Solution 1: A set A is countable if there exsist a bijection $\sigma : \mathbb{N} \mapsto A$.

The typical excample of an uncountable set would be the real numbers.

To show that the real numbers are uncountable we assume the countrary. That there is a bijection $\sigma : \mathbb{N} \mapsto \mathbb{R}$. If such a bijection exsist we may write a list of the real numbers

$$\sigma(1) = n_1 . d_{1,1}\, d_{1,2}\, d_{1,3}\, d_{1,4} \cdots \qquad = n_1 + \sum_{j=1}^{\infty} d_{1,j} \cdot 10^{-j}$$

$$\sigma(2) = n_2 . d_{2,1}\, d_{2,2}\, d_{2,3}\, d_{2,4} \cdots$$

$$\sigma(3) = n_3 . d_{3,1}\, d_{3,2}\, d_{3,3}\, d_{3,4}$$

$$\vdots \qquad \vdots$$

Where we use the decimal notation, that

$$\pi = n . d_1 d_2 d_3 d_4 \ldots \quad \text{where } n=3,\ d_1 = 1,\ d_2 = 4, \text{ etc}$$
$$3 . 1\ 4\ 1\ 5\ 9 \ldots$$

If σ where onto then every number should be on the list. But take the real number

$$r = \sum_{j=1}^{\infty} f(d_{j,j})\, 10^{-j} \quad \text{where} \quad f(d_{j,j}) = \begin{cases} 1 & \text{if } d_{j,j} \neq 1 \\ 2 & \text{if } d_{j,j} = 1 \end{cases}$$

then r differs from every $\sigma(n)$ at the n:th decimal place. It follows that σ is not bijective since it is not onto. This is the contradiction.

Solution 2

1) We aim to show that $f_n \rightrightarrows 0$ in $C^0([-\pi,\pi])$

the metric in $C^0([-\pi,\pi])$ is $d(f,g) = \sup_{[-\pi,\pi]} |f(x)-g(x)|$

so we need to show that

for every $\varepsilon > 0$ there exists an N_ε s.t.

$$\sup_{x\in[-\pi,\pi]} |f_n(x) - 0| < \varepsilon .$$

Since $|\sin(x)| \leq |x|$ it is enough to show that $\left|\frac{x}{1+nx^2}\right| < \varepsilon$ for $n > N_\varepsilon$.

If $|x| < \varepsilon$ then

$$\left|\frac{x}{1+nx^2}\right| \leq |x| < \varepsilon .$$

and if $|x| \geq \varepsilon$ then, for $x \in [-\pi,\pi]$

$\left|\frac{x}{1+nx^2}\right| \leq \frac{\pi}{1+n\varepsilon^2}$ but $\frac{\pi}{1+n\varepsilon^2} < \varepsilon$ if

$n > \frac{\pi}{\varepsilon^3}$.

Therefore

$$n > \frac{\pi}{\varepsilon^3} \implies \sup_{x\in[-\pi,\pi]} |f_n(x)| < \varepsilon .$$

Therefore $f_n \rightrightarrows 0$

2) f_n' does not converge in $C^0([-\pi,\pi])$.

$$f_n'(x) = \frac{\cos(x)}{1+nx^2} - \frac{2nx\sin(x)}{(1+nx^2)^2}.$$

We have for $|x|>\delta>0$ that

$$|f_n'(x)| < \underbrace{\frac{1}{1+n\delta^2}}_{\to 0} + \underbrace{\frac{2n\delta^2}{(1+n\delta^2)^2}}_{\to 0} \to 0$$

And $f_n'(0) = \frac{1}{1+0} - \frac{2n\cdot 0\cdot \sin(0)}{(1+n\cdot 0^2)^2} = 1.$

Therefore $f_n'(x) \to \begin{cases} 1 & x=0 \\ 0 & x\neq 0 \end{cases}$

pointwise. But if f_n' was converging in $C^0([-\pi,\pi])$ then the uniform, and therefore the pointwise limit would have to be a continuous function.

