Exam SF1677/2713 April 3d 2018

Total marks 32: The preliminary relationship between the marks and grades are

A : 30 **B** : 28 **C** : 25 **D** : 22 **E** : 19 **FX** : 18.

A G on the first homework assignment corresponds to full mark (4 marks) on question 1,
a G on the second homework assignment corresponds to full mark (4 marks) on question 2 and
a G on the third homework assignment corresponds to full mark (4 marks) on question 3.

Allowed help: Only writing utensils are allowed, calculators are NOT allowed for this exam.

All your answers should be proved unless otherwise stated.

Question 1: Assume that $f : [-1,1] \mapsto \mathbb{R}$ and $g : [-1,1] \mapsto \mathbb{R}$ are increasing functions and that f is continuous. Assume furthermore that $f(-1) < g(-1)$ and $f(1) > g(1)$. Will the equation $f(x) = g(x)$ have a solution? Note that we do **not** assume that g is continuous. Prove your answer.

(4 marks)

Solution Question 1: Let $f(1) - g(1) = \epsilon > 0$. Then, since f is continuous, there exist a $\delta > 0$ such that if $x \in (1-\delta, 1]$ then

$$f(x) > f(1) - \epsilon = g(1) \geq g(x),$$

where we also used that g is increasing in the last inequality.

Let us define the set

$$S = \{x \in [-1,1];\text{ s.t. } f(y) \geq g(y), \text{for } y \in [x,1]\}.$$

By the previous paragraph $(1-\delta, 1] \subset S$, thus $S \neq \emptyset$, and by definition S is bounded from below. Using the greatest lower bound property of the real numbers we may conclude that $x_0 = glb(S)$ exists.

Next we note that $x_0 > -1$. This follows as in the first paragraph of the proof: by continuity of f if $f(-1)+\hat{\epsilon} = g(-1)$ then there exist a $\hat{\delta}$ such that $f(x) > g(x)$ for all $x \in [-1, -1+\hat{\delta})$ and therefore $[-1, -1+\hat{\delta}) \not\subset S$. We can conclude that $x_0 \in [-1+\hat{\delta}, 1-\delta]$ for some $\delta, \hat{\delta} > 0$.

To finish the proof we show that $f(x_0) = g(x_0)$, that is x_0 solves the desired equation. First we take any sequence $x_j \in S$ s.t. $x_j \to x_0$ and make the following estimate

$$0 \leq f(x_j) - g(x_j) \leq f(x_j) - g(x_0) \to f(x_0) - g(x_0), \tag{1}$$

where we first used that $x_j \in S$, then that g is increasing and finally that $x_j \to x_0$ together with continuity of f.

Similarly we notice that for each $j \in \mathbb{N}$ there is an x_j such that $x_0 - \frac{1}{j} \leq x_j \leq x_0$ and $f(x_j) < g(x_j)$, since x_0 was the greatest lower bound of S. Passing to the limit $j \to \infty$ we may conclude that

$$0 > f(x_j) - g(x_j) \geq f(x_j) - g(x_0) \to f(x_0) - g(x_0), \tag{2}$$

where we again used that g is increasing and f continuous. We can conclude from (2) that $f(x_0) \leq g(x_0)$ and from (1) that $g(x_0) \leq f(x_0)$. It follows that $f(x_0) = g(x_0)$.

Question 2: Let $f_k : (0,1) \mapsto \mathbb{R}$ be a sequence of positive and non-decreasing Riemann integrable functions and that for any $x \in (0,1)$

$$\lim_{N\to\infty} \sum_{k=1}^{N} f_k(x) = f(x),$$

where $f : (0,1) \mapsto \mathbb{R}$. Assume furthermore that

$$\lim_{N\to\infty} \left[\sum_{k=1}^{N} \left(\int_0^1 f_k(x)dx \right) \right] = 1.$$

Will f be Riemann integrable? If so will $\int_0^1 f(x)dx = 1$? Prove your answer.

