Instability of spatially homogeneous solutions in the class of $\mathbb{T}^2$-symmetric solutions to Einstein's vacuum equations

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Abstract

In the subject of cosmology, spatially homogeneous solutions are often used to model the universe. It is therefore of interest to ask what happens when perturbing into the spatially inhomogeneous regime. To this end, we, in the present paper, study the future asymptotics of solutions to Einstein's vacuum equations in the case of $\mathbb{T}^2$-symmetry. It turns out that in this setting, whether the solution is spatially homogeneous or not can be characterized in terms of the asymptotics of one variable appearing in the equations; there is a monotonic function such that if its limit is finite, then the solution is spatially homogeneous and if the limit is infinite, then the solution is spatially inhomogeneous. In particular, regardless of how small the initial perturbation away from spatial homogeneity is, the resulting asymptotics are very different. Using spatially homogeneous solutions as models is therefore, in this class, hard to justify.

1 Introduction

Let us proceed directly to a formulation of the equations we are interested in and then describe the results; the reader interested in a motivation of the choice of equations is referred to Subsection 1.1.

The $\mathbb{T}^2$-symmetric spacetimes. There are various geometric ways of imposing $\mathbb{T}^2$-symmetry; cf., e.g., [7]. However, we shall here simply take the metric form and the equations as given in [7] as a starting point. Setting the constant λ appearing in [7, (2.3), p. 120] to 1 (which can be done without loss of generality), the metric can be written

$$g = e^{2(\nu-U)}(-\alpha dt^2+d\theta^2)+e^{2U}[dx+Qdy+(G+QH)d\theta+(M+QN)dt]^2+e^{-2U}t^2(dy+Hd\theta+Ndt)^2;$$

cf. [7, (2.3), p. 120] (in which we have renamed some of the variables; A to Q etc.). We shall here prefer to use P and λ instead of U and ν, the relation between the different variables being $e^{2U} = te^P$ and $\alpha e^{2(\nu-U)} = t^{-1/2}e^{\lambda/2}$. The metric then takes the form

$$g = t^{-1/2}e^{\lambda/2}(-dt^2+\alpha^{-1}d\theta^2)+te^P[dx+Qdy+(G+QH)d\theta+(M+QN)dt]^2+te^{-P}(dy+Hd\theta+Ndt)^2 \quad (1)$$

on $I \times \mathbb{T}^3$. Here I is an open interval contained in $(0,\infty)$, $t \in I$ and $(\theta, x, y) \in \mathbb{T}^3$. Moreover, the functions appearing only depend on t and θ. In particular, there is a $\mathbb{T}^2$-group of isometries obtained by translation in x and y. The time coordinate t is referred to as an *areal coordinate*, since the area of the orbit obtained by applying the $\mathbb{T}^2$-action to (t,θ,x,y) is proportional to t. There are several subclasses of this class of metrics. To begin with, the set of $\mathbb{T}^3$-*Gowdy symmetric* metrics are obtained by setting $H = G = M = N = 0$ and $\alpha = 1$ (in fact, M and N can be set to zero in general). Moreover, the set of *polarised* metrics are obtained by setting $Q = 0$ (this corresponds to ∂_x and ∂_y being orthogonal). Note that the future asymptotics of the $\mathbb{T}^3$-Gowdy vacuum solutions have already been analysed in some detail in [24]. In the present paper, we shall therefore focus on the general $\mathbb{T}^2$-symmetric case.

Equations. In vacuum, Einstein's equations take the form

$$
\begin{aligned}
P_{tt}+\frac{1}{t}P_t-\alpha P_{\theta\theta} &= \frac{\alpha_\theta}{2}P_\theta+\frac{\alpha_t}{2\alpha}P_t+e^{2P}(Q_t^2-\alpha Q_\theta^2)-\frac{e^{P+\lambda/2}K^2}{2t^{7/2}}, && (2)\\
Q_{tt}+\frac{1}{t}Q_t-\alpha Q_{\theta\theta} &= \frac{\alpha_\theta Q_\theta}{2}+\frac{\alpha_t Q_t}{2\alpha}-2\left(Q_tP_t-\alpha Q_\theta P_\theta\right), && (3)\\
\frac{\alpha_t}{\alpha} &= -\frac{e^{P+\lambda/2}K^2}{t^{5/2}}, && (4)\\
\lambda_t &= t[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)]-\frac{e^{P+\lambda/2}K^2}{t^{5/2}}, && (5)\\
\lambda_\theta &= 2t(P_tP_\theta+e^{2P}Q_tQ_\theta), && (6)\\
\partial_t\left[t\alpha^{-1/2}\left(\lambda_t-2\frac{\alpha_t}{\alpha}-\frac{3}{t}\right)\right] &= \partial_\theta(t\alpha^{1/2}\lambda_\theta)-t\alpha^{-1/2}[P_t^2+e^{2P}Q_t^2-\alpha(P_\theta^2+e^{2P}Q_\theta^2)] && (7)\\
&\quad -2\alpha^{-1/2}\frac{e^{P+\lambda/2}K^2}{t^{5/2}}+\alpha^{-1/2}\lambda_t,
\end{aligned}
$$

where K is a constant (and $K=0$ corresponds to $\mathbb{T}^3$–Gowdy); cf. [7, (2.4) and (2.5), p. 120]. In addition

$$
\begin{aligned}
N_\theta &= H_t-\frac{\alpha^{-1/2}e^{P+\lambda/2}K}{t^{5/2}}, && (8)\\
M_\theta &= G_t+Q(H_t-N_\theta); && (9)
\end{aligned}
$$

cf. [7, (2.6), p. 120]. In (8) and (9), M and N can be considered to be given, and H and G are obtained by integrating the equations.

Considering (2)–(7), it can be verified that (7) is a consequence of (2)–(6). We shall therefore consider (2)–(6) to be the fundamental equations. It is also of interest to note that if we let h be the left hand side of (6) minus the right hand side, and if we assume (2)–(5) to hold, then

$$\partial_t h=\frac{1}{2}\frac{\alpha_t}{\alpha}h.$$

In other words, if we solve (2)–(5) with initial data satisfying (6), then (2)–(6) are satisfied on the maximal interval of existence. In this sense, (6) can be thought of as a constraint. Due to the results of [7], it is known that the existence interval of solutions to (2)–(6) are of the form (t_0,∞), where $t_0\geq 0$.

Pseudo-homogeneity. Considering the equations (2)–(6), it is natural to slightly generalise the notion of spatial homogeneity. In fact, we say that a solution to (2)–(6) is *pseudo-homogeneous* if P, Q and λ are independent of θ; note that α only enters the equations for P, Q and λ in the combination α_t/α (assuming P_θ and Q_θ to be zero), and that α_t/α can be expressed solely in terms of P, λ, K and t. In addition, multiplying α in a pseudo-homogeneous solution by a positive function depending only on θ yields a new pseudo-homogeneous solution. In fact, every pseudo-homogeneous solution can be obtained from a spatially homogeneous solution by performing such an operation; given a pseudo-homogeneous solution, α can be written $\alpha(t,\theta)=f_1(\theta)f_2(t)$ for two positive functions $f_1:\mathbb{S}^1\to\mathbb{R}_+$ and $f_2:(t_0,\infty)\to\mathbb{R}_+$, where $\mathbb{R}_+=(0,\infty)$.

Results. The main result of the paper is a characterisation of the pseudo-homogeneous solutions with $K\neq 0$ in terms of the asymptotics. One particular consequence of the characterisation is that the pseudo-homogeneous solutions are unstable in the class of all solutions. However, before giving a precise formulation of the result, let us put it into context by first describing some subclasses of solutions. To begin with, let us focus on the polarised setting, and let us start with the easiest possible subcase of the above equations: the polarised, spatially homogeneous $\mathbb{T}^3$-Gowdy symmetric setting; i.e., $\alpha=1$, $Q=0$, $K=0$, $G=H=M=N=0$ and vanishing spatial variation. The relevant equation for P is then $\partial_t(tP_t)=0$, with the solutions $P=a\ln t+b$, where $a, b \in \mathbb{R}$. Adding spatial variation, the relevant equation for P is

$$P_{tt} + \frac{1}{t} P_t - P_{\theta\theta} = 0. \tag{10}$$

A solution to this equation can be written

$$P(t,\theta) = a \ln t + b + t^{-1/2}\nu(t,\theta) + \psi(t,\theta), \tag{11}$$

where $a, b \in \mathbb{R}$, ν is a solution to the flat space wave equation with zero average, i.e.,

$$\nu_{tt} - \nu_{\theta\theta} = 0, \quad \int_{\mathbb{S}^1} \nu(\cdot,\theta) d\theta = 0,$$

and ψ and all its derivatives are $O(t^{-3/2})$. Moreover, ψ has zero average. It is also of interest to note that, given a, b and ν satisfying the above conditions, there is a unique solution to (10) with asymptotics of the form (11). In other words, a, b, and ν can be thought of as 'data at future infinity'. The interested reader is referred to [18, 22] for a justification of these statements. In short, the leading order behaviour in the solution (the spatially homogeneous behaviour) is the same, which is not so surprising. However, as we shall explain below, the seemingly minor difference in P causes a significant difference in λ. Turning to the polarised $\mathbb{T}^2$-solutions, the relevant equations are given by (2) and (4)–(6) with Q set to zero. Given a pseudo-homogeneous solution to these equations, there are c_P, $c_\lambda \in \mathbb{R}$, $\alpha_\infty \in C^\infty(\mathbb{S}^1, \mathbb{R}_+)$ and $r_\infty \in (-3, 1)$ such that

$$\lim_{t\to\infty} \alpha(t,\theta) = \alpha_\infty(\theta), \quad \lim_{t\to\infty} |P(t) - r_\infty \ln t - c_P| = 0, \quad \lim_{t\to\infty} |\lambda(t) - r_\infty^2 \ln t - c_\lambda| = 0;$$

cf. Lemma 11. Moreover, this asymptotic information uniquely determines a corresponding polarised pseudo-homogeneous solution; cf. Proposition 14. It is of interest to know what happens when one perturbs the initial data corresponding to a pseudo-homogeneous solution. Unfortunately, we have been unable to derive complete asymptotics, but we have obtained the following partial results.

Proposition 1. *Let $(P_{\rm bg}, \alpha_{\rm bg}, \lambda_{\rm bg})$ be a pseudo-homogeneous polarised solution to (2) and (4)–(6) with $K \neq 0$. Assuming that the relevant existence interval is (t_0, ∞) for some $t_0 \geq 0$, fix $t_a \in (t_0, \infty)$. Let $\mathcal{C}$ be the set consisting of non-pseudo-homogeneous polarised solutions (P, α, λ) to (2) and (4)–(6) such that*

- *(P, α, λ) has the same $K \neq 0$ as $(P_{\rm bg}, \alpha_{\rm bg}, \lambda_{\rm bg})$,*
- *t_a belongs the existence interval of (P, α, λ).*

Then there is an $\epsilon > 0$ such that if $(P, \alpha, \lambda) \in \mathcal{C}$ satisfies

$$\|(P - P_{\rm bg})(t_a,\cdot)\|_{C^1} + \|\partial_t(P - P_{\rm bg})(t_a,\cdot)\|_{C^0} + \|(\alpha - \alpha_{\rm bg})(t_a,\cdot)\|_{C^1} + \|(\lambda - \lambda_{\rm bg})(t_a,\cdot)\|_{C^0} \leq \epsilon,$$

then there is a time sequence $t_k \to \infty$, $k = 1, 2, \ldots$, such that

$$\lim_{t\to\infty} \|\alpha(t,\cdot)\|_{C^0} = 0, \tag{12}$$

$$\lim_{t\to\infty} \left\| \frac{P(t,\cdot)}{\ln t} + 1 \right\|_{C^0} = 0, \tag{13}$$

$$\lim_{k\to\infty} \left\| \frac{\lambda(t_k,\cdot)}{\ln t_k} - 5 \right\|_{C^0} = 0. \tag{14}$$

Remark 2. In related work, Philippe G. LeFloch and Jacques Smulevici have found more detailed asymptotics for solutions in the polarised setting, assuming one starts close enough to the asymptotic regime [19].

The proof of the proposition is to be found at the end of Section 13.

In other words, regardless of how close the initial data are to those of the background, the asymptotic behaviour is quite different (assuming that the initial data are not those of a pseudo-homogeneous solution). It is also of interest to note that if the limit of $P/\ln t$ is r_∞ for a pseudo-homogeneous solution, then the limit of $\lambda/\ln t$ is r_∞^2. Considering (13) and (14), it is clear that the perturbed solutions exhibit quite different behaviour. However, the difference which is, in our opinion, the most important, is that α converges to zero uniformly. It would be desirable to prove that this is a general feature of (not necessarily polarised) solutions to (2)–(6). Unfortunately, we have not been able to obtain such a result. However, we have been able to prove that $\langle\alpha^{-1/2}\rangle \to \infty$ as $t \to \infty$, assuming that the solution is non-pseudo-homogeneous; cf. Remark 4 below for an explanation of the notation. In fact, the main result of the paper is the following.

Theorem 3. *Consider a solution to (2)–(6) on (t_0, ∞) with $K \neq 0$. If $\langle\alpha^{-1/2}\rangle$ is bounded, the solution is pseudo-homogeneous.*

Remark 4. In the statement of the theorem, we use the notation

$$\langle f \rangle = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(\theta) d\theta \tag{15}$$

for functions on $\mathbb{S}^1$. If f is a function on $(t_0, \infty) \times \mathbb{S}^1$, we use the same notation, and consider the result to be a function on (t_0, ∞).

Remark 5. Since $\langle\alpha^{-1/2}(t,\cdot)\rangle$ increases with t, we conclude (in the non-pseudo-homogeneous case) that this quantity converges to ∞ as $t \to \infty$.

Remark 6. This result clearly separates the pseudo-homogeneous solutions from the remaining solutions (in the $K \neq 0$ case). It also separates the $\mathbb{T}^3$-Gowdy symmetric solutions from the general $\mathbb{T}^2$-symmetric solutions.

Remark 7. In the case of a positive cosmological constant, it is very easy to prove that α converges to zero (with an optimal rate); cf. [6].

The proof is to be found at the end of Subsection 12.3.

It is of interest to summarise the conclusions concerning stability that have been obtained in the $\mathbb{T}^3$-Gowdy and $\mathbb{T}^2$-symmetric settings. In the spatially homogeneous $\mathbb{T}^3$-Gowdy symmetric setting, $|P|$ and $|\lambda|$ cannot grow faster than logarithmically, and Q remains bounded. Considering the metric (1), the various metric components thus decay/grow polynomially, and there is no particular component which is preferred in that the expansion/contraction is significantly larger than that of any other (cf. the case of the Kasner metrics). Turning on the slightest bit of spatial variation, however, λ tends to infinity linearly, whereas $|P|$ cannot grow faster than logarithmically. In the inhomogeneous setting, Q can tend to infinity, but not faster than polynomially. Considering (1), it is thus clear that all the expansion is in the θ direction, i.e., the direction in which there is spatial variation (note that $G = H = M = N = 0$ in the $\mathbb{T}^3$-Gowdy symmetric setting). In particular, the spatially homogeneous solutions are an unstable subset of the general solutions, and they can be characterised by the asymptotic behaviour of λ. The reader interested in a justification of these statements is referred to [23, 24]. Considering the general $\mathbb{T}^2$-symmetric setting ($K \neq 0$), the pseudo-homogeneous solutions exhibit a behaviour which is similar to that of spatially homogeneous $\mathbb{T}^3$-Gowdy symmetric solutions. However, there are some limitations on the values of the limit of $P/\ln t$, say r_∞, that are allowed. Changing K from zero to small (but non-zero) could thus, depending on the solution, lead to a significant difference in the asymptotic value of $P/\ln t$. Finally, it is clear that the non-pseudo-homogeneous solutions with $K \neq 0$ exhibit conceptually different asymptotic behaviour from any of the classes of solutions discussed previously. In short, the above, somewhat incomplete, discussion indicates that these classes of solutions are characterised by instabilities.

Comparison with numerics. It is of interest to compare the above results with recent numerical studies of the expanding direction of $\mathbb{T}^2$-symmetric spacetimes; cf. [9]. In the polarised setting, there is a conserved quantity

$$A = \int_{\mathbb{S}^1} t\alpha^{-1/2}\left(P_t + \frac{1}{t}\right) d\theta.$$

Since $\alpha^{-1/2}$ tends to infinity on average, this equality would seem to suggest that $tP_t + 1$ is small on average. As a consequence, it would seem natural to expect P to roughly equal $-\ln t$; this is exactly what is observed in the numerical simulations; cf. also Proposition 1. Turning to (4), it would then seem natural to expect λ to be roughly $5 \ln t$. The reason for this is that if λ were significantly smaller, then $\langle\alpha^{-1/2}\rangle$ could not tend to infinity, and if it were significantly larger, α would tend to zero too quickly (there is of course also the possibility that $\lambda/\ln t$ might oscillate). Again, this seems to be consistent with numerics, as well as Proposition 1. On the other hand, combining all of this information with (5), it is clear that there are some subtleties; either tP_t^2 has to be larger than $1/t$ on average, or $t\alpha P_\theta^2$ has to be larger than tP_t^2 on average. This would seem to suggest that there are oscillations present that separate the averages of the derivatives (possibly weighted with powers of α) from the L^2-norms, just as in the $\mathbb{T}^3$-Gowdy setting.

1.1 Motivation

The study of the class of $\mathbb{T}^2$-symmetric spacetimes goes back to the work of Chruściel; cf. [12], in which he introduced it and discussed its basic properties. In [7], the existence of a preferred (areal) foliation of the maximal Cauchy development was then demonstrated in the vacuum setting. However, the understanding of the asymptotics is quite limited. There are constructions of polarised solutions with prescribed asymptotics in the past direction [17, 1], but, to the best of our knowledge, there are no mathematical results concerning the asymptotic behaviour of general $\mathbb{T}^2$-symmetric solutions in the vacuum setting (though there are results concerning inextendibility to the future [13]). On the other hand, several authors have studied the problem numerically; cf., e.g., [8, 20, 16, 9] and references cited therein. One reason why this class of spacetimes has attracted so much attention is that it is on the borderline of what can be done; the issue of strong cosmic censorship, curvature blow up, etc. has been settled in the case of $\mathbb{T}^3$-Gowdy symmetric vacuum solutions (cf. [25] and references cited therein), but it remains open for $\mathbb{T}^2$-symmetric solutions. However, the perhaps main reason why the $\mathbb{T}^2$-symmetric case has attracted so much attention is that it is expected to be the simplest model in which all the features of the so called BKL conjecture appear; the singularity is 'local', 'spacelike' and 'oscillatory'. This expectation is, roughly speaking, supported by the numerical evidence. Our motivation for studying this class of spacetimes is, however, quite different. The subject of the present paper is the expanding direction, and the motivation for taking an interest in it goes back to a general conjecture concerning the future asymptotics of solutions to Einstein's vacuum equations, which we now describe.

Due to the work of Arthur Fischer and Vincent Moncrief on the one hand, and Michael Anderson on the other, there is a general conjecture concerning the future asymptotic behaviour of vacuum solutions to Einstein's equations in the cosmological setting. In order to formulate the conjecture, assume (M, g) to be a globally hyperbolic solution to Einstein's vacuum equations, foliated by compact constant mean curvature (CMC) hypersurfaces exhausting an interval $[\tau_0, 0)$. Assume, moreover, the interval $[\tau_0, 0)$ to correspond to the future Cauchy development of the Cauchy hypersurface with constant mean curvature τ_0 and (M, g) to be future causally geodesically complete. Denote the hypersurface of constant mean curvature $\tau \in [\tau_0, 0)$ by Σ_τ and denote the Riemannian metric induced on Σ_τ by g_τ. Without rescaling, it is to be expected that the curvature of the metrics g_τ decays to zero and the volume goes to infinity. In order to obtain interesting behaviour, it is therefore natural to rescale the metrics. Fischer and Moncrief consider the metrics $\tau^2 g_\tau$ and Anderson considers $t_\tau^{-2} g_\tau$, where

$$t_\tau = \sup_{p \in \Sigma_\tau} d(p, \Sigma_{\tau_0})$$

and $d(p, \Sigma_{\tau_0})$ denotes the supremum of the length of causal curves from p to Σ_{τ_0}. In addition, Fischer and Moncrief define a quantity which is decaying along the Einstein flow, the so-called *reduced Hamiltonian*: $H_{\text{red}} = |\tau|^3 \text{vol}\Sigma_\tau$, and Anderson defines the quantity

$$\frac{\text{vol}\Sigma_\tau}{t_\tau^3},$$

which also decays along the flow. It is of interest to ask: what are the consequences of assuming one of these monotone quantities to be constant on a time interval, say (τ_1, τ_2)? It turns out that then the solution has to be the *Milne model*, $-dt^2 + t^2 g_H$, on $(0, \infty) \times H$, where H is a compact hyperbolic manifold. This solution can be interpreted as a fixed point of the flow. In fact, due to the work of Lars Andersson and Vincent Moncrief, cf. [4], the Milne model is an attractor of the flow; see also [5] for a generalisation of this result to higher dimensions. In order to formulate the conjecture, let us focus on the rescaled family $\hat{g}_\tau = t_\tau^{-2} g_\tau$ mentioned above, which we shall think of as being defined on a fixed manifold Σ; all of the Σ_τ are diffeomorphic. The expectation is then that Σ can be divided into two pieces, H and G, where H is a finite collection of complete hyperbolic manifolds and G is a finite collection of graph manifolds, and the union is along 2-tori. Moreover, $\hat{g}_\tau$ (or at least suitable subsequences) are then expected to converge to finite volume complete hyperbolic metrics on H and to collapse on G. One interesting aspect of this conjecture is that it yields isotropisation and homogenisation in an averaged sense; normally, physicists impose the assumption of spatial homogeneity and isotropy at the beginning of any discussion of cosmology, but it would be preferable to deduce (a local version of) this conclusion from the evolution. Another remarkable feature of the conjecture is that everything that is known concerning the expanding direction of cosmological vacuum spacetimes fits into it. In particular, the result by Andersson and Moncrief mentioned above demonstrates the stability of the Milne model. Yvonne Choquet-Bruhat and Vincent Moncrief have also studied the case of $U(1)$-symmetry; cf. [10, 11]. In this case, the topology of the spacetime is $\mathbb{R} \times \mathbb{S}^1 \times \Sigma_k$, where Σ_k is a higher genus surface. No spatial variation is allowed in the $\mathbb{S}^1$-direction, and the authors impose a smallness assumption concerning the initial data. They then prove that the corresponding solutions are future causally geodesically complete and admit a CMC foliation of the above type. Rescaling the metrics induced on the CMC hypersurfaces along the lines described above, they also obtain the expected collapse; note that $\mathbb{S}^1 \times \Sigma_k$ is a Seifert fibred space, and therefore, in particular, a graph manifold. Finally, there are results in a higher degree of symmetry; cf. [23] and references cited therein for further details. Even though the above results, in particular the stability of the Milne model, are of importance, the following observation should be kept in mind: the results are either restricted to symmetric situations or demonstrate stability of symmetric metrics. The initial hypersurface must thus admit a symmetric metric. This is typically a severe topological restriction which excludes the possibility of studying the situation where the above mentioned division into hyperbolic pieces and graph manifold pieces is non-trivial. Another approach consists of imposing a priori assumptions on the solution and then deducing the above picture. This has been done by Michael Anderson, see [2]. See also [21]. The above description is a bit brief. We refer the interested reader to [3] for a general description of the two perspectives and to [2, 14, 15] and references cited therein for further details.

The relation of the present work to the above conjecture is the following. Clearly, in the case of hyperbolic spatial geometry, there is a preferred model solution; the Milne model. However, it is less clear if there are any appropriate model solutions in the Seifert fibred/graph manifold setting. The proof of stability of the Milne model and the above mentioned studies of the $U(1)$-symmetric case indicate that it is advantageous to have as much hyperbolic geometry as possible. However, this is not something we can expect to have in general. For this reason, we here focus on $\mathbb{T}^3$-spatial topology, and the long term goal is to find a model solution for the behaviour in this setting. In spatial homogeneity, a natural class of solutions with such spatial topology is that of the so-called Kasner metrics, defined by

$$-dt^2 + \sum_{i=1}^{3} t^{2p_i} dx^i \otimes dx^i$$

on $(0,\infty)\times\mathbb{T}^3$, where $p_i\in\mathbb{R}$ and

$$\sum_{i=1}^{3} p_i = 1, \quad \sum_{i=1}^{3} p_i^2 = 1.$$

However, the results of [24] indicate that the Kasner solutions are unstable in the set of $\mathbb{T}^3$-Gowdy solutions. As a consequence, they are not good models of the behaviour of general solutions. The $\mathbb{T}^3$-Gowdy symmetric solutions are, in their turn, a subset of the $\mathbb{T}^2$-symmetric solutions. However, the results of the present paper indicate that the $\mathbb{T}^3$-Gowdy symmetric solutions constitute an unstable subset of the general $\mathbb{T}^2$-symmetric solutions. In order to obtain a model, it would therefore seem to be necessary to study more general solutions than the $\mathbb{T}^3$-Gowdy symmetric ones. In fact, given the above mentioned instabilities, it would be optimistic to expect the $\mathbb{T}^2$-symmetric solutions to be good models.

