial_\tau u_i - \beta_2 u_i + 2pe^{\tau+\tau_0} g^{lm}\Gamma_{lim} + \frac{4e^{\tau+\tau_0}}{\lambda}\partial_i \psi - \Delta_{0i}, \tag{122}$$

$$\hat{F}_{ij} := \hat{\Box}_g h_{ij} = -(np-1)\partial_\tau h_{ij} - [-2pu + 2\lambda p(np-1)\psi]h_{ij} - \Delta_{ij}, \tag{123}$$

$$\hat{F}_\psi := \hat{\Box}_g \psi = -(np-1)\partial_\tau \psi - 2(np-1)\psi + \frac{2}{\lambda}u - \Delta_\psi, \tag{124}$$

where we have used (61)-(64).

Lemma 12. *Let $p > 1$, $c_1 > 1$, K_0, $0 < \epsilon \leq 1$ and τ_0 be real numbers and $n \geq 3$ and $k_0 > n/2 + 1$ be integers. Assume that (g, ψ) satisfy the main bootstrap assumption on an interval I. Assuming (61)-(64) are satisfied (where h_{ij}, u_i and u are defined in terms of g according to (57)-(59)), we conclude that*

$$\|\hat{F}_0\|_{H^k} \leq Ce^{-a\tau}\hat{H}_k^{1/2}, \tag{125}$$

$$\|\hat{F}_m\|_{H^k} \leq Ce^{p\tau - a\tau + K}\hat{H}_k^{1/2}, \tag{126}$$

$$\|\hat{F}_{ij}\|_{H^k} \leq Ce^{-a\tau+2K}\hat{H}_k^{1/2}, \tag{127}$$

$$\|\hat{F}_\psi\|_{H^k} \leq Ce^{-a\tau}\hat{H}_k^{1/2} \tag{128}$$

on I, where $\hat{F}_0$,...,$\hat{F}_\psi$ are defined in (121)-(124) respectively, $K = K_0 + \tau_0$, $\hat{H}_k$ is defined in (102) and the constants depend on n, p, k and c_1.

Proof. Except for the terms

$$2pe^{\tau+\tau_0} g^{lm}\Gamma_{lim}, \quad -[-2pu + 2\lambda p(np-1)\psi]h_{ij}, \tag{129}$$

the conclusions are immediate consequences of (113)-(116), (104)-(108), the definition of $\hat{H}_k$ and the fact that $\epsilon \leq 1$. In order to deal with the first expression appearing in (129), note that we can apply the algorithm, cf. Subsection 9.1 of [20], with $l_\epsilon = 1$, $l_\partial = 1$ and $l_h = 1$ in order to obtain

$$\|2pe^{\tau+\tau_0} g^{lm}\Gamma_{lim}\|_{H^k} \leq Ce^{\tau+\tau_0}\omega e^{p\tau+K-a\tau}\hat{H}_k^{1/2} = Cpe^{p\tau+K-a\tau}\hat{H}_k^{1/2},$$

which is an estimate of the desired form. In order to deal with the second expression appearing in (129), we can also apply the algorithm with $l_\epsilon = 1$, $l_h = 2$ and $l_\partial = 0$, though in order for this to fit with the conventions of [20], we have to rewrite h_{ij} as $e^{-2p\tau}g_{ij}$. We obtain

$$\|[-2pu + 2\lambda p(np-1)\psi]h_{ij}\|_{H^k} \leq Ce^{-2p\tau}e^{2p\tau+2K-a\tau}\hat{H}_k^{1/2},$$

and the lemma follows. □

Lemma 13. *Let $p > 1$, $c_1 > 1$, K_0, $0 < \epsilon \leq 1$ and τ_0 be real numbers and $n \geq 3$ and $k_0 > n/2 + 1$ be integers. Assume that (g, ψ) satisfy the main bootstrap assumption on an interval I. Assuming (61)-(64) are satisfied (where h_{ij}, u_i and u are defined* *in terms of g according to (57)-(59)), we conclude that, for $0 < |\alpha| \leq k$,*

$$\|[\hat{\Box}_g, \partial^\alpha]u\|_2 \leq C\epsilon e^{-2a\tau}\hat{H}_k^{1/2} \quad (130)$$

$$\|[\hat{\Box}_g, \partial^\alpha]u_m\|_2 \leq C\epsilon e^{p\tau-2a\tau+K}\hat{H}_k^{1/2} \quad (131)$$

$$\|[\hat{\Box}_g, \partial^\alpha]h_{ij}\|_2 \leq C\epsilon e^{-2a\tau+2K}\hat{H}_k^{1/2}, \quad (132)$$

$$\|[\hat{\Box}_g, \partial^\alpha]\psi\|_2 \leq C\epsilon e^{-2a\tau}\hat{H}_k^{1/2} \quad (133)$$

on I. Here, $K = K_0 + \tau_0$, $\hat{H}_k$ is defined in (102) and the constants depend on n, p, k, c_1 and an upper bound on e^{-K_0}.

Remark. Note that $a \leq H$ due to (101).

Proof. The result follows from Lemma 13 in [20]. However, in order to be able to see that, we need to translate the terminology of [20] to the current setting. In [20], the notation $\hat{\Box}_g$ occurs, but this object *does not* coincide with the $\hat{\Box}_g$ used in the current paper. Let us denote the object $\hat{\Box}_g$ that occurs in [20] by $\hat{\Box}_g^{\mathrm{old}}$ to distinguish it from the object considered in the present paper. The relation between the two is then given by

$$\hat{\Box}_g = t^2\hat{\Box}_g^{\mathrm{old}} - g^{00}\partial_\tau.$$

This can be restated as follows:

$$\omega^{-2}\hat{\Box}_g^{\mathrm{old}} = p^{-2}(\hat{\Box}_g + g^{00}\partial_\tau).$$

Expressing the statement of Lemma 13 in [20] in terms of the current terminology, we conclude that if, for some smooth v on $I \times \mathbb{T}^n$,

$$\begin{aligned} p^{-1}\|\partial_\tau v\|_{H^k} + p^{-1}e^{-H\tau-K_0}\|\partial_i v\|_{H^k} &+ p^{-2}\|\hat{\Box}_g v + g^{00}\partial_\tau v\|_{H^k} \\ &\leq Ce^{l_h(p\tau+K)-a\tau}\hat{H}_k^{1/2}, \end{aligned}$$

for some $k > n/2 + 1$, then, for $0 < |\alpha| \leq k$,

$$\|[\hat{\Box}_g + g^{00}\partial_\tau, \partial^\alpha]v\|_2 \leq C\epsilon e^{l_h(p\tau+K)-2a\tau}\hat{H}_k^{1/2},$$

where the constant depends on

$$\sup_{t\in I}\omega^{-1}e^{-\Omega-K+r}, \quad (134)$$

which is assumed to be finite. Note that $\omega^{-1}e^{-\Omega-K+r} = p^{-1}e^{-H\tau+a\tau-K_0}$. In order to be allowed to use Lemma 13 of [20], we thus need to have $0 < a \leq p-1$, which is ensured by (101). Furthermore, the constant depends on e^{-K_0}.

Let us reformulate the assumptions and the conclusions. Note that if we assume

$$\|\partial_\tau v\|_{H^k} \leq Ce^{l_h(p\tau+K)-a\tau}\hat{H}_k^{1/2}, \quad (135)$$

then

$$\begin{aligned} \|g^{00}\partial_\tau v\|_{H^k} &\leq \|\partial_\tau v\|_{H^k} + \|(g^{00}+1)\partial_\tau v\|_{H^k} \\ &\leq C[(1+\|1+g^{00}\|_\infty)\|\partial_\tau v\|_{H^k} + \|\partial_\tau v\|_\infty\|1+g^{00}\|_{H^k}] \\ &\leq Ce^{l_h(p\tau+K)-a\tau}\hat{H}_k^{1/2} \end{aligned}$$

due to the bootstrap assumptions, the algorithm applied to $g^{00}+1$, Sobolev embedding and the fact that $\epsilon \leq 1$. As a conclusion we might as well replace $\hat{\Box}_g v + g^{00}\partial_\tau v$

with $\hat{\Box}_g v$ in the assumptions. Concerning the conclusions, note that for $|\alpha| \leq k$,

$$\begin{aligned}\|[g^{00}\partial_\tau, \partial^\alpha]v\|_2 &\leq C\sum_{i=1}^{n}(\|\partial_i g^{00}\|_\infty \|\partial_\tau v\|_{H^{k-1}} + \|\partial_i g^{00}\|_{H^{k-1}}\|\partial_\tau v\|_\infty) \\ &\leq C\epsilon e^{l_h(p\tau+K)-2a\tau}\hat{H}_k^{1/2},\end{aligned}$$

where we have used (135), (109), the bootstrap assumptions and Sobolev embedding. Thus we might as well replace $\hat{\Box}_g v + g^{00}\partial_\tau v$ with $\hat{\Box}_g v$ in the conclusions.

In order to obtain the desired conclusion, all we need to do is to combine the above result with the estimates (104)-(108) and (125)-(128), with one exception. In the case of h_{ij}, $l_h(\Omega + K)$ should be replaced by $2K$. The argument goes through all the same if we simply let $l_h = 0$ in that case and apply the result to $v = e^{-2K}h_{ij}$. □

6.3. **Estimates for the remainder terms in the energy estimates.** In preparation for the final estimate, let us note that the following estimates hold.

Lemma 14. *Let $p > 1$, $c_1 > 1$, K_0, $0 < \epsilon \leq 1$ and τ_0 be real numbers and $n \geq 3$ and $k_0 > n/2 + 1$ be integers. Assume that (g, ψ) satisfy the main bootstrap assumption on an interval I. Then, on I,*

$$\begin{aligned}\left\|\frac{1}{2}\partial_\tau \hat{g}^{ij} + H\hat{g}^{ij}\right\|_\infty &\leq C\epsilon e^{-2H\tau-2K_0}e^{-a\tau}, \\ \|\partial_\tau g^{00}\|_\infty &\leq C\epsilon e^{-a\tau}.\end{aligned}$$

Proof. Note that $\hat{g}^{ij} = e^{2(\tau+\tau_0)}g^{ij} = t^2 g^{ij}$, so that (recall that $H = p - 1$ and that $\omega = p/t$)

$$\begin{aligned}t\partial_t \hat{g}^{ij} &= -2H\hat{g}^{ij} - t^3 g^{ik}(g^{jl}\partial_t g_{kl} - 2\omega\delta_k^j) - t^3 g^{i0}g^{j0}\partial_t g_{00} - t^3 g^{ik}g^{j0}\partial_t g_{0k} \\ &\quad -t^3 g^{i0}g^{jk}\partial_t g_{0k}.\end{aligned}$$

Moving $-2H\hat{g}^{ij}$ over to the left hand side the objects that remain on the right hand side can be estimated using the algorithm; e.g.

$$\|g^{ik}(g^{jl}\partial_t g_{kl} - 2\omega\delta_k^j)\|_{H^k} \leq C\omega e^{-2(p\tau+K)-a\tau}\hat{H}_k^{1/2},$$

since $l_\epsilon = 1$, $l_h = -2$ and $l_\partial = 1$ in this case. Since $t\partial_t = \partial_\tau$, we get the desired conclusion using the bootstrap assumptions and Sobolev embedding. The second estimate follows by a similar argument. □

Finally, we need the following estimates.

Lemma 15. *Let $p > 1$, $c_1 > 1$, K_0, $0 < \epsilon \leq 1$ and τ_0 be real numbers and $n \geq 3$ and $k_0 > n/2 + 1$ be integers. Assume that (g, ψ) satisfy the main bootstrap assumption on an interval I. Then*

$$\|\Delta_E[\partial^\alpha \hat{\mathbf{u}}]\|_1 \leq C\epsilon e^{-a\tau}H_{\mathrm{lp},k}, \tag{136}$$

$$\|\Delta_{E,\gamma_{\mathrm{s}},\delta_{\mathrm{s}}}[\partial^\alpha u_m]\|_1 \leq C\epsilon e^{-a\tau}H_{\mathrm{s},k}, \tag{137}$$

$$\|\Delta_{E,\gamma_{\mathrm{m}},\delta_{\mathrm{m}}}[\partial^\alpha h_{ij}]\|_1 \leq C\epsilon e^{-a\tau}H_{\mathrm{m},k} \tag{138}$$

on I for $|\alpha| \leq k$, where Δ_E and $\Delta_{E,\gamma,\delta}$ are defined in (85) and (89) respectively. The constants depend on n, p, k, c_1 and an upper bound for e^{-K_0}.

Proof. Due to the algorithm, (112) and Sobolev embedding,

$$\|g^{00}+1\|_\infty \leq C\epsilon e^{-a\tau}.$$

Furthermore, using (110) and (111), we conclude that

$$e^{2H\tau+2K_0}\|\partial_i \hat{g}^{lm}\|_\infty + e^{H\tau+K_0+a\tau}\|\partial_i \hat{g}^{0m}\|_\infty + e^{H\tau+K_0+a\tau}\|\hat{g}^{0m}\|_\infty \leq C\epsilon$$

for all i,l,m, due to Sobolev embedding, the fact that $k_0 > n/2+1$ and the fact that the bootstrap assumptions hold. Recall that $\hat{g}^{0i}$ and $\hat{g}^{ij}$ were defined in Lemma 6. Due to these estimates, (90)-(92) and the estimates of Lemma 14, we conclude that

$$\|\Delta_E[\hat{\mathbf{u}}]\|_1 \leq C\epsilon e^{-a\tau}\mathcal{E}[\hat{\mathbf{u}}],$$

where $\mathcal{E}$ was defined in (82) and the constant depends on an upper bound of e^{-K_0}. Note that in order to obtain this conclusion, we used the fact that $a \leq H$, cf. (101). This proves (136). The other estimates follow in a similar fashion, keeping in mind that $\gamma_{\rm m} = \delta_{\rm m} = 0$. □

7. Differential inequalities

Finally, we are in a position to derive the differential inequalities that will be the core of the proof of global existence.

