On the future stability of cosmological solutions to Einstein's equations with accelerated expansion

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Abstract. The solutions of Einstein's equations used by physicists to model the universe have a high degree of symmetry. In order to verify that they are reasonable models, it is therefore necessary to demonstrate that they are future stable under small perturbations of the corresponding initial data. The purpose of this contribution is to describe mathematical results that have been obtained on this topic. A question which turns out to be related concerns the topology of the universe: what limitations do the observations impose? Using methods similar to ones arising in the proof of future stability, it is possible to construct solutions with arbitrary closed spatial topology. The existence of these solutions indicate that the observations might not impose any limitations at all.

Mathematics Subject Classification (2010). Primary 83C05; Secondary 35Q76.

Keywords. General relativity, stability of solutions, Einstein-Vlasov system.

1. Introduction

In 1915, the interpretation of gravitational forces was fundamentally altered by the introduction of Einstein's general theory of relativity. The underlying mathematical structures were not well understood at the time, and as a consequence, some of the fundamental questions have only recently been phrased in the form of mathematical problems. Since Einstein's equations are not as commonly studied in mathematics as many other equations that appear in physics, we here wish to give a brief description of their origin and of how different perspectives on them have developed since the inception of general relativity. However, the main purpose of this contribution is more specific. Recent observational data indicate that the universe is expanding at an accelerated rate. As a consequence, physicists nowadays use solutions to Einstein's equations with accelerated expansion to model the universe. Since the model solutions are highly symmetric (they are spatially homogeneous and isotropic), a natural question to ask is: are they stable? In order to phrase this question in a more precise way, it is necessary to formulate Einstein's equations (coupled to various matter equations) as an initial value problem. It

*The author would like to acknowledge the support of the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine, and the Swedish Research Council.

turns out that there is a natural and geometric notion of initial data, and that, given initial data, there is a uniquely associated maximal Cauchy development. A more precise formulation of the question of stability is then: given initial data corresponding to one of the standard models, do small perturbations thereof yield maximal Cauchy developments that are globally similar? The currently preferred models have a big bang type singularity and an expanding direction. Proving stability in the direction of the singularity is quite difficult (there are some results in the case of special matter fields), but there are several results on stability in the expanding direction. For that reason we shall focus on the expanding direction here, and we shall think of it as corresponding to the future.

The outline of this contribution is as follows. We begin, in Section 2, by giving a brief description of the origin of the general theory of relativity. Moreover, we explain how the present contribution fits into the general context of mathematical studies of Einstein's equations. In Sections 3 and 4, we then discuss the formulation of the initial value problem, which is needed in order for us to be able to state a stability result. In Section 5, we then discuss the topic of stability in general. We give a rough description of some of the results that have been obtained in the past, as well as of some of the methods. However, we shall only formulate a theorem in the case of the Einstein-Vlasov system. In order to be able to do so, we devote Section 6 to a discussion of this system. In Sections 7 and 8, we then describe the background solutions we are interested in proving stability of, and state the relevant results. Finally, in Section 9, we discuss a construction which indicates that the observations do not impose any restrictions on the topology of the universe.

2. General relativity

In order to discuss the general theory of relativity, it is natural to begin with Einstein's paper on special relativity [6]. The starting point of the paper is the contemporary interpretation of electrodynamics. Noting that this interpretation involves asymmetries, and postulating that the speed of light is independent of inertial observer, Einstein was led to the Poincaré group of transformations, relating the observations of inertial observers. Due to the added insight of Poincaré and Minkowski, it was realized that this group is the group of isometries of Minkowski space; recall that Minkowski space is $\mathbb{R}^4$ with the inner product $\langle x, y\rangle = x^t\eta y$, where $\eta = \mathrm{diag}(-1, 1, 1, 1)$. This interpretation indicates the importance of geometry. As a next step, it is clear that Newtonian gravity has to be modified. Two important principles that guided Einstein in his search for a modified theory were the *equivalence principle* (the equality between inert and gravitational mass; this is roughly speaking the idea that it is not possible to distinguish between a coordinate system at rest in a uniform gravitational field and a uniformly accelerated coordinate system far away from all matter, for example) and the *principle of general covariance*, the idea that the equations should be independent of the choice of coordinate system. By a simple thought experiment involving rotating coordinate systems, it can be argued (heuristically) that acceleration distorts the geometry; cf. [7, pp. 58–59]. Combining this observation with the equivalence principle indicates that gravitation should affect the geometry. In fact, it is not unnatural to equate gravitation with distorsion of the geometry. Since the geometry at a point should be described by the Minkowski metric (with respect to a suitable choice of coordinates), the natural underlying object in general relativity is a Lorentz manifold; in other words, a manifold M on which a smooth symmetric covariant two-tensor field g is defined, where g is such that it, at each point of M, equals the Minkowski metric with respect to suitable coordinates. The standard notions and constructions in Riemannian geometry (Levi-Civita connection, curvature tensor, Ricci tensor, scalar curvature, geodesics etc.) can be defined in the same way in Lorentz geometry, and we shall use them below without further comment. The one question that remains is: what equation should (M, g) satisfy? In some way, the geometry should be related to the matter sources. On the level of special relativity, it was already clear that the matter should be combined into the so-called stress-energy tensor; a symmetric covariant two-tensor field, the exact form of which depends on the specific matter model. Let us denote this object by T. It should be the source term in Einstein's equations (it can be thought of as a generalization of the matter density in Poisson's equation in Newtonian gravity). As a consequence, what remains is to determine what the left hand side of the equation should be. To begin with, it should clearly be symmetric. However, due to the equations for the matter, the stress-energy tensor should be divergence free. As a consequence, the left hand side should be as well. Moreover, it should be such that the resulting equations are independent of the choice of coordinates. Finally, it is natural (for the sake of simplicity, and in analogy with the Poisson equation) to require that the left hand side should contain at most second order derivatives of the gravitational field (i.e., the metric). However, the only equations fulfilling these requirements are the ones of the form

$$G + \Lambda g = \alpha T, \tag{1}$$

where Λ and α are constants and

$$G = \mathrm{Ric} - \frac{1}{2} S g$$

is the *Einstein tensor*, defined in terms of the Ricci tensor, Ric, and the scalar curvature, S, of the metric g (the reader interested in a justification of this statement is referred to the corollary of [15, Theorem 1, p. 500]). In (1), we shall, for simplicity, assume $\alpha = 1$. Moreover, we shall refer to Λ as the *cosmological constant*. The resulting equations are

$$G + \Lambda g = T, \tag{2}$$

and we shall refer to them as *Einstein's equations*.