1.2 Outline of the argument

The proof of Theorem 3 consists of two steps. First, we prove that if α is bounded from below by a positive constant, then the solution is pseudo-homogeneous; this is the main step, which requires most of the effort. In the second step, we prove that if $\langle\alpha^{-1/2}\rangle$ is bounded, then α is bounded from below by a positive constant. In order to develop a feeling for why a positive lower bound on α should imply pseudo-homogeneity, let us first introduce the energy

$$\hat{H} = \int_{\mathbb{S}^1}\left(t^2\alpha^{-1/2}\left[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)\right]+3\alpha^{-1/2}+\frac{\alpha^{-1/2}e^{\lambda/2+P}K^2}{t^{3/2}}\right)d\theta. \tag{16}$$

It can then be computed that $\partial_t\hat{H}\geq 0$, but that $\partial_t(t^{-2}\hat{H})\leq 0$; note that (24) follows from (7). On the other hand, letting

$$f=\alpha^{-1/2}e^{P+\lambda/2}, \quad g=P+\frac{\lambda}{2}-\frac{1}{2}\ln\alpha,$$

Jensen's inequality implies that $\langle f\rangle=\langle e^g\rangle\geq e^{\langle g\rangle}$. Since the boundedness of $t^{-2}\hat{H}$ implies that $\langle f\rangle$ can grow at worst polynomially, we conclude that $\langle g\rangle$ can grow at worst logarithmically (in these arguments, we assume that $K\neq 0$; the Gowdy case is very different). Combining this observation with the equations, one can deduce that there is a constant C such that

$$\int_{\mathbb{S}^1}\int_{t_1}^{\tau} t\left[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)\right]dtd\theta\leq C\ln\tau \tag{17}$$

for all $\tau\geq t_1=t_0+2$ (we here assume the solution to be defined on $(t_0,\infty)\times\mathbb{S}^1$); cf. Lemma 18. Note that this inequality already clearly distinguishes the $\mathbb{T}^2$-symmetric setting from the $\mathbb{T}^3$-Gowdy symmetric setting; a $\mathbb{T}^3$-Gowdy symmetric solution satisfying (17) has to be spatially homogeneous (note also that (17) holds in general; it is not dependent on the assumption that α have a positive lower bound). Moreover, (17) implies that $\langle\lambda\rangle\leq C\ln t$ (an estimate which does not hold for general $\mathbb{T}^3$-symmetric Gowdy solutions) and that $|\langle P\rangle|\leq C\ln t$ for some constant C and all $t\geq t_1$. Due to the positive lower bound on α, including a factor of $\alpha^{-1/2}$ in each of the terms in the integrand of (17) yields a similar upper bound. In fact, it can also be argued that the integral in space and time (as in (17)) of t^{-1} times the last two terms appearing in the integrand in the definition of $\hat{H}$ can be bounded by $C\ln\tau$. As a consequence, there is a constant C such that

$$\int_{t_1}^{\tau} t^{-1}\hat{H}(t)dt\leq C\ln\tau$$

for all $\tau\geq t_1$. Since $\hat{H}$ is an increasing quantity, this estimate implies that $\hat{H}$ is bounded from above, so that, in particular, $\|P-\langle P\rangle\|_{C^0}\leq Ct^{-1}$ etc.; cf. Corollary 32. In other words, the spatial variation is small. It is more intricate to argue that the solution has pseudo-homogeneous asymptotics, but using the boundedness of $\hat{H}$, various monotonicity properties and the conserved quantities A and B, see (22) and (23), it is possible to derive this conclusion. However, it is unfortunately only possible to gradually progress towards the desired conclusion, and quite a large number of steps is required, which makes the argument hard to summarise succinctly. Nevertheless, once this conclusion has been obtained, it can be argued that there is a unique pseudo-homogeneous solution with the corresponding asymptotics; cf. Proposition 14. We shall use a subscript hom to denote this solution. Proving that the difference between the pseudo-homogeneous solution and the original solution is zero is non-trivial, and as a consequence, it is useful to consider a simple special case. Let us therefore discuss the polarised $\mathbb{T}^3$-Gowdy case. Then the relevant equation is given by (10). Let P be a solution converging to a spatially homogeneous solution, say P_{hom}; to be more specific, assume that

$$\lim_{t\to\infty}\int_{\mathbb{S}^1} t^2[(P_t-\partial_t P_{\mathrm{hom}})^2+P_\theta^2]d\theta=0.$$

Note that $t\partial_t P_{\mathrm{hom}}$ is constant for a spatially homogeneous solution. Defining the energy

$$\hat{H}_G=t^2\int_{\mathbb{S}^1}(P_t^2+P_\theta^2)d\theta,$$

we have

$$\frac{d\hat{H}_G}{dt}=2t\int_{\mathbb{S}^1}P_\theta^2 d\theta.$$

Letting $\hat{H}_{G,\mathrm{hom}}$ be the (constant) energy of the spatially homogeneous solution, it is thus clear that

$$\hat{H}_G(t)\le\lim_{t\to\infty}\hat{H}_G(t)=\hat{H}_{G,\mathrm{hom}}. \tag{18}$$

On the other hand, there is a conserved quantity:

$$A=\int_{\mathbb{S}^1}tP_t d\theta.$$

Due to the above assumptions, A has to be the same for P and P_{hom}. Thus

$$\begin{aligned}2\pi\hat{H}_{G,\mathrm{hom}}=&A^2=t^2\left(\int_{\mathbb{S}^1}P_t d\theta\right)^2\le 2\pi t^2\int_{\mathbb{S}^1}P_t^2 d\theta\\ =&2\pi\hat{H}_G-2\pi t^2\int_{\mathbb{S}^1}P_\theta^2 d\theta\le 2\pi\hat{H}_{G,\mathrm{hom}}-2\pi t^2\int_{\mathbb{S}^1}P_\theta^2 d\theta,\end{aligned} \tag{19}$$

where we used Hölder's inequality in the third step and (18) in the fifth. Thus the original solution is spatially homogeneous; a more general argument covering all the $\mathbb{T}^3$-Gowdy cases is to be found in [24].

It would be of interest to generalise the above argument to the $\mathbb{T}^2$-symmetric setting. However, we have not been able to do so; the fact that $K\neq 0$ seems to cause significant complications. The argument that we present in this paper is therefore based on a quantitative version of the above line of reasoning. In fact, we estimate the difference between the solution, say x, and the uniquely associated pseudo-homogeneous solution, say x_{hom}, using two different methods. First, we prove that there are constants C_1 and r_1 such that a suitable energy, say E_1, of the difference between x and x_{hom} satisfies $E_1(t)\ge C_1(t/t_a)^{-r_1}E_1(t_a)$, where $t_1=t_0+2$ and $t\ge t_a\ge t_1$. This estimate is based on the equations and basic energy estimates; cf. Lemma 16. Since we want to prove that $E_1=0$, the idea is then to prove that $E_1(t)\le C_2t^{-r_2}$ for some constants C_2 and r_2 (where $r_2>r_1$) and all $t\ge t_1$; such an estimate would imply that $E_1(t)=0$ for all $t\ge t_1$ and would lead to the conclusion that $x=x_{\mathrm{hom}}$. In order to obtain this second estimate, we use the monotonicity properties of the energy, the conserved quantities etc.; cf. the proofs of

Lemmas 43 and 65. The idea is to argue similarly to the case of polarised Gowdy described above. The difference is that instead of immediately obtaining the conclusion that the spatial variation of P etc. is zero, we obtain an improved estimate of the spatial variation. In most cases, this improvement is not sufficient in order to yield the desired conclusion. It is therefore necessary to iterate the argument in order to improve, in parallel, the estimates for the difference $x - x_{\text{hom}}$ and for the spatial variation of the solution. Naively, one might expect that there is no upper bound to the amount of decay that can be obtained in this way. Unfortunately, this is not true; the iteration yields a system of difference equations for the different decay rates, the asymptotics of the solutions to this system of difference equations can be analysed, and the decay rates converge to specific numbers. All of these argument are on the level of L^2 estimates of derivatives etc. However, in some stages of the argument, it is necessary to derive sup-norm estimates and to derive improvements iteratively, analogously to the L^2-case. Adding up the resulting estimates leads to an estimate of the type $E_1(t) \leq C_2 t^{-r_2}$, cf. above, and we are allowed to conclude that the solution is pseudo-homogeneous. The above description gives a rough idea of the first step of the argument; in practice there are many more details that need to be addressed.

In the second step, we assume $\langle \alpha^{-1/2} \rangle$ to be bounded from above. We then argue that a certain energy has to decay at a specific rate. As opposed to the rest of the arguments, in this case the proof is quite similar to that of the corresponding statement in [24], though there are additional complications. Using the energy decay, one can then conclude that either α decays to zero uniformly, or it is bounded from below by a positive constant. Due to the assumption that $\langle \alpha^{-1/2} \rangle$ is bounded, we conclude that α has a positive lower bound.

1.3 Outline of the paper

General observations. We begin by writing down the conserved quantities, stating Cauchy stability etc. in Section 2. Turning to the actual analysis, we then describe the asymptotics of the pseudo-homogeneous solutions. There are two main reasons for doing this. First of all, this analysis forms the basis for the characterisation of pseudo-homogeneous solutions in terms of their asymptotics. However, it is also useful to have a feeling for the asymptotics of the pseudo-homogeneous solutions before proving that the solutions with a positive lower bound on α have such asymptotics. The proof of the asymptotic characterisation of pseudo-homogeneous solutions consists of two steps; first we derive a general, abstract result, and then we prove that the equations of interest here are such that the abstract result applies; this is the subject of Section 3. In Section 4, we then derive a lower bound on the decay of the energy of the difference between a general solution and a pseudo-homogeneous solution (assuming that they have the same asymptotics). In case the conserved quantity B, cf. (22), is zero, we restrict our attention to polarised solutions; this restriction is only justified much later; cf. Lemma 57. In Section 5, we proceed by deriving some general estimates, such as (17). We also derive some conclusions concerning $\langle P \rangle$ that will be important in taking the step from the assumption of a uniform positive lower bound on α to an upper bound on $\langle \alpha^{-1/2} \rangle$.

Deriving pseudo-homogeneous asymptotics, given a positive lower bound on α. In Section 6, we focus on deriving conclusions that are only dependent on the assumption that α does not converge to zero uniformly. In particular, we demonstrate that it is possible to derive uniform estimates using non-uniform ones, given suitable assumptions concerning the energy $\hat{H}$. In Section 7, we then start deriving conclusions based on the assumption that α is bounded from below by a positive constant. In particular, we prove that for every $C > 0$, there is a $T > t_0$ such that $-3 \ln t + C \leq P(t, \theta) \leq \ln t - C$ for all $(t, \theta) \in [T, \infty) \times \mathbb{S}^1$. Moreover, we demonstrate that $\hat{H}$ is bounded; cf. the above argument. As a first step in proving that the solution has pseudo-homogeneous asymptotics, we prove that if $B \neq 0$, then Q converges, and $P \to \infty$ as $t \to \infty$. In a step by step process, we then gradually improve our knowledge concerning P, Q, λ and the energies. The essential tools in obtaining the conclusions are the conserved quantities, the asymptotic behaviour of the energy, and the monotonicity properties of the energy. However, it is of interest to note that in some of the arguments, it is of crucial importance to keep track of the direction from which certain asymptotic values are attained; estimating the size of the difference is not sufficient in order to obtain the conclusions we need. This becomes particularly clear in the proof of Lemma 35. In this lemma, we demonstrate that $P/\ln t$ converges to a number, say r_∞. It is quite straightforward to prove that $r_\infty \in [-3, 1]$. However, it is of crucial importance to know that $r_\infty \in (-3, 1)$; this is where the direction from which $P/\ln t$ approaches its limit comes into play. In the end, we obtain quite detailed asymptotic information in Lemmas 39 and 46.

Associating a pseudo-homogeneous solution with the given solution. At this stage, it would be desirable to prove that there is a unique pseudo-homogeneous solution with the same asymptotics as the given solution. However, there is one technical problem; in case $B = 0$, we need to know that the solution is polarised. In Section 8, we therefore derive C^1-estimates. In particular, we prove that $tP_t - r_\infty = O(t^{-\delta})$ for some $\delta > 0$. Using this C^1 information, it then turns out to be possible to prove that, for $B = 0$, the energy

$$\mathcal{E}_Q = \int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} (Q_t^2 + \alpha Q_\theta^2) d\theta$$

cannot decay faster to zero than t^{-1}, unless it is zero. Since we have previously proven that it does decay to zero faster than t^{-1}, we conclude that $\mathcal{E}_Q = 0$, so that the solution is polarised. Appealing to Proposition 14, we can then associate a unique pseudo-homogeneous solution with the given solution.

Iterative improvement of the estimates. In Section 10, we give a general argument for how given estimates of the difference between the solution and the pseudo-homogeneous solution can be improved. The corresponding iteration leads to a system of difference equations for various decay rates, and we analyse the asymptotics of solutions to this system in Subsection 10.1. In Subsection 10.2, we then carry out similar arguments for the C^1-estimates. The resulting decay rates are such that the only way for them to be consistent with the estimates derived in Lemma 16 is that the difference between the solution and the pseudeo-homogeneous solution is zero. This finishes the proof of the fact that solutions with an α bounded from below by a positive constant are pseudo-homogeneous.

Characterising pseudo-homogeneous solutions by an upper bound on $\langle \alpha^{-1/2} \rangle$. In order to prove that it is sufficent to assume that $\langle \alpha^{-1/2} \rangle$ is bounded from above in order to conclude that the solution is pseudo-homogeneous, we need to derive energy estimates. The general idea is to prove that if $\langle \alpha^{-1/2} \rangle$ is bounded, then $\hat{H}/t$ is bounded. Appealing to Lemma 26 then yields the conclusion that α is bounded from below by a positive constant. In order to prove the desired energy estimate, we consider the energy H_a; cf. (154). We would like to have an estimate of the form $H_a' \leq -H_a/t$. However, differentiating H_a does not lead to such an estimate; cf. (155). The idea is then to introduce a 'correction' term, say Γ, such that $\mathcal{E} = H_a + \Gamma$ satisfies an estimate of this type, and such that $\mathcal{E}$ and H_a are equivalent. Assuming $\langle \alpha^{-1/2} \rangle$ to be bounded, it turns out to be possible to prove that $\mathcal{E}$ and H_a are equivalent. However, the expression for $\mathcal{E}'$ is not as transparent as one might wish; cf. (162). In order for this estimate to be of use, it is necessary to know that H_a converges to zero. We prove that this is the case in Lemma 74. Combining this information with the expression for $\mathcal{E}'$, it can then be argued that $H_a \leq Ct^{-1}$; however, it is of interest to note that the integrability properties of $\langle P_t \rangle$ derived in Corollary 22 are of crucial importance in the argument. Combining the resulting information, we conclude that if $\langle \alpha^{-1/2} \rangle$ is bounded, then the solution is pseudo-homogeneous. Finally, in Section 13, we discuss the polarised case. In particular, we prove Proposition 1.

2 Basic computations and conventions

As already mentioned, the relevant equations are (2)–(6). As a consequence of these equations

$$\begin{aligned}\partial_t(t\alpha^{-1/2}e^{2P}Q_t) &= \partial_\theta(t\alpha^{1/2}e^{2P}Q_\theta), &(20)\\ \partial_t(t\alpha^{-1/2}P_t) &= \partial_\theta(t\alpha^{1/2}P_\theta)+t\alpha^{-1/2}e^{2P}(Q_t^2-\alpha Q_\theta^2)-\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{2t^{5/2}}. &(21)\end{aligned}$$

Conserved quantities. Due to (20), there is a constant B such that

$$B=\int_{\mathbb{S}^1} t\alpha^{-1/2}e^{2P}Q_t d\theta. \tag{22}$$

Moreover, using (20), (21) and partial integration, it can be calculated that

$$\partial_t\int_{\mathbb{S}^1} t\alpha^{-1/2}(P_t-e^{2P}Q_tQ)d\theta=-\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{2t^{5/2}}d\theta.$$

Due to (4), this equality implies that there is a constant A such that

$$A=\int_{\mathbb{S}^1} t\alpha^{-1/2}\left(P_t+\frac{1}{t}-e^{2P}Q_tQ\right)d\theta. \tag{23}$$

Energies. Letting $\hat{H}$ be defined by (16), equation (7) implies that

$$\frac{d\hat{H}}{dt}=2t\int_{\mathbb{S}^1}\alpha^{1/2}(P_\theta^2+e^{2P}Q_\theta^2)d\theta. \tag{24}$$

Light cone estimates. Let

$$\mathcal{A}_\pm=(\partial_\pm P)^2+e^{2P}(\partial_\pm Q)^2,$$

where $\partial_\pm$ is defined by

$$\partial_\pm=\partial_t\pm\alpha^{1/2}\partial_\theta. \tag{25}$$

In what follows, it is of interest to keep in mind that

$$\begin{aligned}\partial_\pm\partial_\mp P &= -\frac{1}{t}P_t+e^{2P}(Q_t^2-\alpha Q_\theta^2)-\frac{e^{P+\lambda/2}K^2}{2t^{7/2}}+\frac{\alpha_t}{2\alpha}\partial_\mp P, &(26)\\ \partial_\pm\partial_\mp Q &= -\frac{1}{t}Q_t-2(Q_tP_t-\alpha Q_\theta P_\theta)+\frac{\alpha_t}{2\alpha}\partial_\mp Q, &(27)\\ \partial_\pm\mathcal{A}_\mp &= -\left(\frac{2}{t}-\frac{\alpha_t}{\alpha}\right)\mathcal{A}_\mp\mp\frac{2}{t}\sqrt{\alpha}\left(P_\theta\partial_\mp P+e^{2P}Q_\theta\partial_\mp Q\right)-\partial_\mp P\frac{e^{P+\lambda/2}K^2}{t^{7/2}}. &(28)\end{aligned}$$

Cauchy stability. For future reference, let us make the following observation.

Proposition 8. *Consider a pseudo-homogeneous solution to (2)–(6), say $(P_1,Q_1,\alpha_1,\lambda_1)$, defined on $(t_0,\infty)\times\mathbb{S}^1$ for some $t_0\geq 0$. Let $[t_1,t_2]\subset(t_0,\infty)$ and let $\epsilon>0$. Then there is a $\delta>0$ such that if $(P_2,Q_2,\alpha_2,\lambda_2)$ is a solution with the property that*

$$\|p(t_1,\cdot)\|_{C^1}+\|q(t_1,\cdot)\|_{C^1}+\|\partial_t p(t_1,\cdot)\|_{C^0}+\|\partial_t q(t_1,\cdot)\|_{C^0}+\|\varrho(t_1,\cdot)\|_{C^1}+\|\ell(t_1,\cdot)\|_{C^0}\leq\delta, \tag{29}$$

where $(p,q,\varrho,\ell)=(P_2-P_1,Q_2-Q_1,\alpha_2-\alpha_1,\lambda_2-\lambda_1)$, then the same estimate holds with t_1 replaced by t_2 and δ replaced by ϵ.

Remarks 9. Due to the equations and (29), it follows that $\partial_t\varrho$, $\partial_\theta\ell$ and $\partial_t\ell$ are small initially. The restriction to a pseudo-homogeneous 'background' solution is not necessary.

Conventions. In this paper, we consider solutions to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ for some $t_0\geq 0$. Since we are interested in the future asymptotics, we often wish to restrict our attention to some subset which is bounded away from t_0. Assuming a solution of the above type has been introduced, we therefore, in what follows, speak of t_1, and take for granted that $t_1=t_0+2$. The reason for making this choice is that $t_1>t_0$ and that $\ln t\geq\ln 2>0$ for $t\geq t_1$.

3 Asymptotic characterisation of pseudo-homogeneous solutions

Our first goal in this section is to derive asymptotics for pseudo-homogeneous solutions. However, let us begin by making a technical observation on which some of the arguments are based.

Lemma 10. *Consider a function $f \in C^1([T,\infty),\mathbb{R})$ for some $T \geq 1$ such that $f(t) \geq 0$ and*

$$\int_T^\infty t^{-1} f(t)dt < \infty. \tag{30}$$

Assume that there is a constant $C > 0$ such that

$$tf'(t) \geq -C[1 + f^2(t)] \tag{31}$$

for all $t \geq T$. Then

$$\lim_{t\to\infty} f(t) = 0.$$

Proof. Let us reformulate the statement in terms of the time $\tau = \ln t$. Let $g(\tau) = f(e^\tau)$. Then

$$\int_{\ln T}^\infty g(\tau)d\tau < \infty$$

due to (30), and (31) translates into

$$g'(\tau) \geq -C[1 + f^2(e^\tau)] = -C[1 + g^2(\tau)]. \tag{32}$$

Assume, in order to get a contradiction, that there is an $\epsilon > 0$ such that for all T_0, there is a $\tau > T_0$ such that $g(\tau) \geq \epsilon$. Assuming there is a T_1 such that $g(\tau) \geq \epsilon/2$ for all $\tau \geq T_1$ we get a contradiction. As a consequence, we get an infinite number of disjoint intervals $[T_{2,k}, T_{1,k}]$ such that $g(T_{i,k}) = i\epsilon/2$ for $i = 1,2$ and $\epsilon/2 \leq g(\tau) \leq \epsilon$ for $\tau \in [T_{2,k}, T_{1,k}]$. Due to the estimate (32), we conclude that

$$-\frac{\epsilon}{2} = \int_{T_{2,k}}^{T_{1,k}} g'(\tau)d\tau \geq -C(1+\epsilon^2)(T_{1,k} - T_{2,k}).$$

Thus

$$T_{1,k} - T_{2,k} \geq \frac{\epsilon}{2C(1+\epsilon^2)}.$$

Since $g(\tau) \geq \epsilon/2$ in the interval, we obtain a contradiction to integrability. □

Let us now derive asymptotics for pseudo-homogeneous solutions.

Lemma 11. *Consider a pseudo-homogeneous solution to (2)–(6) on $(t_0,\infty) \times \mathbb{S}^1$ with $K \neq 0$. Then there are constants A, B and c_H such that*

$$\begin{aligned} B &= 2\pi t\langle\alpha^{-1/2}\rangle e^{2P} Q_t, && (33)\\ A &= 2\pi\langle\alpha^{-1/2}\rangle(tP_t + 1 - te^{2P}Q_tQ), && (34)\\ c_H &= 2\pi\langle\alpha^{-1/2}\rangle\left(t^2P_t^2 + t^2e^{2P}Q_t^2 + \frac{e^{P+\lambda/2}K^2}{t^{3/2}} + 3\right), && (35) \end{aligned}$$

where $\langle\alpha^{-1/2}\rangle$ is defined in (15). Moreover, there is a smooth positive function α_∞ on $\mathbb{S}^1$ such that

$$\lim_{t\to\infty} \|\alpha(t,\cdot) - \alpha_\infty\|_{C^0} = 0.$$

If $B = 0$, then Q is constant and if $B \neq 0$, then Q converges to a limit. In either case, there is a constant q_∞ such that

$$\lim_{t\to\infty} Q(t) = q_\infty.$$

Let

$$r_\infty = \frac{A + Bq_\infty}{2\pi\langle\alpha^{-1/2}\rangle_\infty} - 1, \tag{36}$$

where $\langle\alpha^{-1/2}\rangle_\infty$ *denotes the limit of* $\langle\alpha^{-1/2}\rangle$. *Then* $r_\infty \in (-3, 1)$ *in case* $B = 0$ *and* $r_\infty \in (0, 1)$ *in case* $B \neq 0$. *Moreover, there are constants* c_P *and* c_λ *such that*

$$\begin{aligned} \lim_{t\to\infty} |P(t) - r_\infty \ln t - c_P| &= 0, \\ \lim_{t\to\infty} |\lambda(t) - r_\infty^2 \ln t - c_\lambda| &= 0. \end{aligned}$$

Finally,

$$c_H = 2\pi\langle\alpha^{-1/2}\rangle_\infty \left(r_\infty^2 + 3\right).$$

Remark 12. Using the asymptotics of the lemma as a starting point, asymptotic expansions can be derived. However, the above conclusions suffice for what we wish to acheive here.

Proof. That the right hand sides of (33)–(35) are independent of t is a consequence of (22), (23) and (24). Since

$$\partial_t\langle\alpha^{-1/2}\rangle = \frac{e^{P+\lambda/2}K^2}{2t^{5/2}}\langle\alpha^{-1/2}\rangle,$$

it is clear that $\langle\alpha^{-1/2}\rangle$ is increasing. On the other hand, (35) implies that $\langle\alpha^{-1/2}\rangle$ is bounded. Consequently, $\langle\alpha^{-1/2}\rangle$ converges; call the limit $\langle\alpha^{-1/2}\rangle_\infty$. Thus there is a smooth positive function α_∞ to which α converges in C^0. Turning to Q, note that it is constant in case $B = 0$. Let us therefore, for the moment, assume that $B \neq 0$. Note that (34) implies that

$$A + BQ = 2\pi\langle\alpha^{-1/2}\rangle(tP_t + 1). \tag{37}$$

However, due to (35), the right hand side is bounded. Thus Q is bounded if $B \neq 0$. On the other hand, (33) implies that Q_t has a sign, assuming $B \neq 0$. Combining these observations, we conclude that Q converges (a conclusion which holds regardless of whether B equals zero or not); call the limit q_∞. Turning to P, note that the fact that Q converges and the fact that (33) holds imply that

$$\int_{t_1}^{\infty} \frac{1}{t} e^{-2P} dt < \infty,$$

assuming $B \neq 0$. Combining this observation with Lemma 10, we conclude that $P \to \infty$ if $B \neq 0$. In order to derive more detailed asymptotics, note that tP_t converges; this is a consequence of the fact that Q and $\langle\alpha^{-1/2}\rangle$ converge and the fact that (37) holds. Denote the limit r_∞ and note that it is given by (36). As a consequence,

$$P(t) = r_\infty \ln t + o(\ln t).$$

Note, for future reference, that $r_\infty \geq 0$ if $B \neq 0$. Note also that, due to (33) and the fact that $P \to \infty$, we know that $te^P Q_t \to 0$ in case $B \neq 0$; in case $B = 0$, we of course have $te^P Q_t = 0$. Thus

$$t^2 P_t^2 + t^2 e^{2P} Q_t^2 \to r_\infty^2,$$

regardless of whether B equals zero or not. Combining this observation with (5) and the fact that α converges to a positive function, we conclude that

$$\lambda(t) = r_\infty^2 \ln t + o(\ln t).$$

Let us now derive restrictions on r_∞. Note that

$$\frac{\alpha_t}{\alpha} = -K^2 \exp\left(\frac{1}{2} r_\infty^2 \ln t + r_\infty \ln t - \frac{5}{2}\ln t + o(\ln t)\right).$$

Since the right hand side is integrable, we have to have $-3 \le r_\infty \le 1$ (and $r_\infty \ge 0$ if $B \neq 0$). It is of interest to exclude the boundary cases.

Let us begin by considering the case $B = 0$, and let us assume that $r_\infty = 1$. Since

$$tP_t = \frac{A}{2\pi\langle\alpha^{-1/2}\rangle} - 1, \tag{38}$$

we have to have $A > 0$. Since $\langle\alpha^{-1/2}\rangle$ is an increasing function, we conclude that

$$tP_t \ge \frac{A}{2\pi\langle\alpha^{-1/2}\rangle_\infty} - 1 = r_\infty = 1.$$

Thus $P \ge \ln t + C$. Similarly, $\lambda \ge \ln t + C$. However, this is not reconcilable with the integrability of α_t/α. We thus obtain a contradiction, so that $r_\infty < 1$. Let us now assume that $r_\infty = -3$. Then $A < 0$, and (38) implies that

$$tP_t \le \frac{A}{2\pi\langle\alpha^{-1/2}\rangle_\infty} - 1 = r_\infty = -3.$$

Thus

$$\frac{1}{2}\lambda_t + P_t - \frac{5}{2t} = t^{-1}\left(\frac{1}{2}t^2 P_t^2 + tP_t - \frac{5}{2}\right) + \frac{\alpha_t}{2\alpha} \ge -t^{-1} + \frac{\alpha_t}{2\alpha}.$$

Again, this estimate is irreconcilable with the integrability of α_t/α. Thus $r_\infty > -3$.

Let us turn to the case $B \neq 0$. We know that $0 \le r_\infty \le 1$. In order to exclude the possibility $r_\infty = 1$, let us assume that this equality holds. Then $A + Bq_\infty > 0$. Moreover (33) implies that $Q_t = O(t^{-2})$. Thus $Q - q_\infty = O(t^{-1})$, so that

$$tP_t = \frac{A + BQ}{2\pi\langle\alpha^{-1/2}\rangle} - 1 = \frac{A + Bq_\infty}{2\pi\langle\alpha^{-1/2}\rangle} - 1 + O(t^{-1}) \ge \frac{A + Bq_\infty}{2\pi\langle\alpha^{-1/2}\rangle_\infty} - 1 + O(t^{-1}) = r_\infty + O(t^{-1}).$$

This estimate leads to a contradiction for reasons similar to ones given above. Thus $r_\infty < 1$. In order to exclude the case $r_\infty = 0$, note that if $r_\infty = 0$, then $\alpha_t = O(t^{-2})$. Thus

$$tP_t = \frac{A + BQ}{2\pi\langle\alpha^{-1/2}\rangle} - 1 = \frac{A + BQ}{2\pi\langle\alpha^{-1/2}\rangle_\infty} - 1 + O(t^{-1}) \le \frac{A + Bq_\infty}{2\pi\langle\alpha^{-1/2}\rangle_\infty} - 1 + O(t^{-1}) = O(t^{-1}),$$

where we have used the fact that BQ is increasing; cf. (33). Thus $P \le C$ for some constant C. However, this estimate contradicts the fact that P tends to infinity. Thus $r_\infty > 0$.