Lemma 16. *Let $p>1$, $c_1>1$, K_0, $0<\epsilon\leq 1$ and τ_0 be real numbers and $n \geq 3$ and $k_0 > n/2+1$ be integers. Assume that (g,ψ) satisfy the main bootstrap assumption on an interval I. Assuming (61)-(64) are satisfied (where h_{ij}, u_i and u are defined in terms of g according to (57)-(59)), we conclude that*

$$\frac{d\hat{H}_{{\rm lp},k}}{d\tau} \leq -2a\hat{H}_{{\rm lp},k} + C\epsilon e^{-a\tau}\hat{H}^{1/2}_{{\rm lp},k}\hat{H}^{1/2}_k \tag{139}$$

$$\frac{d\hat{H}_{{\rm s},k}}{d\tau} \leq -2a\hat{H}_{{\rm s},k} + C\hat{H}^{1/2}_{{\rm s},k}(\hat{H}^{1/2}_{{\rm lp},k}+\hat{H}^{1/2}_{{\rm m},k}) + C\epsilon e^{-a\tau}\hat{H}^{1/2}_{{\rm s},k}\hat{H}^{1/2}_k \tag{140}$$

$$\begin{aligned}\frac{d\hat{H}_{{\rm m},k}}{d\tau} \leq\ & Ce^{-a\tau}\hat{H}_{{\rm m},k} + C\hat{H}^{1/2}_{{\rm lp},k_0}\hat{H}_{{\rm m},k} + C\hat{H}^{1/2}_{{\rm lp},k}\hat{H}^{1/2}_{{\rm m},k} \\ & + C\epsilon e^{-a\tau}\hat{H}^{1/2}_{{\rm m},k}\hat{H}^{1/2}_k\end{aligned} \tag{141}$$

on I, where the constants depend on n, p, k, c_1 and an upper bound on e^{-K_0}.

Proof. Recall that $H_{{\rm lp},k}$ is defined by (97). Due to Corollary 1, where

$$\mathbf{F} = -T^{-1}\begin{pmatrix}\Delta_{00}\\ \Delta_\psi\end{pmatrix}$$

and T is defined in (76), we obtain

$$\frac{dH_{{\rm lp},k}}{d\tau} \leq -2\eta_{\rm lp}H_{{\rm lp},k} + C\epsilon e^{-2a\tau}H^{1/2}_{{\rm lp},k}\hat{H}^{1/2}_k + C\epsilon e^{-a\tau}H_{{\rm lp},k},$$

where we have used (113), (116), (130), (133) and (136). Given the definition of $\hat{H}_{{\rm lp},k}$ and (101), we conclude that (139) holds. Let us turn to $\hat{H}_{{\rm s},k}$. Consider (62). This is an equation for u_i of the form considered in Corollary 2 if we let

$$F_i = 2pe^{\tau+\tau_0}g^{lm}\Gamma_{lim} + \frac{4e^{\tau+\tau_0}}{\lambda}\partial_i\psi - \Delta_{0i}.$$

Due to Corollary 2, (114), (131) and (137), we have

$$\begin{aligned}\frac{dH_{\mathrm{s},k}}{d\tau} &\leq -2\eta_{\mathrm{s}}H_{\mathrm{s},k}+CH_{\mathrm{s},k}^{1/2}\sum_i e^{\tau+\tau_0}(\|g^{lm}\Gamma_{lim}\|_{H^k}+\|\partial_i\psi\|_{H^k})\\ &\quad +C\epsilon e^{p\tau-2a\tau+K}H_{\mathrm{s},k}^{1/2}\hat{H}_k^{1/2}+C\epsilon e^{-a\tau}H_{\mathrm{s},k}.\end{aligned}$$

By (104),

$$e^{\tau+\tau_0}\|\partial_i\psi\|_{H^k}\leq Ce^{p\tau-a\tau+K}\hat{H}_{\mathrm{lp},k}^{1/2}.$$

When estimating $g^{lm}\Gamma_{lim}$ in H^k, it is convenient to divide the terms that appear into two different categories. Due to (107)

$$e^{\tau+\tau_0}\sum_{|\alpha|\leq k}\|g^{lm}\partial^\alpha\partial_j g_{rq}\|_2\leq Ce^{p\tau-a\tau+K}\hat{H}_{\mathrm{m},k}^{1/2}.$$

The second category consists of terms of the form

$$\begin{aligned}e^{\tau+\tau_0}\|\partial^{\alpha_1}\partial_j g^{lm}\partial^{\alpha_2}\partial_i g_{rq}\|_2 &\leq Ce^{\tau+\tau_0}[\|\partial_j g^{lm}\|_\infty\|\partial_i g_{rq}\|_{H^{k-1}}\\ &\quad +\|\partial_j g^{lm}\|_{H^{k-1}}\|\partial_i g_{rq}\|_\infty]\\ &\leq C\epsilon e^{\tau+\tau_0}\hat{H}_k^{1/2},\end{aligned}$$

where $|\alpha_1|+|\alpha_2|\leq k-1$ and we have used (108), (110) and the fact that $k_0>n/2+1$. Due to these observations, the definition of $\hat{H}_{\mathrm{s},k}$ and (101), we obtain the conclusion that (140) holds with a constant depending on an upper bound on e^{-K_0}. Finally, consider $H_{\mathrm{m},k}$ defined by (99). Due to Lemma 7, we have, cf. Corollary 2,

$$\begin{aligned}&\frac{d}{d\tau}\left(\mathcal{E}_{\gamma_{\mathrm{m}},\delta_{\mathrm{m}}}[\partial^\alpha h_{ij}]+\int_{\mathbb{T}^n}e^{-2a\tau}a_\alpha(\partial^\alpha h_{ij})^2dx\right)\\ &\leq -2\eta_{\mathrm{m}}\mathcal{E}_{\gamma_{\mathrm{m}},\delta_{\mathrm{m}}}[\partial^\alpha h_{ij}]+\int_{\mathbb{T}^n}(\partial_\tau\partial^\alpha h_{ij}+\gamma_{\mathrm{m}}\partial^\alpha h_{ij})(\partial^\alpha F_{ij}+[\hat{\Box}_g,\partial^\alpha]h_{ij})dx\\ &\quad +\int_{\mathbb{T}^n}\Delta_{E,\gamma_{\mathrm{m}},\delta_{\mathrm{m}}}[\partial^\alpha h_{ij}]dx-2a\int_{\mathbb{T}^n}e^{-2a\tau}a_\alpha(\partial^\alpha h_{ij})^2dx\\ &\quad +2\int_{\mathbb{T}^n}e^{-2a\tau}a_\alpha\partial^\alpha h_{ij}\partial_\tau\partial^\alpha h_{ij}dx,\end{aligned}$$

where

$$F_{ij}=-[-2pu+2\lambda p(np-1)\psi]h_{ij}-\Delta_{ij}.$$

Due to (101), the fact that $\gamma_{\mathrm{m}}=\delta_{\mathrm{m}}=0$, (115), (132) and (138), we obtain

$$\begin{aligned}\frac{dH_{\mathrm{m},k}}{d\tau} &\leq -2aH_{\mathrm{m},k}+Ce^{-a\tau}H_{\mathrm{m},k}+C\sum_{i,j}[\|uh_{ij}\|_{H^k}+\|\psi h_{ij}\|_{H^k}]H_{\mathrm{m},k}^{1/2}\\ &\quad +C\epsilon e^{-2a\tau+2K}\hat{H}_k^{1/2}H_{\mathrm{m},k}^{1/2}+C\epsilon e^{-a\tau}H_{\mathrm{m},k}.\end{aligned}$$

When estimating uh_{ij} in H^k it is useful to divide the terms into two different categories. Let us first consider

$$\sum_{|\alpha|\leq k}\sum_{i,j}\|h_{ij}\partial^\alpha u\|_2\leq Ce^{2K}H_{\mathrm{lp},k}^{1/2}.$$

The second category consists of terms of the form

$$\sum_{|\alpha_1|+|\alpha_2|\leq k-1}\|\partial^{\alpha_1}\partial_q h_{ij}\partial^{\alpha_2}u\|_2\leq C[\|\partial_q h_{ij}\|_\infty\|u\|_{H^k}+\|u\|_\infty e^{a\tau}H_{\mathrm{m},k}^{1/2}].$$

Since we assume that $k_0 > n/2 + 1$, the bootstrap assumptions imply that

$$\|\partial_q h_{ij}\|_\infty \le C\epsilon e^{2K},$$

cf. (108). Consequently,

$$\|uh_{ij}\|_{H^k} \le C[\hat{H}^{1/2}_{\mathrm{lp},k_0} H^{1/2}_{\mathrm{m},k} + e^{2K} H^{1/2}_{\mathrm{lp},k}].$$

We have a similar estimate for $\|\psi h_{ij}\|_{H^k}$ and consequently we obtain (141). □

8. Global existence

We are now in a position to prove that solutions corresponding to small initial data for (61)-(64) do not become unbounded in finite time. Before we do so, we do, however, need to relate initial data for (24)-(25) to initial data for (61)-(64). A complication arises due to the fact that the background solution we are subtracting has an explicit time dependence. Consequently, we need to determine the starting time based on the data we have. Let $(\varrho, \varsigma, \Phi_a, \Phi_b)$ be given on $\mathbb{T}^n$, where ϱ is a smooth Riemannian metric, ς is a smooth symmetric covariant 2-tensor and Φ_a, Φ_b are smooth functions. Since we wish Φ_a to be close to the background solution, we shall in the end demand that its spatial variation be small. A natural condition to determine the initial time, t_0, is thus

$$\langle \Phi_a \rangle = \frac{2}{\lambda} \ln t_0 - \frac{1}{\lambda} c_0,$$

where $\langle \cdot \rangle$ denotes the mean value over $\mathbb{T}^n$, i.e.

$$\langle \Phi_a \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \Phi_a dx.$$

As a consequence, we make the following definition.

Definition 6. Let $n \ge 3$ be an integer and let $p > 1$. Let $V(\phi)$ be given by (4), where $V_0 > 0$ and λ is given by (11). Let $(\varrho, \varsigma, \Phi_a, \Phi_b)$ be given on $\mathbb{T}^n$, where ϱ is a smooth Riemannian metric, ς is a smooth symmetric covariant 2-tensor and Φ_a, Φ_b are smooth functions. Define the *initial time associated with* $(\varrho, \varsigma, \Phi_a, \Phi_b)$ to be

$$t_0 = \exp\left[\frac{1}{2}(\lambda\langle \Phi_a \rangle + c_0)\right], \tag{142}$$

where c_0 is defined in (12), and define the *initial data for* (61)-(64) *associated with* $(\varrho, \varsigma, \Phi_a, \Phi_b)$ to be

$$u(0,\cdot) = 0, \tag{143}$$
$$(\partial_\tau u)(0,\cdot) = 2np - 2t_0 \mathrm{tr}\varsigma, \tag{144}$$
$$u_i(0,\cdot) = 0, \tag{145}$$
$$(\partial_\tau u_l)(0,\cdot) = \frac{1}{2} t_0 \varrho^{ij}(2\partial_i \varrho_{jl} - \partial_l \varrho_{ij}), \tag{146}$$
$$h_{ij}(0,\cdot) = \varrho_{ij}, \tag{147}$$
$$(\partial_\tau h_{ij})(0,\cdot) = 2t_0 \varsigma_{ij} - 2p\varrho_{ij}, \tag{148}$$
$$\psi(0,\cdot) = \Phi_a - \langle \Phi_a \rangle, \tag{149}$$
$$(\partial_\tau \psi)(0,\cdot) = t_0 \Phi_b - \frac{2}{\lambda}, \tag{150}$$

where all the indices are with respect to the standard frame $\{\partial_i\}$ of the tangent space on $\mathbb{T}^n$.

Lemma 17. *Let $n \geq 3$ be an integer and let $p > 1$. Let $V(\phi)$ be given by (4), where $V_0 > 0$ and λ is given by (11). Let $(\varrho, \varsigma, \Phi_a, \Phi_b)$ be given on $\mathbb{T}^n$, where ϱ is a smooth Riemannian metric, ς is a smooth symmetric covariant 2-tensor and Φ_a, Φ_b are smooth functions. Then $(\varrho, \varsigma, \Phi_a, \Phi_b)$ determine initial data for (24)-(25) according to (28)-(33). Choosing t_0 to be the initial time associated with $(\varrho, \varsigma, \Phi_a, \Phi_b)$, the initial data (28)-(33) for (24)-(25) transform to the initial data (143)-(150) for (61)-(64) under the transformation (57)-(60).*

Proof. The lemma follows by straightforward computations. Note, however, that in the current setting $F_l(t_0, \cdot) = 0$ and $F_0(t_0, \cdot) = n\omega g_{00}(t_0, \cdot) = -np/t_0$. Furthermore, $\phi_0(t_0) = \langle \Phi_a \rangle$ by definition. ☐

In what follows, we shall use the notation

$$K = \ln[4\ell(t_0)], \tag{151}$$

where $\ell(t_0)$ is defined in (13). Note that, using the convention $K = \tau_0 + K_0$, where $\tau_0 = \ln t_0$, we have

$$K_0 = \ln \frac{4}{p-1}.$$

In other words, K_0 only depends on p, so that e^{K_0} and e^{-K_0} can be treated as constants of which we need not keep track.

Theorem 4. *Let $n \geq 3$ be an integer and let $p > 1$. Let $V(\phi)$ be given by (4), where $V_0 > 0$ and λ is given by (11). Let $(\varrho, \varsigma, \Phi_a, \Phi_b)$ be given on $\mathbb{T}^n$, where ϱ is a smooth Riemannian metric, ς is a smooth symmetric covariant 2-tensor and Φ_a, Φ_b are smooth functions. Define initial data for (61)-(64) according to (143)-(150) where $\tau_0 = \ln t_0$ and t_0 is given by (142). Assume there is a constant $c_1 > 2$ such that*

$$\frac{2}{c_1}|v|^2 \leq e^{-2K} h_{ij}(0, x) v^i v^j \leq \frac{c_1}{2}|v|^2 \tag{152}$$

for all $v \in \mathbb{R}^n$ and $x \in \mathbb{T}^n$, where K is given by (151). Let $k_0 > n/2 + 1$ and a be given by (101). There is an $\epsilon_0 > 0$ and a $c_b \in (0, 1)$, where ϵ_0 and c_b should be small enough, depending on n, k_0, p and c_1 such that if

$$\hat{H}_{k_0}^{1/2}(0) \leq c_b \epsilon, \tag{153}$$

for some $\epsilon \leq \epsilon_0$, then the solution to (61)-(64) exists for all future times and (90)-(92) (where η is given by (100)) and

$$\hat{H}_{k_0}^{1/2}(\tau) \leq \epsilon \tag{154}$$

are satisfied for all $\tau \geq 0$.

Remark. Note that a does not appear in $\hat{H}_{k_0}(0)$.

Proof. Note that $p > 1$, $c_1 > 1$, $K_0 = K - \tau_0$, τ_0, $n \geq 3$ and $k_0 > n/2 + 1$ have already been specified. Let $0 < \epsilon \leq 1$ and let $\mathcal{A}$ denote the set of $s \in [0, \infty)$ such that (in the conditions below, we abuse notation by consistently using τ-time, cf. the remark following Lemma 4)

- (g, ψ) satisfy the main bootstrap assumption on $I = [0, s)$.