(4 marks)

Solution Question 2: We will show that f is not necessarily Riemann integrable. Let $g_0(x) = 0$ and

$$g_k(x) = 2\min\left(\frac{1}{\sqrt{1-x}}, k\right).$$

Then $g_{k-1}(x) \leq g_k(x)$ and therefore $f_k(x) = g_k(x) - g_{k-1}(x)$ for all $k = 1, 2, ...$ It is easy to see that f_k is non-decreasing.[1]

We may calculate the sum of the integrals

$$\sum_{k=1}^{N} \int_0^1 f_k(x)dx = \int_0^1 g_k(x)dx = 2\int_0^{\frac{k^2-1}{k^2}} \frac{1}{\sqrt{1-x}}dx + 2\int_{\frac{k^2-1}{k^2}}^1 k dx =$$

$$= 1 - \sqrt{1 - \frac{k^2-1}{k^2}} + \frac{2}{k} \to 1,$$

where we used the standard integration techniques (fundamental theorem of calculus) together with standard limits. Thus the sequence f_k satisfies the conditions of the question.

Furthermore $\sum_{k=1}^{N} f_k(x) = g_N(x) \to \frac{2}{\sqrt{1-x}} = f(x)$ for any $x \in (0,1)$ as $N \to \infty$. We claim that $f(x)$ is not Riemann integrable since f is not bounded. To see this we assume, aiming for a contradiction, that $\int_0^1 f(x)dx = I$. Then there should be a partition $P = \{0 = x_0 < x_1 < ... < x_n = 1\}$ such that

$$\sum_{j=1}^{n} \sup_{x \in (x_{j-1}, x_j)} f(x)(x_j - x_{j-1}) < I + 1.$$

This is not possible since all the terms in the sum are positive and $\sup_{x\in(x_{n-1},x_n)} f(x)(x_n - x_{n-1}) = \infty$ since f is unbounded on (x_{n-1}, x_n); therefore the left side is not bounded by $I + 1$. Thus f is not Riemann integrable even though it satisfies the conditions of the question.

Question 3: Let $f_k : [-1, 1] \mapsto \mathbb{R}$ be a sequence of continuously differentiable functions. Assume furthermore that $f_k \to f$ and that $f'_k \to g$ uniformly on $[-1, 1]$ where $f, g : [-1, 1] \mapsto \mathbb{R}$ are two given continuous functions. Prove that f is differentiable at $x = 0$ and that $f'(0) = g(0)$.

You may, without proof, use any known theorem for continuous functions. However, you may not use any theorem regarding convergence of differentiable functions without proof.

(4 marks)

Solution Question 3: Since $f_k \to f$ and $f'_k \to g$ uniformly on $[-1, 1]$, f_k and f'_k are continuous, it follows that f and g are continuous on $[-1, 1]$.

By the Mean Value Theorem there exist, for any $h \neq 0$, a ξ_k between 0 and h such that

$$\frac{f_k(h) - f_k(0)}{h} = f'_k(\xi_k).$$

Therefore, for any $h \neq 0$,

$$\frac{f(h) - f(0)}{h} = \lim_{k\to\infty} \frac{f_k(h) - f_k(0)}{h} = \lim_{k\to\infty} f'_k(\xi_k). \tag{3}$$

Since $|\xi_k| \leq |h|$ we may choose a sub-sequence $\xi_{k_j} \to \xi_h$ where ξ_h lays between 0 and h.

Since $\xi_{k_j} \to \xi_h$ and $f'_{k_j} \to g$ uniformly it follows that for any $\epsilon > 0$ there is a J_ϵ such that if $j > J_\epsilon$ then

$$|g(\xi_h) - f'_{k_j}(\xi_{k_j})| \leq |g(\xi_h) - g(\xi_{k_j})| + |g(\xi_{k_j}) - f'_{k_j}(\xi_{k_j})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where J_ϵ have been chosen so large that we may estimate each of the the two absolute values by $\epsilon/2$ using continuity of g and uniform convergence. If follows that $f'_{k_j}(\xi_{k_j}) \to g(\xi_h)$. Using this in (3) we can conclude that

$$\frac{f(h) - f(0)}{h} = g(\xi_h).$$

Sending $h \to 0$, using that $|\xi_h| \leq |h|$ and that g is continuous we can conclude that $f'(0) = g(0)$. This finishes the proof.