Turning to more detailed asymptotics, let us introduce the notation

$$\gamma = 2 - \frac{1}{2}(r_\infty + 1)^2. \tag{39}$$

Then $\gamma > 0$, and for every $\epsilon > 0$, there is a C_ϵ such that

$$|\alpha_t| \le C_\epsilon t^{-1-\gamma+\epsilon}$$

for $t \ge t_1$. In particular, α converges at a specific rate. Moreover, the same holds for Q in case $B \neq 0$. Consequently, there is an $\eta > 0$ such that

$$tP_t = r_\infty + O(t^{-\eta}). \tag{40}$$

Thus the stated asymptotics for P hold. The argument concerning λ is similar. □

In order to prove an asymptotic characterisation of the spatially homogeneous solutions, we need the following abstract result.

Lemma 13. *Let $\eta > 0$ and $t_0 \geq 0$ be real numbers, n, m be positive integers, and*

$$
\begin{aligned}
f &: (t_0, \infty) \times \mathrm{B}_2^n(0) \times \mathrm{B}_2^m(0) \to \mathbb{R}^n, \\
g &: (t_0, \infty) \times \mathrm{B}_2^n(0) \times \mathrm{B}_2^m(0) \to \mathbb{R}^m, \\
G &: (t_0, \infty) \times \mathrm{B}_2^n(0) \times \mathrm{B}_2^m(0) \to \mathrm{M}_{m\times n}
\end{aligned}
$$

be smooth functions, where $\mathrm{B}_r^n(p)$ denotes the open ball in $\mathbb{R}^n$ of radius $r > 0$, centered at $p \in \mathbb{R}^n$, and $\mathrm{M}_{n\times m}$ denotes the real $n \times m$-matrices. Assume that f, g, G and their first derivatives with respect to the last $n + m$ variables are uniformly bounded on

$$[t_1, \infty) \times \mathrm{C}_1^n(0) \times \mathrm{C}_1^m(0), \tag{41}$$

where $\mathrm{C}_r^n(p)$ denotes the closed ball in $\mathbb{R}^n$ of radius $r > 0$, centered at $p \in \mathbb{R}^n$, and $t_1 = t_0 + 2$. Then there is a $T > 0$ and unique smooth functions

$$x : (T, \infty) \to \mathrm{C}_1^n(0), \quad y : (T, \infty) \to \mathrm{C}_1^m(0)$$

solving the equations

$$t\dot{x} = t^{-\eta} f(t, x, y), \tag{42}$$

$$t\dot{y} = G(t, x, y)x + t^{-\eta} g(t, x, y) \tag{43}$$

and satisfying the condition

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0.$$

Proof. Let C_f be a constant such that the norm of f and of its first derivative with respect to the last $n + m$ variables is bounded by C_f in the set (41). Define C_g and C_G similarly. Define an iteration by $x_0 = 0$, $y_0 = 0$ and

$$t\dot{x}_{n+1} = t^{-\eta} f(t, x_n, y_n), \quad t\dot{y}_{n+1} = G(t, x_n, y_n)x_{n+1} + t^{-\eta} g(t, x_n, y_n), \tag{44}$$

where we demand that $x_{n+1}, y_{n+1} \to 0$ as $t \to \infty$ (from the arguments given below, it will become clear that this sequence is well defined). Note that

$$|x_1(t)| \leq C_f \eta^{-1} t^{-\eta}, \quad |y_1(t)| \leq C_G C_f \eta^{-2} t^{-\eta} + C_g \eta^{-1} t^{-\eta}.$$

Let T be large enough that

$$C_f \eta^{-1} T^{-\eta} \leq 1, \quad C_G C_f \eta^{-2} T^{-\eta} + C_g \eta^{-1} T^{-\eta} \leq 1. \tag{45}$$

Assume, inductively, that

$$|x_n(t)| \leq 1, \quad |y_n(t)| \leq 1, \quad |x_n(t)| \leq C_f \eta^{-1} t^{-\eta} \tag{46}$$

on $[T, \infty)$. We know these assumptions to hold for $n = 0, 1$. Combining (44) with (46), we conclude that

$$|\dot{x}_{n+1}| \leq C_f t^{-1-\eta}.$$

Combining this estimate with (45), we conclude that the conditions concerning x_n in (46) hold with n replaced by $n + 1$. Thus

$$|\dot{y}_{n+1}| \leq C_G C_f \eta^{-1} t^{-1-\eta} + C_g t^{-1-\eta}.$$

Combining this estimate with (45), we conclude that (46) holds with n replaced by $n+1$. In short, (46) holds for all n. Let us use the notation

$$\|z\|_{C_T} = \sup_{t\in[T,\infty)} |z(t)|$$

for a continuous $\mathbb{R}^l$-valued function z which is bounded on $[T,\infty)$. Note that

$$t|\dot{x}_{n+2}-\dot{x}_{n+1}|\leq C_f t^{-\eta}(\|x_{n+1}-x_n\|_{C_T}+\|y_{n+1}-y_n\|_{C_T}).$$

Thus

$$|x_{n+2}(t)-x_{n+1}(t)|\leq C_f\eta^{-1}t^{-\eta}(\|x_{n+1}-x_n\|_{C_T}+\|y_{n+1}-y_n\|_{C_T})$$

on $[T,\infty)$. Inserting this information into (44), we obtain

$$\begin{aligned} t|\dot{y}_{n+2}-\dot{y}_{n+1}| \leq & 2C_G C_f\eta^{-1}t^{-\eta}(\|x_{n+1}-x_n\|_{C_T}+\|y_{n+1}-y_n\|_{C_T}) \\ & + C_g t^{-\eta}(\|x_{n+1}-x_n\|_{C_T}+\|y_{n+1}-y_n\|_{C_T}). \end{aligned}$$

Thus

$$|y_{n+2}(t)-y_{n+1}(t)|\leq (2C_G C_f\eta^{-2}+C_g\eta^{-1})t^{-\eta}(\|x_{n+1}-x_n\|_{C_T}+\|y_{n+1}-y_n\|_{C_T})$$

on $[T,\infty)$. Letting T be large enough in the above estimates, we conclude that

$$\|x_{n+2}-x_{n+1}\|_{C_T}+\|y_{n+2}-y_{n+1}\|_{C_T}\leq \frac{1}{2}(\|x_{n+1}-x_n\|_{C_T}+\|y_{n+1}-y_n\|_{C_T}).$$

Consequently, both x_n and y_n converge with respect to the norm $\|\cdot\|_{C_T}$. We thus obtain continuous functions x and y on $[T,\infty)$ such that x_n and y_n converge to them. Since $|x_n(t)|\leq Ct^{-\eta}$ and similarly for y_n, where the constant does not depend on n, we conclude that x and y converge to zero. Due to the definition of x_n and y_n, we conclude that x and y solve the original equations. Since f, g and G are smooth, we conclude that x and y are smooth. Finally, uniqueness follows by arguments similar to ones given above. □

Finally, let us state the desired asymptotic characterisation.

Proposition 14. *Let A, B and q_∞ be constants and α_∞ be a positive C^0 function on $\mathbb{S}^1$. Let*

$$\langle\alpha^{-1/2}\rangle_\infty=\langle\alpha_\infty^{-1/2}\rangle,\quad r_\infty=\frac{A+Bq_\infty}{2\pi\langle\alpha^{-1/2}\rangle_\infty}-1.$$

Assume A, B, q_∞ and α_∞ to be such that $r_\infty\in(0,1)$ in case $B\neq 0$ and such that $r_\infty\in(-3,1)$ in case $B=0$. Finally, let c_P and c_λ be constants. Then there is a unique pseudo-homogeneous solution (P,Q,α,λ) to (2)–(6) such that (33), (34) and

$$\lim_{t\to\infty}\left(|\alpha(t,\theta)-\alpha_\infty(\theta)|+|Q(t)-q_\infty|+|P(t)-r_\infty\ln t-c_P|+|\lambda(t)-r_\infty^2\ln t-c_\lambda|\right)=0$$

hold.

Remark 15. The regularity of the solution is the following: P, Q and λ are smooth functions and α is a smooth function of t times α_∞.

Proof. The idea of the proof is to appeal to Lemma 13. To this end, we need to define suitable variables and prove that they satisfy the required type of equations.

Step 1. Let us assume that we have a pseudo-homogeneous solution of the desired type, and let

$$\begin{aligned} \hat{Q} &= Q-q_\infty, \\ \hat{P} &= P-r_\infty\ln t-c_P, \\ \hat{\lambda} &= \lambda-r_\infty^2\ln t-c_\lambda, \\ \hat{\alpha} &= \alpha\langle\alpha^{-1/2}\rangle_\infty^2. \end{aligned}$$

Let, moreover, $x=(\hat{Q},\ln\langle\hat{\alpha}^{-1/2}\rangle)^t$ and $y=(\hat{P},\hat{\lambda})^t$. Note that (33) can be written

$$t\hat{Q}_t=\frac{Be^{-2P}}{2\pi\langle\alpha^{-1/2}\rangle}=\frac{Be^{-2c_P}}{2\pi\langle\alpha^{-1/2}\rangle_\infty}t^{-2r_\infty}\exp\left(-2\hat{P}-\ln\langle\hat{\alpha}^{-1/2}\rangle\right).$$

Let $\eta = \gamma$ (where γ is defined by (39)) in case $B = 0$ and $\eta = \min\{\gamma, 2r_\infty\}$ in case $B \neq 0$. Then we can write this equation

$$t\hat{Q}_t = t^{-\eta} f_1(t, x, y),$$

where

$$f_1(t, x, y) = t^{\eta - 2r_\infty} \frac{Be^{-2c_P}}{2\pi\langle\alpha^{-1/2}\rangle_\infty} \exp\left(-2\hat{P} - \ln\langle\hat{\alpha}^{-1/2}\rangle\right).$$

If $B = 0$, then $f_1 = 0$, and there is nothing to prove. In case $B \neq 0$, it is clear that f_1 satisfies the requirements made in Lemma 13. Moreover, (4) implies

$$t\partial_t \ln\langle\hat{\alpha}^{-1/2}\rangle = \frac{e^{P+\lambda/2}K^2}{2t^{3/2}} = \frac{1}{2} e^{c_P + c_\lambda/2} K^2 t^{-\gamma} e^{\hat{P} + \hat{\lambda}/2}.$$

This equation can be written

$$t\partial_t \ln\langle\hat{\alpha}^{-1/2}\rangle = t^{-\eta} f_2(t, x, y),$$

where

$$f_2(t, x, y) = \frac{1}{2} t^{\eta - \gamma} e^{c_P + c_\lambda/2} K^2 e^{\hat{P} + \hat{\lambda}/2}.$$

Again, f_2 satisfies the relevant conditions. Letting $f = (f_1, f_2)^t$, it is therefore clear that x satisfies an equation of the desired type. Turning to $\hat{P}$, note that (34) and (33) imply

$$\begin{aligned} t\hat{P}_t =& \frac{A + BQ}{2\pi\langle\alpha^{-1/2}\rangle} - 1 - r_\infty = \frac{B}{2\pi\langle\alpha^{-1/2}\rangle}\hat{Q} + \frac{A + Bq_\infty}{2\pi\langle\alpha^{-1/2}\rangle_\infty} \frac{1}{\langle\hat{\alpha}^{-1/2}\rangle} - 1 - r_\infty \\ =& \frac{B}{2\pi\langle\alpha^{-1/2}\rangle}\hat{Q} + (r_\infty + 1)[\exp(-\ln\langle\hat{\alpha}^{-1/2}\rangle) - 1]. \end{aligned}$$

Let

$$G_{P,Q} = \frac{B}{2\pi\langle\alpha^{-1/2}\rangle_\infty} \exp\left(-\ln\langle\hat{\alpha}^{-1/2}\rangle\right), \quad G_{P,\alpha} = (r_\infty + 1)\frac{\exp(-\ln\langle\hat{\alpha}^{-1/2}\rangle) - 1}{\ln\langle\hat{\alpha}^{-1/2}\rangle}.$$

Since $G_{P,Q}$ and $G_{P,\alpha}$ are smooth functions of x only, it is clear that they have the required properties. Letting $g_1 = 0$, it is clear that g_1 has the desired properties. Finally, (4), (5) and (33) can be combined to yield

$$t\hat{\lambda}_t = t^2 P_t^2 + t^2 e^{2P} Q_t^2 + t\frac{\alpha_t}{\alpha} - r_\infty^2 = t^2 \hat{P}_t^2 + 2r_\infty t\hat{P}_t + \frac{B^2 e^{-2P}}{4\pi^2\langle\alpha^{-1/2}\rangle^2} - \frac{e^{P+\lambda/2}K^2}{t^{3/2}}.$$

Let

$$G_{\lambda,Q} = G_{P,Q}^2 \hat{Q} + 2G_{P,Q}G_{P,\alpha} \ln\langle\hat{\alpha}^{-1/2}\rangle + 2r_\infty G_{P,Q}, \quad G_{\lambda,\alpha} = G_{P,\alpha}^2 \ln\langle\hat{\alpha}^{-1/2}\rangle + 2r_\infty G_{P,\alpha}$$

and

$$G = \begin{pmatrix} G_{P,Q} & G_{P,\alpha} \\ G_{\lambda,Q} & G_{\lambda,\alpha} \end{pmatrix}.$$

Then G is a smooth matrix valued function of x only. It is consequently clear that G satisfies the required conditions. Define

$$g_2 = \frac{B^2 e^{-2c_P}}{4\pi^2\langle\alpha^{-1/2}\rangle_\infty^2} t^{\eta - 2r_\infty} \exp\left(-2\hat{P} - 2\ln\langle\hat{\alpha}^{-1/2}\rangle\right) - t^{\eta-\gamma} e^{c_P + c_\lambda/2} K^2 e^{\hat{P} + \hat{\lambda}/2}$$

and $g = (g_1, g_2)^t$. Then y satisfies $t\dot{y} = G(t, x, y)x + t^{-\eta} g(t, x, y)$, and G and g have the desired properties. Since two pseudo-homogeneous solutions satisfying the conditions of the lemma correspond to solutions to (42) and (43) that converge to zero, Lemma 13 implies that they have to coincide.

Step 2. In step 1, we derived equations of the form (42) and (43), assuming we had a pseudo-homogeneous solution. However, let us now use these equations in order to construct a solution. Appealing to Lemma 13, we obtain a unique smooth solution such that x and y converge to zero. Define P, Q, α and λ by

$$P = \hat{P} + r_\infty \ln t + c_P, \quad Q = \hat{Q} + q_\infty, \quad \alpha = \alpha_\infty \langle \hat{\alpha}^{-1/2} \rangle^{-2}, \quad \lambda = \hat{\lambda} + r_\infty^2 \ln t + c_\lambda.$$

Then it is clear that P, Q, α and λ have the correct asymptotics. However, we also need to verify that they satisfy the correct equations. It is clear that α satisfies the equation it should. Note also that

$$\langle \alpha^{-1/2} \rangle = \langle \alpha_\infty^{-1/2} \rangle \langle \hat{\alpha}^{-1/2} \rangle.$$

As a consequence, we know that (33) and (34) are satisfied. Moreover, λ satisfies the equation it should. Turning to Q, note that

$$\langle \alpha^{-1/2} \rangle t e^{2P} Q_t = \frac{B}{2\pi}.$$

Multiplying this equality with $\alpha_\infty^{-1/2} \langle \alpha_\infty^{-1/2} \rangle^{-1}$, we obtain

$$t\alpha^{-1/2} e^{2P} Q_t = \frac{B\alpha_\infty^{-1/2}}{2\pi \langle \alpha_\infty^{-1/2} \rangle}. \tag{47}$$

Time differentiating this equation, keeping in mind that the right hand side only depends on θ, we obtain the equation for Q. Turning to P, we have

$$\langle \alpha^{-1/2} \rangle (tP_t + 1) = \frac{A + BQ}{2\pi}.$$

Thus

$$\alpha^{-1/2}(tP_t + 1) = \alpha_\infty^{-1/2} \langle \alpha_\infty^{-1/2} \rangle^{-1} \frac{A + BQ}{2\pi}.$$

Differentiating with respect to time, we obtain

$$t\alpha^{-1/2} P_{tt} + \alpha^{-1/2} P_t - \frac{\alpha_t}{2\alpha^{3/2}}(tP_t + 1) = B\alpha_\infty^{-1/2} \langle \alpha_\infty^{-1/2} \rangle^{-1} \frac{1}{2\pi} Q_t = t\alpha^{-1/2} e^{2P} Q_t^2,$$

where we have used (47). Thus the equation for P holds. The lemma follows. □

4 Asymptotically pseudo-homogeneous solutions

Lemma 16. *Let (P, Q, α, λ) be a solution and $(P_{\text{hom}}, Q_{\text{hom}}, \alpha_{\text{hom}}, \lambda_{\text{hom}})$ a pseudo-homogeneous solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with conserved quantities (A, B) and $(A_{\text{hom}}, B_{\text{hom}})$ respectively; cf. (22) and (23). Assume that $A = A_{\text{hom}}$, $B = B_{\text{hom}}$ and that the difference between $(P_{\text{hom}}, \lambda_{\text{hom}}, \ln \alpha_{\text{hom}})$ and $(P, \lambda, \ln \alpha)$ converges to zero (in the supremum norm) as $t \to \infty$. Assume, moreover, that if $B = 0$, then $Q = Q_{\text{hom}}$. Let*

$$\hat{\mathcal{H}} = \int_{\mathbb{S}^1} \alpha^{-1/2} [\hat{P}_t^2 + \alpha \hat{P}_\theta^2 + e^{2P} (\hat{Q}_t^2 + \alpha \hat{Q}_\theta^2)] d\theta + \frac{1}{t^{2+\rho}} \int_{\mathbb{S}^1} \left(\hat{\lambda}^2 + \hat{P}^2 \right) d\theta, \tag{48}$$

where

$$\hat{P} = P - P_{\text{hom}}, \quad \hat{Q} = Q - Q_{\text{hom}}, \quad \hat{\lambda} = \lambda - \lambda_{\text{hom}}$$

and $\rho > 0$ is a constant satisfying $\rho < 2\gamma$ in case $B = 0$ and $\rho < 2\min\{\gamma, r_\infty\}$ in case $B \neq 0$; here r_∞ is given by (36), where the quantities appearing on the right hand side are determined by the pseudo-homogeneous solution, and γ is given by (39). Then there is an $r > 0$ and a constant $C > 0$ such that

$$(t/t_a)^r \hat{\mathcal{H}}(t) \geq C \hat{\mathcal{H}}(t_a) \tag{49}$$

for $t \geq t_a \geq t_1$. If $B \neq 0$, then r can be chosen to equal $2(1 + r_\infty)$, and if $B = 0$, then r can be chosen to equal $2 + \rho$.

Remark 17. The assumption that $Q = Q_{\text{hom}}$ in case $B = 0$ may seem unnatural. However, as we shall see below, it follows from the other assumptions that if $B = 0$, then Q has to be constant; cf. Lemma 57. In the end, this assumption is thus unnecessary.

Proof. Let $X = (P, \lambda)$, $x = (P_{\text{hom}}, \lambda_{\text{hom}})$, $\hat{\alpha} = \alpha/\alpha_{\text{hom}}$ and $\hat{X} = X - x$. In what follows, it will be useful to introduce the notation

$$\hat{x} = (P_{\text{hom}} - r_\infty \ln t - c_P, \lambda_{\text{hom}} - r_\infty^2 \ln t - c_\lambda),$$

where r_∞, c_P and c_λ are the constants describing the asymptotics of P_{hom} and λ_{hom}; cf. the statement of Lemma 11. Then

$$\begin{aligned}\frac{\alpha_t}{\alpha} - \frac{\partial_t \alpha_{\text{hom}}}{\alpha_{\text{hom}}} =& -\frac{e^{P+\lambda/2}K^2}{t^{5/2}} + \frac{e^{P_{\text{hom}}+\lambda_{\text{hom}}/2}K^2}{t^{5/2}} = \frac{e^{P_{\text{hom}}+\lambda_{\text{hom}}/2}K^2}{t^{5/2}}\left[1 - \exp(\hat{P} + \hat{\lambda}/2)\right] \\ =& t^{-1-\gamma} f_{\alpha,1}(\hat{x}, \hat{X})(\hat{P} + \hat{\lambda}/2),\end{aligned} \tag{50}$$

where $f_{\alpha,1}$ is a smooth function (depending on c_P, c_λ and K) and γ is the constant defined in (39). Moreover,

$$\begin{aligned}\hat{P}_{tt} + \frac{1}{t}\hat{P}_t - \alpha\hat{P}_{\theta\theta} =& \frac{\alpha_\theta}{2}\hat{P}_\theta + \frac{\alpha_t}{2\alpha}P_t - \frac{\partial_t \alpha_{\text{hom}}}{2\alpha_{\text{hom}}}\partial_t P_{\text{hom}} + e^{2P}(Q_t^2 - \alpha Q_\theta^2) \\ & - e^{2P_{\text{hom}}}[(\partial_t Q_{\text{hom}})^2 - \alpha_{\text{hom}}(\partial_\theta Q_{\text{hom}})^2] - \frac{e^{P+\lambda/2}K^2}{2t^{7/2}} + \frac{e^{P_{\text{hom}}+\lambda_{\text{hom}}/2}K^2}{2t^{7/2}}.\end{aligned}$$

Let us consider the differences appearing on the right hand side, beginning with

$$\frac{\alpha_t}{2\alpha}P_t - \frac{\partial_t \alpha_{\text{hom}}}{2\alpha_{\text{hom}}}\partial_t P_{\text{hom}} = \frac{\alpha_t}{2\alpha}\hat{P}_t + \frac{1}{2}t^{-2-\gamma}(t\partial_t P_{\text{hom}})f_{\alpha,1}(\hat{x}, \hat{X})(\hat{P} + \hat{\lambda}/2),$$

where we have used (50). We also have

$$-\frac{e^{P+\lambda/2}K^2}{2t^{7/2}} + \frac{e^{P_{\text{hom}}+\lambda_{\text{hom}}/2}K^2}{2t^{7/2}} = \frac{1}{2}t^{-2-\gamma}f_{\alpha,1}(\hat{x}, \hat{X})(\hat{P} + \hat{\lambda}/2)$$

due to (50). Finally, consider

$$\begin{aligned}& e^{2P}(Q_t^2 - \alpha Q_\theta^2) - e^{2P_{\text{hom}}}[(\partial_t Q_{\text{hom}})^2 - \alpha_{\text{hom}}(\partial_\theta Q_{\text{hom}})^2] \\ =& e^{2P}(Q_t + \partial_t Q_{\text{hom}})\hat{Q}_t + e^{2P_{\text{hom}}}(\partial_t Q_{\text{hom}})^2(e^{2\hat{P}} - 1) - \alpha e^{2P}\hat{Q}_\theta^2 \\ =& (e^P\hat{Q}_t + 2e^{\hat{P}}e^{P_{\text{hom}}}\partial_t Q_{\text{hom}})e^P\hat{Q}_t + e^{2P_{\text{hom}}}(\partial_t Q_{\text{hom}})^2\frac{e^{2\hat{P}} - 1}{\hat{P}}\hat{P} - \alpha e^{2P}\hat{Q}_\theta^2 \\ =& e^{2P}(\hat{Q}_t^2 - \alpha\hat{Q}_\theta^2) + t^{-1-r_\infty}g_{P,Q,1}(t^{1+r_\infty}e^{P_{\text{hom}}}\partial_t Q_{\text{hom}}, \hat{P})e^P\hat{Q}_t \\ & + t^{-2-2r_\infty}g_{P,P,1}(t^{1+r_\infty}e^{P_{\text{hom}}}\partial_t Q_{\text{hom}}, \hat{P})\hat{P},\end{aligned}$$

where $g_{P,Q,1}$ and $g_{P,P,1}$ are smooth functions which equal zero if $B = 0$ (in particular, if $g_{P,Q,1} \neq 0$, then $r_\infty > 0$, etc.). Let

$$z = (t\partial_t P_{\text{hom}}, t^{1+r_\infty}e^{P_{\text{hom}}}\partial_t Q_{\text{hom}}).$$

Note that z is bounded for $t \geq t_1$; $t\partial_t P_{\text{hom}}$ and $t^{1+r_\infty}e^{P_{\text{hom}}}\partial_t Q_{\text{hom}}$ are bounded due to (35) and (33) respectively, which hold in the pseudo-homogeneous case. For the purposes of the proof of the present lemma, let us call a function *admissible* if it can be written in the form $f(z, \hat{x}, \hat{X})$ for some smooth function f on $\mathbb{R}^6$. We shall also say that a function g of this form is Q-admissible if it vanishes whenever the second argument of z vanishes (i.e., whenever $B = 0$). Due to the above observations,

$$\hat{P}_{tt} + \frac{1}{t}\hat{P}_t - \alpha\hat{P}_{\theta\theta} = \frac{\alpha_\theta}{2}\hat{P}_\theta + \frac{\alpha_t}{2\alpha}\hat{P}_t + e^{2P}(\hat{Q}_t^2 - \alpha\hat{Q}_\theta^2) + f_P, \tag{51}$$

where

$$f_P = t^{-1-r_\infty} g_{P,Q_t} e^P \hat{Q}_t + t^{-2-\gamma} f_{P,P} \hat{P} + t^{-2-\gamma} f_{P,\lambda} \hat{\lambda} + t^{-2-2r_\infty} g_{P,P} \hat{P} \tag{52}$$

and $f_{P,P}$, $f_{P,\lambda}$ are admissible functions and g_{P,Q_t}, $g_{P,P}$ are Q-admissible functions. Concerning Q, we have the equation

$$\hat{Q}_{tt} + \frac{1}{t}\hat{Q}_t - \alpha \hat{Q}_{\theta\theta} = \frac{\alpha_\theta}{2}\hat{Q}_\theta + \frac{\alpha_t}{2\alpha} Q_t - \frac{\partial_t \alpha_{\text{hom}}}{2\alpha_{\text{hom}}} \partial_t Q_{\text{hom}} - 2P_t Q_t + 2\partial_t P_{\text{hom}} \partial_t Q_{\text{hom}} + 2\alpha \hat{P}_\theta \hat{Q}_\theta.$$

As in the case of P,

$$\frac{\alpha_t}{2\alpha} Q_t - \frac{\partial_t \alpha_{\text{hom}}}{2\alpha_{\text{hom}}} \partial_t Q_{\text{hom}} = \frac{\alpha_t}{2\alpha}\hat{Q}_t + \frac{1}{2} t^{-1-\gamma} e^{-P} e^{P_{\text{hom}}} \partial_t Q_{\text{hom}} e^{\hat{P}} f_{\alpha,1}(\hat{x},\hat{X})(\hat{P} + \hat{\lambda}/2).$$