- (g, ψ) constitute a smooth solution to (61)-(64) on $I \times \mathbb{T}^n$ with initial data as specified in (143)-(150) (where h_{ij}, u_i and u are defined in terms of g according to (57)-(59) and ψ is related to ϕ according to (60)).

Note that if $s \in \mathcal{A}$, then the conditions necessary for deriving the different inequalities above are satisfied on $[0, s)$. Note that (61)-(64) are equivalent to (24)-(25) and that initial data specified by (143)-(150) correspond to initial data defined by (28)-(33). Since Proposition 1 of [20] applies to the equations (24)-(25) with initial data given by (28)-(33), we obtain a unique smooth solution to (61)-(64) on some time interval $(T_{\min}, T_{\max})$. Assume $c_b \le 1/2$. Then (154) is satisfied with a margin for $\tau = 0$, so that it is satisfied on an open time interval containing 0. Since (152) holds and since $u(0, \cdot) = 0$ and $u_i(0, \cdot) = 0$, we conclude that (90)-(92) are satisfied on an open interval containing 0. In particular, there is a $T > 0$ such that $T \in \mathcal{A}$.

Assume $0 < T < \infty$ is such that $T \in \mathcal{A}$. Due to the bootstrap assumptions and the equations, we conclude that u, u_i, h_{ij}, and ϕ do not blow up in C^2. Furthermore, g_{00} and the smallest eigenvalue of $\{h_{ij}\}$ stay bounded away from zero due to (90) and (91). Due to Proposition 1 of [20], we conclude that $T < T_{\max}$. As a consequence, we have a smooth solution beyond T, and the bootstrap assumptions (90)-(92) together with (154) hold on $[0, T]$.

The above arguments lead to the conclusion that $\mathcal{A}$ is closed (note that it is connected by definition). All that remains to be proved is that $\mathcal{A}$ is open. This would yield the conclusion that $\mathcal{A} = [0, \infty)$. Let $T \in \mathcal{A}$. That there exists a solution beyond T is clear from the above. We need to prove that we can improve the bootstrap assumptions in $[0, T)$. Due to (154) and Sobolev embedding, we obtain, cf. (107),

$$e^{a\tau - 2K} \|\partial_\tau h_{ij}\|_\infty \le C\epsilon.$$

Consequently,

$$\|e^{-2p\tau - 2K} g_{ij}(\tau, \cdot) - e^{-2K} g_{ij}(0, \cdot)\|_\infty \le C\epsilon a^{-1} \tag{155}$$

for all $\tau \in [0, T)$. By assuming ϵ to be small enough, we obtain an improvement of (90). By assuming ϵ to be small enough, we also obtain improvements of (91) and (92), due to the definition of the energies, (105), (106) and Sobolev embedding. Finally, we need to improve (154). Note that in $[0, T)$, the conditions of Lemma 16 are satisfied so that (139)-(141) hold in this interval. Note also that e^{-K_0} only depends on p. Thus, in $[0, T)$, we have

$$\frac{d\hat{H}_{\mathrm{lp},k_0}}{d\tau} \le -2a\hat{H}_{\mathrm{lp},k_0} + C\epsilon^2 e^{-a\tau} \hat{H}^{1/2}_{\mathrm{lp},k_0}.$$

This inequality implies

$$\hat{H}^{1/2}_{\mathrm{lp},k_0}(\tau) \le e^{-a\tau} \hat{H}^{1/2}_{\mathrm{lp},k_0}(0) + \frac{1}{2} C\tau e^{-a\tau} \epsilon^2$$

for all $\tau \in [0, T)$. We obtain

$$\hat{H}^{1/2}_{\mathrm{lp},k_0}(\tau) \le C_{\mathrm{lp}}(c_b\epsilon + \epsilon^2) e^{-a\tau/2}. \tag{156}$$

In order to get an estimate for $\hat{H}_{\mathrm{m},k_0}$, let us define

$$f = \exp\left[\frac{C}{a}(e^{-a\tau} - 1)\right],$$

where C is the first constant appearing on the right hand side of (141) for $k = k_0$. Note that $\exp(-C/a) \le f \le 1$ for all $\tau \in [0, T)$. Furthermore, since we can assume that $\hat{H}_{\mathrm{m},k_0} \le 1$, we can estimate $\hat{H}_{\mathrm{m},k_0} \le \hat{H}^{1/2}_{\mathrm{m},k_0}$. If we let

$$\tilde{H}_{\mathrm{m},k} = f \hat{H}_{\mathrm{m},k},$$

and use (156), then (141) yields

$$\frac{d\tilde{H}_{\mathrm{m},k_0}}{d\tau} \le [C c_b \epsilon e^{-a\tau/2} + C\epsilon^2 e^{-a\tau/2}] f^{1/2} \tilde{H}^{1/2}_{\mathrm{m},k_0} \le C c_b \epsilon^2 e^{-a\tau/2} + C\epsilon^3 e^{-a\tau/2},$$

so that

$$\hat{H}_{\mathrm{m},k_0}(\tau) \le e^{C/a} \hat{H}_{\mathrm{m},k_0}(0) + C a^{-1} e^{C/a} [\epsilon^3 + c_b \epsilon^2].$$

We obtain

$$\hat{H}_{\mathrm{m},k_0}(\tau) \le C_{\mathrm{m}} (c_b \epsilon + \epsilon^2) \epsilon, \tag{157}$$

assuming $c_b \le 1$. Consider (140). We have

$$\frac{d\hat{H}_{\mathrm{s},k_0}}{d\tau} \le -2a \hat{H}_{\mathrm{s},k_0} + C_{\mathrm{s}} (c_b^{1/2} \epsilon + \epsilon^{3/2}) \hat{H}^{1/2}_{\mathrm{s},k_0}.$$

We see that the right hand side is negative if

$$2a \hat{H}^{1/2}_{\mathrm{s},k_0} > C_{\mathrm{s}} (c_b^{1/2} \epsilon + \epsilon^{3/2}).$$

By assuming c_b and ϵ to be small enough, depending only on C_{lp}, C_{m} and C_{s}, we conclude that

$$\hat{H}_k^{1/2} \le \frac{1}{3} \epsilon$$

holds in $[0, T)$. Consequently, $\mathcal{A}$ is open and the theorem follows. □

Theorem 5. *Consider a solution to (61)-(64) corresponding to smooth initial data satisfying the conditions of Theorem 4, with k_0 given by the smallest integer strictly larger than $n/2 + 1$. Then, for every k, there is a constant C_k such that*

$$\hat{H}_k^{1/2}(\tau) \le C_k \tag{158}$$

for all $\tau \ge 0$.

Proof. Since the conditions required for deriving the differential inequalities are satisfied for the entire future, we have (139)-(141) for all k and all $\tau \ge 0$. Let us define

$$\tilde{H}_{\mathrm{s},k} = e^{-a\tau/2} \hat{H}_{\mathrm{s},k}, \quad \tilde{H}_{\mathrm{lp},k} = e^{a\tau/2} \hat{H}_{\mathrm{lp},k}.$$

Then

$$\begin{aligned} \frac{d\tilde{H}_{\mathrm{s},k}}{d\tau} &\le -2a \tilde{H}_{\mathrm{s},k} + C e^{-a\tau/4} (\hat{H}^{1/2}_{\mathrm{lp},k} + \hat{H}^{1/2}_{\mathrm{m},k}) \tilde{H}^{1/2}_{\mathrm{s},k} + C\epsilon e^{-5a\tau/4} \hat{H}_k^{1/2} \tilde{H}^{1/2}_{\mathrm{s},k}, \\ \frac{d\tilde{H}_{\mathrm{lp},k}}{d\tau} &\le -a \tilde{H}_{\mathrm{lp},k} + C\epsilon^{-3a\tau/4} \tilde{H}^{1/2}_{\mathrm{lp},k} \hat{H}_k^{1/2}. \end{aligned}$$

Due to these inequalities and (141), we obtain

$$\frac{d\mathcal{H}_k}{d\tau} \le C e^{-a\tau/4} \mathcal{H}_k + C \hat{H}^{1/2}_{\mathrm{lp},k_0} \hat{H}_{\mathrm{m},k}, \tag{159}$$

where

$$\mathcal{H}_k = \tilde{H}_{\mathrm{lp},k} + \tilde{H}_{\mathrm{s},k} + \hat{H}_{\mathrm{m},k}.$$

Due to the fact that $\hat{H}^{1/2}_{\mathrm{m},k_0}$ is bounded for all $\tau \geq 0$ and the fact that (159) holds, we conclude that

$$\frac{d\mathcal{H}_{k_0}}{d\tau} \leq Ce^{-a\tau/4}\mathcal{H}^{1/2}_{k_0}.$$

Thus $\mathcal{H}_{k_0}$ is bounded. Consequently, $\hat{H}^{1/2}_{\mathrm{lp},k_0} \leq Ce^{-a\tau/4}$, which, in combination with (159), yields

$$\frac{d\mathcal{H}_{k}}{d\tau} \leq Ce^{-a\tau/4}\mathcal{H}_{k}.$$

Consequently, $\mathcal{H}_k$ is bounded for all k. This leads to the conclusion that $\hat{H}_{\mathrm{lp},k}$ and $\hat{H}_{\mathrm{m},k}$ are both bounded. If we insert this information into (140), we get

$$\frac{d\hat{H}_{\mathrm{s},k}}{d\tau} \leq -2a\hat{H}_{\mathrm{s},k} + Ce^{-a\tau}\hat{H}_{\mathrm{s},k} + C\hat{H}^{1/2}_{\mathrm{s},k}.$$

By assuming τ to be great enough, the second term on the right hand side can be absorbed in the first. The inequality that results immediately implies that $\hat{H}_{\mathrm{s},k}$ is bounded, since it implies that $\hat{H}_{\mathrm{s},k}$ decays as soon as it exceeds a certain value. The theorem follows. □

9. Causal structure

Recall the outline of the proof of Theorem 2 given in the beginning of Subsection 1.6. In the course of the proof of this theorem, we are interested in the future Cauchy development of a subset of the initial data on $\mathbb{T}^n$ on which the constraint equations are satisfied. The purpose of Proposition 1 below is to yield quantitative control of this set, which we referred to as the global patch in the outline. In Proposition 2, we then prove future causal geodesic completeness.

Proposition 1. *Consider a Lorentz manifold of the type constructed in Theorem 4. Let γ be a future directed causal curve with domain $[s_0, s_{\max})$ such that $\gamma^0(s_0) = t_0$, where t_0 is as in Theorem 4. If the ϵ appearing in the assumptions of Theorem 4 is small enough (depending only on n, p and c_1), then $\dot{\gamma}^0 > 0$ and the length of the spatial part of the curve with respect to the metric at $t = t_0$ satisfies*

$$\int_{s_0}^{s_{\max}} [g_{ij}(t_0, \gamma_\flat)\dot{\gamma}^i\dot{\gamma}^j]^{1/2} ds \leq d(\epsilon)\ell(t_0), \tag{160}$$

where $d(\epsilon)$ is independent of γ, $d(\epsilon) \to 1$ as $\epsilon \to 0$, $\ell(t_0)$ is defined in (13) and $\gamma_\flat = \pi \circ \gamma$ where $\pi : [t_0, \infty) \times \mathbb{T}^n \to \mathbb{T}^n$ is given by $\pi(t, x) = x$. Finally, if γ is future inextendible, then $\gamma^0(s) \to \infty$ as $s \to s_{\max}$.

Remark. The time orientation is assumed to be such that ∂_t is future directed and $\dot{\gamma}^\mu$ is defined by the condition that $\dot{\gamma}^\mu \partial_\mu = \dot{\gamma}$, where ∂_μ is the standard frame for the tangent space of $\mathbb{R}_+ \times \mathbb{T}^n$. The statement $d(\epsilon) \to 1$ as $\epsilon \to 0$ can be improved to the statement: for any $\delta > 0$, there is an ϵ_1 depending only on n, p, c_1 and δ such that if $\epsilon \leq \epsilon_1$, then $|d(\epsilon) - 1| \leq \delta$.

Proof. Due to causality, we have

$$g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu \leq 0. \tag{161}$$

The condition that the curve be future directed is equivalent to

$$g_{00}\dot{\gamma}^0 + g_{0i}\dot{\gamma}^i < 0. \tag{162}$$

Let us work out the consequences of this. Due to (92), we have

$$|2g_{0i}\dot{\gamma}^0\dot{\gamma}^i| \leq \eta^{1/2}|\dot{\gamma}^0|^2 + \eta^{-1/2}|g_{0i}\dot{\gamma}^i|^2 \leq \eta^{1/2}|\dot{\gamma}^0|^2 + \eta^{1/2}c_1^{-1}e^{2\Omega+2K-2r}\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j.$$

Note that the t appearing in e.g. Ω is given by $\gamma^0(s)$. Note, furthermore, that, due to (105), (106) and (154), we can replace η in (91) and (92) by $C\epsilon$, where C only depends on n, p and c_1. Since the last term can be bounded by $\eta^{1/2}g_{ij}\dot{\gamma}^i\dot{\gamma}^j$, due to (90), we obtain

$$g_{ij}\dot{\gamma}^i\dot{\gamma}^j \leq c(\eta)\dot{\gamma}^0\dot{\gamma}^0, \tag{163}$$

where $c(\eta) \to 1$ as $\eta \to 0+$ and we have used (91) and (161). Due to (90), we conclude that

$$\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j \leq c_1c(\eta)e^{-2\Omega-2K}\dot{\gamma}^0\dot{\gamma}^0 = c_1c(\eta)(t/t_0)^{-2p}e^{-2K}\dot{\gamma}^0\dot{\gamma}^0. \tag{164}$$

Note that (155) can be rewritten

$$\|(t/t_0)^{-2p}e^{-2K}g_{ij}(t,\cdot) - e^{-2K}g_{ij}(t_0,\cdot)\|_\infty \leq Ca^{-1}\epsilon,$$

where C only depends on n, p and c_1. Combining this observation with (164), we obtain

$$|e^{-2K}g_{ij}(t_0,\gamma_\flat)\dot{\gamma}^i\dot{\gamma}^j - (t/t_0)^{-2p}e^{-2K}g_{ij}\dot{\gamma}^i\dot{\gamma}^j| \leq Ca^{-1}\epsilon c_1c(\eta)(t/t_0)^{-2p}e^{-2K}\dot{\gamma}^0\dot{\gamma}^0.$$