2.1. Historical development. It is of interest to say a few words concerning how different perspectives on these equations have developed over time. In the intial phase, physicists tried to find explicit solutions to the equations. In order to do so, they imposed symmetry assumptions adapted to the physical situation of interest. When considering physical objects such as a star, a galaxy, a globular cluster etc. (i.e., an *isolated system*), a natural first symmetry assumption to make is that of spherical symmetry. This assumption led to the class of Schwarzschild spacetimes, which can be used to model the gravitational field outside a non-rotating star or black hole. Much later, the Kerr family of solutions was found, describing the rotating case. When modelling the universe as a whole, another type of symmetry assumption is required. Guided by the Copernican principle, a natural starting point in this case is the assumption of spatial homogeneity and isotropy; this is the assumption that at 'one moment in time', it is not possible to distinguish between two points in space, nor is it possible to distinguish between two directions. Symmetry assumptions of this type (corresponding to the so-called *cosmological setting*) led to the Friedman-Lemaître-Robertson-Walker metrics, which are still used to this very day when modelling the universe (though the preferred matter models have changed over time). Even though mathematicians nowadays consider significantly less symmetric solutions, the problems considered can still be divided into ones concerning isolated systems and ones concerning the cosmological setting.

In the initial stages of the development of general relativity, when the emphasis was on finding explicit solutions, the geometry remained somewhat obscure. As a consequence, some of the features of, e.g., the Schwarzschild solutions were misunderstood for several decades. In the 50's and 60', the geometry received more attention, and the so-called *singularity theorems* were proven. In order to give an idea of the statements of these results, it is necessary to introduce the notion of causal geodesics. To begin with, a vector v in Minkowski space is said to be *timelike* if $\langle v, v\rangle < 0$; *lightlike* or *null* if $\langle v, v\rangle = 0$ and $v \neq 0$; and *spacelike* if $\langle v, v\rangle > 0$ or $v = 0$. A vector which is either timelike or null is said to be *causal.* These notions can be generalized to Lorentz manifolds. Moreover, it makes sense to speak of timelike curves etc. as well as spacelike hypersurfaces. In particular, the character of a geodesic (timelike, null, spacelike) is preserved, so that it is meaningful to speak of timelike geodesics etc. In the interpretation of general relativity, a causal curve corresponds to an observer that travels at a speed less than or equal to that of light. Moreover, a timelike geodesic corresponds to a freely falling test particle, and a null geodesics corresponds to a light ray. Thus causal geodesics are of particular importance in general relativity. A notion which is also of importance is that of a time orientation. At a given spacetime point, the set of causal vectors based at that point has two components. A continuous choice of component corresponds to a *time orientation* (and we shall, from now on, assume all Lorentz manifolds to be time oriented). Vectors belonging to the chosen component will be referred to as *future oriented.*

The singularity theorems of Hawking and Penrose give general conditions that ensure the existence of incomplete causal geodesics. Since the existence of such a geodesic means that there is a freely falling test particle (or a light ray) which exits the spacetime in finite parameter time, Hawking and Penrose equated causal geodesic incompleteness with the existence of a singularity (examples illustrate that this is not always reasonable). Due to the results, it is to be expected that singularities, in the sense of causal geodesic incompleteness, occur generically in solutions to Einstein's equations. These results changed the perspective concerning the occurrences of singularities. Moreover, due to the methods used to prove them, the importance of the subject of Lorentz geometry became apparent.

In the early 50's, Yvonne Choquet-Bruhat formulated Einstein's equations as an initial value problem [8]. It took a significant amount of time before this perspective became a natural starting point in the subject. Since the initial data cannot be specified freely (they have to satisfy an underdetermined, non-linear system of elliptic PDE's, referred to as the *constraint equations*), and since, given initial data, the evolution problem typically involves proving global existence of solutions to a non-linear system of hyperbolic PDE's, this is perhaps not so surprising. In particular, the relevant PDE tools were not so well developed in the early 50's. Nevertheless, this perspective has become more and more important in the subject. This is, in particular, due to the fact that central questions such as that of stability are most naturally formulated using it.

With the above description in mind, the present contribution can be said to be concerned with the initial value formulation of Einstein's equations in the cosmological setting. Moreover, the precise notion of stability we shall use is highly dependent on a Lorentz geometric interpretation of the outcome of the PDE analysis.

3. On the character of Einstein's equations

In order to justify that it is meaningful to formulate Einstein's equations as an initial value problem, let us begin by focusing on the vacuum equations with a vanishing cosmological constant. Since these equations can be written $\mathrm{Ric} = 0$, it is of interest to know if Ric, considered as a differential operator acting on the components of the metric, has a particular character (elliptic, hyperbolic etc.). Due to the diffeomorphism invariance of the equations, this is not to be expected. On the other hand, it is possible to break the diffeomorphism invariance by making a special choice of coordinates. In fact, choosing coordinates such that the contracted Christoffel symbols vanish, the Ricci tensor (schematically) takes the form

$$\mathrm{Ric}_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + F_{\alpha\beta}(g, \partial g), \tag{3}$$

where F is a quadratic expression in the first derivatives of the metric components. In this equation, we assume Greek indices to range from 0 to n, where $n+1$ is the dimension of the Lorentz manifold, and we tacitly assume that repeated indices are summed over (the Einstein summation convention). With respect to these coordinates, Einstein's vacuum equations can thus be thought of as a system of non-linear wave equations for the metric components. As a consequence, it seems natural to formulate a corresponding inital value problem.