Note also that

$$-2P_t Q_t + 2\partial_t P_{\text{hom}} \partial_t Q_{\text{hom}} = -2P_t \hat{Q}_t - 2\hat{P}_t e^{-P} e^{P_{\text{hom}}} \partial_t Q_{\text{hom}} e^{\hat{P}}.$$

Adding up, we obtain

$$\hat{Q}_{tt} + \frac{1}{t}\hat{Q}_t - \alpha \hat{Q}_{\theta\theta} = \frac{\alpha_\theta}{2}\hat{Q}_\theta + \frac{\alpha_t}{2\alpha}\hat{Q}_t - 2(P_t \hat{Q}_t - \alpha \hat{P}_\theta \hat{Q}_\theta) + e^{-P} f_Q, \tag{53}$$

where

$$f_Q = t^{-1-r_\infty} g_{Q,P_t} \hat{P}_t + t^{-2-\gamma-r_\infty} g_{Q,P} \hat{P} + t^{-2-\gamma-r_\infty} g_{Q,\lambda} \hat{\lambda} \tag{54}$$

and g_{Q,P_t}, $g_{Q,P}$ and $g_{Q,\lambda}$ are Q-admissible functions. Finally, consider λ. We have

$$\begin{aligned}
\hat{\lambda}_t =& t(P_t^2 + e^{2P} Q_t^2) - t[(\partial_t P_{\text{hom}})^2 + e^{2P_{\text{hom}}}(\partial_t Q_{\text{hom}})^2] + t\alpha(\hat{P}_\theta^2 + e^{2P}\hat{Q}_\theta^2) + \frac{\alpha_t}{\alpha} - \frac{\partial_t \alpha_{\text{hom}}}{\alpha_{\text{hom}}} \\
=& 2(t\partial_t P_{\text{hom}})\hat{P}_t + 2te^{P_{\text{hom}}} \partial_t Q_{\text{hom}} e^{\hat{P}} e^P \hat{Q}_t + t[\hat{P}_t^2 + e^{2P}\hat{Q}_t^2 + \alpha(\hat{P}_\theta^2 + e^{2P}\hat{Q}_\theta^2)] \\
&+ (e^{2\hat{P}} - 1) t e^{2P_{\text{hom}}} (\partial_t Q_{\text{hom}})^2 + \frac{\alpha_t}{\alpha} - \frac{\partial_t \alpha_{\text{hom}}}{\alpha_{\text{hom}}}.
\end{aligned}$$

Consequently,

$$\hat{\lambda}_t = t[\hat{P}_t^2 + e^{2P}\hat{Q}_t^2 + \alpha(\hat{P}_\theta^2 + e^{2P}\hat{Q}_\theta^2)] + f_\lambda, \tag{55}$$

where

$$f_\lambda = f_{\lambda,P_t}\hat{P}_t + t^{-r_\infty} g_{\lambda,Q_t} e^P \hat{Q}_t + t^{-1-2r_\infty} g_{\lambda,P}\hat{P} + t^{-1-\gamma} f_{\lambda,P}\hat{P} + t^{-1-\gamma} f_{\lambda,\lambda}\hat{\lambda}. \tag{56}$$

In this equality, f_{λ,P_t}, $f_{\lambda,P}$ and $f_{\lambda,\lambda}$ are admissible functions and g_{λ,Q_t}, $g_{\lambda,P}$ are Q-admissible functions. Due to the above equalities,

$$\begin{aligned}
\partial_t(t\alpha^{-1/2}\hat{P}_t) &= \partial_\theta(t\alpha^{1/2} P_\theta) + t\alpha^{-1/2} e^{2P}(\hat{Q}_t^2 - \alpha\hat{Q}_\theta^2) + t\alpha^{-1/2} f_P, \\
\partial_t(t\alpha^{-1/2} e^{2P}\hat{Q}_t) &= \partial_\theta(t\alpha^{1/2} e^{2P}\hat{Q}_\theta) + t\alpha^{-1/2} e^P f_Q.
\end{aligned}$$

As a consequence, it can be calculated that if

$$\hat{H}_1 = \int_{\mathbb{S}^1} \alpha^{-1/2}[\hat{P}_t^2 + \alpha\hat{P}_\theta^2 + e^{2P}(\hat{Q}_t^2 + \alpha\hat{Q}_\theta^2)]d\theta,$$

then

$$\begin{aligned}
\frac{d\hat{H}_1}{dt} =& -\frac{2}{t}\int_{\mathbb{S}^1} \alpha^{-1/2}(\hat{P}_t^2 + e^{2P}\hat{Q}_t^2)d\theta + \int_{\mathbb{S}^1} \frac{\alpha_t}{2\alpha}\alpha^{-1/2}[\hat{P}_t^2 + \alpha P_\theta^2 + e^{2P}(\hat{Q}_t^2 + \alpha\hat{Q}_\theta^2)]d\theta \\
&- 2(\partial_t P_{\text{hom}})\int_{\mathbb{S}^1} e^{2P}\alpha^{-1/2}(\hat{Q}_t^2 - \alpha\hat{Q}_\theta^2)d\theta + 2\int_{\mathbb{S}^1} \alpha^{-1/2}(\hat{P}_t f_P + e^P\hat{Q}_t f_Q)d\theta.
\end{aligned} \tag{57}$$

Let us now turn to

$$\partial_t \left[\frac{1}{t^{2+\rho}} \int_{\mathbb{S}^1} \left(\hat{\lambda}^2 + \hat{P}^2 \right) d\theta \right] = - \frac{2+\rho}{t} \frac{1}{t^{2+\rho}} \int_{\mathbb{S}^1} \left(\hat{\lambda}^2 + \hat{P}^2 \right) d\theta + \frac{2}{t^{1+\rho}} \int_{\mathbb{S}^1} \hat{\lambda}[\hat{P}_t^2 + e^{2P} \hat{Q}_t^2 + \alpha(\hat{P}_\theta^2 + e^{2P} \hat{Q}_\theta^2)] d\theta + \frac{2}{t^{2+\rho}} \int_{\mathbb{S}^1} \left(\hat{\lambda} f_\lambda + \hat{P} \hat{P}_t \right) d\theta. \quad (58)$$

We wish to estimate the right hand sides of (57) and (58) from below. Before doing so, note that

$$\hat{\mathcal{H}} = \hat{H}_1 + \frac{1}{t^{2+\rho}} \int_{\mathbb{S}^1} \left(\hat{\lambda}^2 + \hat{P}^2 \right) d\theta.$$

Since $\alpha^{1/2}$ is bounded for $t \geq t_1$ and $\hat{\lambda}$ converges to zero, we conclude that there is a constant $C > 0$ such that the second term on the right hand side of (58) is bounded from below by

$$-\frac{C}{t^{1+\rho}} \hat{\mathcal{H}}$$

for all $t \geq t_1$. Using (56), the third term on the right hand side of (58) can be bounded from below by

$$-\frac{C}{t^{1+\rho/2}} \hat{\mathcal{H}} - \frac{D}{t^{1+\rho/2+r_\infty}} \hat{\mathcal{H}} - \frac{D}{t^{1+2r_\infty}} \hat{\mathcal{H}} - \frac{C}{t^{1+\gamma}} \hat{\mathcal{H}},$$

where $D \geq 0$ is a constant which equals zero in case $B = 0$. Regardless of whether $B = 0$ or not, there are thus constants $C, \eta > 0$ such that

$$\partial_t \left[\frac{1}{t^{2+\rho}} \int_{\mathbb{S}^1} \left(\hat{\lambda}^2 + \hat{P}^2 \right) d\theta \right] \geq - \frac{2+\rho}{t} \frac{1}{t^{2+\rho}} \int_{\mathbb{S}^1} \left(\hat{\lambda}^2 + \hat{P}^2 \right) d\theta - \frac{C}{t^{1+\eta}} \hat{\mathcal{H}}$$

for all $t \geq t_1$. Note that an inequality of this form holds regardless of the value of $\rho > 0$. Turning to (57), note that the second term on the right hand side can be estimated from below by $-Ct^{-1-\gamma}\hat{\mathcal{H}}$. In case $B = 0$, the third term on the right hand side is zero, and the sum of the first and third terms can be estimated from below by the first term. If $B \neq 0$, then $t\partial_t P_{\text{hom}} - r_\infty = O(t^{-\eta})$ for some $\eta > 0$ and $r_\infty \in (0, 1)$; cf. (40). The sum of the first and the third terms can thus be estimated from below by

$$-\frac{2}{t} \int_{\mathbb{S}^1} \alpha^{-1/2} \hat{P}_t^2 d\theta - \frac{2(1 + r_\infty)}{t} \int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} \hat{Q}_t^2 d\theta - Ct^{-1-\eta} \hat{\mathcal{H}}$$

for all $t \geq t_1$. Finally, the last term on the right hand side of (57) can be estimated from below by

$$-Dt^{-1-r_\infty+\rho/2} \hat{\mathcal{H}} - Ct^{-1-\gamma+\rho/2} \hat{\mathcal{H}},$$

for $t \geq t_1$; here $C > 0$ and $D \geq 0$ are constants and $D = 0$ if $B = 0$. Fixing $\rho > 0$ to be a constant such that $\rho < 2 \min\{r_\infty, \gamma\}$ in case $B \neq 0$ and such that $\rho < 2\gamma$ in case $B = 0$, we conclude that

$$\frac{d\hat{\mathcal{H}}}{dt} \geq -\frac{2}{t} \int_{\mathbb{S}^1} \alpha^{-1/2} \hat{P}_t^2 d\theta - \frac{2(1 + r_\infty)}{t} \int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} \hat{Q}_t^2 d\theta - \frac{2+\rho}{t} \frac{1}{t^{2+\rho}} \int_{\mathbb{S}^1} \left(\hat{\lambda}^2 + \hat{P}^2 \right) d\theta - \frac{C}{t^{1+\eta}} \hat{\mathcal{H}},$$

for some constants $C, \eta > 0$ and all $t \geq t_1$. Note also that, by assumption, the second term vanishes unless $r_\infty \in (0, 1)$. The lemma follows. □

5 General estimates

In the next few sections, we change perspective and start with the assumption that we have a solution such that α has a positive lower bound. Our goal is to prove that given such a solution, there is a uniquely associated pseudo-homogeneous solution such that the difference converges to zero. Moreover, we wish to prove that the decay rate is such that combining the relevant estimate with Lemma 16 yields the conclusion that the difference between the solution we started with and the pseudo-homogeneous solution is zero. The end result of such an argument is the conclusion that solutions such that α has a positive lower bound are pseudo-homogeneous. However, before we derive conclusions based on the assumption of a positive lower bound for α, let us record some estimates which hold in general.

Lemma 18. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbb{S}^1} \int_{t_1}^{\tau} t \left[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) \right] dt d\theta \leq C \ln \tau \tag{59}$$

for all $\tau \geq t_1$.

Remark 19. The assumption that $K \neq 0$ is necessary here; $\mathbb{T}^3$-Gowdy symmetric solutions satisfying (59) are spatially homogeneous.

Proof. Let

$$f = \alpha^{-1/2} e^{P+\lambda/2}, \quad g = P + \frac{\lambda}{2} - \frac{1}{2} \ln \alpha.$$

Then, due to Jensen's inequality, [26, Theorem 3.3, p. 62], $\langle f \rangle = \langle e^g \rangle \geq e^{\langle g \rangle}$. Furthermore

$$g_t = \frac{1}{2} t \left[\left(P_t + \frac{1}{t} \right)^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) \right] - \frac{1}{2t}.$$

Letting

$$h(t) = \int_{t_a}^{t} \langle g_t \rangle ds,$$

we have $\langle f \rangle(t) \geq \exp[\langle g \rangle(t_a)] \exp[h(t)]$. However, due to the fact that $t^{-2}\hat{H}$ is decreasing, we know that $\langle f \rangle(t) \leq Ct^{7/2}$. As a consequence, $h(t) \leq 7 \ln t/2 + C$ for all $t \geq t_1$. In particular, there is a constant $C > 0$ such that (59) holds for all $\tau \geq t_1$. □

Note that one particular consequence of this estimate is that $\langle P \rangle$ cannot grow faster than logarithmically to the future.

Corollary 20. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Then there is a constant $C > 0$ such that*

$$|\langle P(t, \cdot) \rangle| + \int_{t_1}^{t} \int_{\mathbb{S}^1} |P_t(s, \theta)| d\theta ds \leq C \ln t$$

for all $t \geq t_1$.

Proof. Estimate

$$|\langle P(t, \cdot) \rangle| \leq |\langle P(t_1, \cdot) \rangle| + \frac{1}{2\pi} \int_{t_1}^{t} \int_{\mathbb{S}^1} |P_t(s, \theta)| d\theta ds.$$

On the other hand,

$$\frac{1}{2\pi} \int_{t_1}^{t} \int_{\mathbb{S}^1} |P_t(s, \theta)| d\theta ds \leq \frac{1}{2\pi} \left(\int_{t_1}^{t} 2\pi s^{-1} ds \right)^{1/2} \left(\int_{t_1}^{t} \int_{\mathbb{S}^1} s P_t^2(s, \theta) d\theta ds \right)^{1/2} \leq C \ln t,$$

where we have appealed to Lemma 18. □

In Section 12, it will be of interest to know that $t^{-\rho}\langle P_t\rangle \in L^1([t_1,\infty))$ for constants $\rho > 0$. As a first step in the proof of this fact, let us demonstrate the following.

Lemma 21. *Let $T \geq 1$ and let $f : [T,\infty) \to \mathbb{R}$ be measurable. Assume, moreover, that there is a real constant $C > 0$ such that*

$$\int_T^\tau tf^2(t)dt \leq C \ln \tau$$

for all $\tau \geq T$. Then $t^{1-\rho}f^2 \in L^1([T,\infty))$ for every constant $\rho > 0$. In particular, $t^{-\rho}f \in L^1([T,\infty))$ for every constant $\rho > 0$.

Proof. Let $0 < \rho \in \mathbb{R}$, $1 \leq N \in \mathbb{Z}$, and let us estimate

$$\sum_{n=0}^{N-1} \int_{2^nT}^{2^{n+1}T} t^{1-\rho}f^2(t)dt \leq \sum_{n=0}^{N-1} (2^nT)^{-\rho} \int_{2^nT}^{2^{n+1}T} tf^2(t)dt \leq \sum_{n=0}^{N-1} (2^nT)^{-\rho} C \ln(2^{n+1}T).$$

Since the limit of the right hand side (as $N \to \infty$) is finite, Lebesgue's monotone convergence theorem yields the first conclusion of the lemma. The second follows by appealing to Hölder's inequality. □

Corollary 22. *Consider a solution to (2)–(6) on $(t_0,\infty) \times \mathbb{S}^1$ with $K \neq 0$. Then, for every $0 < \rho \in \mathbb{R}$, $t^{-\rho}\langle P_t\rangle \in L^1([t_1,\infty))$.*

Proof. Let $f = \langle P_t\rangle$. Then

$$\int_{t_1}^\tau tf^2(t)dt \leq \frac{1}{2\pi} \int_{t_1}^\tau t \int_{\mathbb{S}^1} P_t^2 d\theta dt.$$

Due to Lemma 18, it is thus clear that the assumptions of Lemma 21 are fulfilled. The corollary follows. □

6 Pointwise considerations

In the end, we shall assume α to be bounded from below by a positive constant. However, it is possible to deduce some of the desired conclusions assuming only that α does not converge to zero uniformly. We shall therefore start by making such assumptions. Moreover, in the proof of the fact that $\langle \alpha^{-1/2}\rangle$ converges to infinity in the non-pseudo-homogeneous setting, we need to be able to derive uniform bounds given non-uniform ones. This is the main purpose of the present section.

Lemma 23. *Consider a solution to (2)–(6) on $(t_0,\infty) \times \mathbb{S}^1$. Assume that α does not converge uniformly to zero as $t \to \infty$. Then there is a $\theta_0 \in \mathbb{S}^1$ and an $\epsilon > 0$ such that $\alpha(t,\theta_0) \geq \epsilon$ for all $t \in (t_0,\infty)$.*

Proof. Since α does not converge to zero uniformly, there is an $\epsilon > 0$ and, for every $T > t_0$, a (t,θ) such that $t \geq T$ and $\alpha(t,\theta) \geq \epsilon$. As a consequence, there is a sequence (t_k,θ_k) with $t_k \to \infty$ such that $\alpha(t_k,\theta_k) \geq \epsilon$. We can assume that θ_k converges to, say, $\theta_* \in \mathbb{S}^1$. Let $t \in (t_0,\infty)$. Then

$$\alpha(t,\theta_*) = \lim_{k\to\infty} \alpha(t,\theta_k) \geq \liminf_{k\to\infty} \alpha(t_k,\theta_k) \geq \epsilon,$$

where we have used the fact that α is monotonically decaying and the fact that $t_k \geq t$ for k large enough. The lemma follows. □

Let us derive some preliminary bounds on P, assuming α to have a pointwise positive lower bound.

Lemma 24. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that there is a $\theta_0\in\mathbb{S}^1$ and an $\epsilon>0$ such that $\alpha(t,\theta_0)\geq\epsilon$ for all $t\in(t_0,\infty)$. For every $C>0$, there is then a T_C such that*

$$-3\ln t + C \leq P(t,\theta_0)\leq \ln t - C \tag{60}$$

for all $t\geq T_C$. Moreover,

$$\lim_{t\to\infty}\left(\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{3/2}}\right)(t,\theta_0)=0. \tag{61}$$

Remark 25. The conclusions should be compared with the statement of Lemma 11.

Proof. Due to the assumptions and (4),

$$\int_{t_1}^{\infty}\frac{e^{P+\lambda/2}K^2}{t^{5/2}}dt<\infty,$$

where we take it to be understood that $\theta=\theta_0$. Since α has a positive lower bound, we also know that

$$\int_{t_1}^{\infty}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{5/2}}dt<\infty.$$

Let

$$f(t)=\left(\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{3/2}}\right)(t,\theta_0).$$

Then

$$\int_{t_1}^{\infty}t^{-1}f(t)dt<\infty \tag{62}$$

and f is smooth on (t_0,∞). Note, moreover, that

$$\begin{aligned}\partial_t(\alpha^{-1/2}e^{P+\lambda/2}) &= \left(\frac{1}{2}t[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)]+P_t\right)\alpha^{-1/2}e^{P+\lambda/2}\\ &=\left(\frac{1}{2}t[(P_t+t^{-1})^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)]-\frac{1}{2t}\right)\alpha^{-1/2}e^{P+\lambda/2}\\ &\geq -\frac{1}{2t}\alpha^{-1/2}e^{P+\lambda/2}.\end{aligned}$$

Thus

$$f'\geq -\frac{3}{2t}f-\frac{1}{2t}f=-\frac{2}{t}f. \tag{63}$$

Combining (62), (63) and Lemma 10, we conclude that (61) holds. In order to derive conclusions concerning P from this observation, let us note that

$$(\alpha^{-1/2}e^{P+\lambda/2})(t,\theta_0)=(\alpha^{-1/2}e^{\lambda/2})(t_1,\theta_0)\exp\left(\int_{t_1}^{t}\frac{1}{2}s[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)]ds+P\right).$$

Combining this observation with (61), we obtain

$$\lim_{t\to\infty}\left(\int_{t_1}^{t}\frac{1}{2}s[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)]ds+P-\frac{3}{2}\ln t\right)=-\infty, \tag{64}$$

where $\theta=\theta_0$. Let $C>0$ and assume that there is a sequence $t_k\to\infty$ such that $P(t_k,\theta_0)\geq a\ln t_k - C$ for some constant $a>0$. Then

$$a\ln t_k - C - P(t_1,\theta_0)\leq\int_{t_1}^{t_k}P_t(s,\theta_0)ds\leq\left(\int_{t_1}^{t_k}\frac{1}{s}ds\right)^{1/2}\left(\int_{t_1}^{t_k}sP_t^2(s,\theta_0)ds\right)^{1/2}.$$

There is consequently a constant c_1 (depending on C) such that

$$a^2 \ln t_k - c_1 \leq \int_{t_1}^{t_k} s P_t^2(s, \theta_0) ds$$

for k large enough. Thus

$$a \ln t_k + \frac{1}{2} a^2 \ln t_k - \frac{1}{2} c_1 - C \leq \int_{t_1}^{t_k} \frac{1}{2} s P_t^2(s, \theta_0) ds + P(t_k, \theta_0).$$

If $a \geq 1$, this estimate contradicts (64). As a consequence, there is a $T_C > t_0$ such that

$$P(t, \theta_0) \leq \ln t - C \tag{65}$$

for $t \geq T_C$. In other words, we obtain the upper bound in (60).

Fix $C > 0$. Let us assume that for every $T_C > t_0$, there is a $t \geq T_C$ such that $P(t, \theta_0) \leq -3 \ln t + C$. Then there is a sequence $t_k \to \infty$ and a sequence $a_k \leq -3$ such that $P(t_k, \theta_0) = a_k \ln t_k + C$. As a consequence,

$$-a_k \ln t_k - C + P(t_1, \theta_0) = -\int_{t_1}^{t_k} P_t(s, \theta_0) ds \leq \ln^{1/2} t_k \left(\int_{t_1}^{t_k} s P_t^2(s, \theta_0) ds \right)^{1/2}.$$

Thus

$$a_k^2 \ln t_k - c_1 \leq \int_{t_1}^{t_k} s P_t^2(s, \theta_0) ds$$

for k large enough, so that

$$\frac{1}{2} a_k^2 \ln t_k + a_k \ln t_k + C - \frac{1}{2} c_1 \leq \frac{1}{2} \int_{t_1}^{t_k} s P_t^2(s, \theta_0) ds + P(t_k, \theta_0).$$

Due to (64), we obtain a contradiction. Consequently, the lower bound in (60) holds. □

It is of interest to derive uniform bounds on P of the above type. However, in order to be able to do so, we unfortunately need a uniform positive lower bound on α. It would be preferable to avoid making such an assumption. Let us therefore briefly discuss some ways in which this assumption might be avoided.

Lemma 26. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that α does not converge to zero uniformly, and that $\hat{H} \leq Ct$ for some constant $C > 0$ and all $t \geq t_1$. Then there is an $\alpha_0 > 0$ such that $\alpha(t, \theta) \geq \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$.*

Proof. Due to Lemma 23, we know that there is a $\theta_0 \in \mathbb{S}^1$ and an $\epsilon > 0$ such that $\alpha(t, \theta_0) \geq \epsilon$. In particular, we know that

$$\int_{t_1}^{\infty} \frac{e^{P + \lambda/2} K^2}{t^{5/2}} dt < \infty$$

for $\theta = \theta_0$. It is of interest to consider the spatial variation of P and λ. Note that

$$\|P - \langle P \rangle\|_{C^0} \leq \frac{1}{2} \int_{\mathbb{S}^1} |P_\theta| d\theta \leq \frac{1}{2} \left(\int_{\mathbb{S}^1} \alpha^{1/2} P_\theta^2 d\theta \right)^{1/2} \left(\int_{\mathbb{S}^1} \alpha^{-1/2} d\theta \right)^{1/2} \leq \frac{1}{4\sqrt{3}} t^{-1} \hat{H} \leq C,$$

due to our assumptions. Moreover,

$$\|\lambda - \langle \lambda \rangle\|_{C^0} \leq \frac{1}{2} \int_{\mathbb{S}^1} |\lambda_\theta| d\theta \leq \frac{1}{2} t^{-1} \hat{H} \leq C.$$

As a consequence of the above observations, we have the following conclusion:

$$\int_{t_1}^{\infty} \frac{e^{P(t,\theta)+\lambda(t,\theta)/2}K^2}{t^{5/2}} dt \leq \int_{t_1}^{\infty} \frac{e^{P(t,\theta_0)+\lambda(t,\theta_0)/2+C}K^2}{t^{5/2}} dt \leq e^C \int_{t_1}^{\infty} \frac{e^{P(t,\theta_0)+\lambda(t,\theta_0)/2}K^2}{t^{5/2}} dt.$$

Since the integral on the right hand side has a uniform upper bound (independent of θ), we obtain a positive uniform lower bound on α. □

In particular, we have the following consequence of the above observation.

Corollary 27. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that α does not converge to zero uniformly, and that there is a constant $C>0$ such that*

$$\alpha^{-1/2}(t,\theta) \leq C\frac{t}{\ln t} \tag{66}$$

for all $(t,\theta)\in[t_1,\infty)\times\mathbb{S}^1$. Then there is an $\alpha_0>0$ such that $\alpha(t,\theta)\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$.

Proof. As a consequence of (59),

$$\begin{aligned}
&\int_{\mathbb{S}^1}\int_{t_1}^{\tau} t\alpha^{-1/2}\left[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)\right] dtd\theta\\
\leq&\int_{\mathbb{S}^1}\int_{t_1}^{\tau} Ct^2(\ln t)^{-1}\left[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)\right] dtd\theta\\
\leq& C\tau(\ln\tau)^{-1}\int_{\mathbb{S}^1}\int_{t_1}^{\tau} t\left[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)\right] dtd\theta \leq C\tau,
\end{aligned}$$

where we have used (66). In particular,

$$\int_{\mathbb{S}^1}\int_{t_1}^{\tau} t\alpha^{1/2}(P_\theta^2+e^{2P}Q_\theta^2)dtd\theta \leq C\tau.$$

Combining this estimate with (24) yields

$$\hat{H}(\tau)\leq\hat{H}(t_1)+2\int_{\mathbb{S}^1}\int_{t_1}^{\tau} t\alpha^{1/2}(P_\theta^2+e^{2P}Q_\theta^2)dtd\theta \leq C\tau.$$

We thus obtain the desired conclusion by appealing to Lemma 26. □

7 Positive lower bound on α

Let us now derive uniform bounds on P, given a positive lower bound on α.

Lemma 28. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then, for every $C>0$, there is a T_C such that*

$$-3\ln t + C \leq P(t,\theta) \leq \ln t - C \tag{67}$$

for all $(t,\theta)\in[T_C,\infty)\times\mathbb{S}^1$.

Remark 29. Here and below, α_0 is a constant.

Proof. Due to the lower bound on α and the boundedness of the energy $H=\hat{H}/t^2$, cf. (24), there is a constant C_0 such that

$$\|P-\langle P\rangle\|_{C^0}\leq C_0$$

for all $t\geq t_1$. Combining this observation with (60), we obtain the desired conclusion. □

Let us now consider the energies.