This observation, together with (163), yields

$$e^{-2K}g_{ij}(t_0,\gamma_\flat)\dot{\gamma}^i\dot{\gamma}^j \leq d^2(\epsilon)(t/t_0)^{-2p}e^{-2K}\dot{\gamma}^0\dot{\gamma}^0, \tag{165}$$

where $d(\epsilon) \to 1$ as $\epsilon \to 0+$ (note that $\eta \to 0+$ as $\epsilon \to 0+$). Consider (162). Note that

$$|g_{0i}\dot{\gamma}^i| \leq [e^{-2\Omega-2K}\delta^{ij}g_{0i}g_{0j}]^{1/2}[e^{2\Omega+2K}\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j]^{1/2} \leq \xi(\epsilon)|\dot{\gamma}^0|,$$

where $\xi(\epsilon) \to 0$ as $\epsilon \to 0+$, due to (92) and (164). Assuming ϵ to be small enough (depending only on n, p and c_1), we conclude that $\dot{\gamma}^0 > 0$, which yields the first conclusion of the proposition. Combining this observation with (165), we obtain (160). Finally, let γ be future inextendible and assume γ^0 does not tend to ∞. Since $\dot{\gamma}^0 > 0$, γ^0 has to converge to a finite number and thus, since we have (164), $\gamma_\flat$ has to converge to a point on $\mathbb{T}^n$. We have a contradiction. □

Proposition 2. *Consider a spacetime of the type constructed in Theorem 4. Assuming the ϵ appearing in the assumptions of Theorem 4 to be small enough (depending only on n, p and c_1), this spacetime is future causally geodesically complete.*

Proof. Let γ be a future directed causal geodesic and assume $(s_{\min}, s_{\max})$ to be the maximal existence interval. In other words, γ is a map from $(s_{\min}, s_{\max})$ into the spacetime satisfying $\gamma'' = 0$, and $(s_{\min}, s_{\max})$ is the maximal existence interval of solutions to the corresponding equation. We shall use the notation $t = \gamma^0(s)$. Due to the equation for a geodesic, we have

$$\ddot{\gamma}^0 + \Gamma^0_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu = 0. \tag{166}$$

Due to (154) and the algorithm, cf. Subsection 9.1 of [20],

$$|\Gamma^0_{00}| \leq C\epsilon\omega e^{-a\tau}, \quad |\Gamma^0_{0i}| \leq C\epsilon\omega e^{p\tau+K-a\tau}, \quad |\Gamma^0_{ij} - \omega g_{ij}| \leq C\epsilon\omega e^{2p\tau+2K-a\tau}.$$

Consequently, $\Gamma^0_{ij}\dot{\gamma}^i\dot{\gamma}^j \geq 0$ for t large enough (or ϵ small enough). Due to these estimates and (164), we conclude that

$$|\Gamma^0_{00}\dot{\gamma}^0\dot{\gamma}^0| + 2|\Gamma^0_{0i}\dot{\gamma}^0\dot{\gamma}^i| \leq C\epsilon\omega e^{-a\tau}|\dot{\gamma}^0|^2,$$

where C only depends on n, p and c_1. Combining these pieces of information with (166), we obtain

$$\ddot{\gamma}^0 \leq C\epsilon\omega e^{-a\tau}\dot{\gamma}^0\dot{\gamma}^0 = C\epsilon p t^{-1}\left(\frac{t}{t_0}\right)^{-a}\dot{\gamma}^0\dot{\gamma}^0$$

for $s \geq s_1$. Since $\dot{\gamma}^0 > 0$, assuming ϵ to be small enough (depending only on n, p and c_1), we can divide by $\dot{\gamma}^0$ in this equation and integrate in order to obtain (recall that $t = \gamma^0(s)$)

$$\ln\frac{\dot{\gamma}^0(s)}{\dot{\gamma}^0(s_1)} \leq C\epsilon p\int_{s_1}^{s} t^{-1}\left(\frac{t}{t_0}\right)^{-a}\dot{\gamma}^0 ds = C\epsilon p\int_{\gamma^0(s_1)}^{\gamma^0(s)} t^{-1}\left(\frac{t}{t_0}\right)^{-a} dt \leq C\epsilon p a^{-1},$$

if we assume s_1 to be large enough that $\gamma^0(s_1) \geq t_0$. Thus $\dot{\gamma}^0$ is bounded to the future. Consequently,

$$\gamma^0(s) - \gamma^0(s_0) = \int_{s_0}^{s}\dot{\gamma}^0(s)ds \leq C|s - s_0|.$$

Since $\gamma^0(s) \to \infty$ as $s \to s_{\max}$, we conclude that $s_{\max} = \infty$. □

10. Asymptotic expansions

Let us derive conclusions concerning the asymptotic behaviour which are more detailed than (158).

Proposition 3. *Consider a spacetime of the type constructed in Theorem 4. Then, assuming ϵ to be small enough in this construction (depending on n, p, c_1 and k_0), there is a positive constant $\alpha > 0$, a smooth Riemannian metric ρ on $\mathbb{T}^n$ and, for every $l \geq 0$, a constant K_l (depending on n, l, p and c_1) such that for all $t \geq t_0$,*

$$\left\|\phi(t,\cdot) - \frac{2}{\lambda}\ln t + \frac{1}{\lambda}c_0\right\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha}, \tag{167}$$

$$\left\|(t\partial_t\phi)(t,\cdot) - \frac{2}{\lambda}\right\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha}, \tag{168}$$

$$\|(g_{00}+1)(t,\cdot)\|_{C^l} + \|(t\partial_t g_{00})(t,\cdot)\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha}, \tag{169}$$

$$\begin{aligned}&\left\|t^{-1}g_{0i}(t,\cdot) - \frac{1}{np-2p+1}\rho^{jm}\gamma_{jim}\right\|_{C^l}\\ &+\|[t\partial_t(t^{-1}g_{0i})](t,\cdot)\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha},\end{aligned} \tag{170}$$

$$\begin{aligned}&\|(t/t_0)^{-2p}e^{-2K}g_{ij}(t,\cdot) - \rho_{ij}\|_{C^l}\\ &+\|(t/t_0)^{-2p}e^{-2K}t\partial_t g_{ij}(t,\cdot) - 2p\rho_{ij}\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha},\end{aligned} \tag{171}$$

$$\|(t/t_0)^{2p}e^{2K}g^{ij}(t,\cdot) - \rho^{ij}\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha}, \tag{172}$$

$$\|(t/t_0)^{-2p}e^{-2K}tk_{ij}(t,\cdot) - p\rho_{ij}\|_{C^l} \leq K_l\,(t/t_0)^{-\alpha}, \tag{173}$$

where γ_{jim} are the Christoffel symbols associated with the metric ρ and $k_{ij}(t,\cdot)$ are the components of the second fundamental form induced on the hypersurface $\{t\}\times\mathbb{T}^n$ with respect to the standard vectorfields on $\mathbb{T}^n$. Here $\|\cdot\|_{C^l}$ denotes the C^l norm on $\mathbb{T}^n$.

Proof. Let us begin by observing that (due to (158), (104), (105) and (107)) u, ψ, u_τ, ψ_τ and $e^{-2K}\partial_\tau h_{ij}$ are decaying in any C^l norm as $e^{-a\tau}$. As a consequence, (167)-(169) hold and there are smooth functions ρ_{ij} on $\mathbb{T}^n$ such that for every $k \geq 0$, there is a constant K_l such that

$$\|e^{-2K}h_{ij}(\tau,\cdot) - \rho_{ij}\|_{C^l} \leq K_l e^{-a\tau}$$

for all $\tau \geq 0$. This leads to the conclusion that (171) holds. Furthermore, $e^{-2K}h_{ij}$ is bounded in any C^l norm. Consider

$$e^{2K}\partial_\tau(e^{2p\tau}g^{ij}) = 2pe^{2p\tau+2K}g^{ij} - e^{2p\tau+2K}g^{i\mu}g^{j\nu}\partial_\tau g_{\mu\nu}.$$

Using the algorithm, one can conclude that the right hand side is bounded by $e^{-a\tau}$ in any C^l norm. In other words, there are smooth functions ρ^{ij} on $\mathbb{T}^n$ such that

$$\|e^{2p\tau+2K}g^{ij}(\tau,\cdot) - \rho^{ij}\|_{C^l} \leq K_l e^{-a\tau}.$$

From the above, we conclude that $\rho^{ij}\rho_{jk} = \delta^i_k$ (note that $g^{i0}g_{0m}$ converges to zero due to the algorithm), so that ρ_{ij} are the components of a Riemannian metric on $\mathbb{T}^n$ and ρ^{ij} are the components of the inverse of the matrix with components ρ_{ij}. Furthermore, (172) holds. If we let γ_{lim} denote the Christoffel symbols of ρ, we obtain, in particular, that

$$\|(g^{lm}\Gamma_{lim})(\tau,\cdot) - \rho^{lm}\gamma_{lim}\|_{C^l} \leq K_l e^{-a\tau}. \tag{174}$$

We wish to improve our knowledge concerning u_i. Consider (62). Note that a term of the form

$$-2pe^{\tau+\tau_0}g^{lm}\Gamma_{lim}$$

appears in this equation. Since, by the above observations, $g^{lm}\Gamma_{lim}$ converges to something which is not necessarily zero, it is clear that this object may tend to infinity. It therefore seems natural to rescale the equation and to introduce

$$\hat{u}_i = e^{-\tau-\tau_0}u_i.$$

Using (62), we obtain

$$\begin{aligned} &\hat{\Box}_g\hat{u}_i + \hat{\alpha}_2\partial_\tau\hat{u}_i + \hat{\beta}_2\hat{u}_i \\ &-2pg^{lm}\Gamma_{lim} - \frac{4}{\lambda}\partial_i\psi + e^{-\tau-\tau_0}\Delta_{0i} - (g^{00}+1)(2\partial_\tau\hat{u}_i + \hat{u}_i) - 2\hat{g}^{0j}\partial_j\hat{u}_i = 0, \end{aligned} \tag{175}$$

where

$$\hat{\alpha}_2 = \alpha_2 + 2 = np + 1, \quad \hat{\beta}_2 = \beta_2 + \alpha_2 + 1 = p(n-2)(2p-1) + np.$$

Note that the first three terms on the left hand side of (175) are such that Lemma 7 applies. In particular, there are strictly positive constants $\hat{\gamma}_{\rm s}$, $\hat{\delta}_{\rm s}$, $\hat{\zeta}_{\rm s}$ and $\hat{\eta}_{\rm s}$ as specified in Lemma 7. Assuming ϵ to be small enough in the original construction of the development, we are allowed to use the conclusions of Lemma 7 as well as the conclusions of Corollary 2. In particular, we can define an energy as described in Corollary 2,

$$\mathfrak{E}_l = \sum_i \sum_{|\alpha|\leq l} \mathcal{E}_{\hat{\gamma}_{\rm s},\hat{\delta}_{\rm s}}[\partial^\alpha \hat{u}_i].$$

Note that $\hat{u}_i(0,\cdot) = 0$ and that

$$(\partial_\tau\hat{u}_i)(0,\cdot) = e^{-\tau_0}(\partial_\tau u_i)(0,\cdot).$$

Considering (146), it is clear that this object is small in H^{k_0}. As a consequence, $\mathfrak{E}_{k_0}(0)$ is also small. Due to Lemma 7, we have

$$\hat{\zeta}_{\mathrm{s}} \sum_i \sum_{|\alpha|\leq l} \int_{\mathbb{T}^n} [(\partial^\alpha \partial_\tau \hat{u}_i)^2 + \hat{g}^{lm} \partial^\alpha \partial_l \hat{u}_i \partial^\alpha \partial_m \hat{u}_i + (\partial^\alpha \hat{u}_i)^2] dx \leq \mathfrak{E}_l.$$

Due to Corollary 2, we have

$$\frac{d\mathfrak{E}_l}{d\tau} \leq -2\hat{\eta}_{\mathrm{s}} \mathfrak{E}_l + C\mathfrak{E}_l^{1/2} \sum_{|\alpha|\leq l} \sum_i \|\partial^\alpha \hat{F}_i + [\hat{\Box}_g, \partial^\alpha]\hat{u}_i\|_2 + C\epsilon e^{-a\tau} \mathfrak{E}_l, \tag{176}$$

where the constant depends on an upper bound on e^{-K_0} and we have argued similarly to the proof of Lemma 15 to deal with the term arising from $\Delta_{E,\hat{\gamma}_{\mathrm{s}},\hat{\delta}_{\mathrm{s}}}[\partial^\alpha \hat{u}_i]$ (note that the only difference between proving the estimate needed for (176) and proving (137) is that the constants γ and δ are different, something which does not affect the arguments) and $\hat{F}_i$ is given by

$$\hat{F}_i = 2pg^{lm}\Gamma_{lim} + \frac{4}{\lambda}\partial_i \psi - e^{-\tau-\tau_0}\Delta_{0i} + (g^{00}+1)(2\partial_\tau \hat{u}_i + \hat{u}_i) + 2\hat{g}^{0j}\partial_j \hat{u}_i.$$

Note that the first and the second terms in $\hat{F}_i$ are bounded in any C^l norm (in fact, the second term is exponentially decaying in any C^l norm). Since

$$\|(g^{00}+1)(\tau,\cdot)\|_{C^m} \leq C_m e^{-a\tau}, \quad \|\hat{g}^{0j}(\tau,\cdot)\|_{C^m} \leq C_m e^{-H\tau - K_0 - a\tau}$$

for any m, we conclude that the fourth and fifth terms in $\hat{F}_i$ can be bounded by

$$Ce^{-a\tau}\mathfrak{E}_l^{1/2}. \tag{177}$$

Thus

$$\|\hat{F}_i(\tau,\cdot)\|_{H^l} \leq K_l(1 + e^{-a\tau}\mathfrak{E}_l^{1/2}) + e^{-\tau-\tau_0}\|\Delta_{0i}(\tau,\cdot)\|_{H^l}$$

and

$$\|\hat{F}_i(\tau,\cdot) - 2pg^{lm}\Gamma_{lim}\|_{H^l} \leq K_l e^{-a\tau}(1 + \mathfrak{E}_l^{1/2}) + e^{-\tau-\tau_0}\|\Delta_{0i}(\tau,\cdot)\|_{H^l}.$$

Consider (67). The first term appearing on the right hand side of this expression can, after multiplication with $e^{-\tau-\tau_0}$, be estimated by the expression appearing in (177), so that

$$e^{-\tau-\tau_0}\|\Delta_{0i}(\tau,\cdot)\|_{H^l} \leq Ce^{-a\tau}\mathfrak{E}_l^{1/2} + e^{\tau+\tau_0}\|\tilde{\Delta}_{0i}(\tau,\cdot)\|_{H^l}.$$