It is of interest to note that the above issues arise not only in general relativity, but also in Riemannian geometry and in Ricci flow. In Riemannian geometry, it is sometimes convenient to think of the Ricci tensor as an elliptic differential operator acting on the components of the metric; this yields good control of the metric components, given information concerning the Ricci tensor. It is therefore of interest to consider the so-called *harmonic coordinates*, defined by the condition that the contracted Christoffel symbols vanish. The reason for referring to these coordinates as harmonic is that their defining requirement is equivalent to

$$\Delta_g x^\mu = 0,$$

where Δ_g is the scalar covariant Laplacian associated with the metric g and x^μ are the components of the coordinate system. The analogous coordinates in the Lorentzian setting are sometimes, by analogy, referred to as harmonic coordinates (and sometimes as wave coordinates). In Ricci flow, the relevant equation is $\partial_t g = -2\mathrm{Ric}[g]$, and when dealing with this equation analytically, it would be convenient if Ric were an elliptic operator. Hamilton's original idea concerning how to prove local existence was to appeal to the Nash-Moser inverse function theorem. However, later proofs instead relied on breaking the diffeomorphism invariance in order to obtain a strictly parabolic equation.

4. The initial value problem

With the above observations in mind, it seems natural to formulate an initial value problem. However, it is not so clear what the initial data should be, nor where they should be specified. It turns out that there are several ways of proceeding, but we shall here focus on the perspective that arises in analogy with the standard Cauchy problem for the ordinary wave equation. In that setting, the initial data are specified on a $t = \mathrm{const}$ hypersurface in Minkowski space. These hypersurfaces are special in several ways. First of all, they are *spacelike*, meaning that the induced metric is Riemannian (in this particular case, they are in fact the ordinary Euclidean metric). Moreover, they are intersected exactly once by every inextendible causal curve; cf. the above terminology. Hypersurfaces in Lorentz manifolds which are intersected exactly once by every inextendible casual curve are referred to as *Cauchy hypersurfaces*. They are natural surfaces on which to specify initial data, since given initial data on a Cauchy hypersurface (for the linear wave equation on the Lorentz manifold), there is a unique corresponding solution. A Lorentz manifold which admits a Cauchy hypersurface is called *globally hyperbolic*.

Turning to the choice of the intial data, it would seem natural to specify the metric components and their normal derivative at the initial hypersurface (keeping (3) in mind). However, since Einstein's equations are geometric in nature, the initial data should be geometric as well. On the other hand, the induced metric and second fundamental form are geometric in nature and correspond to a part of the desired information; with respect to local coordinates, they yield some of the metric components and the normal derivative of some metric components.

The induced metric and second fundamental form would thus seem to constitute minimal information needed in order to contruct a solution. On the other hand, it unfortunately turns out that these initial data cannot be specified freely. In order to be more specific, let Σ be a spacelike hypersurface in a Lorentz manifold on which Einstein's equations (2) are satisfied. Contracting the equations twice with respect to the future directed unit normal, say N, yields

$$\frac{1}{2}[\bar{S} - \bar{k}_{ij}\bar{k}^{ij} + (\mathrm{tr}_{\bar{g}}\bar{k})^2] = \rho + \Lambda, \tag{4}$$

where $\bar{g}$ and $\bar{k}$ are the induced metric and second fundamental form on the hypersuface Σ respectively; cf. [18, Proposition 13.3, p. 149]. Moreover, $\bar{S}$ is the scalar curvature of the metric $\bar{g}$, indices are raised and lowered with $\bar{g}$ and $\rho = T(N, N)$. In particular, all the ingredients in (4) are intrinsic to the hypersurface. Contracting (2) once with respect to the future directed unit normal and once with respect to a tangential vector yields the equation

$$\overline{\nabla}^j \bar{k}_{ji} - \overline{\nabla}_i \mathrm{tr}_{\bar{g}}\bar{k} = -J_i, \tag{5}$$

where $\overline{\nabla}$ is the Levi-Civita connection associated with the metric $\bar{g}$ and J is the one-form field defined by $J = -T(N, \cdot)$; cf. [18, Proposition 13.3, p. 149]. Again, the ingredients of (5) are intrinsic to the hypersurface Σ. Clearly, the initial data have to satisfy (4) and (5), which are referred to as the *Hamiltonian* and *momentum constraints* respectively; collectively, we shall refer to them as the *constraint equations*. It is natural to ask whether the constraint equations are sufficient in order to guarantee the existence of a corresponding development. In the vacuum setting, this question was settled in the seminal result of Yvonne Choquet-Bruhat [8], which we now formulate.

Theorem 4.1. *Let $(\Sigma, \bar{g}, \bar{k})$ be initial data for Einstein's vacuum equations; i.e., Σ is an n-dimensional manifold, $\bar{g}$ is a Riemannian metric and $\bar{k}$ is a symmetric covariant 2-tensor field satisfying the vacuum constraint equations; i.e., (4) and (5) with $\Lambda = 0$, $\rho = 0$ and $J = 0$. Then there is a globally hyperbolic development of the initial data. In other words, a Lorentz manifold (M, g) satisfying Einstein's vacuum equations and an embedding $i : \Sigma \to M$ such that $i^*g = \bar{g}$ and $i^*\kappa = \bar{k}$, where κ is the second fundamental form of $i(\Sigma)$ in (M, g). Moreover, $i(\Sigma)$ is a Cauchy hypersurface in (M, g).*

This result has been generalized to include many different types of matter models. We shall not list them, but for all the matter models discussed in this contribution, there is a result analogous to Theorem 4.1.

