Exam SF1677 January 2017

Total marks 35: The relationship between the marks and grades are

A : 31 **B** : 27 **C** : 23 **D** : 19 **E** : 15 **FX** : 14.

A G^+ on the first homework assignment corresponds to full mark (5 marks) on question 1 and a G^+ on the second homework assignment corresponds to full mark (5 marks) on question 2.

All your answers should be proved unless otherwise stated.

Question 1: Define what it means for a set to be uncountable. Then prove that there exists uncountable sets, for instance by providing an example and proving that your example is uncountable.

(5 marks)

Question 2: Let

$$f_n(x) = \frac{\sin(x)}{1+nx^2} \quad \text{on } [-\pi, \pi].$$

1. Will $f_n(x)$ converge in $C^0([-\pi, \pi])$?
2. Will the derivative $f_n'(x)$ converge in $C^0([-\pi, \pi])$?

(5 marks)

Question 3: Let a_j be a bounded sequence of real numbers and define

$$\limsup_{j\to\infty} a_j = \lim_{j\to\infty}\left(\sup_{k>j} a_k\right).$$

Prove, by using the definition of convergence, that $\limsup_{j\to\infty} a_j$ exists. Make sure to specify where you use the least upper bound property for the real numbers.

(5 marks)

Question 4: Let $g_n(x)$ be a sequence of Riemann integrable functions on $[0, 1]$ and $|g_n(x)| \leq 1$ for all n and $x \in [0, 1]$. Define the function

$$G_n(x) = \int_0^x g_n(t)dt \quad \text{for } x \in [0, 1].$$

Prove that there is a subsequence $G_{n_k}(x) \rightrightarrows G(x)$, as $k \to \infty$, for some function $G(x)$ (here $\rightrightarrows$ means converges uniformly). You may use any theorem from the course as long as you state it correctly and show that the assumptions are satisfied.

(5 marks)

Question 5: Let $F : C^0([0, 1]) \mapsto C^0([0, 1])$ be defined by

$$F(f) = \cos(x) + \int_0^1 \sin\left(\frac{y + e^y + x}{10}\right) f(y)dy.$$

Prove that F has a fixed point: that is, there exist an $f \in C^0([0, 1])$ such that $F(f) = f$. You may use any theorem from the course without providing proof; except fixed point theorems, these you have to prove.

(5 marks)

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Question 6: In elementary calculus one uses the formula

$$\int\int_{M_1(0,0)} f(x,y)dxdy = \int_0^1 \int_0^{2\pi} f(r\cos(\phi), r\sin(\phi))rd\phi dr,$$

where $M_1(0,0) = \{(x,y);\ \sqrt{x^2+y^2} < 1\}$.

Prove this formula for bounded and continuous functions f defined on $\mathbb{R}^2$. You may use any theorem form the course without proof, as long as you state it correctly and show that the assumptions are satisfied.

(5 marks)

Question 7: Does it exist a bounded non-decreasing function $f : [0,1] \mapsto [0,1]$ such that $f(x)$ is discontinuous at every $x \in [0,1] \cap \mathbb{Q}$? Decide whether your example of such a function (if they indeed exist) is Riemann integrable or not. If you use a theorem from the course you should sketch the proof.

(5 marks)

Good Luck!