We thus focus on $e^{\tau+\tau_0}\tilde{\Delta}_{0i}$. The expression $\tilde{\Delta}_{0i}$ is given in (52). The third term on the right hand side is, after multiplication with $t = e^{\tau+\tau_0}$, given by $-2\partial_\tau\psi\partial_i\psi$, an object which decays exponentially. The fourth term, after multiplication by t, is given by

$$2p(np-1)\lambda\psi\hat{u}_i,$$

so that it can be estimated by $Ce^{-a\tau}\mathfrak{E}_l^{1/2}$. Since the fifth term can be written, after multiplication by t,

$$-\frac{4t^2\Delta_{E,\phi}}{n-1}\hat{u}_i$$

and $t^2\Delta_{E,\phi}$ is exponentially decaying with respect to any C^m norm as $e^{-2a\tau}$, we get a similar estimate for it. Thus

$$e^{\tau+\tau_0}\|\tilde{\Delta}_{0i}(\tau,\cdot)\|_{H^l} \leq K_l e^{-a\tau}(1 + \mathfrak{E}_l^{1/2}) + 2t\|\Delta_{A,0i} + \Delta_{C,0i}\|_{H^l},$$

where $\Delta_{A,0i}$ and $\Delta_{C,0i}$ are given in (87) and (93) of [20] respectively. Assume a term in $t\Delta_{A,0i}$ or $t\Delta_{C,0i}$ contains a factor g_{0i}. If we extract $t^{-1}g_{0i} = \hat{u}_i$ from this term, what remains is t^2 times an expression to which we can apply the algorithm with $l_\epsilon \geq 1$, $l_h = 0$, $l_\partial = 2$ (sometimes l_∂ may be less than 2, but this will then be compensated for by a corresponding number of factors of ω). By the algorithm, the factor multiplying $\hat{u}_i$ can thus be estimated by

$$t^2 K_m \omega^2 e^{-a\tau} \hat{E}_m^{1/2} \leq C e^{-a\tau}$$

in H^m, and the corresponding term can be estimated by

$$Ce^{-a\tau}\mathfrak{E}_l^{1/2}. \tag{178}$$

Note that

$$g^{0i} = -\frac{1}{g_{00}} g^{ij} g_{0j},$$

so that a term appearing in $t\Delta_{A,0i}$ or $t\Delta_{C,0i}$ which contains a factor of g^{0i} can also be estimated as in (178). Assume a term in $t\Delta_{A,0i}$ or $t\Delta_{C,0i}$ contains a factor

$$\partial_t g_{0i} = \partial_\tau \hat{u}_i + \hat{u}_i.$$

What remains of this term after extracting $\partial_t g_{0i}$ is then t times something to which the algorithm can be applied with $l_\epsilon \geq 1$, $l_h = 0$ and $l_\partial = 1$ (with the same caveat as before). Applying the algorithm, one sees that the original term can be estimated by (178). If a term contains a factor of the form $\partial_j g_{0i}$, one can argue similarly to the above to conclude that it is bounded by

$$Ce^{-H\tau - K_0 - a\tau}\|\partial_j \hat{u}_i\|_{H^l},$$

which in its turn is bounded by (178). If a term contains a factor of the form $\partial_i g_{00}$, we can extract this term, and conclude, by the algorithm, that what remains is exponentially decaying so that the term we started with had to be exponentially decaying in any C^m norm. The argument to deal with terms containing a factor of the form $\partial_i g_{lm}$ is similar. Since all the terms in $\Delta_{A,0i}$ and $\Delta_{C,0i}$ are such that each term falls into one of the categories described above, cf. (87) and (93) of [20], we obtain

$$2t\|\Delta_{A,0i} + \Delta_{C,0i}\|_{H^l} \leq Ce^{-a\tau}(1 + \mathfrak{E}_l^{1/2}).$$

To conclude,

$$\|\hat{F}_i(\tau,\cdot)\|_{H^l} \leq K_l + K_l e^{-a\tau}(1 + \mathfrak{E}_l^{1/2}). \tag{179}$$

Note also that

$$\|\hat{F}_i(\tau,\cdot) - 2p(g^{lm}\Gamma_{lim})(\tau,\cdot)\|_{H^l} \leq K_l e^{-a\tau}(1 + \mathfrak{E}_l^{1/2}). \tag{180}$$

What remains to be estimated is

$$\|[\hat{\Box}_g, \partial^\alpha]\hat{u}_i\|_2$$

for $|\alpha| \leq k$. Estimate, using (158) and (111),

$$\|\partial^{\alpha_1}(\partial_j \hat{g}^{0m})\partial^{\alpha_2}\partial_\tau\partial_m \hat{u}_i\|_2 \leq Ce^{-H\tau - K_0 - a\tau}\|\partial_\tau \hat{u}_i\|_{H^l} \leq Ce^{-a\tau}\mathfrak{E}_l^{1/2}$$

where $|\alpha_1| + |\alpha_2| = |\alpha| - 1$ and the constant depends on an upper bound for e^{-K_0}. Similarly, we get an estimate

$$\|\partial^{\alpha_1}(\partial_j \hat{g}^{lm})\partial^{\alpha_2}\partial_l\partial_m \hat{u}_i\|_2 \leq Ce^{-a\tau}\mathfrak{E}_l^{1/2}.$$

Consider

$$\|\partial^{\alpha_1}(\partial_j g^{00})\partial^{\alpha_2}\partial_\tau^2 \hat{u}_i\|_2 \leq Ce^{-a\tau}\|\partial_\tau^2 \hat{u}_i\|_{H^{l-1}}.$$

We have

$$\partial_\tau^2 \hat{u}_i = -\frac{1}{g^{00}} \left(2\hat{g}^{0m}\partial_\tau\partial_m \hat{u}_i + \hat{g}^{lm}\partial_l\partial_m \hat{u}_i - \hat{\alpha}_2\partial_\tau\hat{u}_i - \hat{\beta}_2\hat{u}_i + \hat{F}_i \right).$$

Due to the estimates given above, we obtain

$$\|\partial^{\alpha_1}(\partial_j g^{00})\partial^{\alpha_2}\partial_\tau^2\hat{u}_i\|_2 \le Ce^{-a\tau}(1+\mathfrak{E}_l^{1/2}).$$

Thus

$$\|[\hat{\Box}_g, \partial^\alpha]\hat{u}_i\|_2 \le Ce^{-a\tau}(1+\mathfrak{E}_l^{1/2}). \tag{181}$$

Inserting (179) and (181) into (176), we get

$$\frac{d\mathfrak{E}_l}{d\tau} \le -2\hat{\eta}_{\mathrm{s}}\mathfrak{E}_l + C\mathfrak{E}_l^{1/2} + Ce^{-a\tau}\mathfrak{E}_l,$$

which leads to the conclusion that $\mathfrak{E}_l$ is bounded for all l, since it implies that $\mathfrak{E}_l$ is decreasing after it has exceeded a certain value. Let us introduce

$$\tilde{u}_i(\tau,\cdot) = \hat{u}_i(\tau,\cdot) - \frac{2p}{\hat{\beta}_2}\rho^{lm}\gamma_{lim}.$$

Then,

$$\hat{\Box}_g\tilde{u}_i + \hat{\alpha}_2\partial_\tau\tilde{u}_i + \hat{\beta}_2\tilde{u}_i = \tilde{F}_i, \tag{182}$$

where

$$\tilde{F}_i = \hat{F}_i - 2p\rho^{rq}\gamma_{riq} + \hat{g}^{lm}\partial_l\partial_m\left[\frac{2p}{\hat{\beta}_2}\rho^{rq}\gamma_{riq}\right],$$

so that

$$\|\tilde{F}_i\|_{C^m} \le C_m e^{-a\tau}$$

for all m due to (174) and (180). Note also that as a consequence of (182), the fact that $\hat{u}_i$ and $\partial_\tau\hat{u}_i$ are bounded in any C^m norm and the fact that $\hat{g}^{ij}$ and $\hat{g}^{0i}$ are exponentially decaying in any C^m norm as $e^{-H\tau}$, we have

$$\|\hat{g}^{lq}\partial_l\partial_q\tilde{u}_i\|_{C^m} + \|\hat{g}^{0l}\partial_l\partial_\tau\tilde{u}_i\|_{C^m} + \|(g^{00}+1)\partial_\tau^2\tilde{u}_i\|_{C^m} \le Ce^{-a\tau}.$$

Combining this observation with (182), we conclude that

$$\partial_\tau^2\tilde{u}_i + \hat{\alpha}_2\partial_\tau\tilde{u}_i + \hat{\beta}_2\tilde{u}_i = \tilde{\mathcal{F}}_i,$$

where $\tilde{\mathcal{F}}_i$ satisfies the same kind of estimate as $\tilde{F}_i$. By arguments similar to those used to prove Lemma 7, one can prove that $\tilde{u}_i$ is exponentially decaying in every C^m norm as well as $\partial_\tau\hat{u}_i = \partial_\tau\tilde{u}_i$. We obtain (170).

Finally, let us turn to the second fundamental form. Note that the future directed unit normal is given by

$$N = -(-g^{00})^{-1/2}g^{0\mu}\partial_\mu.$$

Thus

$$k_{ij} = \langle\nabla_{\partial_i}N, \partial_j\rangle = -\partial_i[(-g^{00})^{-1/2}g^{0\mu}]g_{\mu j} - (-g^{00})^{-1/2}g^{0\mu}\Gamma_{ij\mu}.$$

With the exception of

$$\frac{1}{2}(-g^{00})^{1/2}\partial_t g_{ij},$$

all the terms appearing in k_{ij} can be estimated using the algorithm with $l_h = 2$, $l_\partial = 1$ and $l_\epsilon \ge 1$, i.e. by

$$C\omega e^{2p\tau+2K-a\tau}\hat{H}_l^{1/2},$$

so that for every $l \geq 0$, there is a constant C_l such that

$$\left\| (t/t_0)^{-2p} e^{-2K} \left[t k_{ij} - \frac{1}{2}(-g^{00})^{1/2} t \partial_t g_{ij} \right] (t, \cdot) \right\|_{C^l} \leq C_l \left(\frac{t}{t_0} \right)^{-\alpha}.$$

Since $g^{00} + 1$ is exponentially decaying and (171) holds, we conclude that (173) holds. □

11. Proof of the main theorem

Proof of Theorem 2. Consider $\mathbb{T}^n$ to be $[-\pi, \pi]^n$ with the ends identified.

Construction of a global (in time) patch. Let us start by constructing a patch of spacetime which is essentially the development of the piece of the data over which we have some control. Let $f_c \in C_0^\infty[B_1(0)]$ be such that $f_c(p) = 1$ for $|p| \leq 15/16$ and $0 \leq f_c \leq 1$. In order to apply Theorem 4, we need to define a Riemannian metric on $\mathbb{T}^n$, a symmetric covariant 2-tensor and two functions. We define them by

$$\begin{aligned} \varrho_{ij} &= f_c \rho_{ij} \circ x^{-1} + (1 - f_c) e^{2K} \delta_{ij} \\ \varsigma_{ij} &= f_c \kappa_{ij} \circ x^{-1} + \frac{p}{t_0}(1 - f_c) e^{2K} \delta_{ij} \\ \Phi_a &= f_c \phi_a \circ x^{-1} + (1 - f_c)\langle \phi_a \rangle \\ &\quad -\langle f_c(\phi_a \circ x^{-1} - \langle \phi_a \rangle)\rangle \langle 1 - f_c \rangle^{-1}(1 - f_c) \\ \Phi_b &= f_c \phi_b \circ x^{-1} + (1 - f_c)\frac{2}{\lambda t_0}, \end{aligned} \tag{183}$$

where t_0 and K are given by (15) and where the indices on the right hand side refer to the coordinates x assumed to exist in the statement of the theorem, δ_{ij} are the components of the Kronecker delta and the indices on the left hand side refer to the standard coordinates on $\mathbb{T}^n$. The choice (183) requires some motivation. The last term is there to ensure that

$$\langle \Phi_a \rangle = \langle \phi_a \rangle \tag{184}$$

while, at the same time, ensuring that Φ_a equals $\phi_a \circ x^{-1}$ in the set of interest. The reason it is of importance to have (184) is that it ensures that t_0 defined in Theorem 2 coincides with t_0 defined in Theorem 4. We can view $(\varrho, \varsigma, \Phi_a, \Phi_b)$ as initial data on $\mathbb{T}^n$. Given these data, we can define initial data for (61)-(64) by (143)-(150). Due to (16),

$$\|e^{-2K}\varrho - \delta\|_{H^{k_0+1}} = \left\| f_c\{e^{-2K}\rho \circ x^{-1} - \delta\} \right\|_{H^{k_0+1}} \leq C\epsilon. \tag{185}$$

Furthermore, due to (16),

$$\|e^{-2K} t_0 \varsigma - p\delta\|_{H^{k_0}} = \|f_c\{t_0 e^{-2K}\kappa \circ x^{-1} - p\delta\}\|_{H^{k_0}} \leq C\epsilon,$$

which implies

$$\|2e^{-2K} t_0 \varsigma - 2pe^{-2K}\varrho\|_{H^{k_0}} \leq C\epsilon. \tag{186}$$

Since the object inside the norm in (186) corresponds to $e^{-2K}\partial_\tau h_{ij}(0, \cdot)$, cf. (148), the estimates (185) and (186) imply that

$$\hat{H}^{1/2}_{\mathrm{m},k_0}(0) \leq C\epsilon.$$

Note also that due to (185), we have (152) for some suitable $c_1 > 2$. Let us turn to $\hat{H}_{\mathrm{lp},k_0}(0)$. Since $u(0,\cdot) = 0$, we only need concern ourselves with $\partial_\tau u(0,\cdot)$ and the initial data for ψ. The initial data for $\partial_\tau u$ is given by (144). Note that (185) implies that

$$\|e^{2K}\varrho^{ij} - \delta^{ij}\|_{H^{k_0}} \leq C\epsilon, \tag{187}$$

assuming ϵ to be small enough, where ϱ^{ij} are the components of the inverse of the matrix with components ϱ_{ij}. Combining this observation with (186), we conclude that

$$\|t_0\varrho^{ij}\varsigma_{ij} - np\|_{H^{k_0}} \leq C\epsilon.$$

Thus the part of $\hat{H}^{1/2}_{\mathrm{lp},k_0}$ coming from u is bounded by $C\epsilon$, since the object appearing inside the norm is, up to a numerical factor, the right hand side of (144). Turning to ψ,

$$\psi(0,\cdot) = \Phi_a - \langle\Phi_a\rangle = f_c(\phi_a \circ x^{-1} - \langle\phi_a\rangle) - \langle f_c(\phi_a \circ x^{-1} - \langle\phi_a\rangle)\rangle\langle 1 - f_c\rangle^{-1}(1 - f_c),$$

so that, due to (16) (recall that $\langle\phi_a\rangle = \langle\Phi_a\rangle = \phi_0(t_0)$),

$$\|\psi(0,\cdot)\|_{H^{k_0+1}} \leq C\epsilon.$$

Consider

$$\partial_\tau\psi(0,\cdot) = t_0\Phi_b - \frac{2}{\lambda} = f_c\left(t_0\phi_b \circ x^{-1} - \frac{2}{\lambda}\right).$$

Due to (16), we have

$$\|\partial_\tau\psi(0,\cdot)\|_{H^{k_0}} \leq C\epsilon.$$

The above estimates together imply

$$\hat{H}_{\mathrm{lp},k_0}(0) \leq C\epsilon.$$

What remains to be considered is $\hat{H}^{1/2}_{\mathrm{s},k_0}$. Since $u_i(0,\cdot) = 0$, we need only estimate

$$e^{-K}\partial_\tau u_i(0,\cdot) = \frac{p-1}{4t_0}\frac{1}{2}t_0\varrho^{lj}(2\partial_l\varrho_{ji} - \partial_i\varrho_{jl}),$$

cf. (146). Due to (185) and (187), we get

$$\|e^{-K}\partial_\tau u_i(0,\cdot)\|_{H^{k_0}} \leq C\epsilon,$$

so that

$$\hat{H}^{1/2}_{\mathrm{s},k_0}(0) \leq C\epsilon.$$

To conclude,

$$\hat{H}^{1/2}_{k_0}(0) \leq C\epsilon$$

where the constant depends on n and p. Note, furthermore, that c_1 is numerical in the current setting, that K_0 only depends on p and that k_0 only depends on n. As a consequence, we get the conclusions of Theorem 4, assuming ϵ to be small enough depending only on n and p. In particular, we get a solution, say $(\bar{g}, \Phi)$, on $(t_-, \infty) \times \mathbb{T}^n$. Note that we also get asymptotics as in the statement of Proposition 3.