Even though Theorem 4.1 is important, it does have one deficiency; there is no uniqueness statement. Given initial data, there are infinitely many inequivalent globally hyperbolic developments associated with it. In order to obtain uniqueness, it is necessary to require some sort of maximality. In fact, the fundamental result, due to Yvonne Choquet-Bruhat and Robert Geroch [4], is the following.

Theorem 4.2. *Let $(\Sigma, \bar{g}, \bar{k})$ be initial data for Einstein's vacuum equations. Then there is a unique maximal globally hyperbolic development.*

Due to this theorem, it is clear that the notion of initial data introduced in the statement of Theorem 4.1 is meaningful. Unfortunately, there are examples of maximal globally hyperbolic developments that are extendible in the class of all (not necessarily globally hyperbolic) developments. In fact, there might even be inequivalent maximal developments, indicating that the general theory of relativity is not deterministic. Since the examples are very special, one is led to the strong cosmic censorship conjecture. However, that is not the main topic of this contribution. In fact, we shall here be content with the maximal globally hyperbolic development as *the* development of the intial data.

5. Stability

Since Einstein's equations can be formulated as an initial value problem, it is possible to phrase the stability question: Given initial data corresponding to a specific solution, do small perturbations thereof yield maximal globally hyperbolic developments which are globally similar? The question is still somewhat vague, since we have not specified what is meant by globally similar, nor what is meant by small perturbations. However, the precise meaning in practice depends on the particular solution under consideration, and even for a given solution it is sometimes possible to take different perspectives.

Turning to the stability results that have been obtained in the past, the first one is due to Helmut Friedrich; cf. [9], which contains a proof of stability of de Sitter space. In the same paper, he also proved future stability of Minkowski space, starting with hyperboloidal initial data. Later on, Demetrios Christodoulou and Sergiu Klainerman proved stability of Minkowski space [5]. That stability holds when using harmonic coordinates was only demonstrated much later by Hans Lindblad and Igor Rodnianski [13]. Another perspective on the stability of Minkowski space is given by the work of Lydia Bieri; cf. [3]. Even though all of the references [9, 5, 13, 3] pertain to the problem of stability of Minkowski space, they are very different in nature; the assumptions and conclusions are different in all of these references, and the results correspond to different notions of 'smallness' and 'global similarity'.

Minkowski space is a natural solution to start with when one is interested in isolated systems. However, the topic of the present contribution is cosmology. Of the references mentioned above, the one which is of interest in that setting is [9], in which Friedrich proves stability of de Sitter space. For an appropriate value of the cosmological constant, Λ, the metric of de Sitter space is given by

$$g_{\mathrm{dS}} = -dt \otimes dt + \cosh^2(t)\bar{g}_{\mathbb{S}^3}$$

on $\mathbb{R} \times \mathbb{S}^3$, where $\bar{g}_{\mathbb{S}^3}$ is the standard metric on $\mathbb{S}^3$. The de Sitter space is a solution to Einstein's vacuum equations with a positive cosmological constant; i.e.,

$$G + \Lambda g = 0.$$

The result of Friedrich is peculiar to $3+1$-dimensions, but Michael Anderson later generalized it to $n+1$-dimensions, with n odd; cf. [1]. Friedrich also generalized [9] to include matter of Maxwell and Yang-Mills type; cf. [10]. All of these references yield stability of cosmological solutions with accelerated expansion. They thus belong to the class of results we wish to discuss here. It is of interest to note that the proofs given in [9, 10, 1] are based on conformal reformulations of the equations. The idea is to first rescale the background spacetime by a conformal factor, so that what corresponds to past and future infinity in the physical spacetime is at a finite distance away with respect to the rescaled metric. The second step is then to derive a suitable system of equations for the rescaled metric and conformal factor (in reality, the variables might be quite different). This step can be expected to be very difficult, and the only cases in which it is known to be possible is when the matter sources have suitable conformal invariance properties. However, when it is possible, the problem of global existence and stability becomes an issue of continuous dependence on initial data. Assuming the conformally rescaled equations admit a well posed initial value problem (with respect to an appropriately chosen gauge; i.e., an appropriate choice of how to break the diffeomorphism invariance), this is, however, immediate, so that the desired result follows. It is of interest to note that, even though [1] yields stability in the case of higher dimensions, it can be used to prove stability in $3+1$-dimensions for spacetimes with a special type of matter source; cf. [12]. The results mentioned above are appealing due the the geometric nature of the arguments involved. However, the methods used seem to suffer from a lack of robustness. This leads us to the different perspective developed in [17].

In [17], we considered the case of Einstein's equations coupled to a non-linear scalar field. The relevant stress-energy tensor in that case is

$$T = d\phi \otimes d\phi - \left[\frac{1}{2} g(\mathrm{grad}\phi, \mathrm{grad}\phi) + V(\phi)\right] g, \tag{6}$$

where ϕ is a scalar valued function on the manifold (the so-called *scalar field*) and V is a smooth function on $\mathbb{R}$ referred to as the *potential.* The relevant matter field equations are

$$\Box_g \phi - V' \circ \phi = 0, \tag{7}$$

where $\Box_g$ is the scalar wave operator associated with g (defined in the same way as the scalar Laplacian in the case of Riemannian geometry). Note that (6) is divergence free if (7) holds. In [17], it is assumed that $V(0) > 0$, $V'(0) = 0$ and $V''(0) > 0$; in other words, that 0 is a positive non-degenerate local minimum of the potential. Moreover, the scalar field is assumed to be small initially.

The motivation for studying non-linear scalar fields is partly due to their interest in physics. Once it became clear that the observational data indicate that the universe is expanding at an accelerated rate, it was natural to try to find matter models that induce accelerated expansion. One possibility is to include a positive cosmological constant. Another is to add matter of non-linear scalar field type. The types of potentials considered above specialize to the case of a positive cosmological constant when demanding that $\phi = 0$ (a case which can already be handled using conformal methods). However, they are more general, and only in the case of special relations between $V(0)$ and $V''(0)$ do the conformal methods seem to work; cf. [11].