Lemma 30. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then there is a constant $0\leq c_E\in\mathbb{R}$ such that*

$$\lim_{t\to\infty}\int_{\mathbb{S}^1} t^2\alpha^{-1/2}\left[P_t^2+e^{2P}Q_t^2+\alpha(P_\theta^2+e^{2P}Q_\theta^2)\right]d\theta=c_E. \tag{68}$$

Moreover,

$$\int_{t_1}^{\infty} t\int_{\mathbb{S}^1}\alpha^{1/2}(P_\theta^2+e^{2P}Q_\theta^2)d\theta dt<\infty. \tag{69}$$

Proof. Recall (24). As a consequence of this equality, $\hat{H}$ is increasing. It is of interest to prove that there is a $C\in\mathbb{R}$ such that

$$\int_{t_1}^{t} s^{-1}\hat{H}(s)ds\leq C\ln t \tag{70}$$

for all $t\geq t_1$. Note, to this end, that $t^{-1}\hat{H}(t)$ is given by

$$\int_{\mathbb{S}^1}\left[t\alpha^{-1/2}[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)]+\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{5/2}}+3t^{-1}\alpha^{-1/2}\right]d\theta.$$

Due to Lemma 18 and the positive lower bound on α, we conclude that the integral in time of the quantity arising from the first term in the integrand is bounded by $C\ln t$. Moreover, due to the lower bound on α, the integral of the quantity arising from the last two terms in the integrand is bounded by $C\ln t$ for some constant C. In fact, due to the lower bound on α and (4),

$$\int_{T}^{\infty}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{5/2}}d\theta dt<\infty \tag{71}$$

for each $T>t_0$. To conclude, (70) holds for some C. On the other hand, $\hat{H}$ is increasing. If $\hat{H}(t)>C$ for some t, we thus obtain a contradiction. Since $\hat{H}$ is increasing and bounded from above, there is an $\hat{H}_\infty$ such that $\hat{H}(t)\to\hat{H}_\infty$. However,

$$\int_{\mathbb{S}^1}3\alpha^{-1/2}d\theta$$

is increasing and converges to a limit. Moreover,

$$\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{3/2}}d\theta$$

converges to zero. In order to prove this statement, note that if we let

$$f(t)=\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{3/2}}d\theta,$$

then

$$\int_{t_1}^{\infty}\frac{1}{t}f(t)dt<\infty.$$

Moreover, by an argument similar to the derivation of (63), we have $f'\geq -2f/t$. We are thus in a position to apply Lemma 10 in order to obtain the desired conclusion. Combining the above observations, we conclude that there is a constant $c_E\geq 0$ such that (68) holds. Moreover, the boundedness of $\hat{H}$ and (24) imply that (69) holds. □

It is of interest to derive some basic conclusions concerning the spatial variation of P, Q and λ.

Lemma 31. *Consider a solution to (2)–(6) on* $(t_0,\infty)\times\mathbb{S}^1$ *with* $K\neq 0$. *Assume that there is an* $\alpha_0>0$ *such that* $\alpha\geq\alpha_0$ *for all* $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. *Then there is a constant* $C>0$ *such that*

$$\|P-\langle P\rangle\|_{C^0} \leq C\left(\int_{\mathbb{S}^1}\alpha P_\theta^2 d\theta\right)^{1/2}, \tag{72}$$

$$\|e^P(Q-\langle Q\rangle)\|_{C^0} \leq C\left(\int_{\mathbb{S}^1}\alpha e^{2P} Q_\theta^2 d\theta\right)^{1/2}, \tag{73}$$

$$\|\lambda-\langle\lambda\rangle\|_{C^0} \leq C\left(\int_{\mathbb{S}^1}\alpha (P_\theta^2+e^{2P}Q_\theta^2) d\theta\right)^{1/2} \tag{74}$$

for all $t\geq t_1$.

Proof. That (72) holds is obvious. As a consequence, $\|P-\langle P\rangle\|_{C^0}\leq Ct^{-1}$. Thus

$$\begin{aligned}\|e^P(Q-\langle Q\rangle)\|_{C^0} \leq& Ce^{\langle P\rangle}\|Q-\langle Q\rangle\|_{C^0}\leq Ce^{\langle P\rangle}\int_{\mathbb{S}^1}|Q_\theta|d\theta\\ \leq& Ce^{\langle P\rangle}\left(\int_{\mathbb{S}^1}\alpha Q_\theta^2 d\theta\right)^{1/2}\leq C\left(\int_{\mathbb{S}^1}\alpha e^{2P}Q_\theta^2 d\theta\right)^{1/2}.\end{aligned}$$

Thus (73) holds. Finally, note that

$$\begin{aligned}\|\lambda-\langle\lambda\rangle\|_{C^0} \leq& \frac{1}{2}\int_{\mathbb{S}^1}|\lambda_\theta|d\theta\leq t\int_{\mathbb{S}^1}|P_tP_\theta+e^{2P}Q_tQ_\theta|d\theta\\ \leq& t\int_{\mathbb{S}^1}\left(\alpha^{-1/2}P_t^2+\alpha^{-1/2}e^{2P}Q_t^2\right)^{1/2}\left(\alpha^{1/2}P_\theta^2+\alpha^{1/2}e^{2P}Q_\theta^2\right)^{1/2}d\theta\\ \leq& t\left(\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t^2+e^{2P}Q_t^2\right)d\theta\right)^{1/2}\left(\int_{\mathbb{S}^1}\alpha^{1/2}\left(P_\theta^2+e^{2P}Q_\theta^2\right)d\theta\right)^{1/2}\\ \leq& C\left(\int_{\mathbb{S}^1}\alpha(P_\theta^2+e^{2P}Q_\theta^2)d\theta\right)^{1/2}.\end{aligned} \tag{75}$$

Thus (74) holds. □

Corollary 32. *Consider a solution to (2)–(6) on* $(t_0,\infty)\times\mathbb{S}^1$ *with* $K\neq 0$. *Assume that there is an* $\alpha_0>0$ *such that* $\alpha\geq\alpha_0$ *for all* $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. *Then there is a constant* $C>0$ *such that*

$$\|P-\langle P\rangle\|_{C^0}+\|e^P(Q-\langle Q\rangle)\|_{C^0}+\|\lambda-\langle\lambda\rangle\|_{C^0}\leq Ct^{-1} \tag{76}$$

for all $t\geq t_1$.

One immediate consequence of the above observations is a sup-norm estimate of α_t/α.

Corollary 33. *Consider a solution to (2)–(6) on* $(t_0,\infty)\times\mathbb{S}^1$ *with* $K\neq 0$. *Assume that there is an* $\alpha_0>0$ *such that* $\alpha\geq\alpha_0$ *for all* $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. *Then*

$$\lim_{t\to\infty}t\left\|\frac{\alpha_t}{\alpha}(t,\cdot)\right\|_{C^0}=0.$$

Proof. Let us estimate, for $t\geq t_1$,

$$t\left\|\frac{\alpha_t}{\alpha}(t,\cdot)\right\|_{C^0}\leq C\frac{e^{\langle P\rangle+\langle\lambda\rangle/2}K^2}{t^{3/2}}\leq C\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{3/2}}d\theta.$$

However, the right hand side converges to zero as $t\to\infty$; cf. the proof of Lemma 30. □

In what follows, we shall use the fact that $\|\alpha_t/\alpha\|_{C^0} = O(t^{-1})$ without further comment. Let us now turn to more detailed conclusions concerning the asymptotics.

Lemma 34. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$. Assume, moreover, that the conserved quantity B defined in (22) is non-zero. Then there is a q_∞ such that*

$$\lim_{t \to \infty} \|Q(t, \cdot) - q_\infty\|_{C^0} = 0.$$

Moreover,

$$\lim_{t \to \infty} \langle P(t, \cdot) \rangle = \infty. \tag{77}$$

Proof. Note that

$$B^2 e^{-2\langle P \rangle} \leq t^2 \int_{\mathbb{S}^1} \alpha^{-1/2} d\theta \int_{\mathbb{S}^1} \alpha^{-1/2} e^{4P - 2\langle P \rangle} Q_t^2 d\theta \leq C.$$

There is thus a constant C such that $P(t, \theta) \geq C$ for all $(t, \theta) \in [t_1, \infty) \times \mathbb{S}^1$.

Consider the conserved quantity A; cf. (23). Note that

$$\int_{\mathbb{S}^1} t\alpha^{-1/2} e^{2P} Q_t Q d\theta = B\langle Q \rangle + \int_{\mathbb{S}^1} t\alpha^{-1/2} e^{2P} Q_t (Q - \langle Q \rangle) d\theta.$$

Due to (76), our bounds on the basic energy and Hölder's inequality, we know that the second term on the right hand side is $O(t^{-1})$. As a consequence,

$$A = \int_{\mathbb{S}^1} t\alpha^{-1/2} (P_t + t^{-1}) d\theta - B\langle Q \rangle + O(t^{-1}). \tag{78}$$

Since we know that the first term on the right hand side is bounded, we conclude that $\langle Q \rangle$ is bounded. Consider

$$\partial_t \left[\frac{1}{e^{\langle P \rangle} \langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} e^{\langle P \rangle} \alpha^{-1/2} (Q - \langle Q \rangle) d\theta \right].$$

Note that when the derivative hits a factor involving $\langle P \rangle$ or $\alpha^{-1/2}$, then the resulting term is $O(t^{-2} e^{-\langle P \rangle})$. Thus

$$\begin{aligned}
&\partial_t \left[\frac{1}{e^{\langle P \rangle} \langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} e^{\langle P \rangle} \alpha^{-1/2} (Q - \langle Q \rangle) d\theta \right] \\
=&\frac{1}{e^{\langle P \rangle} \langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} e^{\langle P \rangle} \alpha^{-1/2} (Q_t - \langle Q_t \rangle) d\theta + O(t^{-2} e^{-\langle P \rangle}) \\
=&\frac{1}{e^{\langle 2P \rangle} \langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} e^{\langle 2P \rangle} \alpha^{-1/2} Q_t d\theta - 2\pi \langle Q_t \rangle + O(t^{-2} e^{-\langle P \rangle}) \\
=&\frac{1}{e^{\langle 2P \rangle} \langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} (e^{\langle 2P \rangle} - e^{2P}) \alpha^{-1/2} Q_t d\theta + \frac{1}{e^{\langle 2P \rangle} \langle \alpha^{-1/2} \rangle} t^{-1} B - 2\pi \langle Q_t \rangle + O(t^{-2} e^{-\langle P \rangle}) \\
=&\frac{1}{e^{\langle 2P \rangle} \langle \alpha^{-1/2} \rangle} t^{-1} B - 2\pi \langle Q_t \rangle + O(t^{-2} e^{-\langle P \rangle}).
\end{aligned} \tag{79}$$

Since $B \neq 0$, we know that $\langle P \rangle$ is bounded from below by a constant, and we can integrate the above equality in order to obtain

$$c_1 + O(t^{-1}) = \int_{t_1}^{t} \frac{1}{e^{\langle 2P \rangle} \langle \alpha^{-1/2} \rangle} s^{-1} B ds - 2\pi \langle Q \rangle.$$

Since $\langle Q \rangle$ is bounded, we conclude that

$$\int_{t_1}^{\infty} t^{-1} e^{-2\langle P \rangle} dt < \infty,$$

so that $\langle Q \rangle$ converges to a limit, say q_∞. Since $t\langle P_t \rangle$ is bounded, we can, moreover, appeal to Lemma 10 in order to conclude that $\langle P \rangle \to \infty$. □

In the next lemma, we prove that $P/\ln t$ converges to a limit. Moreover, we prove that the limit, say r_∞, belongs to $(-3, 1)$ in case $B = 0$ and that it belongs to $[0, 1)$ in case $B \neq 0$. Most of the effort in the proof is in excluding the cases $r_\infty = 1$ and $r_\infty = -3$. At first sight, this might seem to be a technical issue. Let us therefore justify the effort spent in achieving this goal. In Lemma 38, we shall be able to prove that $\lambda = r_\infty^2 \ln t + o(\ln t)$. Combining this information with Lemma 35, we conclude that, for every $\epsilon > 0$, $\alpha_t/\alpha = O(t^{-1-\gamma+\epsilon})$, where γ is defined in (39). Clearly, this equality is only useful if $\gamma > 0$; in that case, we obtain $\alpha(t, \theta) = \alpha_\infty(\theta) + O(t^{-\eta})$ for some $\eta > 0$. However, we are only allowed to conclude that $\gamma > 0$ if we are able to exclude the extreme cases $r_\infty = 1$ and $r_\infty = -3$. It is also of interest to exclude $r_\infty = 0$ in case $B \neq 0$ (that would yield a decay rate for $Q - q_\infty$). However, we shall only be able to do so later; cf. Lemma 39.

Lemma 35. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$. Then there is an $r_\infty \in (-3, 1)$ such that*

$$\lim_{t\to\infty} \left\|(\ln t)^{-1} P - r_\infty\right\|_{C^0} = 0. \tag{80}$$

Moreover, if $B \neq 0$, then $r_\infty \in [0, 1)$.

Remark 36. For future reference, it is of interest to keep the following consequence of the proof in mind:

$$2\pi(r_\infty + 1) = \frac{A + Bq_\infty}{\langle\alpha^{-1/2}\rangle_\infty},$$

where $\langle\alpha^{-1/2}\rangle_\infty$ is the limit of $\langle\alpha^{-1/2}\rangle$, q_∞ is given by the statement of Lemma 34 in case $B \neq 0$, and Bq_∞ should be replaced by zero in case $B = 0$.

Proof. Let us begin by computing that

$$\begin{aligned}\partial_t \left[\frac{1}{\langle\alpha^{-1/2}\rangle} \int_{\mathbb{S}^1} \alpha^{-1/2}(P - \langle P\rangle) d\theta\right] =& O(t^{-2}) + \frac{1}{\langle\alpha^{-1/2}\rangle} \int_{\mathbb{S}^1} \alpha^{-1/2}(P_t + t^{-1}) d\theta - 2\pi\langle P_t + t^{-1}\rangle \\ =& \frac{1}{\langle\alpha^{-1/2}\rangle}\left(\frac{A}{t} + \frac{B}{t}\langle Q\rangle\right) - 2\pi\langle P_t + t^{-1}\rangle + O(t^{-2}),\end{aligned} \tag{81}$$

where we have used (78). It is of interest to consider the quantity

$$\frac{1}{\langle\alpha^{-1/2}\rangle}(A + B\langle Q\rangle). \tag{82}$$

From the conclusions we have already derived, this quantity converges. Call the limit $2\pi(r_\infty + 1)$. As a consequence,

$$\langle P\rangle = r_\infty \ln t + o(\ln t). \tag{83}$$

In order to obtain more information, it is of interest to determine how the limit $2\pi(r_\infty + 1)$ is approached.

The case $B = 0$. When $B = 0$,

$$\frac{1}{\langle\alpha^{-1/2}\rangle}(A + B\langle Q\rangle) = \frac{1}{\langle\alpha^{-1/2}\rangle} A.$$

Note that this is a decreasing quantity if $A \geq 0$ and an increasing quantity if $A \leq 0$. Assuming $A \geq 0$, we thus have

$$\partial_t \left[\frac{1}{\langle\alpha^{-1/2}\rangle} \int_{\mathbb{S}^1} \alpha^{-1/2}(P - \langle P\rangle) d\theta\right] \geq 2\pi(r_\infty + 1)t^{-1} - 2\pi\langle P_t + t^{-1}\rangle + O(t^{-2}).$$

Thus

$$c + O(t^{-1}) \geq 2\pi(r_\infty + 1)\ln t - 2\pi\langle P + \ln t\rangle,$$

so that

$$\langle P\rangle \geq r_\infty \ln t + c + O(t^{-1}), \tag{84}$$

assuming $r_\infty \geq -1$. Assuming $A \leq 0$, we have

$$\partial_t \left[\frac{1}{\langle\alpha^{-1/2}\rangle}\int_{\mathbb{S}^1}\alpha^{-1/2}(P-\langle P\rangle)d\theta\right] \leq 2\pi(r_\infty+1)t^{-1} - 2\pi\langle P_t + t^{-1}\rangle + O(t^{-2}).$$

Thus

$$c + O(t^{-1}) \leq 2\pi(r_\infty+1)\ln t - 2\pi\langle P + \ln t\rangle,$$

so that

$$\langle P\rangle \leq r_\infty \ln t + c + O(t^{-1}), \tag{85}$$

assuming $r_\infty \leq -1$. From the estimate (67), we know that $-3 \leq r_\infty \leq 1$. In fact, combining this estimate with (84) and (85), we obtain $-3 < r_\infty < 1$.

The case $B \neq 0$. In this case, we know that $0 \leq r_\infty \leq 1$; this conclusion is due to (67), (77) and (83). Consequently, we have $A + Bq_\infty \geq 0$. In order to proceed, it is of interest to analyse how $\langle Q\rangle$ approaches q_∞. Consider, to this end, (79). Multiplying this equality by B and integrating from t to infinity, we obtain

$$O(t^{-1}) = \int_t^\infty \frac{1}{e^{\langle 2P\rangle}\langle\alpha^{-1/2}\rangle}s^{-1}B^2 ds - 2\pi B(q_\infty - \langle Q\rangle). \tag{86}$$

Since we are mainly interested in excluding $r_\infty = 1$, let us assume that this equality holds. Then $\langle 2P\rangle = 2\ln t + o(\ln t)$, so that

$$B\langle Q\rangle = Bq_\infty + O(t^{-1}).$$

Thus,

$$\frac{1}{\langle\alpha^{-1/2}\rangle}(A + B\langle Q\rangle) = \frac{1}{\langle\alpha^{-1/2}\rangle}(A + Bq_\infty) + O(t^{-1}) \geq 2\pi(r_\infty+1) + O(t^{-1}),$$

where we have used the fact that $A + Bq_\infty \geq 0$. We can thus argue as above in order to conclude that

$$\langle P\rangle \geq \ln t + c + O(t^{-1}).$$

However, this estimate contradicts (67). Thus $r_\infty \in [0,1)$. □

7.1 Integrals of the energies

Let us consider the energies in greater detail. The following lemma may seem somewhat technical, but it almost immediately gives asymptotic estimates for λ; cf. Lemma 38.

Lemma 37. *Consider a solution to (2)–(6) on* $(t_0,\infty)\times\mathbb{S}^1$ *with* $K \neq 0$. *Assume that there is an* $\alpha_0 > 0$ *such that* $\alpha \geq \alpha_0$ *for all* $(t,\theta) \in (t_0,\infty)\times\mathbb{S}^1$. *Then*

$$\int_{t_1}^\infty \int_{\mathbb{S}^1} t\alpha^{-1/2}[\alpha P_\theta^2 + \alpha e^{2P}Q_\theta^2 + e^{2P}Q_t^2]d\theta dt < \infty. \tag{87}$$

Moreover,

$$\int_{t_1}^t \int_{\mathbb{S}^1} s\alpha^{-1/2}[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)]d\theta ds = 2\pi\langle\alpha^{-1/2}\rangle_\infty r_\infty^2 \ln t + o(\ln t), \tag{88}$$

where

$$\langle\alpha^{-1/2}\rangle_\infty = \lim_{t\to\infty}\langle\alpha^{-1/2}\rangle$$

and r_∞ *is the quantity defined by (80).*

Proof. Compute

$$\begin{aligned}\int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}e^{2P}Q_t^2 d\theta ds =& \int_{t_1}^{t}\left[\int_{\mathbb{S}^1} s\alpha^{-1/2}e^{2P}Q_t(Q_t-\langle Q_t\rangle)d\theta + B\langle Q_t\rangle\right]ds\\ =& \int_{t_1}^{t}\left[\partial_s\left\{\int_{\mathbb{S}^1} s\alpha^{-1/2}e^{2P}Q_t(Q-\langle Q\rangle)d\theta + B\langle Q\rangle\right\}\right]ds\\ &- \int_{t_1}^{t}\int_{\mathbb{S}^1}\partial_s(s\alpha^{-1/2}e^{2P}Q_t)(Q-\langle Q\rangle)d\theta ds\\ =& \left[\int_{\mathbb{S}^1} s\alpha^{-1/2}e^{2P}Q_t(Q-\langle Q\rangle)d\theta + B\langle Q\rangle\right]_{t_1}^{t}\\ &+ \int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{1/2}e^{2P}Q_\theta^2 d\theta ds.\end{aligned}$$

Note, in particular, that the right hand side is bounded, so that

$$\int_{t_1}^{\infty}\int_{\mathbb{S}^1} t\alpha^{-1/2}e^{2P}Q_t^2 d\theta dt < \infty,$$

an estimate which, together with (69), proves (87). Let us turn to

$$\begin{aligned}\int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}P_t^2 d\theta ds =& \int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}\left(P_t+\frac{1}{s}\right)P_t d\theta ds - \int_{t_1}^{t}\int_{\mathbb{S}^1}\alpha^{-1/2}P_t d\theta ds\\ =& \int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}\left(P_t+\frac{1}{s}\right)(P_t-\langle P_t\rangle)d\theta ds\\ &+ \int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}\left(P_t+\frac{1}{s}\right)\langle P_t\rangle d\theta ds - \int_{t_1}^{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{s}\right)d\theta ds\\ &+ \int_{t_1}^{t}\int_{\mathbb{S}^1}\frac{1}{s}\alpha^{-1/2}d\theta ds.\end{aligned} \tag{89}$$

Note that the first term on the far right hand side can be written

$$\begin{aligned}&\left[\int_{\mathbb{S}^1} s\alpha^{-1/2}\left(P_t+\frac{1}{s}\right)(P-\langle P\rangle)d\theta\right]_{t_1}^{t} - \int_{t_1}^{t}\int_{\mathbb{S}^1}\partial_s\left[s\alpha^{-1/2}\left(P_t+s^{-1}\right)\right](P-\langle P\rangle)d\theta ds\\ =& c_0 + O(t^{-1}) - \int_{t_1}^{t}\int_{\mathbb{S}^1}\left[\partial_\theta\left(s\alpha^{1/2}P_\theta\right) + s\alpha^{-1/2}e^{2P}(Q_t^2-\alpha Q_\theta^2)\right](P-\langle P\rangle)d\theta ds\\ =& c_0 + O(t^{-1}) + \int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{1/2}P_\theta^2 d\theta ds;\end{aligned}$$

cf. (21). The second term on the far right hand side of (89) can be written

$$\begin{aligned}&\int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}\left(P_t+\frac{1}{s}-e^{2P}Q_tQ\right)d\theta\langle P_t\rangle ds + \int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}e^{2P}Q_t(Q-\langle Q\rangle)d\theta\langle P_t\rangle ds\\ &+ \int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}e^{2P}Q_t\langle Q\rangle d\theta\langle P_t\rangle ds\\ =& \int_{t_1}^{t}(A+B\langle Q\rangle)\langle P_t\rangle ds + c_0 + O(t^{-1}).\end{aligned}$$

By a similar argument, the third term on the far right hand side of (89) can be written

$$-\int_{t_1}^{t}\frac{1}{s}(A+B\langle Q\rangle)ds + c_0 + O(t^{-1}).$$

Adding up the above observations, we conclude that

$$\int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}P_t^2 d\theta ds = \int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{1/2}P_\theta^2 d\theta ds + \int_{t_1}^{t}(A+B\langle Q\rangle)\left(\langle P_t\rangle - \frac{1}{s}\right)ds + \int_{t_1}^{t}\int_{\mathbb{S}^1}\frac{1}{s}\alpha^{-1/2}d\theta dt + c_0 + O(t^{-1}).$$

As a consequence,

$$\begin{aligned}\int_{t_1}^{t}\int_{\mathbb{S}^1} s\alpha^{-1/2}[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)]d\theta ds =&\langle\alpha^{-1/2}\rangle_\infty 2\pi(r_\infty+1)\left(\langle P\rangle - \ln t\right)\\ &+2\pi\langle\alpha^{-1/2}\rangle_\infty \ln t + o(\ln t)\\ =&2\pi\langle\alpha^{-1/2}\rangle_\infty[(r_\infty^2-1)+1]\ln t + o(\ln t)\\ =&2\pi\langle\alpha^{-1/2}\rangle_\infty r_\infty^2 \ln t + o(\ln t),\end{aligned} \tag{90}$$

where we have used the fact that $2\pi(r_\infty+1)$ is the limit of (82). The lemma follows. □

7.2 Asymptotics

Let us now turn to the asymptotics of λ.

Lemma 38. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then*

$$\lim_{t\to\infty}\|(\ln t)^{-1}\lambda - r_\infty^2\|_{C^0}=0,$$

where r_∞ is the real number defined by (80).

Proof. Compute

$$\partial_t\langle\alpha^{-1/2}(\lambda-\langle\lambda\rangle)\rangle = O(t^{-2}) + \langle\alpha^{-1/2}\partial_t\lambda\rangle - \langle\alpha^{-1/2}\rangle\langle\lambda_t\rangle.$$

Integrating this equality, we obtain

$$\langle\alpha^{-1/2}\rangle_\infty\langle\lambda\rangle = \int_{t_1}^{t}\langle\alpha^{-1/2}\partial_t\lambda\rangle dt + o(\ln t) = \langle\alpha^{-1/2}\rangle_\infty r_\infty^2\ln t + o(\ln t), \tag{91}$$

where we, in the last step, used (88) and the fact that α_t/α is integrable. Thus

$$\langle\lambda\rangle = r_\infty^2\ln t + o(\ln t).$$

The lemma follows. □

Given the above information, we obtain a decay rate for $\alpha-\alpha_\infty$. As a consequence, we can exclude $r_\infty=0$ in case $B\neq 0$, so that we obtain a decay rate for $Q-q_\infty$. Combining these two pieces of information, we obtain an estimate of the form (92) below. However, the estimate we obtain for λ does not include a decay rate. This is due to the fact that we do not have sufficient information concerning the L^2-norm of P_θ etc. However, we will improve our knowledge concerning these norms in Lemma 43 below.

Lemma 39. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then there is an $r_\infty\in(-3,1)$ and constants c_P and $C,\delta>0$ such that*

$$\|P(t,\cdot)-r_\infty\ln t - c_P\|_{C^0}\leq Ct^{-\delta} \tag{92}$$

for $t \geq t_1$. Moreover, if $B \neq 0$, then $r_\infty \in (0,1)$. Let γ be the real number defined in (39). Then there is an $\alpha_\infty \in C^0(\mathbb{S}^1, \mathbb{R}_+)$ and a constant C such that

$$\|\alpha(t,\cdot) - \alpha_\infty\|_{C^0} \leq Ct^{-\gamma} \tag{93}$$

for all $t \geq t_1$. Furthermore, if $B \neq 0$,

$$\|Q(t,\cdot) - q_\infty\|_{C^0} \leq Ct^{-2r_\infty} \tag{94}$$

for all $t \geq t_1$. Finally, there is a constant c_λ such that

$$\lim_{t\to\infty} \left\|\lambda(t,\cdot) - r_\infty^2 \ln t - c_\lambda\right\|_{C^0} = 0. \tag{95}$$

Remarks 40. That $\gamma > 0$ follows from the fact that $r_\infty \in (-3,1)$.

Remark 41. It is possible to use (79) and the above estimates in order to prove that (94) is optimal. Note, however, that

$$\|(Q - \langle Q\rangle)(t,\cdot)\|_{C^0} \leq Ct^{-1-r_\infty}.$$

Remark 42. Using the above asymptotics together with (81), it can be concluded that δ in (92) can be chosen to equal $\min\{\gamma, 1\}$ in case $B = 0$ and to equal $\min\{\gamma, 1, 2r_\infty\}$ in case $B \neq 0$. Note also that $\delta < 1$ in case $B \neq 0$. In fact, assuming $\delta = 1$, we conclude that $\gamma \geq 1$ and $2r_\infty \geq 1$. On the other hand, the latter estimate implies that

$$\gamma = 2 - \frac{1}{2}(r_\infty + 1)^2 \leq 2 - \frac{1}{2}\frac{9}{4} = \frac{7}{8} < 1,$$

contradicting the fact that $\gamma \geq 1$. In other words, $\delta = \min\{\gamma, 2r_\infty\}$ in case $B \neq 0$, and $\delta < 1$.