Question 4: Given a set $A \subset \mathbb{R}$ we define the set

$$\mathcal{S}_A = \{\sin(ax);\ a \in A\}.$$

State a condition on the set A such that $\mathcal{S}_A$ is equicontinuous if and only if A satisfies the stated condition. Prove your answer.

(4 marks)

[1] As a matter of fact f_k will equal 0 on $(0, 1 - (k-1)^{-2}]$ and $f(x) = 2$ on $[1 - k^{-2}, 1)$ and $2(1-x)^{-1/2} - 2k + 2$ which has strictly positive derivative on the interval between.

Solution Question 4: Se claim that $\mathcal{S}_A$ is equicontinuous if and only if A is bounded.

Step 1: *If A is bounded then $\mathcal{S}_A$ is equicontinuous.*

Let us assume that A is bounded by M; that is $a \in A$ implies that $|a| \leq M$. Let $f(x) = \sin(ax) \in \mathcal{S}_A$. Then $|f'(x)| \leq M$. From the Mean Value Theorem it follows that if $|x - y| < \delta = \epsilon/M$ then

$$|f(x) - f(y)| < \delta |f'(\xi)| < \epsilon.$$

Since δ is independent of both f and x it follows that $\mathcal{S}_A$ is equicontinuous.

Step 2: *If $\mathcal{S}_A$ is equicontinuous then A is bounded.*

We will use a converse argument and assume that there is a sequence $a_j \in A$, $|a_j| \to \infty$, and show that then $\mathcal{S}_A$ is not equicontinuous.

Pick an arbitrary $0 < \epsilon < 1$. We need to show that for every $\delta > 0$ there exist an $f \in \mathcal{S}_A$ and $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and

$$|f(x) - f(y)| > \epsilon.$$

To that end we pick an arbitrary $\delta > 0$ and j so large that $\left|\frac{\pi}{2a_j}\right| < \delta$, this is always possible since $|a_j| \to \infty$. Then $f = \sin(a_j x) \in \mathcal{S}_A$ and with $x = \frac{\pi}{2a_j}$ we have that $|x - 0| < \delta$ and

$$|f(x) - f(0)| = |\sin(a_j \frac{\pi}{2a_j}) - \sin(0)| = 1 > \epsilon.$$

It follows that $\mathcal{S}_A$ is not equicontinuous if A is not bounded. This finishes the proof.

Question 5: Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuously differentiable function and also assume that $D_{12}f$ and $D_{21}f$ exist and are continuous; here $D_{ij}f = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Prove that $D_{12}f(x, y) = D_{21}f(x, y)$.

HINT: *You may, without proof, use the following result from Rudin (Theorem 9.40): If Q is the cube $[a, a + h] \times [b, b + k] \subset \mathbb{R}^2$ and*

$$\Delta(f, Q) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$

then there exist a point $(x, y) \in Q$ such that

$$\Delta(f, Q) = hkD_{21}f(x, y).$$

(4 marks)

Solution Question 5: Clearly, by symmetry, the hint is also valid for $D_{21}f$ in place of $D_{12}f$.

Pick an arbitrary $(a, b) \in \mathbb{R}^2$ and let $h_j = k_j = \frac{1}{j}$. Then, using the hint, there exist $(x_j, y_j), (\hat{x}_j, \hat{y}_j) \in Q_j = [a, a + 1/j] \times [b, b + 1/j]$ such that

$$0 = |\Delta(f, Q_j) - \Delta(f, Q_j)| = \frac{1}{j^2} |D_{21}f(x_j, y_j) - D_{12}f(\hat{x}_j, \hat{y}_j)|. \tag{4}$$

Using that $D_{12}f$ and $D_{21}f$ are continuous and that $(x_j, y_j) \to (a, b)$ and $(\hat{x}_j, \hat{y}_j) \to (a, b)$ as $j \to \infty$ (the last convergence follows from that $(x_j, y_j) \in Q_j$ implies that $a \leq x_j \leq a + 1/j$ and $b \leq y_j \leq b + 1/j$ and similarly for $(\hat{x}_j, \hat{y}_j)$) it follows that

$$|D_{21}f(a, b) - D_{12}f(a, b)| = \lim_{j \to \infty} |D_{21}f(x_j, y_j) - D_{12}f(\hat{x}_j, \hat{y}_j)| = \lim_{j \to \infty} 0 = 0,$$

where we used (4) in the second equality. It follows that $D_{21}f(a, b) = D_{12}f(a, b)$ from the last displayed formula.