Following the appearance of [17], there were several results obtained using similar methods; cf. [19] (treating the case of an exponential potential and generalizing the results of [12]), [23, 16] (in which an electromagnetic field was added), [21, 22] (in which the Euler-Einstein system was considered). However, the situation we focus on in what follows is the Einstein-Vlasov setting, discussed in [20].

6. The Einstein-Vlasov system

The physical situation matter of Vlasov type is supposed to represent is that of a gas. The fundamental assumption is that collisions are sufficiently rare that they can be negelected (including binary collisions would, e.g., lead to the Boltzmann equation, which we do not consider here). In the case of general relativity, each particle in the gas can thus be expected to behave as a freely falling test particle. A test particle with a non-zero rest mass (also referred to as a *massive* particle) can thus be expected to travel along a timelike geodesic. In the case of a zero rest mass (i.e., a *massless*) particle, the relevant curves are the null geodesics. On the other hand, the particles collectively generate a gravitational field which, in its turn, affects the geometry (and thereby the geodesics). In order to describe the gas, it is convenient to use a distribution function. The natural space on which this function is defined is the space of states of particles. Assuming all the particles to have rest mass 1, the space of states is given by the set of future directed unit timelike vectors. We shall denote this set by P, and we shall refer to it as the *mass shell*. The *distribution function*, say f, is then a function from P to the non-negative real numbers.

In order to couple Vlasov matter to Einstein's equations, it is necessary to explain how to construct a stress-energy tensor, given a distribution function. Moreover, it is necessary to formulate an evolution equation for the distribution function. The relevant stress-energy tensor is defined by

$$T|_{T_\xi M \times T_\xi M} = \int_{P_\xi} f(p) p^\flat \otimes p^\flat \mu_{P_\xi}(p). \tag{8}$$

In this equation, ξ is a spacetime point (i.e., an element of the spacetime manifold M); P_ξ is the mass shell above ξ (i.e., the elements of P based at ξ); p is an element of P_ξ; $p^\flat$ is the one-form metrically associated with p (i.e., $p^\flat(X) = g(p, X)$ for $X \in T_\xi M$); and μ_{P_ξ} is a volume form defined on P_ξ in the following way: the metric g induces a Lorentz metric g_ξ on $T_\xi M$, the Lorentz metric g_ξ induces a Riemannian metric on P_ξ, and this Riemannian metric induces a volume form on P_ξ (which we denote by μ_{P_ξ}). It is important to note that it is necessary to impose fall-off conditions on the distribution function in order for (8) to make sense. Often the requirement of compact support in the momentum directions is imposed, but we here prefer to demand that the distribution function belong to Sobolev spaces with appropriate weights in the momentum directions.

Turning to the equation for the distribution function, it is given by

$$\mathcal{L}f = 0, \tag{9}$$

and it is referred to as the *Vlasov equation*. Here $\mathcal{L}$ is a vector field on P defined as follows. Given an element of P, say v, there is a unique geodesic γ such that $\dot{\gamma}(0) = v$. Moreover, $\dot{\gamma}(s)$ is a curve in P, and its tangent vector at 0 (considered as a curve in P) is $\mathcal{L}_v$, the vector field $\mathcal{L}$ at the point v. Note that the Vlasov equation is equivalent to the requirement that $f(\dot{\gamma})$ be constant for each geodesic γ with initial values on P. Moreover, this requirement corresponds to the assumption that collisions can be neglected, so that the particles travel along timelike geodesics. It is of interest to note that (9) implies that the stress energy tensor defined by (8) is divergence free (regardless of whether Einstein's equations are satisfied or not).

Summing up the above discussion, the *Einstein-Vlasov* system with a positive cosmological constant is given by the equations

$$\begin{aligned} G + \Lambda g =& T, \\ \mathcal{L}f =& 0, \end{aligned}$$

where T is defined by (8). It is also possible to couple this system to a non-linear scalar field, but we shall focus on the above equations in what follows. There are results corresponding to Theorems 4.1 and 4.2 in this setting. We shall not write them down in detail, but it is of some interest to clarify what the initial data are.

Initial data for the Einstein-Vlasov system. For the geometry, the relevant initial data are the induced metric and second fundamental form, just as before. Since the Vlasov equation is a first order equation, we only need one initial datum for the distribution function. In order to explain how it is related to the spacetime picture, let us assume that we have a solution (M, g, f) and a spacelike hypersurface Σ in (M, g). Then there is a diffeomorphism from P_Σ (the mass shell above Σ) to $T\Sigma$ obtained by projecting orthogonally to the normal of Σ. Let us denote it by proj_Σ. The initial datum for the distribution function is given by $\bar{f} = f \circ \mathrm{proj}_\Sigma^{-1}$, and it is defined on $T\Sigma$. In the case of the Einstein-Vlasov system, the relevant initial data are $(\Sigma, \bar{g}, \bar{k}, \bar{f})$, where Σ is an n-dimensional manifold, $\bar{g}$ is a Riemannian metric on Σ, $\bar{k}$ is a symmetric covariant 2-tensor field on Σ and $\bar{f}$ is a smooth, non-negative function on $T\Sigma$. Moreover, these data should satisfy the constraint equations (4) and (5) (where the matter quantities should be expressed in terms of $\bar{g}$ and $\bar{f}$, something which can be done; cf. [20, (7.20) and (7.21), p. 92]). In order to phrase a stability result, we also need a notion of distance between initial data sets.