Solution 3

Define $A_j = \sup_{k>j} a_k$. Then

$\limsup_{j\to\infty} a_j = \lim_{j\to\infty} A_j$ so it is enough to show that A_j converges.

Since a_j is bounded $\sup_j |a_j| \le M$ for some M and therefore $|A_j| \le M$.

Thus A_j is a bounded sequence.

Moreover

$$A_j = \sup_{k>j} a_k \ge \max\left(a_{j+1}, \sup_{k>j+1} a_k\right) = \max(a_{j+1}, A_{j+1}) \ge A_{j+1}$$

so A_j is decreasing. Since A_j is bounded there exist, by the least upper bound property, a greatest lower bound $a = \inf(A_j) = -\sup(-A_j)$.

We claim that $A_j \to a$. Fix an $\varepsilon > 0$ then $a+\varepsilon$ is not a greatest lower bound for the set $\{A_j ; j \in \mathbb{N}\}$ therefore $\exists J_\varepsilon$ s.t. $A_{J_\varepsilon} < a+\varepsilon$.
It follows that

$$j > J_\varepsilon \Rightarrow \cancel{a \le A_{J_\varepsilon} \le A_j} \quad a+\varepsilon > A_{J_\varepsilon} \underset{\text{decreasing}}{\ge} A_j \underset{a \text{ is glb.}}{\ge} a$$

So $j > J_\varepsilon \Rightarrow |A_j - a| < \varepsilon$.

$\blacksquare$

Solution 4

We aim to show that G_n forms an equicontinuous set. That is for any $\varepsilon > 0$ there exsist a $\delta > 0$ s.t. $|x-y| < \delta$ implies that

$$|G_n(x) - G_n(y)| = \left| \int_0^x g_n(t)\,dt - \int_0^y g_n(t)\,dt \right| \leq \left| \int_x^y g_n(t)\,dt \right| \leq |x-y| < \varepsilon$$

where we used that $|g_n(t)| \leq 1$ and that

$\left| \int_a^b f(x)\,dx \right| \leq |b-a| \sup_{[a,b]} |f(x)|$ in the last inequality.

Clearly, by choosing $\varepsilon = \delta$, the family G_n is equicontinuous.

The Arzela-Ascoli theorem states that for any sequence, bounded and equicontinuous, there exsist a sub-sequence converging in $C^0([0,1])$.

We therefore only have to show that G_n are bounded. But that follows from

$$|G_n(x)| = \left| \int_0^x g_n(t)\,dt \right| \leq \int_0^x |g_n(t)|\,dt \leq |x| \leq 1 \quad \text{for } x \in [0,1]$$

So G_n forms a bounded & equicontinuous sequence of functions and has therefore a convergent, in $C^0([0,1])$, sub-sequence.

Solution 5.

Theorem: If $F: M \mapsto M$, M complete metric space, and $|F(x) - F(y)| \leq \lambda |x-y|$ for some $\lambda < 1$, then F has a fixed point $x \in M$; $F(x) = x$.

Proof: Pick an $x^0 \in M$ arbitrarily and define $x^{j+1} = F(x^j)$. Then

$$|F(x^{j+1}) - F(x^j)| \leq \lambda | x^{j+1} - x^j| =$$

$$= \lambda \,|F(x^j) - F(x^{j-1})| \leq \lambda^2 |x^j - x^{j-1}| \leq \lambda^2 |F(x^{j-1}) - F(x^{j-2})|$$

$$\leq \ldots \leq \lambda^j \underbrace{|F(x^0) - x^0|}_{= C} = C\lambda^j.$$

This implies that

$$|x^k - x^m| \leq \sum_{j=m}^{k-1} |x^{j+1} - x^j| \leq C \sum_{j=m}^{k-1} \lambda^{j-1} \leq \frac{C}{1-\lambda} \lambda^{m-1}.$$

So if $m, k > N$, where N is chosen large enough so that $\frac{C}{1-\lambda} \lambda^{N-1} < \varepsilon$, which is always possible since $\lim_{N \to +\infty} \lambda^N = 0$ since $\lambda < 1$, then

$|x^k - x^m| < \varepsilon$. So x^j forms a Cauchy sequence in M. But M is complete so $x^j \to x^0 \in M$.