Proof. By combining Lemmas 35 and 38, we have

$$P + \frac{1}{2}\lambda = \left(\frac{1}{2}r_\infty^2 + r_\infty\right)\ln t + o(\ln t).$$

Thus

$$\frac{e^{P+\lambda/2}K^2}{t^{5/2}} = t^{-1-\gamma}\exp[o(\ln t)].$$

Let us use this observation in order to prove that $r_\infty > 0$ in case $B \neq 0$. Assume, to this end, that $r_\infty = 0$ and that $B \neq 0$. Then $\gamma = 3/2$, so that $\alpha_t = O(t^{-2})$. As a consequence,

$$\frac{1}{\langle\alpha^{-1/2}\rangle} - \frac{1}{\langle\alpha^{-1/2}\rangle_\infty} = O(t^{-1}).$$

Thus

$$\frac{1}{\langle\alpha^{-1/2}\rangle}(A + B\langle Q\rangle) = \frac{1}{\langle\alpha^{-1/2}\rangle_\infty}(A + B\langle Q\rangle) + O(t^{-1}).$$

On the other hand, due to (86), we have

$$Bq_\infty = \int_t^\infty \frac{1}{2\pi e^{\langle 2P\rangle}\langle\alpha^{-1/2}\rangle} s^{-1}B^2 ds + B\langle Q\rangle + O(t^{-1}) \geq B\langle Q\rangle + O(t^{-1}).$$

Combining these two observations, we conclude that

$$\frac{1}{\langle\alpha^{-1/2}\rangle}(A + B\langle Q\rangle) \leq \frac{1}{\langle\alpha^{-1/2}\rangle_\infty}(A + Bq_\infty) + O(t^{-1}).$$

Combining this observation with (81), we obtain

$$\partial_t \left[\frac{1}{\langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} \alpha^{-1/2}(P - \langle P \rangle) d\theta \right] \leq 2\pi(r_\infty + 1)\frac{1}{t} - 2\pi\langle P_t + t^{-1} \rangle + O(t^{-2}).$$

Integrating this inequality and keeping in mind that $r_\infty = 0$, we conclude that

$$\langle P \rangle \leq c + O(t^{-1}),$$

in contradiction with (77). To conclude: if $B \neq 0$, then $r_\infty \in (0, 1)$.

Due to the above observation, we obtain a rate of convergence of $\langle Q \rangle$ to q_∞. In fact, due to (79) and the fact that $r_\infty > 0$, there is an $\eta > 0$ such that

$$\langle Q \rangle - q_\infty = O(t^{-\eta}).$$

Combining the above observations, we conclude that there is a $\delta > 0$ such that

$$\frac{1}{\langle \alpha^{-1/2} \rangle}(A + B\langle Q \rangle) = 2\pi(r_\infty + 1) + O(t^{-\delta}).$$

We can thus integrate (81) in order to conclude that (92) holds for some $\delta > 0$ and some $c_P \in \mathbb{R}$. Moreover, we are in a position to improve (90) to

$$\int_{t_1}^{t} \int_{\mathbb{S}^1} s\alpha^{-1/2}[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] d\theta ds = 2\pi\langle \alpha^{-1/2} \rangle_\infty r_\infty^2 \ln t + c_0 + o(1).$$

Using this equality, (91) can be improved to

$$\langle \lambda \rangle = r_\infty^2 \ln t + c_0 + o(1).$$

As a consequence, (95) holds. Due to (92) and (95),

$$\frac{e^{P+\lambda/2}K^2}{t^{5/2}} = t^{-1-\gamma} \exp[c_0 + o(1)].$$

Thus (93) holds. Finally, integrating (79) from t to infinity, keeping the above in mind, we obtain (94). □

It would of course be of interest to improve the estimate for λ; we would like to have a rate of convergence. However, it is then necessary to obtain a rate of convergence to zero for

$$\int_{t}^{\infty} \int_{\mathbb{S}^1} s\alpha^{-1/2}[\alpha P_\theta^2 + \alpha e^{2P} Q_\theta^2 + e^{2P} Q_t^2] d\theta ds.$$

This is the next topic of interest. Note also that the proof of the next lemma constitutes the model case of how to derive estimates for the the L^2-norms of P_θ and $e^P Q_\theta$ using the conserved quantities and the monotonicity properties of the energies; cf. the discussion in the introduction.

Lemma 43. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbb{S}^1} t^2 \alpha^{-1/2} \left[\alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) \right] d\theta \leq Ct^{-\delta}, \quad (96)$$

$$\left| \int_{\mathbb{S}^1} t^2 \alpha^{-1/2} P_t^2 d\theta - 2\pi\langle \alpha^{-1/2} \rangle_\infty r_\infty^2 \right| \leq Ct^{-\delta}, \quad (97)$$

for all $t \geq t_1$.

Remark 44. The constant δ can be chosen as in Remark 42.

Remark 45. Due to the estimate (96), several of the conclusions derived previously can be improved.

Proof. **Step 1.** The first step of the proof is to use the conserved quantities to estimate the integral of $\alpha^{-1/2}P_t^2$ from below. Consider, to this end,

$$A = \int_{\mathbb{S}^1} t\alpha^{-1/2}P_t d\theta + 2\pi\langle\alpha^{-1/2}\rangle - \int_{\mathbb{S}^1} t\alpha^{-1/2}e^{2P}Q_t(Q-\langle Q\rangle)d\theta - B\langle Q\rangle. \tag{98}$$

This equality can be written

$$A + Bq_\infty - 2\pi\langle\alpha^{-1/2}\rangle_\infty + O(t^{-\delta}) = \int_{\mathbb{S}^1} t\alpha^{-1/2}P_t d\theta.$$

Since

$$A + Bq_\infty = 2\pi\langle\alpha^{-1/2}\rangle_\infty(r_\infty + 1),$$

we conclude that

$$2\pi\langle\alpha^{-1/2}\rangle_\infty r_\infty + O(t^{-\delta}) = \int_{\mathbb{S}^1} t\alpha^{-1/2}P_t d\theta. \tag{99}$$

Squaring this equality, we obtain

$$\langle\alpha^{-1/2}\rangle_\infty^2 r_\infty^2 + O(t^{-\delta}) = \left(\frac{1}{2\pi}\int_{\mathbb{S}^1} t\alpha^{-1/2}P_t d\theta\right)^2.$$

On the other hand,

$$\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} t\alpha^{-1/2}|P_t|d\theta\right)^2 \le \langle\alpha^{-1/2}\rangle\langle t^2\alpha^{-1/2}P_t^2\rangle \le \langle\alpha^{-1/2}\rangle_\infty\langle t^2\alpha^{-1/2}P_t^2\rangle.$$

Combining these observations, we obtain

$$\langle\alpha^{-1/2}\rangle_\infty r_\infty^2 + O(t^{-\delta}) \le \langle t^2\alpha^{-1/2}P_t^2\rangle.$$

Thus there is a constant $C > 0$ such that

$$-Ct^{-\delta} \le \int_{\mathbb{S}^1} t^2\alpha^{-1/2}P_t^2 d\theta - 2\pi\langle\alpha^{-1/2}\rangle_\infty r_\infty^2. \tag{100}$$

Step 2. The idea of the second step is to combine the lower bound on the P_t-energy with the monotonicity of the energy to derive upper bounds on the remainder of the PQ-energy. Estimate

$$\begin{aligned}
&\int_{\mathbb{S}^1} t^2\alpha^{-1/2}\left[\alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)\right]d\theta \\
=&\hat{H} - \int_{\mathbb{S}^1} t^2\alpha^{-1/2}P_t^2 d\theta - \int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{3/2}}d\theta - \int_{\mathbb{S}^1} 3\alpha^{-1/2}d\theta \\
\le&\hat{H}_\infty - 6\pi\langle\alpha^{-1/2}\rangle_\infty - 2\pi\langle\alpha^{-1/2}\rangle_\infty r_\infty^2 + Ct^{-\delta}.
\end{aligned}$$

However, due to Lemmas 30 and 37, we know that

$$\hat{H}_\infty = 6\pi\langle\alpha^{-1/2}\rangle_\infty + 2\pi\langle\alpha^{-1/2}\rangle_\infty r_\infty^2.$$

Combining the above observations, we obtain (96). Due to (24), we thus have

$$0 \le \hat{H}_\infty - \hat{H}(t) = \int_t^\infty 2s\int_{\mathbb{S}^1}\alpha^{1/2}(P_\theta^2 + e^{2P}Q_\theta^2)d\theta ds \le Ct^{-\delta}.$$

As a consequence of this and earlier observations,

$$\int_{\mathbb{S}^1} t^2\alpha^{-1/2}\left[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)\right]d\theta - 2\pi\langle\alpha^{-1/2}\rangle_\infty r_\infty^2 = O(t^{-\delta}). \tag{101}$$

Combining this estimate with (96), we conclude that (97) holds. □

We are now in a position to improve the estimates concerning λ.

Lemma 46. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then there are constants $C>0$ and c_λ such that*

$$\left\|\lambda(t,\cdot)-r_\infty^2\ln t-c_\lambda\right\|_{C^0}\leq Ct^{-\delta} \tag{102}$$

for $t\geq t_1$.

Remark 47. The constant δ can be chosen as in Remark 42.

Proof. To begin with, we can improve (91) to

$$c_0+O(t^{-\delta})=\int_{t_1}^t\langle\alpha^{-1/2}\lambda_t\rangle dt-\langle\alpha^{-1/2}\rangle_\infty\langle\lambda\rangle.$$

However,

$$\int_{t_1}^t\langle\alpha^{-1/2}\lambda_t\rangle dt=\frac{1}{2\pi}\int_{t_1}^t\int_{\mathbb{S}^1}s\alpha^{-1/2}\left[P_t^2+\alpha P_\theta^2+e^{2P}(Q_t^2+\alpha Q_\theta^2)\right]d\theta ds+c_0+O(t^{-\delta}).$$

Combining this observation with (96) and (97), we conclude that (102) holds. □

7.3 Auxiliary observations

Let us record some auxiliary observations that we shall need when deriving sup-norm estimates.

Lemma 48. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then there is a constant C such that*

$$\left|\frac{1}{\langle\alpha^{-1/2}\rangle}\langle\alpha^{-1/2}P_t\rangle-\frac{r_\infty}{t}\right| \leq Ct^{-1-\delta}, \tag{103}$$

$$\left\|\partial_t\left(\frac{\alpha^{-1/2}}{\langle\alpha^{-1/2}\rangle}\right)\right\|_{C^0} \leq Ct^{-2-\gamma-\delta/2} \tag{104}$$

for all $t\geq t_1$.

Proof. Combining (99) with (93), we obtain (103). Turning to (104), note that

$$\partial_t\left(\frac{\alpha^{-1/2}}{\langle\alpha^{-1/2}\rangle}\right)=-\frac{1}{2}\frac{\alpha^{-1/2}}{\langle\alpha^{-1/2}\rangle}\left(\frac{\alpha_t}{\alpha}-\frac{1}{\langle\alpha^{-1/2}\rangle}\left\langle\alpha^{-1/2}\frac{\alpha_t}{\alpha}\right\rangle\right). \tag{105}$$

On the other hand,

$$\begin{aligned}\frac{\alpha_t}{\alpha}-\frac{1}{\langle\alpha^{-1/2}\rangle}\left\langle\alpha^{-1/2}\frac{\alpha_t}{\alpha}\right\rangle=&\frac{e^{\langle P\rangle+\langle\lambda\rangle/2}(1-e^{P-\langle P\rangle+(\lambda-\langle\lambda\rangle)/2})K^2}{t^{5/2}}\\&+\frac{1}{\langle\alpha^{-1/2}\rangle}\left\langle\alpha^{-1/2}\left(\frac{e^{\langle P\rangle+\langle\lambda\rangle/2}(e^{P-\langle P\rangle+(\lambda-\langle\lambda\rangle)/2}-1)K^2}{t^{5/2}}\right)\right\rangle.\end{aligned} \tag{106}$$

Appealing to (72), (74) and (96), we conclude that (104) holds. □

8 Light cone estimates

In the present section, we derive sup-norm bounds on the first derivatives. However, as a preliminary step, it is of interest to derive estimates for the diffeomorphisms of the circle generated by the characteristics. Fix, to this end, a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Define

$$\Phi_\pm : (t_0, \infty) \times \mathbb{R} \to (t_0, \infty) \times \mathbb{R}$$

by requiring

$$\Phi_\pm(t_1, \theta) = (t_1, \theta), \quad (\partial_t \Phi_\pm)(t, \theta) = [1, \pm\sqrt{\alpha} \circ \Phi_\pm(t, \theta)]. \tag{107}$$

Note that $\Phi_\pm$ are smooth maps that can be written

$$\Phi_\pm(t, \theta) = [t, \bar{\Phi}_\pm(t, \theta)].$$

Note also that $\bar{\Phi}_\pm(t, \theta + n2\pi) = \bar{\Phi}_\pm(t, \theta) + n2\pi$ for $n \in \mathbb{Z}$. In this sense, $\Phi_\pm$ can be considered to be a smooth function on $(t_0, \infty) \times \mathbb{S}^1$ and $\bar{\Phi}_\pm(t, \cdot)$ can be considered to be a smooth map from the circle to itself. In what follows, it will be of interest to change variables from θ to $\bar{\Phi}_\pm(t, \theta)$. As a consequence, it is of interest to estimate $\bar{\Phi}'_\pm$, the θ-derivative of $\bar{\Phi}_\pm$.

Lemma 49. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$. Then $\bar{\Phi}_\pm(t, \cdot)$ is a diffeomorphism of $\mathbb{S}^1$ for each $t \in (t_0, \infty)$. Moreover, there is a constant $C > 1$ such that*

$$\frac{1}{C} \leq \bar{\Phi}'_\pm(t, \theta) \leq C \tag{108}$$

for all $(t, \theta) \in [t_1, \infty) \times \mathbb{R}$.

Proof. Due to the equation defining $\Phi_\pm$,

$$\frac{\partial \bar{\Phi}'_\pm}{\partial t} = \pm \left(\frac{\alpha_\theta}{2\sqrt{\alpha}} \right) \circ \Phi_\pm(t, \theta) \cdot \bar{\Phi}'_\pm.$$

Since $\bar{\Phi}'_\pm(t_1, \theta) = 1$, this equation implies that $\bar{\Phi}'_\pm$ is always positive. Thus $\bar{\Phi}_\pm(t, \cdot)$ is a diffeomorphism of $\mathbb{S}^1$ for each $t \in (t_0, \infty)$. Moreover,

$$\begin{aligned}
\ln \bar{\Phi}'_\pm(t, \theta) &= \pm \int_{t_1}^{t} \left(\frac{\alpha_\theta}{2\sqrt{\alpha}} \right) \circ \Phi_\pm(s, \theta) ds = \int_{t_1}^{t} \left(\frac{\alpha_\theta}{2\alpha} \right) \circ \Phi_\pm(s, \theta) \cdot \frac{\partial \bar{\Phi}_\pm}{\partial t}(s, \theta) ds \\
&= \int_{t_1}^{t} \left[\left(\frac{\alpha_t}{2\alpha} \right) \circ \Phi_\pm(s, \theta) + \left(\frac{\alpha_\theta}{2\alpha} \right) \circ \Phi_\pm(s, \theta) \cdot \frac{\partial \bar{\Phi}_\pm}{\partial t}(s, \theta) - \left(\frac{\alpha_t}{2\alpha} \right) \circ \Phi_\pm(s, \theta) \right] ds \\
&= \frac{1}{2} \ln \frac{\alpha \circ \Phi_\pm(t, \theta)}{\alpha(t_1, \theta)} - \int_{t_1}^{t} \left(\frac{\alpha_t}{2\alpha} \right) \circ \Phi_\pm(s, \theta) ds,
\end{aligned}$$

where we used (107) in the second step. Since the right hand side is bounded, we conclude that $\bar{\Phi}'_\pm(t, \theta)$ is uniformly bounded from above and below for $t \geq t_1$. □

Let us now prove a non-optimal estimate.

Lemma 50. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$. Then, for every $0 < a < 2$, there is a constant $C_a > 0$ such that*

$$\| P_t^2 + e^{2P} Q_t^2 + \alpha(P_\theta^2 + e^{2P} Q_\theta^2) \|_{C^0} \leq C_a t^{-a} \tag{109}$$

for all $t \geq t_1$.

Proof. Using (28), it can be computed that

$$\partial_\pm(t\mathcal{A}_\mp) = -\partial_- P\partial_+ P - e^{2P}\partial_- Q\partial_+ Q + t\frac{\alpha_t}{\alpha}\mathcal{A}_\mp - \partial_\mp P\frac{e^{P+\lambda/2}K^2}{t^{5/2}}.$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t}\left[(t\mathcal{A}_\mp)_{\circ\pm}\right] =& -(\partial_\mp P)_{\circ\pm}(\partial_\pm P)_{\circ\pm} - e^{2P_{\circ\pm}}(\partial_\mp Q)_{\circ\pm}(\partial_\pm Q)_{\circ\pm} + t\left(\frac{\alpha_t}{\alpha}\right)_{\circ\pm}(\mathcal{A}_\mp)_{\circ\pm} \\ & -(\partial_\mp P)_{\circ\pm}\frac{e^{P_{\circ\pm}+\lambda_{\circ\pm}/2}K^2}{t^{5/2}},\end{aligned} \tag{110}$$

where we use the notation $f_{\circ\pm} = f\circ\Phi_\pm$. It is of interest to estimate the integral of the right hand side. Before doing so, let us, however, introduce the quantities

$$F_\pm = t\mathcal{A}_\pm + \partial_\pm P(P - \langle P\rangle) + e^{2\langle P\rangle}\partial_\pm Q(Q - \langle Q\rangle) + 1. \tag{111}$$

It may seem unnatural to introduce a quantity such as $F_\pm$, since (as we shall show below) it does not decay and since we would like to prove that $t\mathcal{A}_\pm$ converges to zero. For example, it might seem more natural to to replace the 1 on the right hand side of (111) by t raised to an appropriate negative power. However, the problem is that we are not able to derive the desired decay by studying the derivative of the resulting object (along the appropriate characteristics). The idea is therefore to

- prove that $F_\pm\circ\Phi_\mp$ converges to some function, say ψ, at a certain rate,
- use the L^2-bounds we have together with the bounds (108) in order to prove that $\psi = 1$.

Using this information, it can then be argued that $t\mathcal{A}_\pm$ decays. The argument can then be iterated to yield the conclusion of the lemma.

In order to proceed, note that, since

$$\begin{aligned}|\partial_\pm P(P - \langle P\rangle)| &\leq Ct^{-1-\delta/2}|\partial_\pm P| \leq Ct^{-3/2-\delta/2}[1 + t(\partial_\pm P)^2], \\ |e^{2\langle P\rangle}\partial_\pm Q(Q - \langle Q\rangle)| &\leq Ct^{-1-\delta/2}e^{2\langle P\rangle - 2P}e^P|\partial_\pm Q| \leq Ct^{-3/2-\delta/2}[1 + te^{2P}(\partial_\pm Q)^2],\end{aligned}$$

we have that $F_\pm$ is equivalent to $t\mathcal{A}_\pm + 1$ for t large enough. Define

$$E_\pm(t) = \sup_{\theta\in\mathbb{S}^1} F_\pm(t,\theta), \quad E(t) = E_+(t) + E_-(t).$$

Then there is a $T > t_0$ and a $C > 1$ such that

$$C^{-1}E(t) \leq t\|P_t^2 + e^{2P}Q_t^2 + \alpha(P_\theta^2 + e^{2P}Q_\theta^2)\|_{C^0} + 1 \leq CE(t)$$

for all $t \geq T$. Let

$$G_\pm = F_\pm\circ\Phi_\mp.$$

Then, due to (110) and the definition of $F_\pm$,

$$\begin{aligned}\partial_t G_\mp =&(e^{2\langle P\rangle} - e^{2P_{\circ\pm}})(\partial_\mp Q)_{\circ\pm}(\partial_\pm Q)_{\circ\pm} + (\partial_\pm\partial_\mp P)_{\circ\pm}(P_{\circ\pm} - \langle P\rangle) - (\partial_\mp P)_{\circ\pm}\langle P_t\rangle \\ & + 2\langle P_t\rangle e^{2\langle P\rangle}(\partial_\mp Q)_{\circ\pm}(Q_{\circ\pm} - \langle Q\rangle) + e^{2\langle P\rangle}(\partial_\pm\partial_\mp Q)_{\circ\pm}(Q_{\circ\pm} - \langle Q\rangle) \\ & - e^{2\langle P\rangle}(\partial_\mp Q)_{\circ\pm}\langle Q_t\rangle + t\left(\frac{\alpha_t}{\alpha}\right)_{\circ\pm}(\mathcal{A}_\mp)_{\circ\pm} - (\partial_\mp P)_{\circ\pm}\frac{e^{P_{\circ\pm}+\lambda_{\circ\pm}/2}K^2}{t^{5/2}}.\end{aligned} \tag{112}$$

Note that

$$\|(e^{2\langle P\rangle} - e^{2P_{\circ\pm}})(\partial_\mp Q)_{\circ\pm}(\partial_\pm Q)_{\circ\pm}\|_{C^0} \leq Ct^{-2-\delta/2}E,$$

where we have used the definition of E, (72) and (96). We also have

$$
\begin{aligned}
\|(\partial_{\mp}P)_{\circ\pm}\langle P_t\rangle\|_{C^0} &\leq Ct^{-3/2}E,\\
\|(\partial_{\pm}\partial_{\mp}P)_{\circ\pm}(P_{\circ\pm}-\langle P\rangle)\|_{C^0} &\leq Ct^{-2-\delta/2}E,
\end{aligned}
$$

where we have used (26) in the second estimate. The remaining terms appearing in (112) can similarly be controlled by appealing to earlier estimates. It is of interest to note that the worst term appearing on the right hand side of (112) is the second to last term (assuming γ to be small). On the other hand, this term has a good sign. However, we are here interested in the decay rate of $G_{\pm}$ to its limiting value. As a consequence, we need to estimate the absolute value of the right hand side. To conclude, we have

$$
\|\partial_t G_{\mp}\|_{C^0} \leq C(t^{-3/2}+t^{-1-\gamma})E \tag{113}
$$

for $t \geq T$. Due to this inequality, we obtain

$$
G_{\mp}(t) \leq E_{\mp}(t_a) + \int_{t_a}^{t} C(s^{-3/2}+s^{-1-\gamma})Eds,
$$

assuming $t_a \geq T$. Taking the supremum with respect to θ and summing the two resulting estimates, we obtain

$$
E(t) \leq E(t_a) + \int_{t_a}^{t} C(s^{-3/2}+s^{-1-\gamma})Eds.
$$

Applying Grönwall's lemma, we conclude that E is bounded. Combining this observation with (113), we obtain

$$
\|\partial_t G_{\mp}\|_{C^0} \leq Ct^{-3/2}+Ct^{-1-\gamma}.
$$

Thus $G_{\pm}$ converges in C^0 to a limit function, say $G_{\pm,\infty}$. In fact, there is a constant C such that

$$
\|G_{\pm}(t,\cdot)-G_{\pm,\infty}\|_{C^0} \leq Ct^{-1/2}+Ct^{-\gamma}. \tag{114}
$$

Thus

$$
\begin{aligned}
\int_{\mathbb{S}^1} |G_{\pm,\infty}-1|d\theta &= \lim_{t\to\infty}\int_{\mathbb{S}^1}|G_{\pm}(t,\theta)-1|d\theta = \lim_{t\to\infty}\int_{\mathbb{S}^1}|F_{\pm}(t,\bar{\Phi}_{\mp}(t,\theta))-1|d\theta\\
&= \lim_{t\to\infty}\int_{\mathbb{S}^1}|F_{\pm}(t,\phi)-1|\frac{1}{\bar{\Phi}'_{\mp}(t,\theta_t(\phi))}d\phi = 0
\end{aligned}
$$

(for some suitable function θ_t), where we used (108) in the last step. Since $G_{\pm,\infty}$ is a continuous function, we conclude that it equals 1. Combining this observation with (114), we obtain

$$
\|F_{\pm}(t,\cdot)-1\|_{C^0} \leq Ct^{-1/2}+Ct^{-\gamma}.
$$

Letting

$$
\mathcal{E} = \sup_{\theta\in\mathbb{S}^1} t\mathcal{A}_+ + \sup_{\theta\in\mathbb{S}^1} t\mathcal{A}_-,
$$

this estimate yields

$$
\mathcal{E} \leq Ct^{-1/2}+Ct^{-\gamma}. \tag{115}
$$

By iterating the above procedure, we can improve this estimate. In order to justify this statement, assume that

$$
\mathcal{E} \leq C_b t^{-b} \tag{116}
$$

for $t \geq t_a$ and some $0 < b < 1$. Assuming $b \leq 1-2\epsilon$ for some $0 < \epsilon < \gamma$ and returning to (112), it can then be estimated that

$$
\|\partial_t G_{\pm}\|_{C^0} \leq Ct^{-3/2-b/2}+Ct^{-1-\gamma-b}
$$

Arguing, again, as above, we conclude that

$$\mathcal{E} \leq Ct^{-(1-b)/2-b} + Ct^{-\gamma-b}.$$

In other words, we have improved the estimate (116) by at least ϵ. We can iterate this procedure until $b > 1-\epsilon$. To conclude, for every $0 < b < 1$, there is a constant C_b such that (116) holds. The lemma follows. □

Let us now derive an optimal estimate.

Lemma 51. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then there is a constant C such that*

$$t^2\mathcal{A}_\pm \leq r_\infty^2 + Ct^{-\delta}$$

for all $t \geq t_1$.

Remark 52. It should be possible to use the methods described in the proof of Lemma 53 iteratively in order to obtain the same conclusion, as well as more detailed information. However, the argument presented here (besides being different) yields the optimal estimate immediately, and we have therefore chosen to present it separately.

Proof. Before going into the details, let us say a few words concerning the intuition behind the argument. We would like to estimate $\mathcal{A}_\pm$. One way of doing so is by integrating equalities such as (118) below. However, it is then necessary to know, e.g., what the integrals of $(t\mathcal{A}_\pm)_{\circ\pm}$ and $(t\mathcal{A}_\mp)_{\circ\pm}$ are. Furthermore, we need to be able to control the remaining terms in (118). On the other hand, $\partial_\pm\lambda = t\mathcal{A}_\pm + \alpha_t/\alpha$. Using the knowledge we have concerning the asymptotic behaviour of λ, and the fact that we have good control of α_t/α in the sup norm, we obtain good control over the integral of $(t\mathcal{A}_\pm)_{\circ\pm}$. Since the last two terms on the right hand side of (118) are under good control due to previous estimates, what remains is the second and the third terms. However, the third term has an advantageous sign, and what remains can (roughly speaking) be interpreted as an equation for the integral of $(t\mathcal{A}_\mp)_{\circ\pm}$. Using a Grönwall's lemma type argument and previous estimates, it can then be deduced that the desired estimate holds.