Distance between initial data sets. Let us assume Σ to be a closed manifold. Then we can use ordinary Sobolev norms on manifolds to measure the distance between two metrics and between two symmetric covariant 2-tensor fields. Since the tangent space of Σ is non-compact, we do, however, need a different norm to measure the difference between initial data for the distribution function. We shall use

$$\|\bar{f}\|_{H^l_{\mathrm{Vl},\mu}} = \left(\sum_{i=1}^{j}\sum_{|\alpha|+|\beta|\leq l}\int_{\bar{\mathsf{x}}_i(U_i)\times\mathbb{R}^n}\langle\bar{\varrho}\rangle^{2\mu+2|\beta|}\bar{\chi}_i(\bar{\xi})(\partial_{\bar{\xi}}^{\alpha}\partial_{\bar{\varrho}}^{\beta}\bar{\mathsf{f}}_{\bar{\mathsf{x}}_i})^2(\bar{\xi},\bar{\varrho})d\bar{\xi}d\bar{\varrho}\right)^{1/2}. \tag{10}$$

In this expression, $(U_i, \bar{\mathsf{x}}_i)$, $i = 1, \dots, j$, is a covering of Σ by coordinate neighbourhoods, and $\{\bar{\chi}_i\}$ is a partition of unity subordinate to the covering $\{U_i\}$. The expression $\bar{\mathsf{f}}_{\bar{\mathsf{x}}_i}$ is the distribution function expressed with respect to the local coordinates on $T\Sigma$ induced by $(U_i, \bar{\mathsf{x}}_i)$; in particular, it is a function on $\bar{\mathsf{x}}_i(U_i) \times \mathbb{R}^n$, where the $\mathbb{R}^n$-factor corresponds to the tangential directions. Finally, we use the notation

$$\langle\bar{\varrho}\rangle = (1 + |\bar{\varrho}|^2)^{1/2}.$$

Considering the norm (10), there are two contributions to the power of the weight $\langle\bar{\varrho}\rangle$; 2μ and $2|\beta|$. The reason for including μ is that it yields an overall decay (assuming it to be positive). In fact, for $\mu > n/2+1$, the relevant matter quantities are well defined, assuming the right hand side of (10) to be bounded (for a high enough l). The reason for including $2|\beta|$ is that it ensures that the notion of smallness obtained using (10) is geometrically meaningful; the exact value of the right hand side of (10) depends on the coordinates and the partition of unity, but different choices lead to equivalent norms, assuming we include $2|\beta|$ in the power of the weight. We shall refer to the space of functions $\bar{f}$ such that (10) is bounded for all l by $\bar{\mathfrak{D}}^{\infty}_{\mu}(T\Sigma)$ (this space can also be defined in case Σ is not compact; we then only require the integrals appearing in the definition of the norm to be bounded on compact subsets of Σ). The reader interested in a more detailed discussion of norms such as (10) is referred to [20]. In this reference, there is also a description of the relevant function spaces for the corresponding distribution functions on the maximal globally hyperbolic development associated with the initial data. The final ingredient we need before phrasing a stability result is a description of the relevant background solutions. We turn to this topic next.

7. Background solutions

Let us begin by describing the class of solutions to Einstein's equations which is currently preferred by physicists when modelling the universe. The geometry is taken to be spatially homogeneous and isotropic, as well as spatially flat. In other words, the relevant metrics take the form

$$g_{\mathrm{model}} = -dt \otimes dt + a^2(t)\bar{g}_{\mathrm{E}}$$

on $I \times \mathbb{R}^3$ (or $I \times \mathbb{T}^3$), where I is an open interval, $\bar{g}_{\mathrm{E}}$ is the standard Euclidean metric on $\mathbb{R}^3$, and a is a positive smooth function on I. Concerning the matter sources, they are usually taken to be a combination of so-called *perfect fluids*. In the case of a perfect fluid (and the above type of symmetry conditions), the stress energy tensor is of the form

$$T = (\rho + p)dt \otimes dt + pg_{\text{model}}.$$

Here the functions ρ and p are referred to as the *energy density* and the *pressure* respectively. In order to obtain evolution equations for p and ρ, it is common to introduce an *equation of state*, giving p in terms of ρ. The condition that T be divergence free then yields an evolution equation for ρ. Two equations of state that are often used by physicists are *dust* (in which case $p = 0$) and *radiation* (in which case $p = \rho/3$). In fact, the early universe is expected to have been radiation dominated, and at late times, the matter is expected to behave as dust. Physicists often study one of these situations at at time, and then they include only dust or only radiation. However, it is possible to include both at the same time, and we shall take the matter content of the standard model to consist of a radiation fluid and dust. The corresponding stress energy tensors are required to be divergence free individually, and this yields evolution equations for the corresponding energy densities. Finally, a mechanism is required in order to produce the observed accelerated expansion. One possibility is to include a non-linear scalar field, but we shall here simply add a positive cosmological constant Λ to the above description. The relevant equations are then

$$\begin{aligned} G + \Lambda g_{\text{model}} =& T_{\text{rad}} + T_{\text{dust}}, \\ T_{\text{rad}} =& (\rho_{\text{rad}} + p_{\text{rad}})dt \otimes dt + p_{\text{rad}} g_{\text{model}}, \\ T_{\text{dust}} =& \rho_{\text{dust}} dt \otimes dt, \\ \dot{\rho}_{\text{rad}} =& -4\frac{\dot{a}}{a}\rho_{\text{rad}}, \\ \dot{\rho}_{\text{dust}} =& -3\frac{\dot{a}}{a}\rho_{\text{dust}}, \end{aligned}$$

where $p_{\text{rad}} = \rho_{\text{rad}}/3$ and G is the Einstein tensor of g_{model}. It should be pointed out that solutions of the above type are only relevant models after decoupling (i.e., the time at which matter and radiation decoupled). In particular, inflationary phases etc. are not included. The above matter models are not of Vlasov type. However, it turns out to be possible to approximate the above solutions arbitrarily well with solutions to the Einstein-Vlasov system with a positive cosmological constant and the above type of symmetry; cf. [20, Chapter 28]. Moreover, Vlasov matter is such that it naturally behaves as a radiation fluid close to the singularity and as dust in the expanding direction. In other words, it is not necessary to put in a dust and a radiation fluid by hand; Vlasov matter is such that this emerges naturally. Finally, Vlasov matter is conceptually natural in the later part of the evolution of the universe. As a consequnce, we shall prefer it here.