Question 6: Let $\mathcal{X}$ be the metric space consisting of all functions $f : \mathbb{N} \mapsto \mathbb{R}$ such that $\lim_{n \to \infty} f(n) = 0$ equipped with the metric:

$$d(f, g) = \sup_{n \in \mathbb{N}} |f(n) - g(n)|.$$

Is $\mathcal{X}$ complete? Prove your answer. (You do not need to prove that $\mathcal{X}$ is a metric space.)

(4 marks)

Solution Question 6: We need to show that if f_k is a Cauchy sequence, that is for every $\epsilon > 0$ there exist an N such that if $k, l > N$ $d(f_k, f_l) < \epsilon$, then there exist an $f \in \mathcal{X}$ such that $\lim_{k \to \infty}(d(f_k, f)) = 0$.

For every $n \in \mathbb{N}$, using that f_k is Cauchy, then there exist an N such that if $k, l > N$ then

$$|f_k(n) - f_l(n)| \leq \sup_{n \in \mathbb{N}} |f_k(n) - f_l(n)| < \epsilon. \tag{5}$$

Therefore, for every $n \in \mathbb{N}$ the sequence of real numbers $f_k(n)$ is a Cauchy sequence and by the completeness of the real numbers it follows that $f_k(n)$ converges. We may define the function $f : \mathbb{N} \mapsto \mathbb{R}$ according to

$$f(n) = \lim_{k\to\infty} f_k(n).$$

Next we show that $\lim_{k\to\infty} d(f_k, f) = 0$, without claiming that $f \in \mathcal{X}$. This follows from taking the limit in (5), assuming that $k > N$,

$$\sup_{n\in\mathbb{N}} |f_k(n) - f(n)| = \sup_{n\in\mathbb{N}} \lim_{l\to\infty} |f_k(n) - f_l(n)| \leq \sup_{n\in\mathbb{N}} (\sup_{l>k} |f_k(n) - f_l(n)|) \leq \epsilon. \tag{6}$$

We may conclude that $d(f_k, f) \to 0$, if not then we would be able to find a subsequence, f_{k_j}, such that $d(f_{k_j}, f) = 2\epsilon > 0$ contradicting (6).

Next we need to show that $f \in \mathcal{X}$. To that end we pick a k large enough so that $d(f, f_k) < \epsilon/2$. Also since $f_k \in \mathcal{X}$ there is an M such that $|f_k(n)| < \epsilon/2$ for $n > M$. We may conclude that for $n > M$

$$|f(n)| \leq |f(n) - f_k(n)| + |f_k(n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves that $\lim_{n\to\infty} f(n) = 0$ and thus that $f \in \mathcal{X}$.

Question 7: Let $f : [a, b] \mapsto \mathbb{R}$, $0 < f \leq M$, be a function such that the following integral exist

$$\int_a^b \frac{1}{f(x)} dx.$$

Is f integrable over $[a, b]$? Prove your answer.

(4 marks)

Solution Question 7: Notice that if $f(x) > f(y) > 0$ then

$$f(x) - f(y) = \frac{f(x)f(y)}{f(x)} - \frac{f(x)f(y)}{f(x)} \leq M^2 \left(\frac{1}{f(y)} - \frac{1}{f(x)} \right).$$

It follows that, for any $a \leq x_{k-1} < x_k \leq b$

$$M^2 \left(\sup_{x\in(x_{k-1},x_k)} \frac{1}{f(x)} - \inf_{x\in(x_{k-1},x_k)} \frac{1}{f(x)} \right) \geq \sup_{x\in(x_{k-1},x_k)} f(x) - \inf_{x\in(x_{k-1},x_k)} f(x).$$

Let $\epsilon > 0$ be arbitrary. Since $\frac{1}{f(x)}$ is integrable there is a partition $P = \{a = x_0 < x_1 < ... < x_n = b\}$ such that

$$\epsilon > M^2 \sum_{k=1}^{n} \left(\sup_{x\in(x_{k-1},x_k)} \frac{1}{f(x)} - \inf_{x\in(x_{k-1},x_k)} \frac{1}{f(x)} \right) (x_k - x_{k-1}) \geq$$
$$\geq \sum_{k=1}^{n} \left(\sup_{x\in(x_{k-1},x_k)} f(x) - \inf_{x\in(x_{k-1},x_k)} f(x) \right) (x_k - x_{k-1}).$$

Since $\epsilon > 0$ is arbitrary it follows that f is Riemann integrable.