But the condition $|F(x) - F(y)| \leq \lambda |x-y|$ implies that F is continuous so $F(x^0) = \lim_{j \to +\infty} F(x^j) = \lim_{j \to +\infty} x^{j+1} = x^0$ $\boxed{}$

We now claim that the F given in the question is a Contraction, such as in the Thm, on $C^0([0,1])$. To see this we calculate

$$|F(f)-F(g)| = \left| \int_0^1 \sin\left(\frac{y+e^y+x}{10}\right)(f(y)-g(y))\,dy \right| \leq$$

$$\leq \sup|f(y)-g(y)| \left| \int_0^1 \sin\left(\frac{y+e^y+x}{10}\right) dy \right| . \qquad (*)$$

But $|\sin(x)| \leq |x|$ so $\left|\sin\left(\frac{y+e^y+x}{10}\right)\right| \leq \left|\frac{y+e^y+x}{10}\right|$

where both $y, x \in [0,1]$ so $\left|\frac{y+e^y+x}{10}\right| \leq \left|\frac{2+e^1}{10}\right| < \frac{1}{2}$

From $(*)$ we see that

$$|F(f)-F(g)| \leq \sup|f-g| \cdot \left| \int_0^1 \frac{1}{2}\,dy \right| = \frac{1}{2}\sup|f-g| .$$

So F is a contraction with $\lambda = \frac{1}{2} < 1$.

By the Theorem it follows that F has a fixed point.

Answer 6.

This is obviously an application of the Change of variables" formula.

Theorem: Let $e : U \longmapsto \mathbb{R}^2$ be a diffeomorphism, that is

1) e is bijective onto its image $e(U)$

2) e and e^{-1} are C^1 functions on U and $e(U)$ respectively

and ~~$Jac_x e = \det(De)_{(x)}$~~, $Jac_{(\phi,r)} e = \det(De)_{(\phi,r)}$

assume furthermore that $\overline{R} \subset U$ is a rectangle and that f is integrable on $e(R)$. Then

$$\int_R f \circ e \, |Jac_{(\phi,r)} e| \, dr\, d\phi = \int_{e(R)} f(x,y)\, dx\, dy.$$

Since f, in the exercise, is continuous it is integrable on the disk $M_1(0,0)$.

Furthermore

$$e : [0, 2\pi] \times [0,1] \longmapsto M_1(0,0).$$

When $e(\phi, r) = (r\cos\phi, r\sin\phi)$. However, e is not a diffeomorphism on $[0,2\pi] \times [0,1]$. In particular e is not injective since the entire set $[0,2\pi] \times \{0\}$ is mapped to $(0,0)$.

We claim that e is a diffeomorphism from

$[\varepsilon, 2\pi-\varepsilon]\times[\varepsilon,1]$ onto its image.

To see this we calculate the jacobian

$$(De)_{(\phi,r)} = \begin{bmatrix} -r\sin\phi & r\cos\phi \\ \cos\phi & \sin\phi \end{bmatrix} \Rightarrow \left|\det (De)_{(\phi,r)}\right| = r$$

so $(De)_{(\phi,r)}$ is invertable at every point $(\phi,r)\in[\varepsilon,2\pi-\varepsilon]\times[\varepsilon,1]$ since $r\geq\varepsilon$ on that set. Since r, $\sin(\phi)$ and $\cos(\phi)$ are continuously differentiable the inverse function then implies that e^{-1} is a C^1 function.