Spatial homogeneity. It is of interest to put the above example into a slightly bigger context, namely that of spatially homogeneous solutions. In [24], Robert Wald presented general ideas for how to analyze the future asymptotics of spatially homogeneous solutions to Einstein's equations with a positive cosmological constant (assuming the matter sources satisfy certain energy conditions). He did not address the issue of future global existence; this was taken for granted. However, he did obtain quite general results. The most fundamental ingredient of the argument is the Hamiltonian constraint (4). This equation can be written

$$(\mathrm{tr}_{\bar{g}}\bar{k})^2 = -\frac{3}{2}\bar{S} + \frac{3}{2}\bar{\sigma}_{ij}\bar{\sigma}^{ij} + 3\rho + 3\Lambda, \tag{11}$$

where $\bar{\sigma}_{ij}$ are the components of the trace free part of the second fundamental form. Assuming the matter to satisfy the *dominant energy condition* (i.e., the requirement that $T(u,v) \geq 0$ for future directed timelike vectors), the energy density ρ is non-negative. Considering (11), it is thus clear that the only term on the right hand side which might be negative is the first one. However, the sign of the scalar curvature of the metric induced on the hypersurfaces of spatial homogeneity is intimately connected with the symmetry type. Before describing this connection in detail, let us give a formal definition of a spatially homogeneous spacetime: it is the maximal globally hyperbolic development of homogeneous initial data (we assume the relevant matter model to be such that the initial value problem is well posed). Initial data, given by a manifold Σ, a metric $\bar{g}$, a symmetric covariant 2-tensor field, as well as matter fields, are said to be homogeneous if there is a smooth transitive Lie group action on Σ which leaves the initial data invariant. In the 3-dimensional case, there are two possibilities. Focusing on the simply connected setting for simplicity, Σ is either a Lie group or $\mathbb{S}^2 \times \mathbb{R}$. The latter case is referred to as Kantowski-Sachs in the physics community, and we shall ignore it in what follows, since the corresponding metrics have positive scalar curvature; cf. (11). That is not to say that it is not possible to obtain results in the Kantowski-Sachs setting, but rather that the statements of the corresponding results would be more involved. Turning to the Lie group setting, SU(2) constitutes a particular case; it is the only simply connected 3-dimensional Lie group which admits a left invariant metric with positive scalar curvature. Again, there are results in the SU(2) setting, but the statements of the results are more involved. Ignoring Kantowski-Sachs and SU(2) for the moment, the remaining symmetry types are such that the corresponding invariant metrics have non-positive scalar curvature. Returning to (11), we conclude that the right hand side has a positive lower bound. On the other hand, since the left hand side is zero when the volume is at a local maximum (or minimum), this indicates that there is no local maximum or minimum. Naively, one would then expect that there is a big bang in one time direction and infinite expansion in the other. In fact, the corresponding solutions all have an expanding direction (which we shall refer to as the future). Moreover, it is possible to say something concerning the future asymptotics: the solution isotropizes and the matter content becomes irrelevant.

Spatially homogeneous solutions to the Einstein-Vlasov system. As mentioned above, the results in [24] were based on the assumption that the solution exists globally to the future. When studying a particular case, it thus has to be verified that this holds. In the case of the Einstein-Vlasov equations with a positive cosmological constant, this was done in [14] (the result was later extended to the case of non-compact support in the momentum directions in [20]). Moreover, asymoptotic information concerning the solution was obtained. In what follows, we wish to describe a stability result for these solutions.

8. Stability in the Einstein-Vlasov setting

Before stating the main stability result, let us define the relevant background initial data; the definition below is a specialization of [20, Definition 7.21, p. 107] to the case of the Einstein-Vlasov system with a positive cosmological constant.

Definition 8.1. Let G be a 3–dimensional Lie group and $5/2 < \mu \in \mathbb{R}$. Let $\bar{g}$ and $\bar{k}$ be a left invariant Riemannian metric and a left invariant symmetric covariant 2–tensor field on G respectively. Furthermore, let $\bar{f} \in \bar{\mathfrak{D}}^{\infty}_{\mu}(TG)$ be left invariant; in other words, if $h \in G$, then $\bar{f} \circ L_{h*} = \bar{f}$. Then $(G, \bar{g}, \bar{k}, \bar{f})$ are referred to as *Bianchi initial data* for the Einstein–Vlasov system with a positive cosmological constant, assuming they constitute initial data in the ordinary sense.

As discussed in the previous section, the corresponding solutions have an expanding direction (if the universal covering group of the Lie group is not isomorphic to SU(2)), and it is of interest to prove global non-linear stability in that direction. It is also important to keep in mind that by letting $G = \mathbb{R}^3$ (or $G = \mathbb{T}^3$); taking $\bar{g}$ and $\bar{k}$ to be suitable multiples of the standard Euclidean metric; and by making an appropriate choice of $\bar{f}$, one obtains initial data corresponding to a solution which is consistent with observations. Future stability of solutions consistent with observations is thus a corollary of the result below. The following theorem is a specialization of [20, Theorem 7.22, p. 108] to the case of the Einstein-Vlasov system with a positive cosmological constant.