Turning to the details, note that

$$\partial_t\lambda_{\circ\pm} = (\partial_\pm\lambda)_{\circ\pm} = (t\mathcal{A}_\pm)_{\circ\pm} + \left(\frac{\alpha_t}{\alpha}\right)_{\circ\pm}.$$

Consequently, there is a continuous function f_λ on $\mathbb{S}^1$ such that

$$\left\|\int_{t_1}^t (s\mathcal{A}_\pm)_{\circ\pm}(s,\cdot)ds - r_\infty^2\ln t - f_\lambda\right\|_{C^0} \leq Ct^{-\delta} \tag{117}$$

for all $t \geq t_1$. On the other hand,

$$\begin{aligned}\partial_\pm(t^2\mathcal{A}_\mp) =& t\mathcal{A}_\mp + t\partial_\pm(t\mathcal{A}_\mp) = \mp 2t\sqrt{\alpha}(P_tP_\theta + e^{2P}Q_tQ_\theta) + 2t\alpha(P_\theta^2 + e^{2P}Q_\theta^2) + \frac{\alpha_t}{\alpha}t^2\mathcal{A}_\mp \\ &- \partial_\mp P\frac{e^{P+\lambda/2}K^2}{t^{3/2}} \\ =& -\frac{1}{2}t\mathcal{A}_\pm + \frac{1}{2}t\mathcal{A}_\mp + 2t\alpha(P_\theta^2 + e^{2P}Q_\theta^2) + \frac{\alpha_t}{\alpha}t^2\mathcal{A}_\mp - \partial_\mp P\frac{e^{P+\lambda/2}K^2}{t^{3/2}}.\end{aligned} \tag{118}$$

Due to the fact that, given any $0 < a < 2$, (109) holds, the last two terms on the far right hand side are integrable. In fact, there is an $0 < \eta < \gamma$ and a continuous function f on $\mathbb{S}^1$ such that the integral from t_1 to t of the last two terms (composed with $\Phi_\pm$) is given by $f + O(t^{-\eta})$. As a consequence (keeping (117) in mind), letting $0 < \eta < \delta$, there is a continuous function $f_\pm$ on $\mathbb{S}^1$ and a constant $C > 0$ such that

$$\left\|(t^2\mathcal{A}_\mp)_{\circ\pm} + \frac{1}{2}r_\infty^2 \ln t - \frac{1}{2}\int_{t_1}^t (s\mathcal{A}_\mp)_{\circ\pm} ds - 2\int_{t_1}^t s[\alpha(P_\theta^2 + e^{2P}Q_\theta^2)]_{\circ\pm} ds - f_\pm\right\|_{C^0} \le Ct^{-\eta} \quad (119)$$

for $t \ge t_1$. In particular,

$$(t^2\mathcal{A}_\mp)_{\circ\pm} \ge -\frac{1}{2}r_\infty^2 \ln t + \frac{1}{2}\int_{t_1}^t (s\mathcal{A}_\mp)_{\circ\pm} ds + 2\int_{t_1}^t s[\alpha(P_\theta^2 + e^{2P}Q_\theta^2)]_{\circ\pm} ds + f_\pm - Ct^{-\eta}$$

for all $(t, \theta) \in [t_1, \infty) \times \mathbb{S}^1$. Denoting the right hand side by $g_\pm$, we obtain

$$\begin{aligned}\partial_t g_\pm =& -\frac{1}{2t}r_\infty^2 + \frac{1}{2}(t\mathcal{A}_\mp)_{\circ\pm} + 2t[\alpha(P_\theta^2 + e^{2P}Q_\theta^2)]_{\circ\pm} + \eta C t^{-1-\eta} \\ \ge& -\frac{1}{2t}r_\infty^2 + \frac{1}{2t}g_\pm + 2t[\alpha(P_\theta^2 + e^{2P}Q_\theta^2)]_{\circ\pm} + \eta C t^{-1-\eta}.\end{aligned}$$

Consequently,

$$\partial_t(t^{-1/2}g_\pm) \ge -\frac{1}{2t^{3/2}}r_\infty^2.$$

Integrating from $t_a \ge t_1$ to $t_b \ge t_a$, we obtain

$$t_b^{-1/2}g_\pm(t_b, \theta) \ge t_a^{-1/2}g_\pm(t_a, \theta) + t_b^{-1/2}r_\infty^2 - t_a^{-1/2}r_\infty^2.$$

Assume that there is a $\theta \in \mathbb{S}^1$ and a $t_a \ge t_1$ such that

$$t_a^{-1/2}g_\pm(t_a, \theta) - t_a^{-1/2}r_\infty^2 > 0.$$

Then there is a constant $c > 0$ and a $\theta \in \mathbb{S}^1$ such that

$$t_b^{-1/2}g_\pm(t_b, \theta) \ge c$$

for all $t_b \ge t_a$. On the other hand, due to previous estimates, we know that the left hand side tends to zero. Thus we have to have

$$g_\pm(t, \theta) \le r_\infty^2$$

for all $(t, \theta) \in [t_1, \infty) \times \mathbb{S}^1$. Due to the definition of $g_\pm$, this inequality can be written

$$\frac{1}{2}\int_{t_1}^t (s\mathcal{A}_\mp)_{\circ\pm} ds + 2\int_{t_1}^t s[\alpha(P_\theta^2 + e^{2P}Q_\theta^2)]_{\circ\pm} ds + f_\pm \le \frac{1}{2}r_\infty^2 \ln t + r_\infty^2 + Ct^{-\eta}.$$

Combining this estimate with (119), we obtain

$$(t^2\mathcal{A}_\mp)_{\circ\pm} \le r_\infty^2 + Ct^{-\eta}.$$

Since the estimate is independent of θ, we conclude that

$$\|P_t^2 + e^{2P}Q_t^2 + \alpha(P_\theta^2 + e^{2P}Q_\theta^2)\|_{C^0} \le Ct^{-2}$$

for all $t \ge t_1$. Returning to the above estimates, it can then be seen that η can be chosen to equal δ. In fact, the statement of the lemma follows. □

Finally, let us separate the different parts of the energy.

Lemma 53. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \ge \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$. Then there is a constant C such that*

$$\|e^{2P}(Q_t^2 + \alpha Q_\theta^2)\|_{C^0} \le Ct^{-2-\delta}$$

and such that

$$\|tP_t(t, \cdot) - r_\infty\|_{C^0} + \|tP_\theta(t, \cdot)\|_{C^0} \le Ct^{-\delta}$$

for all $t \ge t_1$.

Proof. Before writing down the details, let us give the idea behind the proof. First of all, it is of interest to note that we first consider the energy associated with Q, and then the energy associated with P. The reason for carrying out the argument in this order is that we need an estimate of the Q-part of the energy before being able to estimate the P-part. When deriving the estimate for the Q-energy, we consider $\partial_\pm(t^{1/2}e^P\partial_\mp Q)$. The reason for considering the derivative of $t^{1/2}\partial_\mp Q$ is that the combination of

- the term that results when the derivative hits $t^{1/2}$ and
- the first term on the right hand side of (27) (times $t^{1/2}$)

is given by

$$\frac{1}{2}t^{-1/2}\partial_\mp Q - t^{-1/2}Q_t = -\frac{1}{2}t^{-1/2}\partial_\pm Q.$$

When deriving estimates, we are interested in composing this expression with $\Phi_\pm$ and integrating with respect to t. In that context, it is useful to have $\partial_\pm Q$ as opposed to $\partial_\mp Q$, since $(\partial_\pm Q)_{\circ\pm} = \partial_t Q_{\circ\pm}$; the latter expression (when multiplied by other functions) can be integrated partially in order to obtain an improved estimate. On the other hand, in order for these estimates to be useful, we need to know that $e^P\partial_\pm Q = o(t^{-1/2})$; one particular consequence of earlier estimates. In order to control what remains after carrying out the partial integration, it is convenient to rewrite $\partial_\pm Q$ as $\partial_\pm Q = \partial_\pm(Q - \langle Q\rangle) + \langle Q_t\rangle$. The expressions resulting from the first term on the right hand side can be estimated after a partial integration (using prior knowledge concerning the spatial variation of Q) and the expressions resulting from $\langle Q_t\rangle$ can be estimate due to our earlier L^2-estimates of the derivatives of Q.

Turning to the details, it can be computed that

$$\begin{aligned}
\partial_\pm(t^{1/2}e^P\partial_\mp Q) =& -\frac{1}{2}t^{-1/2}e^P\partial_\pm Q - t^{1/2}e^P\partial_\mp P\partial_\pm Q + t^{1/2}\frac{\alpha_t}{2\alpha}e^P\partial_\mp Q \\
=& -\frac{1}{2}t^{-1/2}e^P\langle Q_t\rangle - \frac{1}{2}t^{-1/2}e^P\partial_\pm(Q - \langle Q\rangle) - t^{1/2}e^P\partial_\mp P\partial_\pm(Q - \langle Q\rangle) \\
& - t^{1/2}e^P\partial_\mp P\langle Q_t\rangle + t^{1/2}\frac{\alpha_t}{2\alpha}e^P\partial_\mp Q \\
=& -\frac{1}{2}t^{-1/2}e^P\partial_\pm(Q - \langle Q\rangle) - t^{1/2}e^P\partial_\mp P\partial_\pm(Q - \langle Q\rangle) + O(t^{-3/2-\delta/2}),
\end{aligned} \tag{120}$$

where the constant implicit in the expression $O(t^{-3/2-\delta/2})$ is independent of θ. Thus

$$\begin{aligned}
\partial_t[t^{1/2}e^{P_{\circ\pm}}(\partial_\mp Q)_{\circ\pm}] =& -\frac{1}{2}t^{-1/2}e^{P_{\circ\pm}}\partial_t(Q_{\circ\pm} - \langle Q\rangle) \\
& - t^{1/2}e^{P_{\circ\pm}}(\partial_\mp P)_{\circ\pm}\partial_t(Q_{\circ\pm} - \langle Q\rangle) + O(t^{-3/2-\delta/2}),
\end{aligned}$$

where we use the notation $f_{\circ\pm} = f\circ\Phi_\pm$. Integrating this equality from t_a to t_b (and integrating partially in the integrals that arise from the integral of the first two terms on the right hand side), we obtain

$$t_b^{1/2}[e^{P_{\circ\pm}}(\partial_\mp Q)_{\circ\pm}](t_b,\theta) - t_a^{1/2}[e^{P_{\circ\pm}}(\partial_\mp Q)_{\circ\pm}](t_a,\theta) = O(t_a^{-1/2-\delta/2}),$$

where the constant implicit in the expression $O(t^{-1/2-\delta/2})$ is independent of θ. Letting t_b tend to infinity, we obtain

$$\|e^{2P}(Q_t^2 + \alpha Q_\theta^2)\|_{C^0} \le Ct^{-2-\delta}$$

for $t \geq t_1$. Let us turn to P. Compute

$$\begin{aligned}
\partial_\pm(t^{1/2}\partial_\mp P) =& \frac{1}{2t^{1/2}}\partial_\mp P - \frac{1}{t^{1/2}}P_t + t^{1/2}e^{2P}(Q_t^2 - \alpha Q_\theta^2) - \frac{e^{P+\lambda/2}K^2}{2t^3} + t^{1/2}\frac{\alpha_t}{2\alpha}\partial_\mp P \\
=& -\frac{1}{2}t^{-1/2}\partial_\pm P + O(t^{-3/2-\delta}) \\
=& -\frac{1}{2}t^{-1/2}\frac{1}{\langle\alpha^{-1/2}\rangle}\langle\alpha^{-1/2}P_t\rangle - \frac{1}{2}t^{-1/2}\partial_\pm\left(P - \frac{1}{\langle\alpha^{-1/2}\rangle}\langle\alpha^{-1/2}P\rangle\right) \\
& -\frac{1}{2}t^{-1/2}\left\langle\partial_t\left(\frac{\alpha^{-1/2}}{\langle\alpha^{-1/2}\rangle}\right)P\right\rangle + O(t^{-3/2-\delta}),
\end{aligned}$$

where the constant implicit in $O(t^{-3/2-\delta})$ is independent of θ. Combining this equality with (103) and (104), we conclude that

$$\partial_\pm(t^{1/2}\partial_\mp P) = -\frac{1}{2}r_\infty t^{-3/2} - \frac{1}{2}t^{-1/2}\partial_\pm\left(P - \frac{1}{\langle\alpha^{-1/2}\rangle}\langle\alpha^{-1/2}P\rangle\right) + O(t^{-3/2-\delta}).$$

This equality implies that

$$\frac{\partial}{\partial t}[t^{1/2}(\partial_\mp P)_{\circ\pm}] = -\frac{1}{2}r_\infty t^{-3/2} - t^{-1/2}\frac{\partial}{\partial t}\left(P_{\circ\pm} - \frac{1}{\langle\alpha^{-1/2}\rangle}\langle\alpha^{-1/2}P\rangle\right) + O(t^{-3/2-\delta}).$$

Integrating this equality from t_a to t_b (and integrating the expression arising from the second term on the right hand side partially), we obtain

$$t_b^{1/2}(\partial_\mp P)_{\circ\pm}(t_b,\theta) - t_a^{1/2}(\partial_\mp P)_{\circ\pm}(t_a,\theta) = r_\infty t_b^{-1/2} - r_\infty t_a^{-1/2} + O(t_a^{-1/2-\delta}).$$

Letting $t_b \to \infty$ in this equality, we conclude that

$$\|t(\partial_\mp P)(t,\cdot) - r_\infty\|_{C^0} \leq Ct^{-\delta}$$

for $t \geq t_1$. The lemma follows. □

Corollary 54. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t,\theta) \in (t_0,\infty)\times\mathbb{S}^1$. Then there is a constant C such that*

$$\|t\lambda_t - r_\infty^2\|_{C^0} + \|t\lambda_\theta\|_{C^0} \leq Ct^{-\delta}$$

for all $t \geq t_1$. Moreover, α_∞ (the limit of α) is C^1, and

$$\|\partial_\theta[\ln\alpha(t,\cdot) - \ln\alpha_\infty]\|_{C^0} \leq Ct^{-1-\gamma-\delta}$$

for all $t \geq t_1$.

Proof. The estimates concerning λ are immediate consequences of Lemma 53, the equations for λ_t and λ_θ, as well as previous estimates. In order to derive the stated conclusions for α, note that

$$\partial_t\partial_\theta\ln\alpha = -\left(P_\theta + \frac{1}{2}\lambda_\theta\right)\frac{e^{P+\lambda/2}K^2}{t^{5/2}}.$$

Due to this identity and previous estimates, we obtain the desired conclusion for α. □

9 Asymptotics uniquely determining a pseudo-homogeneous solution

Due to the asymptotic information obtained in the previous section, we are in a position to prove the following statement: given a solution such that α is bounded from below by a positive constant, there is a uniquely associated pseudo-homogeneous solution. To begin with, it is, however, of interest to note that if $B = 0$ and α is bounded from below by a positive constant, then the solution has to be polarised. As a preliminary step, let us define the following object:

$$\Gamma_2 = \frac{1}{t}\int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} Q_t (Q - \langle Q \rangle) d\theta. \tag{121}$$

Lemma 55. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Then*

$$\frac{d\Gamma_2}{dt} = -\frac{2}{t}\Gamma_2 - \frac{1}{t}\int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} \left(\alpha Q_\theta^2 - Q_t^2\right) d\theta - \frac{1}{t^2} B \langle Q_t \rangle. \tag{122}$$

Remark 56. It is important to note that this calculation holds in general; it is not dependent on α having a positive lower bound.

Proof. Compute

$$\begin{aligned} \frac{d\Gamma_2}{dt} &= -\frac{2}{t}\Gamma_2 + \frac{1}{t^2}\int_{\mathbb{S}^1} \partial_t \left(t\alpha^{-1/2} e^{2P} Q_t\right)(Q - \langle Q \rangle) d\theta + \frac{1}{t}\int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} Q_t (Q_t - \langle Q_t \rangle) d\theta \\ &= -\frac{2}{t}\Gamma_2 - \frac{1}{t^2}\int_{\mathbb{S}^1} t\alpha^{1/2} e^{2P} Q_\theta^2 d\theta + \frac{1}{t}\int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} Q_t^2 d\theta - \frac{1}{t}\int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} Q_t d\theta \langle Q_t \rangle. \end{aligned}$$

The lemma follows. □

Lemma 57. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$. Assume, moreover, that $B = 0$. Then Q is constant; i.e., the solution is polarised.*

Proof. Define

$$\mathcal{E}_Q = \int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} (Q_t^2 + \alpha Q_\theta^2) d\theta.$$

It can be computed that

$$\begin{aligned} \partial_t \mathcal{E}_Q = &-\frac{2}{t}\int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} Q_t^2 d\theta + \int_{\mathbb{S}^1} \frac{\alpha_t}{2\alpha} e^{2P} (\alpha^{-1/2} Q_t^2 + \alpha^{1/2} Q_\theta^2) d\theta \\ &- 2\int_{\mathbb{S}^1} P_t e^{2P} (\alpha^{-1/2} Q_t^2 - \alpha^{1/2} Q_\theta^2) d\theta, \end{aligned}$$

so that

$$\partial_t \mathcal{E}_Q = -\frac{2(1 + r_\infty)}{t}\int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} Q_t^2 d\theta + \frac{2r_\infty}{t}\int_{\mathbb{S}^1} \alpha^{1/2} e^{2P} Q_\theta^2 d\theta + O(t^{-1-\delta})\mathcal{E}_Q,$$

where we have used the fact that $\alpha_t/\alpha = O(t^{-1-\gamma})$ and the fact that $P_t = r_\infty/t + O(t^{-1-\delta})$; cf. Lemma 53. Let

$$E_Q = \mathcal{E}_Q + (1 + 2r_\infty)\Gamma_2,$$

where Γ_2 is defined in (121). Note that $|\Gamma_2| \leq C\mathcal{E}_Q/t$, so that there is a T such that

$$\frac{1}{2}\mathcal{E}_Q \leq E_Q \leq 2\mathcal{E}_Q$$

for all $t \geq T$. Note, in particular, that $|\Gamma_2|/t \leq Ct^{-1-\delta}E_Q$. Combining the above observations, we have

$$\partial_t E_Q \geq -\frac{1}{t}E_Q - \frac{C}{t^{1+\delta}}E_Q$$

for $t \geq T$. In particular,

$$E_Q(t) \geq CE_Q(T)t^{-1}$$

for $t \geq T$. On the other hand, we know that $E_Q(t) \leq Ct^{-2-\delta}$ due to Lemma 43. Combining these observations, we conclude that $E_Q = 0$. The lemma follows. □

Lemma 58. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t,\theta) \in (t_0,\infty)\times\mathbb{S}^1$. Then there is a unique pseudo-homogeneous solution $(P_{\text{hom}}, Q_{\text{hom}}, \lambda_{\text{hom}}, \alpha_{\text{hom}})$ having the same conserved quantities A and B and satisfying*

$$\lim_{t\to\infty}(\|\alpha - \alpha_{\text{hom}}\|_{C^0} + \|P - P_{\text{hom}}\|_{C^0} + \|Q - Q_{\text{hom}}\|_{C^0} + \|\lambda - \lambda_{\text{hom}}\|_{C^0}) = 0.$$

Proof. Let us begin by considering the case $B = 0$. Due to Lemma 57, we know that the solution is polarised. We can therefore choose $Q_{\text{hom}} = Q$. Due to Corollary 54, we know that α converges to α_∞ in C^1. Letting $q_\infty = Q$, it is clear that Q converges to q_∞ in C^0. Moreover, due to (92) and (102), we know that there are constants c_P and c_λ such that

$$\lim_{t\to\infty}(\|P - r_\infty \ln t - c_P\|_{C^0} + \|\lambda - r_\infty^2 \ln t - c_\lambda\|_{C^0}) = 0.$$

Finally, due to Lemma 39, we know that $r_\infty \in (-3, 1)$. As a consequence, we are allowed to appeal to Proposition 14, and the lemma follows in the case $B = 0$. The case $B \neq 0$ is similar, but slightly simpler. □

Due to this observation, we are in a position to prove that if $B = 0$, then the solution is pseudo-homogeneous.

Theorem 59. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K \neq 0$ and $B = 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t,\theta) \in (t_0,\infty)\times\mathbb{S}^1$. Then the solution is pseudo-homogeneous.*

Proof. Due to Lemma 53, (92) and (102), it can be estimated that

$$\hat{\mathcal{H}}(t) \leq Ct^{-2-2\delta} \tag{123}$$

for all $t \geq t_1$ (the estimate being independent of the choice of ρ in the definition of $\hat{\mathcal{H}}$). In this estimate, $C > 0$ is a constant and $\hat{\mathcal{H}}$ was defined in the statement of Lemma 16. Choosing $\rho = \delta$ in the definition of $\hat{\mathcal{H}}$, the estimate (49) implies that

$$\hat{\mathcal{H}}(t_a) \leq C(t/t_a)^{2+\delta}\hat{\mathcal{H}}(t) \leq Ct^{-\delta}$$

for all $t \geq t_a \geq t_1$, where we used (123) in the last step. Letting $t \to \infty$ we conclude that $\hat{\mathcal{H}}(t_a) = 0$ for all $t_a \geq t_1$. Thus $P = P_{\text{hom}}$, $\lambda = \lambda_{\text{hom}}$ and $Q = Q_{\text{hom}}$. Since $\alpha_t/\alpha = \partial_t\alpha_{\text{hom}}/\alpha_{\text{hom}}$, and since α and α_{hom} converge to the same function, we conclude that $\alpha = \alpha_{\text{hom}}$. □

10 Improved estimates of the difference between the solution and the associated pseudo-homogeneous solution

Given a solution such that α is bounded from below by a positive constant, Lemma 58 ensures that there is a uniquely associated pseudo-homogeneous solution. In the present section, we wish to improve our estimates of the difference between the solution and the associated pseudo-homogeneous solution (since we already know that the difference is zero in case $B = 0$, we assume

$B \neq 0$ here). Let us denote the relevant pseudo-homogeneous solution by the same letters as the original solution, but with the subscript hom. Note that Lemmas 39 and 46 apply to the pseudo-homogeneous solution and that K, A, B, r_∞, c_P and c_λ are the same for the two solutions. In particular, we thus have

$$\begin{aligned} \|P(t,\cdot) - P_{\mathrm{hom}}(t)\|_{C^0} + \|\lambda(t,\cdot) - \lambda_{\mathrm{hom}}(t)\|_{C^0} &\leq Ct^{-\delta}, \\ \|\alpha(t,\cdot) - \alpha_{\mathrm{hom}}(t,\cdot)\|_{C^0} &\leq Ct^{-\gamma}, \end{aligned}$$

for all $t \geq t_1$; note that α_{hom} is allowed to depend on θ. Let us now, for the sake of argument, assume that

$$\|P(t,\cdot) - P_{\mathrm{hom}}(t)\|_{C^0} + \|\lambda(t,\cdot) - \lambda_{\mathrm{hom}}(t)\|_{C^0} \leq Ct^{-a}, \tag{124}$$

$$\int_{\mathbb{S}^1} \alpha^{1/2}(P_\theta^2 + e^{2P}Q_\theta^2)d\theta \leq Ct^{-2b}, \tag{125}$$

$$\int_{\mathbb{S}^1} \alpha^{-1/2}e^{2P}Q_t^2 d\theta \leq Ct^{-2c}, \tag{126}$$

for some constants $a, b, c > 0$ and all $t \geq t_1$. Note that we know these estimates to hold with $\delta = a$ and $b = c = 1 + \delta/2$. In what follows, we shall assume $1 < c \leq b$.

Let us compare $\langle \alpha^{-1/2} \rangle$ with $\langle \alpha_{\mathrm{hom}}^{-1/2} \rangle$.

Lemma 60. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t,\theta) \in (t_0,\infty) \times \mathbb{S}^1$. Then*

$$\langle \alpha^{-1/2} \rangle = \langle \alpha_{\mathrm{hom}}^{-1/2} \rangle [1 + O(t^{-\gamma - a})], \tag{127}$$

assuming that (124) holds.

Proof. Compute

$$\partial_t \langle \alpha^{-1/2} \rangle = \left\langle -\frac{1}{2}\alpha^{-1/2}\frac{\alpha_t}{\alpha} \right\rangle = \left\langle \alpha^{-1/2}\frac{e^{P+\lambda/2}K^2}{2t^{5/2}} \right\rangle = -\frac{1}{2}\langle \alpha^{-1/2} \rangle \frac{\partial_t \alpha_{\mathrm{hom}}}{\alpha_{\mathrm{hom}}} + O(t^{-1-\gamma-a});$$

note that $\partial_t \alpha_{\mathrm{hom}}/\alpha_{\mathrm{hom}}$ is independent of θ even though α_{hom} need not be. Consequently,

$$\partial_t \left(\frac{\langle \alpha^{-1/2} \rangle}{\langle \alpha_{\mathrm{hom}}^{-1/2} \rangle} \right) = O(t^{-1-\gamma-a}).$$

The lemma follows. ∎

Before proceeding to the asymptotics for Q, it is of interest to make the following observation.

Lemma 61. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t,\theta) \in (t_0,\infty) \times \mathbb{S}^1$. Then there is a constant $C > 0$ such that*

$$\left\| \partial_t \left(\frac{\alpha^{-1/2}}{\langle \alpha^{-1/2} \rangle} \right) \right\|_{C^0} \leq Ct^{-1-\gamma} \left(\|P - \langle P \rangle\|_{C^0} + \|\lambda - \langle \lambda \rangle\|_{C^0} \right) \tag{128}$$

for all $t \geq t_1$. In particular,

$$\left\| \partial_t \left(\frac{\alpha^{-1/2}}{\langle \alpha^{-1/2} \rangle} \right) \right\|_{C^0} \leq Ct^{-1-\gamma-b} \tag{129}$$

for all $t \geq t_1$, assuming (125) to hold.

Proof. Due to (105) and (106), the estimate (128) holds. Combining this estimate with (72), (74) and (125), we obtain (129). ∎

Given this information, let us turn to the asymptotics for Q.

Lemma 62. *Consider a solution to (2)–(6) on* $(t_0,\infty)\times\mathbb{S}^1$ *with* $K\neq 0$ *and* $B\neq 0$*. Assume that there is an* $\alpha_0>0$ *such that* $\alpha\geq\alpha_0$ *for all* $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$*. Then*

$$\|Q-Q_{\mathrm{hom}}\|_{C^0}\leq Ct^{-a-2r_\infty}+Ct^{-b-r_\infty}, \tag{130}$$

for $t\geq t_1$*, assuming (124)–(126) to hold.*

Proof. Consider (79). It is of interest to improve the estimates implicit in this equation slightly. Due to (129),

$$\begin{aligned}\partial_t\left[\frac{1}{\langle\alpha^{-1/2}\rangle}\int_{\mathbb{S}^1}\alpha^{-1/2}(Q-\langle Q\rangle)d\theta\right]=&\frac{1}{\langle\alpha^{-1/2}\rangle}\int_{\mathbb{S}^1}\alpha^{-1/2}(Q_t-\langle Q_t\rangle)d\theta+O(t^{-1-\gamma-2b}e^{-\langle P\rangle})\\=&\frac{1}{e^{2\langle P\rangle}\langle\alpha^{-1/2}\rangle}t^{-1}B-2\pi\langle Q_t\rangle+O(t^{-b-c}e^{-\langle P\rangle}).\end{aligned} \tag{131}$$

Moreover,

$$\frac{1}{e^{2\langle P\rangle}\langle\alpha^{-1/2}\rangle}t^{-1}B=2\pi\frac{\langle\alpha_{\mathrm{hom}}^{-1/2}\rangle}{\langle\alpha^{-1/2}\rangle}e^{2(P_{\mathrm{hom}}-\langle P\rangle)}\partial_tQ_{\mathrm{hom}}=2\pi\partial_tQ_{\mathrm{hom}}+O(t^{-1-a-2r_\infty}).$$

Thus

$$\langle Q\rangle=Q_{\mathrm{hom}}+O(t^{-a-2r_\infty})+O(t^{-b-r_\infty}).$$

The lemma follows. ☐

Let us turn to P.