We also need to show that e is bijective onto its image. Surjectivity is obvious so we only need to show that e is injective, but that is clear, if $e(\phi_1,r_1)=e(\phi_2,r_2)$ then

$$|e(\phi_1,r_1)| = |e(\phi_2,r_2)| \Rightarrow r_1=r_2$$

and
$$\left.\begin{array}{l} \sin(\phi_1)=\sin(\phi_2) \Rightarrow \phi_1 = \pi-\phi_2 \\ \cos(\phi_1)=\cos(\phi_2) \Rightarrow \phi_1=\pm\phi_2 \end{array}\right\} \Rightarrow \phi_1=\phi_2.$$

It follows from the Change of Variables formula that, with $R=R_\varepsilon=[\varepsilon,2\pi-\varepsilon]\times[\varepsilon,1]$ and $U=(\frac{\varepsilon}{2},2\pi-\frac{\varepsilon}{2})\times(\frac{\varepsilon}{2},1+\frac{\varepsilon}{2})$,

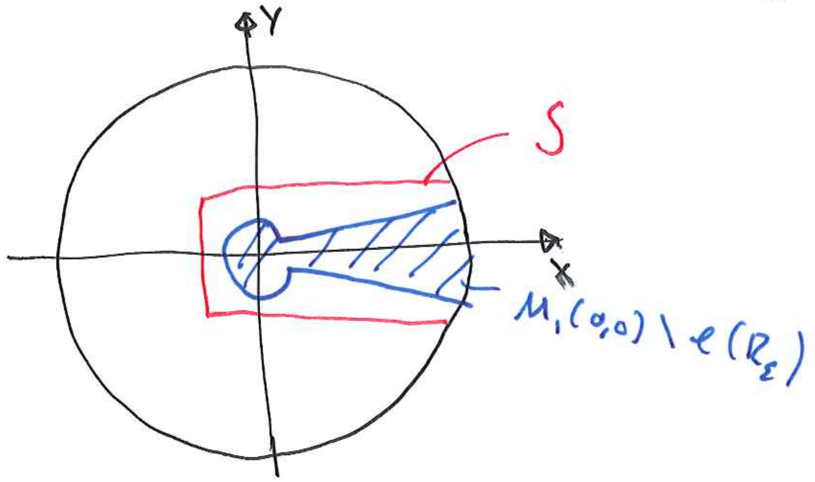
$$\int_{R_\varepsilon} f(r\cos(\phi),r\sin(\phi))\,r\,dr\,d\phi = \int_{e(R_\varepsilon)} f(x,y)\,dx\,dy$$

Therefore

$$\left| \iint_{M_1(0,0)} f(x,y)\,dx\,dy - \int_0^1\int_0^{2\pi} f(r\cos(\phi), r\sin(\phi))\, r\,dr\,d\phi \right| \leq \qquad (*)$$

$$\underbrace{\left| \int_{e(R_\varepsilon)} f\,dx\,dy - \int_{R_\varepsilon} f(r\cos(\phi), r\sin\phi)\, r\,dr\,d\phi \right|}_{=0 \text{ by change of var formula}} + \left| \int_{M_1\setminus e(R_\varepsilon)} f\,dx\,dy - \int_{R\setminus R_\varepsilon} f(r\cos, r\sin)\,dr\,d\phi \right|$$

But the set $M_1 \setminus e(R_\varepsilon) \subset \{(x,y);\, -\varepsilon < x \leq 1,\ |y| \leq \varepsilon\} = S$



And therefore

$$\left| \int_{M_1\setminus e(R_\varepsilon)} f\,dx\,dy \right| \leq \sup_{\overline{M}_1} |f| \left| \int_S dx\,dy \right| <$$

$$\leq \sup_{\overline{M}_1} |f| \cdot 2\varepsilon(1+\varepsilon).$$

Similarly

$$\left| \int_{R\setminus R_\varepsilon} f(r\cos(\phi), r\sin\phi)\, r\,dr\,d\phi \right| \leq \sup_{R\setminus R_\varepsilon} |f| \cdot (\varepsilon(2\pi+1))$$

Using this in $(*)$ we see that

$$\left| \int_{M_1(0,0)} f(x,y)\,dx\,dy - \int_0^1\int_0^{2\pi} f(r\cos(\phi), r\sin(\phi))\, r\,d\phi\,dr \right| \leq \sup|f| \cdot \left(2(1+\varepsilon) + (2\pi+1)\right)\varepsilon$$

Since $\varepsilon > 0$ is arbitrary the desired equality follows.