Note that the variables used in (34)-(37) are related to the variables in (61)-(64) according to (57)-(60). Using these relations, we get solutions to the original equations (34)-(37). Furthermore, on $B_{15/16}(0)$, the constraint equations are satisfied, and we have chosen the initial data in such a way that $\mathcal{D}_\mu|_{t=t_0} = 0$ (cf. Lemma 17 and the comments made in Subsection 2.1). Due to standard local existence and uniqueness results, cf. Proposition 1 of [20], we conclude that in $D[\{t_0\}\times B_{15/16}(0)]$, the solution $(\bar{g},\Phi)$ satisfies (5) and (6). If ϵ is small enough, Proposition 1 implies that

$$(t_-,\infty)\times B_{5/8}(0)\subseteq D[\{t_0\}\times B_{29/32}(0)], \tag{188}$$

where we increase t_- if necessary. The reason for this is that, first of all, (185) and Sobolev embedding yield (here $\bar{g}_{ij}(t_0,\cdot)=\varrho_{ij}$)

$$[4\ell(t_0)]^2|v|^2\leq d_1^2(\epsilon)\bar{g}_{ij}(t_0,\cdot)v^iv^j$$

for all $v\in\mathbb{R}^n$, where $d_1(\epsilon)\to 1$ as $\epsilon\to 0$. Due to (160), we then obtain

$$4\ell(t_0)\int_{s_0}^{s_{\max}}[\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j]^{1/2}ds\leq d(\epsilon)d_1(\epsilon)\ell(t_0).$$

For ϵ small enough, we thus get

$$\int_{s_0}^{s_{\max}}[\delta_{ij}\dot{\gamma}^i\dot{\gamma}^j]^{1/2}ds\leq\frac{9}{32},$$

which implies (188). Note that due to Lemma 3 of [20], see also the proof corresponding to the present one in Section 16 of [20], the sets

$$\begin{aligned} U_{0,\text{exc}} &= D[\{t_0\}\times B_{15/16}(0)],\quad U_{1,\text{exc}}=D[\{t_0\}\times B_{29/32}(0)],\\ U_{2,\text{exc}} &= D[\{t_0\}\times \bar{B}_{29/32}(0)], \end{aligned}$$

are open, open and closed subsets of $(t_-,\infty)\times x(U)$ respectively. Consequently, $W_{i,\text{exc}}=(\text{Id}\times x^{-1})(U_{i,\text{exc}})$ for $i=0,1,2$ are also open, open and closed respectively.

Construction of a reference metric. In order to prove that the patches that we construct fit together to form a globally hyperbolic development, it is convenient to construct a reference metric. Let

$$\tilde{g}=(1-f_c\circ x)(-dt^2+\rho)+(f_c\circ x)(\text{Id}\times x)^*\bar{g}.$$

Here ρ is the Riemannian metric on Σ given by the initial data. Note that ∂_t is timelike with respect to $\bar{g}$ so that ∂_t is timelike with respect to $\tilde{g}$. The hypersurfaces $\{s\}\times\Sigma$ are spacelike with respect to $-dt^2+\rho$ and with respect to $(\text{Id}\times x)^*\bar{g}$ for $s\in(t_-,\infty)$ (where this metric is defined), so that they are spacelike with respect to $\tilde{g}$. As a consequence, $\tilde{g}$ is a Lorentz metric on $(t_-,\infty)\times\Sigma$, cf. Lemma 2.

End of the proof. The argument required to finish the proof is essentially identical to the end of the corresponding proof in [20] and need not be repeated here (at one stage $V'(0)=0$ is used, but this can easily be circumvented by multiplying the corresponding term by a cut-off function). □

12. Stability of locally spatially homogeneous spacetimes

Let us first consider the case in which the background initial data are given by (G,g,k,ϕ_a,ϕ_b), where G is a simply connected unimodular Lie group and the isometry group of the initial data contains the left translations in G. Many of the arguments are quite similar to the ones presented in [20], and we shall therefore sometimes only sketch them. One can define an orthonormal basis $\{e_i'\}$ (with respect to the metric g) of the Lie algebra such that the components of k with respect to this basis, say k_{ij}, are diagonal and such that there is a diagonal matrix ν^{ij} with the property that

$$[e_j', e_k'] = \epsilon_{jkl}\nu^{li}e_i',$$

where ϵ_{jkl} is antisymmetric in all of its indices and $\epsilon_{123} = 1$. The reader interested in the details is referred to Section 17 of [20] (the momentum constraint (8) corresponds to the same condition as in [20] since $D_i\phi_a = 0$). Define $n(0) = \nu$, $\theta(0) = \mathrm{tr}_g k$, $\sigma_{ij}(0) = k_{ij} - \theta(0)\delta_{ij}/3$, $\phi(0) = \phi_a$ and $\dot{\phi}(0) = \phi_b$. Define n, θ, σ, ϕ to be the solution to

$$\dot{\theta} = -\frac{3}{2}\sigma^2 + \frac{1}{2}R - \frac{3}{2}\dot{\phi}^2, \tag{189}$$

$$\ddot{\phi} = -\theta\dot{\phi} - V'(\phi), \tag{190}$$

$$\dot{\sigma}_{lm} = -\theta\sigma_{lm} - s_{lm}, \tag{191}$$

$$\dot{n}_{ij} = 2\sigma^k{}_{(i}n_{j)k} - \frac{1}{3}\theta n_{ij}, \tag{192}$$

where a parenthesis among indices denotes symmetrization and

$$s_{lm} = b_{lm} - \frac{1}{3}(\mathrm{tr}b)\delta_{lm}, \tag{193}$$

$$b_{lm} = 2n_m{}^i n_{il} - (\mathrm{tr}n)n_{lm}, \tag{194}$$

$$R = -n_{ij}n^{ij} + \frac{1}{2}[\mathrm{tr}n]^2, \tag{195}$$

$$\sigma^2 = \sigma_{ij}\sigma^{ij}, \tag{196}$$

$$\mathrm{tr}n = \delta^{ij}n_{ij}. \tag{197}$$

In these equations, indices are raised and lowered with δ_{ij}. In other words, there is no difference between indices upstairs and downstairs. Let (t_-, t_+) be the maximal existence interval. Note that (7) is equivalent to

$$\frac{2}{3}\theta^2 - \sigma^2 + R = \dot{\phi}^2 + 2V(\phi), \tag{198}$$

so that this equation holds for $t = 0$. Due to (191)-(192), the off diagonal components of n and σ, collected into one vector, say v, satisfy an equation of the form $\dot{v} = Cv$, so that n and σ remain diagonal in all of (t_-, t_+). Collecting all the terms in (198) on one side and differentiating, using (189)-(197), one obtains zero as a result, so that (198) is satisfied for all $t \in (t_-, t_+)$. Finally, σ remains trace free.

Using the above information, we can construct a spacetime metric as in [20],

$$\bar{g} = -dt^2 + \sum_{i=1}^{3} a_i^2(t)\xi^i \otimes \xi^i, \tag{199}$$

on $M = (t_-, t_+) \times G$ where the ξ^i are the duals of the e_i'. Here $a_i(0) = 1$ and

$$\frac{\dot{a}_i}{a_i} = \sigma_i + \frac{1}{3}\theta,$$

where σ_i are the diagonal components of σ_{ij}. Define $e_i = a_i^{-1}e_i'$. Then $e_0 = \partial_t$ and e_i constitute and orthonormal frame for $(M, \bar{g})$. Similarly to Section 17 of [20], one can prove that

$$g(\nabla_{e_i}e_0, e_j) = \sigma_{ij} + \frac{1}{3}\theta\delta_{ij}$$

and that if γ^i_{jk} is defined by $[e_j, e_k] = \gamma^i_{jk} e_i$, then

$$\gamma^i_{jk} = \epsilon_{jkl} n^{il}.$$

We refer the interested reader to [20], Section 17, for a proof of these facts, cf. also the proof of Lemma 21.2 of [19]. Given this information, one can compute that the scalar curvature of the hypersurfaces $\{t\} \times G$ is given by (195). The Ricci curvature can be expressed in terms of the quantities n_{ij} and θ_{ij}. In fact, in the current setting, the 00 components and the lm components of (5) read

$$-\dot{\theta} - \theta^{ij}\theta_{ij} = \dot{\phi}^2 - V(\phi), \tag{200}$$

$$\dot{\theta}_{lm} + \theta\theta_{lm} + 2n_m{}^i n_{il} - n^{ij} n_{ij}\delta_{lm} + \frac{1}{2}(\mathrm{tr}n)^2\delta_{lm} - (\mathrm{tr}n)n_{lm} = V(\phi)\delta_{lm}. \tag{201}$$

In fact, in these equations, the left hand side of the first equation is the 00 component of Ric and the left hand side of the second equation represents the lm components of Ric. The $0l$-components of the left and right hand sides of (5) vanish identically due to the setup; the $0l$ equations correspond to the momentum constraint (8) and in the current setting the momentum constraint is equivalent to the matrices with components n_{ij} and θ_{ij} commuting, which is an immediate consequence of the fact that both these matrices are diagonal.

Let us prove that $(M, \bar{g}, \phi)$ is a solution of (5) and (6). That (6) holds is an immediate consequence of (190) due to the current geometric setup. To prove that (5) is satisfied all we need to prove is that (200)-(201) are satisfied. However, (200) is a consequence of (189) and (198); one simply uses (198) to eliminate R in (189). The equation (201) on the other hand can be divided into its trace part and its trace free part. Due to (191), we see that the equation corresponding to the trace free part of (201) holds. Furthermore, the equation corresponding to the trace part is a consequence of (189) and (198); one simply uses (198) to eliminate the expressions involving σ_{ij} in (189). Thus (5) and (6) are satisfied. That all the hypersurfaces $\{t\} \times G$ are Cauchy hypersurfaces in $(M, \bar{g})$ follows by an argument which is identical to the proof of Lemma 21.4 of [19]. Finally, the initial data induced on $\{0\} \times G$ correspond to the data we started with.

Analyzing the asymptotics. The asymptotics were already analyzed in [10], see also [11] for the situation with matter of Vlasov type, but since the analysis is, for our purposes, in some respects incomplete, we prefer to give a different analysis here.

Definition 7. We refer to initial data for (189)-(192) satisfying (198) as *Bianchi class A initial data* if σ and n are diagonal matrices. If all the diagonal elements of n are non-zero and have the same sign, we shall say that the initial data are of Bianchi type IX.

We shall here be interested in the case that the potential is given by

$$V(\phi) = V_0 e^{-\lambda\phi}, \tag{202}$$

where $V_0 > 0$ and $\lambda \in (0, \sqrt{2})$ are constants. We shall furthermore restrict our attention to Bianchi class A initial data and exclude Bianchi type IX (Bianchi IX corresponds to the universal covering group of the Lie group under consideration being isomorphic to SU(2)), so that

$$R = -n_{ij}n^{ij} + \frac{1}{2}(\mathrm{tr}n)^2 \leq 0, \tag{203}$$

where we use $R(t)$ to denote the scalar curvature of the hypersurface $\{t\} \times G$, and G is the unimodular Lie group under consideration. For convenience, we shall also drop the argument t most of the time.

Lemma 18. *Consider Bianchi class A initial data for (189)-(192) at $t = 0$ which is not of Bianchi type IX. If $\theta(0) > 0$ and the maximal existence interval of the corresponding solution to (189)-(192) is (t_-, t_+), then $t_+ = \infty$.*

Proof. Due to (189) and (203), we see that $\dot{\theta} \leq 0$. Due to (189) and (198), we see that $\dot{\theta}/\theta^2$ is bounded. Assuming $t_1 \in (0, t_+)$ to be the first time such that $\theta(t_1) = 0$, we get, for $t_2 \in (0, t_1)$,

$$\left| \frac{1}{\theta(0)} - \frac{1}{\theta(t_2)} \right| \leq C|t_2|.$$

As $t_2 \to t_1-$, the left hand side blows up whereas the right hand side is bounded. As a consequence, $\theta(t) > 0$ for all $t \in (t_-, t_+)$. Due to (198), σ_{ij} and $\dot{\phi}$ are bounded to the future, so that ϕ cannot blow up in finite time. Considering (192), keeping the fact that θ and σ_{ij} are bounded in mind, we see that the n_{ij} cannot blow up in finite time. Global existence follows. □

It will be of interest to note that many of the conclusions hold using only (189), (190), (198) and the assumptions that $R \leq 0$ and that $\sigma^2 \geq 0$.