Theorem 8.2. *Let $5/2 < \mu \in \mathbb{R}$ and $(G, \bar{g}_{\rm bg}, \bar{k}_{\rm bg}, \bar{f}_{\rm bg})$ be Bianchi initial data for the Einstein–Vlasov system with a positive cosmological constant, where*

- *the universal covering group of G is not isomorphic to* SU(2)*,*
- $\mathrm{tr}\bar{k}_{\rm bg} = \bar{g}^{ij}_{\rm bg}\bar{k}_{{\rm bg},ij} > 0$*.*

Assume that there is a cocompact subgroup Γ of the isometry group of the initial data. Let Σ be the compact quotient. Then the initial data induce initial data on Σ which, by abuse of notation, will be denoted by the same symbols. Make a choice of Sobolev norms $\|\cdot\|_{H^l}$ on tensor fields on Σ and a choice of norms $\|\cdot\|_{H^l_{{\rm Vl},\mu}}$. Then there is an $\epsilon > 0$ such that if $(\Sigma, \bar{g}, \bar{k}, \bar{f})$ are initial data for the Einstein–Vlasov system with a positive cosmological constant with the property that

$$\|\bar{g} - \bar{g}_{\rm bg}\|_{H^5} + \|\bar{k} - \bar{k}_{\rm bg}\|_{H^4} + \|\bar{f} - \bar{f}_{\rm bg}\|_{H^4_{{\rm Vl},\mu}} \leq \epsilon,$$

then the maximal globally hyperbolic development of $(\Sigma, \bar{g}, \bar{k}, \bar{f})$ is future causally geodesically complete.

It is perhaps worth commenting on the requirement that there be a cocompact subgroup of the isometry group of the initial data. We expect this requirement to be unnecessary (though we have not proven this statment). However, it would then be necessary to introduce a more complicated notion of distance between initial data sets in the formulation of stability. The reason for focusing on future causal geodesic completeness in the conclusions is the physical interpretation that freely falling test particles (light) follow timelike (null) geodesics. Future causal geodesic completeness thus implies that freely falling test particles do not exit the spacetime in finite proper time to the future. In this geometric sense, the solution is thus future global. It is of course also of interest to write down estimates characterizing the asymptotic behaviour. This has been done in [20, Theorem 7.16, p. 104–106]; cf. [20, Theorem 7.22, p. 108]. We shall not repeat the technical details here.

In the presence of a positive cosmological constant, solutions are expected to homogenize and isotropize at late times. In fact, they are expected to appear de Sitter like, and this rough expectation goes under the name of the *cosmic no-hair conjecture*. A more precise formulation of this expectation is given in [2, Definition 8, p. 7]. We shall not write down the formal definition here, as it requires a somewhat technical discussion of the causal structure of solutions (the main point is to focus on the parts of the spacetime that can actually be seen by observers). However, the solutions that arise as a result of Theorem 8.2 become de Sitter like asymptotically to the future, in the sense of [2, Definition 8, p. 7].

Even though we have excluded Lie groups whose universal covers are isomorphic to SU(2), there are results in that setting. However, it is then necessary to impose additional conditions. An example of a result which holds when perturbing isotropic solutions is given in [20, Theorem 7.28, p. 109].

The $\mathbb{T}^3$-Gowdy symmetric setting. Beyond the above stability results concerning spatially homogeneous solutions, there are results in the $\mathbb{T}^3$-Gowdy symmetric setting. The main assumption that characterizes this symmetry class is the requirement that the initial data be invariant under a 2-torus action. In practice, the effective number of spacetime dimensions is thus 2. On the other hand, the symmetry class admits both inhomogeneities and anisotropies. Nevertheless, it turns out that solutions to the Einstein-Vlasov system in the $\mathbb{T}^3$-Gowdy symmetric setting homogeneize and isotropize. In fact, they are future asymptotically de Sitter like. Moreover, perturbing the initial data corresponding to a $\mathbb{T}^3$-Gowdy symmetric solution in the class of all solutions yields maximal globally hyperbolic developments with the same properties. The reader interested in a more detailed description is referred to [2].

9. On the topology of the universe

In Section 7, we described the solutions that physicists normally use to model the universe. Note that the justification for using them is based not only on observations, but also on the philosophical idea that all observers should see something which is roughly similar (an assumption which cannot be tested). In practice, the assumption that leads to the standard models is that every observer sees exactly the same spatially homogeneous and isotropic solution. Clearly, this is asking too much, since what we see is not exactly spatially homogeneous and isotropic. An assumption which would be slightly more reasonable would be to fix a standard model and to say that every observer should see something which is very close to that standard model. It is of interest to ask what limitations on the topology such an assumption imposes; note that the standard perspective, which implies a locally homogeneous and isotropic spatial geometry, is only consistent with a topology which is the 3-sphere, hyperbolic space or Euclidean space, or a quotient thereof. However, using methods similar to ones on which the future global non-linear stability result is based, it turns out to be possible to prove that, given

- a closed 3-manifold, say Σ,
- a standard solution (with flat spatial geometry and $\mathbb{R}^3$ spatial topology),
- a time t_0 in the existence interval of the standard solution (note that the matter models discussed here are only valid after decoupling, and we shall think of t_0 as representing decoupling),
- a choice of norm (say C^k-norm) and $\epsilon > 0$,

there is a solution to the Einstein-Vlasov system with a positive cosmological constant, such that

- it is the maximal globally hyperbolic development of initial data,
- it is future causally geodesically complete,
- it has spatial topology Σ (globally hyperbolic Lorentz manifolds have topology $\mathbb{R} \times \Sigma$, where Σ is a Cauchy hypersurface; Σ is referred to as the *spatial topology*),
- every observer considers the solution to be at distance ϵ away from the chosen standard solution to the future of t_0 and with respect to the chosen C^k-norm,
- the solution is stable with all these properties (in other words, if we perturb the corresponding initial data, we obtain a maximal globally hyperbolic development with the same properties).

Under the given assumptions, it is thus not possible to draw any conclusions concerning the topology of the universe.

The statement is still somewhat imprecise; it is not so clear how to measure the distance (as percieved by an observer) between the solution and the background solution. This is a somewhat technical issue, and we refer the interested reader to [20, Section 7.9] for a discussion.

The above description is somewhat brief, and we refer the reader interested in more details to [20, Section 7.9] for a mathematical statement of the result, and to [20, Chapter 34] for a proof.

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