Question 8: Let $f : \mathbb{R}^5 \mapsto \mathbb{R}^3$ be a C^1-map and assume that $f(0,0,0,0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and that

$$Df(0) = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Prove that there exist a function $g = (g_1, g_2, g_3) : \mathbb{R}^2 \mapsto \mathbb{R}^3$ such that $f(x_1, x_2, g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x})) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for every $\mathbf{x} = (x_1, x_2)$ close enough to $\mathbf{x} = (x_1, x_2) = (0, 0)$.

You may use any aspect of the Banach fixed point theorem without proof.

(4 marks)

Solution Question 8: This is a direct application of the implicit function theorem. Making a Taylor expansion of $f(x_1,x_2,y_1,y_2,y_3)$ we see that

$$f(x_1,x_2,y_1,y_2,y_3) = \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + R(x_1,x_2,y_1,y_2,y_3).$$

For a given $\mathbf{x}$ to find a solution $(y_1,y_2,y_3) = (g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x}))$ is equivalent to solving

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - R(x_1,x_2,y_1,y_2,y_3),$$

which is the same as, for every $\mathbf{x} = (x_1,x_2)^T$ finding a fixed point to the mapping

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \mapsto F(\mathbf{y}) = -\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - R(x_1,x_2,y_1,y_2,y_3),$$

where the equality to the right defines $F(\mathbf{y})$.

Therefore we let $\mathbf{y} = (y_1,y_2,y_3)^T$ and $\mathbf{z} = (z_1,z_2,z_3)^T$ be two points close to the origin. Then

$$|F(\mathbf{y}) - F(\mathbf{z})| = |R(\mathbf{x},\mathbf{y}) - R(\mathbf{x}.\mathbf{z})|.$$

Since the Jacobian $J_R(\mathbf{x},\mathbf{y}) \to 0$ as $(\mathbf{x},\mathbf{y}) \to 0$ there is a small $\delta > 0$ such that if $|\mathbf{x}|, |\mathbf{y}| < \delta$ then $\|J_R(\mathbf{x},\mathbf{y})\| \le 1/2$, where $\|\cdot\|$ denotes the operator norm. It follows from the mean value theorem that, for $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ close to the origin,

$$|F(\mathbf{x}.\mathbf{y}) - F(\mathbf{x},\mathbf{z})| \le \frac{1}{2}|\mathbf{y} - \mathbf{z}|,$$

that is F is a contraction for small enough $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

Arguing as in Banach's fixed point Theorem we let $\mathbf{y}_0 = 0$ and $\mathbf{y}_{k+1} = F(\mathbf{y}_k)$ it follows that

$$|\mathbf{y}_k - \mathbf{y}_0| \le |\mathbf{y}_1 - \mathbf{y}_2| \sum_{j=0}^{k-1} \frac{1}{2^j} \le 2|\mathbf{y}_0| = 2|R(\mathbf{x},0)|.$$

Thus, if $\mathbf{x}$ is so small that $|R(\mathbf{x},0)| < \delta/2$ and $|\mathbf{x}| < \delta$, then, arguing as in the Banach Fixed Point Theorem, $|F(\mathbf{x},\mathbf{y}_k) - F(\mathbf{x},\mathbf{y}_{k+1})| < \frac{1}{2}|\mathbf{y}_k - \mathbf{y}_{k+1}|$ which implies that $\mathbf{y}_k \to \mathbf{y}$ as $k \to \infty$. We may conclude that for every $\mathbf{x}$ s.t. $|\mathbf{x}|, |R(\mathbf{x},0)| < \delta/2$ there is a unique $\mathbf{y}$ such that $\mathbf{y} = F(\mathbf{x},\mathbf{y})$.