Answer 7 There is a function f on $[0,1]$ that is bounded non-decreasing with a discontinuity at exactly every $x \in [0,1] \cap \mathbb{Q}$, furthermore that function is Riemann integrable.

To see this we ~~define~~ let q_j be an enumeration of $[0,1] \cap \mathbb{Q}$ and define

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{2^j} \chi_{(q_j,1]}(x) \quad \text{where } \chi_{(q_j,1]}(x) = \begin{cases} 0 & x \le q_j \\ 1 & x > q_j \end{cases}$$

Then, for each $x \in [0,1]$, $\sum_{j=1}^{N} \frac{1}{2^j} \chi_{(q_j,1]}(x) \le \sum_{j=1}^{\infty} \frac{1}{2^j} \le 1$

is bounded and also increasing in N. It follows that f is well defined since bounded & increasing sequences converge.

To see that f is non-decreasing we just notice that limits of non-decreasing functions has to be non-decreasing

$$y > x \Rightarrow f(y) - f(x) = \lim_{N \to \infty} \sum_{j=1}^{N} \frac{1}{2^j} \underbrace{\left(\chi_{(q_j,1]}(y) - \chi_{(q_j,1]}(x) \right)}_{\ge 0} \ge 0.$$

By a similar argument it follows that

$f(x) - \frac{1}{2^j} \chi_{(q_j,1]}(x)$ is non-decreasing which implies that

$$\lim_{x \to q_j^-} \left(f(x) - \frac{1}{2^j} \chi_{(q_j,1]}(x) \right) \le \lim_{j \to q_j^+} \left(f(x) - \frac{1}{2^j} \chi_{(q_j,1]}(x) \right)$$

That is

$$\lim_{x \to q_j^+} f(x) - \lim_{x \to q_j^-} f(x) \ge \lim_{x \to q_j^+} \frac{1}{2^j} \chi_{(q_j,1]}(x) - \lim_{x \to q_j^-} \frac{1}{2^j} \chi_{(q_j,1]}(x) = \frac{1}{2^j}$$

Thus $f(x)$ is discontinuous at every q_j.

To see that $f(x)$ is integrable we fix an $\varepsilon > 0$ and let N be so large that

$$\left| f(x) - \sum_{j=1}^{N} \frac{1}{2^j} \chi_{(q_j,1]}(x) \right| \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j} < \varepsilon/4$$

Then $$\sum_{j=1}^{N} \frac{1}{2^j} \chi_{(q_j,1]}(x) \leq f(x) \leq \sum_{j=1}^{N} \frac{1}{2^j} \chi_{(q_j,1]}(x) + \varepsilon/4 .$$

But the functions $\sum_{j=1}^{N} \frac{1}{2^j} \chi_{(q_j,1]}(x)$ and $\sum_{j=1}^{N} \frac{1}{2^j} \chi_{(q_j,1]}(x) + \varepsilon/4$

are step functions since they only attain finitely many values, and each value on an interval (q_j, q_k).
Thus if we use $0, 1$ and all $q_1 \ldots q_N$ in our partition P in the definition of Riemann integration it follows that

$$U(f, P) - L(f, P) \leq U\left(\sum_{j=1}^{N} \frac{1}{2^j} \chi_{(q_j,1]} + \frac{\varepsilon}{4}, P\right) - L\left(\sum_{j=1}^{N} \frac{1}{2^j} \chi_{(q_j,1]}, P\right) =$$

$$= \varepsilon/4 .$$

Since ε was arbitrary it follows that f is Riemann integrable - and yes there is no point in choosing $\varepsilon/4$ I was just first planning to use a different argument....