Exam SF1677/2713 April 3d 2018

Total marks 32: The preliminary relationship between the marks and grades are

A : 30 **B** : 28 **C** : 25 **D** : 22 **E** : 19 **FX** : 18.

A G on the first homework assignment corresponds to full mark (4 marks) on question 1,
a G on the second homework assignment corresponds to full mark (4 marks) on question 2 and
a G on the third homework assignment corresponds to full mark (4 marks) on question 3.

Allowed help: Only writing utensils are allowed, calculators are NOT allowed for this exam.

All your answers should be proved unless otherwise stated.

Question 1: Assume that $f : [-1, 1] \mapsto \mathbb{R}$ and $g : [-1, 1] \mapsto \mathbb{R}$ are increasing functions and that f is continuous. Assume furthermore that $f(-1) < g(-1)$ and $f(1) > g(1)$. Will the equation $f(x) = g(x)$ have a solution? Note that we do **not** assume that g is continuous. Prove your answer.

(4 marks)

Question 2: Let $f_k : (0, 1) \mapsto \mathbb{R}$ be a sequence of positive and non-decreasing Riemann integrable functions and that for any $x \in (0, 1)$

$$\lim_{N\to\infty} \sum_{k=1}^{N} f_k(x) = f(x),$$

where $f : (0, 1) \mapsto \mathbb{R}$. Assume furthermore that

$$\lim_{N\to\infty} \left[\sum_{k=1}^{N} \left(\int_0^1 f_k(x)dx \right) \right] = 1.$$

Will f be Riemann integrable? If so will $\int_0^1 f(x)dx = 1$? Prove your answer.

(4 marks)

Question 3: Let $f_k : [-1, 1] \mapsto \mathbb{R}$ be a sequence of continuously differentiable functions. Assume furthermore that $f_k \to f$ and that $f_k' \to g$ uniformly on $[-1, 1]$ where $f, g : [-1, 1] \mapsto \mathbb{R}$ are two given continuous functions. Prove that f is differentiable at $x = 0$ and that $f'(0) = g(0)$.

You may, without proof, use any known theorem for continuous functions. However, you may not use any theorem regarding convergence of differentiable functions without proof.

(4 marks)

Question 4: Given a set $A \subset \mathbb{R}$ we define the set

$$\mathcal{S}_A = \{\sin(ax);\ a \in A\}.$$

State a condition on the set A such that $\mathcal{S}_A$ is equicontinuous if and only if A satisfies the stated condition. Prove your answer.

(4 marks)

Question 5: Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuously differentiable function and also assume that $D_{12}f$ and $D_{21}f$ exist and are continuous; here $D_{ij}f = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Prove that $D_{12}f(x, y) = D_{21}f(x, y)$.

HINT: *You may, without proof, use the following result from Rudin (Theorem 9.40): If Q is the cube $[a, a+h] \times [b, b+k] \subset \mathbb{R}^2$ and*

$$\Delta(f, Q) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$

then there exist a point $(x, y) \in Q$ such that

$$\Delta(f, Q) = hkD_{21}f(x, y).$$

(4 marks)

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Question 6: Let $\mathcal{X}$ be the metric space consisting of all functions $f : \mathbb{N} \mapsto \mathbb{R}$ such that $\lim_{n\to\infty} f(n) = 0$ equipped with the metric:

$$d(f, g) = \sup_{n\in\mathbb{N}} |f(n) - g(n)|.$$

Is $\mathcal{X}$ complete? Prove your answer. (You do not need to prove that $\mathcal{X}$ is a metric space.)

(4 marks)

Question 7: Let $f : [a, b] \mapsto \mathbb{R}$, $0 < f \le M$, be a function such that the following integral exist

$$\int_a^b \frac{1}{f(x)} dx.$$

Is f integrable over $[a, b]$? Prove your answer.

(4 marks)

Question 8: Let $f : \mathbb{R}^5 \mapsto \mathbb{R}^3$ be a C^1-map and assume that $f(0, 0, 0, 0, 0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and that

$$Df(0) = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Prove that there exist a function $g = (g_1, g_2, g_3) : \mathbb{R}^2 \mapsto \mathbb{R}^3$ such that $f(x_1, x_2, g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x})) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for every $\mathbf{x} = (x_1, x_2)$ close enough to $\mathbf{x} = (x_1, x_2) = (0, 0)$.

You may use any aspect of the Banach fixed point theorem without proof.

(4 marks)

Good Luck!