Lemma 19. *Assume we have a solution to (189), (190), (198) on (t_-, ∞) where $t_- < 0$ and R and σ^2 are functions satisfying $R \leq 0$ and $\sigma^2 \geq 0$ on this interval. If, furthermore, $\theta(0) > 0$, then $0 < \theta(t) \leq \theta(0)$ for all $t \geq 0$, there is a $T \geq 0$ such that $\dot{\phi}(t) > 0$ for all $t \geq T$, $\theta \notin L^1([0, \infty))$ and*

$$\lim_{t \to \infty} \phi(t) = \infty.$$

Proof. The proof that θ has to remain positive is identical to the one presented in the proof of Lemma 18. Since $\dot{\theta} \leq 0$ due to (189) and the assumptions, the first conclusion follows. Note that $V'(\phi) < 0$ so that if $\dot{\phi} \leq 0$, then, due to (190), $\ddot{\phi} > 0$. Since $-V'(\phi)$ has a positive lower bound on sets of the form $(-\infty, \varphi_0)$ for $\varphi_0 \in \mathbb{R}$, we conclude that $\dot{\phi}$ must, sooner or later, become positive and then, due to (190), it will stay positive. Assuming ϕ to be bounded from above, we conclude that it has to converge to a finite number. As a consequence,

$$-V'(\phi) = \lambda V(\phi) \geq c_{\min} > 0.$$

As long as

$$\dot{\phi} < \frac{c_{\min}}{\theta(0)},$$

we get $\ddot{\phi} > 0$, so that $\dot{\phi}$ will in the end have a positive uniform lower bound. We conclude that $\phi \to \infty$, a contradiction. Thus ϕ is not bounded from above. In fact, $\phi \to \infty$.

Note that (198) and the assumptions imply that $\dot{\phi}$ is bounded. Since θ and $V'(\phi)$ are bounded, (190) thus implies that $\ddot{\phi}$ is bounded. Since, due to (189), $\dot{\phi}^2$ is integrable, we conclude that $\dot{\phi}$ converges to zero. Let T be chosen so that $\dot{\phi}(t) > 0$ for $t \geq T$ and let

$$q(t) = \int_0^t \theta(s)ds.$$

Then, due to (190),

$$\frac{d}{dt}\left(e^q \dot{\phi}\right) = e^q(\ddot{\phi} + \theta\dot{\phi}) = -V'(\phi)e^q > 0,$$

so that

$$(e^q\dot{\phi})(t) \geq (e^q\dot{\phi})(T) > 0$$

for all $t \geq T$. Since $\dot{\phi}$ converges to zero, we conclude that $\theta \notin L^1([0,\infty))$. □

Lemma 20. *Assume we have a solution to (189), (190), (198) on (t_-,∞) where $t_- < 0$ and R and σ^2 are functions satisfying $R \leq 0$ and $\sigma^2 \geq 0$ on this interval. If, furthermore, $\theta(0) > 0$, then*

$$\lim_{t\to\infty} \frac{\sigma^2 - R}{\theta^2} = 0, \tag{204}$$

$$\lim_{t\to\infty} \frac{\dot{\phi}}{\theta} = \frac{\lambda}{3}, \tag{205}$$

$$\lim_{t\to\infty} \frac{V}{\theta^2} = \frac{1}{3} - \frac{\lambda^2}{18}. \tag{206}$$

Proof. Let T_0 be such that $\dot{\phi}(t) > 0$ for all $t \geq T_0$. In the present proof, we shall consistently assume that $t \geq T_0$. Using (189) and (198), we obtain (one simply uses (198) to eliminate the expression involving R)

$$\frac{d}{dt}\left(\frac{V}{\theta^2}\right) = \dot{\phi}\frac{V}{\theta^2}\left[-\lambda + 2\frac{\dot{\phi}}{\theta} + 2\frac{\theta}{\dot{\phi}}\left(\frac{1}{3} - \frac{V}{\theta^2} + \frac{\sigma^2}{\theta^2}\right)\right],$$

an equation which should be compared with (11) of [17], cf. also the proof of Theorem 4, pp. 1660–1661 of [17]. Since

$$2x + \frac{2\alpha}{x} \geq 4\sqrt{\alpha}$$

for all $\alpha \geq 0$ and $x > 0$, we obtain (note that $1/3 - V/\theta^2 \geq 0$ due to (198))

$$\frac{d}{dt}\left(\frac{V}{\theta^2}\right) \geq \dot{\phi}\frac{V}{\theta^2}\left[-\lambda + 4\left(\frac{1}{3} - \frac{V}{\theta^2} + \frac{\sigma^2}{\theta^2}\right)^{1/2}\right].$$

Say, for the sake of argument, that

$$\frac{V}{\theta^2} \leq \frac{1}{3} - \frac{\lambda^2}{16} - \epsilon$$

for some $\epsilon > 0$ and for all $t \geq T$. Then

$$\frac{1}{3} - \frac{V}{\theta^2} + \frac{\sigma^2}{\theta^2} \geq \frac{\lambda^2}{16} + \epsilon$$

for all $t \geq T$, so that there is a constant $C(\epsilon) > 0$ such that

$$\frac{d}{dt}\left(\frac{V}{\theta^2}\right) \geq C(\epsilon)\dot{\phi}\frac{V}{\theta^2}$$

for all $t \geq T$. Integrating this differential inequality, we obtain

$$\left(\frac{V}{\theta^2}\right)(t) \geq \left(\frac{V}{\theta^2}\right)(T)\exp\{C(\epsilon)[\phi(t)-\phi(T)]\}.$$

Due to Lemma 19, $\phi \to \infty$, so that $V/\theta^2 \to \infty$, which contradicts (198). Due to the above arguments, once V/θ^2 has exceeded $1/3 - \lambda^2/16 - \epsilon$ it will not decay below that to the future. To conclude: for any $\epsilon > 0$, there is a T such that

$$\frac{V}{\theta^2} \geq \frac{1}{3} - \frac{\lambda^2}{16} - \epsilon \tag{207}$$

holds for $t \geq T$.

Using (189), (190) and (198) (in the expression that appears, one simply uses (198) to eliminate σ^2), we obtain

$$\frac{d}{dt}\left(\frac{\dot{\phi}}{\theta}\right) = \theta\frac{V}{\theta^2}\left(\lambda - 3\frac{\dot{\phi}}{\theta}\right) + \theta\frac{\dot{\phi}}{\theta}\frac{R}{\theta^2}. \tag{208}$$

Since $R \leq 0$ by assumption, we conclude that if

$$\frac{\dot{\phi}}{\theta} \geq \frac{\lambda}{3} + \epsilon$$

for some $\epsilon > 0$ and for all $t \geq T$, then

$$\frac{d}{dt}\left(\frac{\dot{\phi}}{\theta}\right) \leq -3\theta\frac{V}{\theta^2}\epsilon$$

for all $t \geq T$. Since V/θ^2 has a uniform positive lower bound, due to (207) and the fact that $\lambda^2 < 2$, and since $\theta \notin L^1([0,\infty))$, due to Lemma 19, this implies that $\dot{\phi}/\theta \to -\infty$, contradicting (198). Since the time derivative of $\dot{\phi}/\theta$ is negative for $\dot{\phi}/\theta > \lambda/3$, there is, for every $\epsilon > 0$, a T such that

$$\frac{\dot{\phi}}{\theta} \leq \frac{\lambda}{3} + \epsilon$$

for all $t \geq T$.

Define

$$S = \left(\frac{2}{3}\theta^2 - \dot{\phi}^2 - 2V\right)e^{\lambda\phi},$$

a quantity which should be compared with $\tilde{S}$ defined in (3.1) of [10]. Then, using (198) to eliminate R from (189),

$$\frac{dS}{dt} = -\theta\left(\frac{2}{3} - \lambda\frac{\dot{\phi}}{\theta}\right)S - \frac{4}{3}\theta\sigma^2 e^{\lambda\phi}.$$

Since $\dot{\phi}/\theta \leq \lambda/3 + \epsilon$ and $\lambda^2 < 2$, we have

$$\frac{2}{3} - \lambda\frac{\dot{\phi}}{\theta} \geq \frac{2}{3} - \frac{\lambda^2}{3} - \lambda\epsilon = \eta_\epsilon > 0$$

for ϵ small enough. Thus

$$\frac{dS}{dt} \leq -\eta_\epsilon\theta S \tag{209}$$

for $t \geq T$ so that $S \to 0$ since $\theta \notin L^1([0,\infty))$. Note that V/θ^2 is bounded from below and from above by positive constants. Thus the same is true of $\theta^2 e^{\lambda\phi}$. Since S converges to zero, we thus conclude that

$$\lim_{t\to\infty}\left(\frac{2}{3}-\frac{\dot{\phi}^2}{\theta^2}-\frac{2V}{\theta^2}\right)=0.$$

Combining this with (198), we conclude that (204) holds. Combining this observation with (208), the fact that V/θ^2 has a positive lower bound and the fact that $\theta \notin L^1([0,\infty))$, we conclude that (205) must hold. Combining (198), (204) and (205), we obtain (206). □

Lemma 21. *Assume we have a solution to (189), (190), (198) on* (t_-,∞) *where* $t_- < 0$ *and* R *and* σ^2 *are functions satisfying* $R \leq 0$ *and* $\sigma^2 \geq 0$ *on this interval. If, furthermore,* $\theta(0) > 0$, *then there are constants* C, c_{a_i} *and* $\beta > 0$ *such that for* $t \geq 1$,

$$\left|\phi-\frac{2}{\lambda}\ln t+\frac{c_0}{\lambda}\right| \leq Ct^{-\beta}, \tag{210}$$

$$\left|t\dot{\phi}-\frac{2}{\lambda}\right| \leq Ct^{-\beta}, \tag{211}$$

where c_0 *is the constant defined in (12). Assuming, furthermore, that*

$$\dot{a}_i=\left(\sigma_i+\frac{1}{3}\theta\right)a_i, \tag{212}$$

where the σ_i *are functions such that*

$$\sum_{i=1}^{3}\sigma_i^2 \leq \sigma^2, \tag{213}$$

we have

$$\left|\ln\frac{a_i(t)}{a_i(0)}-\frac{2}{\lambda^2}\ln t-c_{a_i}\right| \leq Ct^{-\beta}, \tag{214}$$

$$\left|\frac{t\dot{a}_i}{a_i}-\frac{2}{\lambda^2}\right| \leq Ct^{-\beta}. \tag{215}$$

Proof. Let us introduce a new time coordinate

$$\tau(t)=\int_0^t\theta(s)ds. \tag{216}$$

Note that $\tau \to \infty$ as $t \to \infty$ due to Lemma 19. Furthermore

$$\frac{d\tau}{dt}=\theta.$$

Due to (209), we conclude that S converges to zero exponentially in τ-time. In other words, there are constants C and $\alpha > 0$ such that

$$\left|\frac{2}{3}-\frac{\dot{\phi}^2}{\theta^2}-\frac{2V}{\theta^2}\right| \leq Ce^{-\alpha\tau}$$

for $\tau \geq 0$. Combining this fact with (208) and (198), we conclude that $\dot{\phi}/\theta$ converges to $\lambda/3$ exponentially in τ-time. To see this, derive an equation for $e^{\alpha\tau}(\dot{\phi}/\theta-\lambda/3)$;

for $\alpha > 0$ small enough the resulting equation implies that this quantity has to converge to zero. As a consequence, V/θ^2 converges to $1/3 - \lambda^2/18$ exponentially. Compute

$$\frac{d}{d\tau}\ln\theta = \frac{\dot{\theta}}{\theta^2} = -\frac{3}{2}\frac{\sigma^2}{\theta^2} + \frac{1}{2}\frac{R}{\theta^2} - \frac{3}{2}\left(\frac{\dot{\phi}^2}{\theta^2} - \frac{\lambda^2}{9}\right) - \frac{\lambda^2}{6}. \tag{217}$$

By the above observations, we have

$$\left|\ln\frac{\theta(\tau)}{\theta(0)} + \frac{\lambda^2}{6}\tau - c_\theta\right| \le Ce^{-\alpha\tau}$$

for some suitably chosen c_θ, where we have abused notation by writing $\theta(\tau)$ when we should in fact write $\tilde{\theta}(\tau)$, where $\tilde{\theta}$ is the function such that $\tilde{\theta}[\tau(t)] = \theta(t)$. Letting $r(\tau)$ be the expression inside the absolute value signs, we obtain

$$\theta(\tau) = \theta(0)\exp\left(-\frac{\lambda^2}{6}\tau + c_\theta + r(\tau)\right).$$

Since $dt/d\tau = 1/\theta$, this leads to

$$t(\tau) = \frac{1}{\theta(0)}\int_0^\tau \exp\left(\frac{\lambda^2}{6}s - c_\theta - r(s)\right)ds.$$

Combining $\theta(0)$ and c_θ into one constant, say c_1, this leads to

$$t(\tau) = \int_0^\tau \exp\left(\frac{\lambda^2}{6}s + c_1 - r(s)\right)ds = \frac{6}{\lambda^2}\exp\left(\frac{\lambda^2}{6}\tau + c_1\right)\left[1 + O(e^{-\alpha\tau})\right]$$

for $\tau \ge 0$, where $0 < \alpha < \lambda^2/6$. As a consequence,

$$\tau = \frac{6}{\lambda^2}\ln t + c_2 + O(t^{-\beta})$$

for $t \ge 1$ and some constants $\beta > 0$ and c_2. Since $\phi_\tau = \dot{\phi}/\theta$ converges to $\lambda/3$ exponentially, we conclude that

$$\phi = \frac{2}{\lambda}\ln t + c_3 + O(t^{-\beta}) \tag{218}$$

for $t \ge 1$ and some constant c_3. Note that, cf. (217),

$$\begin{aligned}\frac{d}{d\tau}(t\theta) &= 1 + t\theta\frac{\dot{\theta}}{\theta^2} = 1 + t\theta\left[-\frac{\lambda^2}{6} + O(e^{-\alpha\tau})\right] \\ &= \left(t\theta - \frac{6}{\lambda^2}\right)\left[-\frac{\lambda^2}{6} + O(e^{-\alpha\tau})\right] + O(e^{-\alpha\tau}).\end{aligned}$$

Thus $t\theta$ converges to $6/\lambda^2$ exponentially so that (211) holds for $t \ge 1$, since $\dot{\phi}/\theta$ converges to $\lambda/3$ exponentially. Due to (212) and (213), we have

$$\begin{aligned}\ln\frac{a_i(t)}{a_i(0)} &= \int_0^t \left(\sigma_i + \frac{1}{3}\theta\right)ds = \int_0^\tau \left(\frac{\sigma_i}{\theta} + \frac{1}{3}\right)d\tau = \frac{1}{3}\tau + c_{1,i} + O(e^{-\alpha\tau}) \\ &= \frac{2}{\lambda^2}\ln t + c_{2,i} + O(t^{-\beta}),\end{aligned}$$

yielding (214), and (215) follows from

$$\frac{t\dot{a}_i}{a_i} = t\theta\frac{\sigma_i}{\theta} + \frac{1}{3}t\theta = \frac{2}{\lambda^2} + O(t^{-\beta})$$

for $t \geq 1$. What remains to be proved is (210). Consider (190). Let us introduce

$$\psi = \phi - \frac{2}{\lambda}\ln t + \frac{c_0}{\lambda}.$$

At this stage, we only know that ψ converges to a constant, with an error of the form $O(t^{-\beta})$, cf. (218), and that (211) holds. Compute, using the fact that $p = 2/\lambda^2$ in the present situation,

$$\begin{aligned} t^2\ddot{\psi} &= t^2\left(-\theta\dot{\psi} - \theta\frac{2}{\lambda t} + \lambda V_0 t^{-2}\frac{2(3p-1)p}{2V_0}e^{-\lambda\psi} + \frac{2}{\lambda t^2}\right) \\ &= -t\theta t\dot{\psi} - \frac{2}{\lambda}(t\theta - 1) + \lambda(3p-1)pe^{-\lambda\psi} \\ &= -t\theta t\dot{\psi} - \frac{2}{\lambda}\left(t\theta - \frac{6}{\lambda^2}\right) - \frac{2}{\lambda}\left(\frac{6}{\lambda^2} - 1\right)(1 - e^{-\lambda\psi}). \end{aligned}$$

The first two terms on the right hand side are $O(t^{-\beta})$ due to (211) and the fact that $t\theta$ converges to $6/\lambda^2$ with an error of the order of magnitude $t^{-\beta}$. If the constant c_3 appearing in (218) is $-c_0/\lambda$ we are done, so let us assume not. Then the above shows that

$$t^2\ddot{\psi} = \alpha_0 + O(t^{-\beta})$$

for some $\alpha_0 \neq 0$. Since $t\dot{\psi} = O(t^{-\beta})$, we conclude that

$$t\partial_t(t\dot{\psi}) = \alpha_0 + O(t^{-\beta}).$$

Integrating this equality from $T \geq 1$, we get

$$t\dot{\psi}(t) = T\dot{\psi}(T) + \alpha_0 \ln\frac{t}{T} + O(1).$$

Since everything in this equation is bounded except for $\ln(t/T)$, we get a contradiction, and the lemma follows. □

Let us assume the initial data are specified on $\mathbb{H}^3$ and that they are invariant under the isometry group of the corresponding canonical metric. By arguments similar to those given in Section 17 of [20], the initial data for the metric and second fundamental form can be assumed to be of the form $g = \alpha^2 g_{\mathbb{H}^3}$ and $k = \alpha\beta g_{\mathbb{H}^3}$ for positive constants α and β and it is enough to consider metrics of the form

$$\bar{g} = -dt^2 + a^2(t)g_{\mathbb{H}^3} \tag{219}$$

on $I \times \mathbb{H}^3$ for some open interval I. Using the formulas (1)-(3), p. 211 of [14] to compute the Ricci tensor, one concludes that (5) and (6) in the current situation are equivalent to

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\dot{\phi}^2 + \frac{1}{3}V(\phi), \quad 6\left(\frac{\dot{a}}{a}\right)^2 - \frac{6}{a^2} = \dot{\phi}^2 + 2V(\phi), \quad \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0. \tag{220}$$

The first and the last of these equations can be used as evolution equations given initial data. Collecting all the terms in the middle equation on the right hand side and denoting the result f, one can compute, using the first and the last equation, that $\dot{f}$ is a multiple of f. Since $f(0) = 0$ (this is simply the Hamiltonian constraint), one obtains $f = 0$ where the solution exists. Letting $R = -6/a^2$ (this is simply the scalar curvature of the hypersurfaces $\{t\} \times \mathbb{H}^3$), $\theta = 3\dot{a}/a$ (this is simply the trace of the second fundamental form of the hypersurfaces $\{t\} \times \mathbb{H}^3$) and $\sigma^2 = 0$, one can compute, using (220), that (189), (190) and (198) hold in the present setting. By arguments similar to the proof of Lemma 18, one can prove that future global existence holds. Since, in our case, $\dot{a} = \theta a/3$, we are thus allowed to use the conclusions of Lemma 21 with $a_i = a$ and $\sigma_i = 0$.

Finally, consider the case that the initial data are specified on $\mathbb{H}^2 \times \mathbb{R}$ and are invariant under the isometry group of the corresponding canonical metric. Then, by the same argument that was presented in Section 17 of [20], the initial data can be assumed to be of the form

$$g = a_0^2 g_{\mathbb{H}^2} + b_0^2 dz^2, \quad k = a_1 a_0 g_{\mathbb{H}^2} + b_1 b_0 dz^2,$$

and it is enough to consider metrics of the form

$$\bar{g} = -dt^2 + a^2(t) g_{\mathbb{H}^2} + b^2(t) dz^2. \tag{221}$$

When computing the Ricci curvature of (221), it is convenient to note that the spacetime $(I \times \mathbb{H}^2 \times \mathbb{R}, \bar{g})$, where I is an open interval, can be viewed as a warped product with warping function a and

$$B = I \times \mathbb{R}, \quad g_B = -dt^2 + b^2(t) dz^2, \quad F = \mathbb{H}^2, \quad g_F = g_{\mathbb{H}^2},$$

using the terminology of [14], pp. 204-211. One can compute that (5) is equivalent to

$$\frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} + \left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = V(\phi), \quad \frac{\ddot{b}}{b} + 2\frac{\dot{a}\dot{b}}{ab} = V(\phi), \tag{222}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}\dot{b}}{ab} - \frac{1}{a^2} = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{223}$$

The equation (6) for the scalar field turns into

$$\ddot{\phi} + \theta\dot{\phi} + V'(\phi) = 0, \tag{224}$$

where $\theta = 2\dot{a}/a + \dot{b}/b$ is the trace of the second fundamental form of the hypersurfaces $\{t\} \times \mathbb{H}^2 \times \mathbb{R}$ (note that we shall assume $\theta(0) > 0$ in what follows). We evolve the initial data using the evolution equations (222) and (224). If we collect all the terms in (223) on the left hand side, denote the resulting function f, then (222) and (224) imply that $\dot{f} = -2\theta f$, so that f vanishes where the solution is defined, since $f(0) = 0$ due to the fact that the initial data satisfy the Hamiltonian constraint (7). As a consequence, the development satisfies (5) and (6). Let us introduce $\sigma = \sqrt{2}(\dot{a}/a - \dot{b}/b)/\sqrt{3}$ and $R = -2/a^2$ (this is the scalar curvature of the hypersurfaces $\{t\} \times \mathbb{H}^2 \times \mathbb{R}$). Then (223) takes the form (198). Using (222) and (223), one can prove that (189) holds. Finally, note that (224) and (190) coincide in the current setting. In order to prove future global existence, one proceeds similarly to Lemma 18. Due to the above observations, we are allowed to use the conclusions of Lemma 21 with $a_1 = a_2 = a$, $a_3 = b$, $\sigma_1 = \sigma_2 = \sigma/\sqrt{6}$ and $\sigma_3 = -\sqrt{2}\sigma/\sqrt{3}$.

Proof of Theorem 3. Let us assume we have a metric of the form (199) on $I \times G$, where G is a 3-dimensional Lie group, I is an open interval containing (t_0, ∞) for t_0 large enough and ξ^i are the duals of a basis $\{e_i\}$ for the Lie algebra (the metrics (219) and (221) can be written in this form due to the fact that hyperbolic space can be considered as a Lie group with a left invariant metric, cf. Section 17 of [20]). Assume furthermore that

$$\lim_{t\to\infty} t^{-2/\lambda^2} a_i(t) = \alpha_i, \quad \lim_{t\to\infty} \frac{t\dot{a}_i}{a_i} = \frac{2}{\lambda^2},$$

for some $\alpha_i > 0$ and that (210) and (211) hold. Note that these assumptions hold in the cases of interest here, due to the arguments given at the beginning of the present section. Assume finally that there is a group of diffeomorphisms Γ acting freely and properly discontinuously on G such that $\{\mathrm{Id}\} \times \Gamma$ is a group of isometries of $\bar{g}$ and such that the quotient of G under Γ is compact (it is clear that the groups under consideration in the theorem are of this type in the unimodular case, due to our assumptions, and in the remaining cases due to the forms of the metrics in these cases, cf. (219) and (221)). Let Σ denote the quotient and let $\pi : G \to \Sigma$ be the covering projection. Let us define a reference metric

$$h = \sum_{i=1}^{3} \alpha_i^2 \xi^i \otimes \xi^i$$

on G. Note that since

$$\hat{h} = t^{-4/\lambda^2} \sum_{i=1}^{3} a_i^2(t) \xi^i \otimes \xi^i$$

converges to the metric h as $t \to \infty$ and Γ is a group of isometries of $\hat{h}$, Γ is a group of isometries of h. Consequently, h induces a metric on Σ. In what follows it will be useful to compare ∂_{y^i} for some coordinates y with the basis e_i. Unfortunately, we cannot assume that the e_i are well defined on Σ, since the group Γ may contain diffeomorphisms that do not map e_i to itself. On the other hand, there is an $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$ and $q \in \Sigma$, then $B_\epsilon(q)$ (measured with respect to the metric h) is such that $\pi^{-1}[B_\epsilon(q)]$ consists of a disjoint collection of open sets such that π, restricted to any connected member of the disjoint union, is an isometry onto $B_\epsilon(q)$. One can use one of these isometries to push the basis e_i (and thus ξ^i) forward to $B_\epsilon(q)$. However, the result will in general depend on the choice of connected member of $\pi^{-1}[B_\epsilon(q)]$; below we shall speak of a choice of ξ^i on $B_\epsilon(q)$. In [20], we proved that there is an $\epsilon > 0$ and a $K > 0$ such that for every $q \in \Sigma$, there are normal coordinates y^i on $B_\epsilon(q)$ with respect to the metric h, and a choice of ξ^i such that if $\zeta^i_j = \xi^i(\partial_{y^j})$, then all the derivatives of ζ^i_j with respect to y^l up to order $k_0 + 1$ are bounded by K in the sup norm on $B_\epsilon(q)$ (cf. pp. 204-205 of [20]).

Let $\epsilon > 0$ and $K > 0$ be as above and $q \in \Sigma$. Let y^i be normal coordinates on $B_\epsilon(q)$ with respect to the metric h, and make a choice of ξ^i such that if $\zeta^i_j = \xi^i(\partial_{y^j})$, then all the derivatives of ζ^i_j with respect to y^l up to order $k_0 + 1$ are bounded by K in the sup norm on $B_\epsilon(q)$. The initial data induced on the hypersurface $\{t\} \times G$ are given by

$$g = \sum_{i=1}^{3} a_i^2(t) \xi^i \otimes \xi^i, \quad k = \sum_{i=1}^{3} \dot{a}_i(t) a_i(t) \xi^i \otimes \xi^i, \quad \phi(t), \quad \dot{\phi}(t).$$

Let us introduce coordinates $x^i = [4\ell(t)]^{-1} t^{2/\lambda^2} y^i$. For t large enough, the range of x^i contains the ball of radius 1 (recall that $\lambda^2 < 2$). Note that

$$g_{ij} = g(\partial_{x^i}, \partial_{x^j}) = [4\ell(t)]^2 \sum_{l=1}^{3} t^{-4/\lambda^2} a_l^2(t) (\xi^l \otimes \xi^l)(\partial_{y^i}, \partial_{y^j}).$$

Since $t^{-2/\lambda^2} a_i(t) \to \alpha_i$ as $t \to \infty$, $h(\partial_{y^i}, \partial_{y^j}) = \delta_{ij}$ at q, the derivatives of $\xi^l(\partial_{y^i})$ with respect to y^j are bounded by K on $B_\epsilon(q)$ and the ball of radius 1 with respect to the x^i coordinates corresponds to a ball of an arbitrarily small radius with respect to the y^i coordinates for t large enough, we conclude that for t large enough (the bound being independent of q),

$$[4\ell(t)]^{-2}g_{ij} - \delta_{ij} \tag{225}$$

is arbitrarily small in the ball of radius 1 with respect to the x^i coordinates. Since

$$\frac{\partial}{\partial x^i} = 4\ell(t)t^{-2/\lambda^2}\frac{\partial}{\partial y^i},$$

and $\xi^i(\partial_{y^j})$ is bounded in C^{k_0+1}, the spatial derivatives of the expression appearing in (225) with respect to x^l are arbitrarily small for t large enough (independent of q). Similarly,

$$k_{ij} = k(\partial_{x^i}, \partial_{x^j}) = [4\ell(t)]^2 \sum_{l=1}^{3} t^{-4/\lambda^2}\dot{a}_l(t)a_l(t)(\xi^l \otimes \xi^l)(\partial_{y^i}, \partial_{y^j}).$$

Since, in addition to the above observations,

$$\lim_{t\to\infty} t^{-2/\lambda^2} t\dot{a}_i(t) = \frac{2}{\lambda^2}\alpha_i,$$

we conclude that (recall that $p = 2/\lambda^2$)

$$tp^{-1}[4\ell(t)]^{-2}k_{ij} - \delta_{ij} \tag{226}$$

is arbitrarily small in a ball of radius 1 with respect to the x^i-coordinates. Furthermore, the derivatives of the expression appearing in (226) with respect to ∂_{x^l} are arbitrarily small. There is one problem with the above argument of course; in Theorem 2, the time t_0 used is determined by the mean value of the scalar field. In fact, instead of $\ell(t)$, we should use $\ell[t_0(t)]$ in (225), where

$$t_0(t) = \exp\left[\frac{1}{2}(\lambda\phi(t) + c_0)\right]$$

and similarly in (226). Due to (210), we have

$$t_0(t) = t[1 + O(t^{-\beta})].$$

As a consequence

$$\frac{[4\ell(t)]^{-2}}{\{4\ell[t_0(t)]\}^{-2}} = 1 + O(t^{-\beta}), \quad \frac{tp^{-1}[4\ell(t)]^{-2}}{t_0(t)p^{-1}\{4\ell[t_0(t)]\}^{-2}} = 1 + O(t^{-\beta}).$$

In other words, whether we use t or $t_0(t)$ does not make any difference as far as the conclusions are concerned. Note that, by definition, $\phi(t) - \phi_0[t_0(t)]$ is zero, and by (211) and the above observations,

$$t_0(t)\dot{\phi}(t) - t_0(t)\dot{\phi}_0[t_0(t)]$$

converges to zero. Since this object is spatially homogeneous, we are allowed to conclude that for t large enough, (16) is satisfied with ϵ replaced by $\epsilon/2$, where the coordinates are of the form described above (regardless of the point q). Combining Theorem 2 of the present paper with Theorem 7 of [20], we get the desired stability statement. □

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Department of Mathematics, KTH, 100 44 Stockholm, Sweden