Lemma 63. *Consider a solution to (2)–(6) on* $(t_0,\infty)\times\mathbb{S}^1$ *with* $K\neq 0$ *and* $B\neq 0$*. Assume that there is an* $\alpha_0>0$ *such that* $\alpha\geq\alpha_0$ *for all* $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$*. Then*

$$\|P-P_{\mathrm{hom}}\|_{C^0}\leq Ct^{-b}+Ct^{-a-\delta} \tag{132}$$

for $t\geq t_1$*, assuming (124)–(126) to hold.*

Proof. Due to (129), we have

$$\begin{aligned}\partial_t\left[\frac{1}{\langle\alpha^{-1/2}\rangle}\int_{\mathbb{S}^1}\alpha^{-1/2}(P-\langle P\rangle)d\theta\right]=&\frac{1}{\langle\alpha^{-1/2}\rangle}\int_{\mathbb{S}^1}\alpha^{-1/2}(P_t-\langle P_t\rangle)d\theta+O(t^{-1-\gamma-2b})\\=&\frac{1}{\langle\alpha^{-1/2}\rangle}\frac{1}{t}(A+B\langle Q\rangle)-2\pi\langle P_t+t^{-1}\rangle\\&+\frac{1}{\langle\alpha^{-1/2}\rangle}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}Q_t(Q-\langle Q\rangle)d\theta+O(t^{-1-\gamma-2b})\\=&\frac{1}{\langle\alpha^{-1/2}\rangle}\frac{1}{t}(A+B\langle Q\rangle)-2\pi\langle P_t+t^{-1}\rangle+O(t^{-b-c}).\end{aligned} \tag{133}$$

Note that

$$\frac{1}{\langle\alpha^{-1/2}\rangle}\frac{1}{t}(A+B\langle Q\rangle)=\frac{1}{\langle\alpha^{-1/2}\rangle}\frac{1}{t}B(\langle Q\rangle-Q_{\mathrm{hom}})+2\pi\frac{\langle\alpha_{\mathrm{hom}}^{-1/2}\rangle}{\langle\alpha^{-1/2}\rangle}(\partial_tP_{\mathrm{hom}}+t^{-1}).$$

Thus

$$\frac{1}{\langle\alpha^{-1/2}\rangle}\frac{1}{t}(A+B\langle Q\rangle)=2\pi(\partial_tP_{\mathrm{hom}}+t^{-1})+O(t^{-1-a-\delta})+O(t^{-1-b-r_\infty}).$$

Integrating (133), using this information, we conclude that

$$\langle P\rangle = P_{\mathrm{hom}} + O(t^{-b}) + O(t^{-a-\delta}).$$

The lemma follows. □

Given our improved knowledge concerning P, we are in a position to improve the above estimates for Q.

Lemma 64. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$ and $B\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then*

$$\|Q-Q_{\mathrm{hom}}\|_{C^0} \leq Ct^{-b-r_\infty} + Ct^{-a-\delta-2r_\infty} \tag{134}$$

for all $t\geq t_1$, assuming (124)–(126) to hold.

Proof. Compute

$$\frac{1}{e^{2\langle P\rangle}\langle\alpha^{-1/2}\rangle}t^{-1}B = 2\pi\partial_t Q_{\mathrm{hom}} + O(t^{-1-b-2r_\infty}) + O(t^{-1-a-\delta-2r_\infty}).$$

As a consequence,

$$\int_t^\infty \frac{1}{e^{2\langle P\rangle}\langle\alpha^{-1/2}\rangle}\frac{1}{s}Bds = 2\pi(q_\infty - Q_{\mathrm{hom}}) + O(t^{-b-2r_\infty}) + O(t^{-a-\delta-2r_\infty}).$$

Integrating (131) from t to ∞, we thus obtain the conclusion of the lemma. □

Next, it is natural to turn to the energies.

Lemma 65. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$ and $B\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Assume, moreover, (124)–(126) to hold. Then*

$$\int_{\mathbb{S}^1} t^2\alpha^{1/2}(P_\theta^2 + e^{2P}Q_\theta^2)d\theta \leq Ct^{-a-\gamma} + Ct^{1-b-c} + Ct^{-b-r_\infty} + Ct^{-a-\delta-2r_\infty} \tag{135}$$

for all $t\geq t_1$. Moreover,

$$\begin{aligned}\int_{\mathbb{S}^1} t^2\alpha^{-1/2}P_t^2 d\theta &= \int_{\mathbb{S}^1} t^2\alpha_{\mathrm{hom}}^{-1/2}(\partial_t P_{\mathrm{hom}})^2 d\theta \\ &\quad + O(t^{-a-\gamma}) + O(t^{1-b-c}) + O(t^{-b-r_\infty}) + O(t^{-a-\delta-2r_\infty}),\end{aligned} \tag{136}$$

$$\begin{aligned}\int_{\mathbb{S}^1} t^2\alpha^{-1/2}e^{2P}Q_t^2 d\theta &= \int_{\mathbb{S}^1} t^2\alpha_{\mathrm{hom}}^{-1/2}e^{2P_{\mathrm{hom}}}(\partial_t Q_{\mathrm{hom}})^2 d\theta \\ &\quad + O(t^{-a-\gamma}) + O(t^{1-b-c}) + O(t^{-b-r_\infty}) + O(t^{-a-\delta-2r_\infty})\end{aligned} \tag{137}$$

for all $t\geq t_1$.

Proof. It is natural to begin by estimating the integrals of $\alpha^{-1/2}P_t^2$ and $\alpha^{-1/2}e^{2P}Q_t^2$ from below.

Q-energy. Let us begin by considering the Q-energy. Note, to this end, that

$$|B| \leq \int_{\mathbb{S}^1} t\alpha^{-1/2}e^{2P}|Q_t|d\theta \leq t\left(\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}d\theta\right)^{1/2}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}Q_t^2 d\theta\right)^{1/2}.$$

Consequently,

$$\frac{B^2}{t}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}d\theta\right)^{-1} \leq t\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}Q_t^2 d\theta.$$

On the other hand,

$$\int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} d\theta = \int_{\mathbb{S}^1} \alpha_{\mathrm{hom}}^{-1/2} e^{2P_{\mathrm{hom}}} d\theta \; [1 + O(t^{-b}) + O(t^{-a-\delta})]$$

and

$$B^2 = 4\pi^2 t^2 \langle \alpha_{\mathrm{hom}}^{-1/2} \rangle^2 e^{4P_{\mathrm{hom}}} (\partial_t Q_{\mathrm{hom}})^2.$$

Combining the above observations, we conclude that

$$\int_{\mathbb{S}^1} \alpha_{\mathrm{hom}}^{-1/2} e^{2P_{\mathrm{hom}}} (\partial_t Q_{\mathrm{hom}})^2 d\theta \; [1 + O(t^{-b}) + O(t^{-a-\delta})] \leq \int_{\mathbb{S}^1} \alpha^{-1/2} e^{2P} Q_t^2 d\theta. \tag{138}$$

P-energy. Let us turn to the P-energy. To begin with,

$$\int_{\mathbb{S}^1} t\alpha^{-1/2} P_t d\theta = A - 2\pi \langle \alpha^{-1/2} \rangle + B\langle Q \rangle + O(t^{1-b-c}).$$

As a consequence of this equality and previous estimates,

$$\int_{\mathbb{S}^1} t\alpha^{-1/2} P_t d\theta = \int_{\mathbb{S}^1} t\alpha_{\mathrm{hom}}^{-1/2} \partial_t P_{\mathrm{hom}} d\theta + O(t^{-a-\gamma}) + O(t^{1-b-c}) + O(t^{-b-r_\infty}) + O(t^{-a-\delta-2r_\infty}).$$

Squaring this equality, we obtain

$$\begin{aligned} &2\pi \langle \alpha_{\mathrm{hom}}^{-1/2} \rangle t^2 \int_{\mathbb{S}^1} \alpha_{\mathrm{hom}}^{-1/2} (\partial_t P_{\mathrm{hom}})^2 d\theta + O(t^{-a-\gamma}) + O(t^{1-b-c}) + O(t^{-b-r_\infty}) + O(t^{-a-\delta-2r_\infty}) \\ \leq &2\pi \langle \alpha^{-1/2} \rangle t^2 \int_{\mathbb{S}^1} \alpha^{-1/2} P_t^2 d\theta, \end{aligned}$$

so that

$$\begin{aligned} &t^2 \int_{\mathbb{S}^1} \alpha_{\mathrm{hom}}^{-1/2} (\partial_t P_{\mathrm{hom}})^2 d\theta + O(t^{-a-\gamma}) + O(t^{1-b-c}) + O(t^{-b-r_\infty}) + O(t^{-a-\delta-2r_\infty}) \\ \leq &t^2 \int_{\mathbb{S}^1} \alpha^{-1/2} P_t^2 d\theta. \end{aligned} \tag{139}$$

Thus

$$\begin{aligned} &t^2 \int_{\mathbb{S}^1} \alpha_{\mathrm{hom}}^{-1/2} [(\partial_t P_{\mathrm{hom}})^2 + e^{2P_{\mathrm{hom}}} (\partial_t Q_{\mathrm{hom}})^2] d\theta \\ &+ O(t^{1-b-c}) + O(t^{-a-\gamma}) + O(t^{-b-r_\infty}) + O(t^{-a-\delta-2r_\infty}) \\ \leq &t^2 \int_{\mathbb{S}^1} \alpha^{-1/2} (P_t^2 + e^{2P} Q_t^2) d\theta. \end{aligned}$$

α-energy. Turning to the α-contribution to the energy, note that

$$\int_{\mathbb{S}^1} \frac{\alpha^{-1/2} e^{P+\lambda/2} K^2}{t^{3/2}} d\theta + 3\int_{\mathbb{S}^1} \alpha^{-1/2} d\theta = \int_{\mathbb{S}^1} \frac{\alpha_{\mathrm{hom}}^{-1/2} e^{P_{\mathrm{hom}}+\lambda_{\mathrm{hom}}/2} K^2}{t^{3/2}} d\theta + 3\int_{\mathbb{S}^1} \alpha_{\mathrm{hom}}^{-1/2} d\theta + O(t^{-\gamma-a}). \tag{140}$$

Since the $\hat{H}$-energy for the pseudo-homogeneous solution is constant and equal to the $\hat{H}_\infty$-energy for the solution under consideration, we conclude that

$$\begin{aligned} &\hat{H}_\infty + O(t^{-a-\gamma}) + O(t^{1-b-c}) + O(t^{-b-r_\infty}) + O(t^{-a-\delta-2r_\infty}) \\ \leq &t^2 \int_{\mathbb{S}^1} \alpha^{-1/2} (P_t^2 + e^{2P} Q_t^2) d\theta + \int_{\mathbb{S}^1} \frac{\alpha^{-1/2} e^{P+\lambda/2} K^2}{t^{3/2}} d\theta + 3\int_{\mathbb{S}^1} \alpha^{-1/2} d\theta. \end{aligned} \tag{141}$$

Total energy. Due to (141) and the fact that $\hat{H}$ is increasing, we conclude that

$$\begin{aligned}
&\int_{\mathbb{S}^1} t^2\alpha^{1/2}(P_\theta^2+e^{2P}Q_\theta^2)d\theta\\
=&\hat{H}-\int_{\mathbb{S}^1} t^2\alpha^{-1/2}(P_t^2+e^{2P}Q_t^2)-\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{3/2}}d\theta-\int_{\mathbb{S}^1}3\alpha^{-1/2}d\theta\\
\leq&Ct^{-a-\gamma}+Ct^{1-b-c}+Ct^{-b-r_\infty}+Ct^{-a-\delta-2r_\infty}.
\end{aligned}$$

Thus (135) holds. Combining (135) with (24), we obtain

$$0\leq\hat{H}_\infty-\hat{H}(t)=\int_t^\infty 2s\int_{\mathbb{S}^1}\alpha^{1/2}(P_\theta^2+e^{2P}Q_\theta^2)d\theta ds\leq Ct^{-a-\gamma}+Ct^{1-b-c}+Ct^{-b-r_\infty}+Ct^{-a-\delta-2r_\infty}.$$

Combining this estimate with (135) and (140), we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{S}^1} t^2\alpha_{\mathrm{hom}}^{-1/2}(\partial_t P_{\mathrm{hom}})^2d\theta-\int_{\mathbb{S}^1}t^2\alpha^{-1/2}P_t^2d\theta\right)\\
&+\left(\int_{\mathbb{S}^1} t^2\alpha_{\mathrm{hom}}^{-1/2}e^{2P_{\mathrm{hom}}}(\partial_t Q_{\mathrm{hom}})^2d\theta-\int_{\mathbb{S}^1}t^2\alpha^{-1/2}e^{2P}Q_t^2d\theta\right)\\
=&O(t^{-a-\gamma})+O(t^{1-b-c})+O(t^{-b-r_\infty})+O(t^{-a-\delta-2r_\infty}).
\end{aligned}$$

On the other hand, due to (138) and (139), we know that the terms inside the parantheses on the left hand side have upper bound of the same order as the right hand side. For that reason, the corresponding lower bounds also hold. The lemma follows. □

Finally, we are in a position to improve our knowledge concerning the asymptotics for λ.

Lemma 66. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$ and $B\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Assume, moreover, (124)–(126) to hold. Then*

$$\|\lambda-\lambda_{\mathrm{hom}}\|_{C^0}\leq Ct^{-b}+Ct^{-a-\gamma}+Ct^{-a-\delta-2r_\infty}\tag{142}$$

for all $t\geq t_1$.

Proof. Due to (74) and (125), $\|\lambda-\langle\lambda\rangle\|_{C^0}\leq Ct^{-b}$. Due to (129),

$$\partial_t\left[\frac{1}{\langle\alpha^{-1/2}\rangle}\langle\alpha^{-1/2}(\lambda-\langle\lambda\rangle)\rangle\right]=O(t^{-1-\gamma-b}\|\lambda-\langle\lambda\rangle\|_{C^0})+\frac{1}{\langle\alpha^{-1/2}\rangle}\langle\alpha^{-1/2}\lambda_t\rangle-\langle\lambda_t\rangle.\tag{143}$$

On the other hand,

$$\begin{aligned}
\langle\alpha^{-1/2}\lambda_t\rangle=&\frac{1}{2\pi}\int_{\mathbb{S}^1}t\alpha^{-1/2}[P_t^2+e^{2P}Q_t^2+\alpha(P_\theta^2+e^{2P}Q_\theta^2)]d\theta-\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{5/2}}d\theta\\
=&\frac{1}{2\pi}\int_{\mathbb{S}^1}t\alpha_{\mathrm{hom}}^{-1/2}[(\partial_t P_{\mathrm{hom}})^2+e^{2P_{\mathrm{hom}}}(\partial_t Q_{\mathrm{hom}})^2]d\theta-\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\alpha_{\mathrm{hom}}^{-1/2}e^{P_{\mathrm{hom}}+\lambda_{\mathrm{hom}}/2}K^2}{t^{5/2}}d\theta\\
&+O(t^{-1-a-\gamma})+O(t^{-b-c})+O(t^{-1-b-r_\infty})+O(t^{-1-a-\delta-2r_\infty}).
\end{aligned}$$

Combining this observation with (127), we conclude that

$$\frac{\langle\alpha^{-1/2}\lambda_t\rangle}{\langle\alpha^{-1/2}\rangle}=\partial_t\lambda_{\mathrm{hom}}+O(t^{-1-a-\gamma})+O(t^{-b-c})+O(t^{-1-b-r_\infty})+O(t^{-1-a-\delta-2r_\infty}).$$

The lemma follows. □

10.1 Iterative improvement of the C^0-estimates

It is of interest to iterate the above procedure. This leads to the following result.

Lemma 67. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$ and $B\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\geq\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then, for every $\eta<2+\min\{\gamma,r_\infty\}$, there is a constant C_η such that (124) and (125) hold for $t\geq t_1$, with $a=b=\eta$ and C replaced by C_η.*

Proof. There are two possibilities. Either $\delta=2r_\infty$ or $\delta=\gamma$. Let us begin by considering the case $\delta=2r_\infty$. Let $a_0=\delta=2r_\infty$ and $b_0=1+\delta/2=1+r_\infty$. Combining Lemmas 63 and 66, we conclude that the estimate (124) can be improved in that a can be replaced by

$$a_{n+1}=\min\{b_n,a_n+2r_\infty,1+(a_n+\gamma)/2,1+2r_\infty+a_n/2\},\tag{144}$$

assuming (124) to hold with a replaced by a_n and (125) to hold with b replaced by b_n (note that we actually obtain a better bound, $a_{n+1}=\min\{b_n,a_n+2r_\infty\}$, but as we shall see, this improvement does not lead to an improvement of the end result, and it is more difficult to obtain the desired conclusion using the better bound). Under the same assumptions, we obtain

$$b_{n+1}=\min\{1+(b_n+r_\infty)/2,1+(a_n+\gamma)/2,1+2r_\infty+a_n/2\}\tag{145}$$

due to Lemma 65; note that $c=1+r_\infty$. Due to (145), we have

$$b_{n+1}\leq 1+r_\infty/2+b_n/2.$$

Defining

$$\hat{b}_n=b_n-2-r_\infty,$$

we have $\hat{b}_{n+1}\leq\hat{b}_n/2$, whence $\hat{b}_n\leq 2^{-n}\hat{b}_0=-2^{-n}$. In particular, $b_n<2+r_\infty$ for all n. Due to (144), we conclude that $a_n<2+r_\infty$ (note that this conclusion holds even if we improve the estimate (144) as described above). Note that $b_0\geq a_0$. Moreover, there are three possibilities for b_{n+1}. Either $b_{n+1}=(2+r_\infty+b_n)/2>b_n\geq a_{n+1}$, or $b_{n+1}=1+(a_n+\gamma)/2\geq a_{n+1}$, or $b_{n+1}=1+2r_\infty+a_n/2\geq a_{n+1}$. Thus $b_n\geq a_n$ for all n. By similar arguments, one can therefore prove that $a_{n+1}\geq a_n$. Thus a_n is an increasing sequence of numbers which is bounded from above. Thus a_n converges to some number, say a_*. Assume that $a_*<2+r_\infty$. Then

$$\min\{a_n+2r_\infty,1+(a_n+\gamma)/2,1+2r_\infty+a_n/2\}>a_*$$

for n large enough. Consequently, $a_{n+1}=b_n$ for n large enough. On the other hand, $b_{n+1}>a_*$ for n large enough (note that $b_n\geq a_n$), so that we obtain a contradiction. Thus $a_*=2+r_\infty$. As a consequence, the improvement of (144) described above does not lead to any improvement of the end result. To conclude, if $\delta=2r_\infty$, then (124) and (125) hold with a and b replaced by any number strictly less than $2+r_\infty$.

Let us now consider the case $\delta=\gamma\leq 2r_\infty$. Similarly to the above, we have

$$a_{n+1} = \min\{b_n,a_n+\gamma\},\tag{146}$$
$$b_{n+1} = \min\{(1+b_n+c)/2,1+(a_n+\gamma)/2,1+(b_n+r_\infty)/2\}.\tag{147}$$

We know that this estimate holds with $c=1+\gamma/2$. Thus, we can certainly choose

$$\begin{aligned}a_{n+1} &= \min\{b_n,a_n+\gamma,1+(a_n+\gamma)/2\},\\ b_{n+1} &= \min\{1+(b_n+\gamma/2)/2,1+(a_n+\gamma)/2\},\end{aligned}$$

where we have used the fact that $\gamma\leq 2r_\infty$ (for reasons similar to ones given above, there is no loss in including $1+(a_n+\gamma)/2$ in the formula for a_{n+1}). As above, we can argue that $b_n<2+\gamma/2$, $a_n<2+\gamma/2$, $a_n\leq b_n$, a_n is an increasing sequence and a_n converges to $2+\gamma/2$. Returning to

(137), we conclude that (126) holds with $c = 1 + r_\infty$ (note that this estimate is optimal in case $B \neq 0$). Returning to (146) and (147), we conclude that

$$\begin{aligned} a_{n+1} &= \min\{b_n, a_n + \gamma\}, \\ b_{n+1} &= \min\{1 + (b_n + r_\infty)/2, 1 + (a_n + \gamma)/2\}. \end{aligned}$$

Just as before, we conclude that $a_n, b_n < 2 + r_\infty$. Note that we also have $b_{n+1} \leq 1 + (b_{n-1} + \gamma)/2$. Since $b_0, b_1 < 2 + \gamma$, we conclude that $b_n < 2 + \gamma$, so that $a_n < 2 + \gamma$. Thus

$$a_n, b_n < \min\{2 + r_\infty, 2 + \gamma\}.$$

As above, we also have the iteration

$$\begin{aligned} a_{n+1} &= \min\{b_n, a_n + \gamma, 1 + (a_n + \gamma)/2\}, \\ b_{n+1} &= \min\{1 + (b_n + r_\infty)/2, 1 + (a_n + \gamma)/2\}. \end{aligned}$$

By arguments similar to ones given above, $b_n \geq a_n$ for all n and a_n is an increasing sequence which is bounded from above. Thus a_n converges to, say, a_*. Assuming $a_* < 2 + \min\{r_\infty, \gamma\}$, we obtain a contradiction, as above. □

10.2 Iterative improvement of the C^1-estimates

Using the knowledge we have concerning the C^0-distance between the solution and the associated pseudo-homogeneous solution, it turns out to be possible to improve the C^1-estimates.

Lemma 68. *Consider a solution to (2)–(6) on $(t_0, \infty) \times \mathbb{S}^1$ with $K \neq 0$ and $B \neq 0$. Assume that there is an $\alpha_0 > 0$ such that $\alpha \geq \alpha_0$ for all $(t, \theta) \in (t_0, \infty) \times \mathbb{S}^1$. Let $\eta < 3 + \min\{\gamma, r_\infty\}$. Then there is a constant C_η such that*

$$\|P_t - \partial_t P_{\rm hom}\|_{C^0} + \|e^P (Q_t - \partial_t Q_{\rm hom})\|_{C^0} + \|P_\theta\|_{C^0} + \|e^P Q_\theta\|_{C^0} \leq C_\eta t^{-\eta}$$

for all $t \geq t_1$.

Proof. Let us assume that there is an $1/2 < a_{\rm s} \in \mathbb{R}$ and a constant C such that

$$\|P_t - \partial_t P_{\rm hom}\|_{C^0} + \|e^P (Q_t - \partial_t Q_{\rm hom})\|_{C^0} + \|P_\theta\|_{C^0} + \|e^P Q_\theta\|_{C^0} \leq C t^{-a_{\rm s}} \tag{148}$$

for all $t \geq t_1$. Note that this estimate holds with $a_{\rm s} = 1$ due to Lemma 53. Let us compute

$$\begin{aligned} \partial_\pm (t^{1/2} e^P \partial_\mp Q - t^{1/2} e^{P_{\rm hom}} \partial_\mp Q_{\rm hom}) = &-\frac{1}{2} t^{-1/2} (e^P \partial_\pm Q - e^{P_{\rm hom}} \partial_\pm Q_{\rm hom}) \\ &- t^{1/2} (e^P \partial_\mp P \partial_\pm Q - e^{P_{\rm hom}} \partial_\mp P_{\rm hom} \partial_\pm Q_{\rm hom}) \\ &+ t^{1/2} \left(\frac{\alpha_t}{2\alpha} e^P \partial_\mp Q - \frac{\partial_t \alpha_{\rm hom}}{2\alpha_{\rm hom}} e^{P_{\rm hom}} \partial_\mp Q_{\rm hom} \right); \end{aligned} \tag{149}$$

cf. (120). Let us consider the terms on the right hand side one by one. The first term can be written

$$\begin{aligned} &-\frac{1}{2} t^{-1/2} e^P \partial_\pm (Q - Q_{\rm hom}) - \frac{1}{2} t^{-1/2} (e^{P - P_{\rm hom}} - 1) e^{P_{\rm hom}} \partial_\pm Q_{\rm hom} \\ = &-\frac{1}{2} t^{-1/2} e^P \partial_\pm (Q - Q_{\rm hom}) + O(t^{-3/2 - a - r_\infty}), \end{aligned}$$

where we have assumed that

$$\|\lambda - \lambda_{\rm hom}\|_{C^0} + \|P - P_{\rm hom}\|_{C^0} + \|e^P (Q - Q_{\rm hom})\|_{C^0} \leq C t^{-a}$$

for all $t \geq t_1$; note that such an estimate holds due to Lemmas 64 and 67, and that $a < 2 + \min\{\gamma, r_\infty\}$ can be chosen to be as close to $2 + \min\{\gamma, r_\infty\}$ as we wish. Turning to the second term on the right hand side of (149), it can be written

$$
\begin{aligned}
& - t^{1/2}\partial_\mp P e^P \partial_\pm (Q - Q_{\text{hom}}) - t^{1/2}\partial_\mp (P - P_{\text{hom}}) e^P \partial_\pm Q_{\text{hom}} \\
& - t^{1/2}(e^{P-P_{\text{hom}}} - 1)\partial_\mp P_{\text{hom}} e^{P_{\text{hom}}} \partial_\pm Q_{\text{hom}} \\
= & - t^{1/2}\partial_\mp P e^P \partial_\pm (Q - Q_{\text{hom}}) + O(t^{-1/2-a_{\text{s}}-r_\infty}) + O(t^{-3/2-a-r_\infty}).
\end{aligned}
$$

Finally, the third term on the right hand side of (149) can be written

$$
\begin{aligned}
& t^{1/2}\frac{\alpha_t}{2\alpha} e^P \partial_\mp (Q - Q_{\text{hom}}) + t^{1/2}\frac{\alpha_t}{2\alpha}(e^{P-P_{\text{hom}}} - 1)e^{P_{\text{hom}}}\partial_\mp Q_{\text{hom}} \\
& + \frac{1}{2} t^{1/2}\left(\frac{\alpha_t}{\alpha} - \frac{\partial_t \alpha_{\text{hom}}}{\alpha_{\text{hom}}}\right) e^{P_{\text{hom}}}\partial_\mp Q_{\text{hom}} = O(t^{-1/2-a_{\text{s}}-\gamma}) + O(t^{-3/2-\gamma-a-r_\infty}).
\end{aligned}
$$

Adding up the above observations, we conclude that

$$
\begin{aligned}
& \partial_t \left[t^{1/2}(e^P \partial_\mp Q - e^{P_{\text{hom}}}\partial_\mp Q_{\text{hom}})_{\circ\pm}\right] \\
= & - \frac{1}{2} t^{-1/2} e^{P_{\circ\pm}} \partial_t (Q - Q_{\text{hom}})_{\circ\pm} - t^{1/2}(\partial_\mp P)_{\circ\pm} e^{P_{\circ\pm}} \partial_t (Q - Q_{\text{hom}})_{\circ\pm} \\
& + O(t^{-1/2-a_{\text{s}}-\delta_{\text{s}}}) + (t^{-3/2-a-r_\infty}),
\end{aligned}
\tag{150}
$$

where $\delta_{\text{s}} = \min\{r_\infty, \gamma\}$. It is of interest to integrate this equality from, say, t_a to t_b. Let us begin by considering the integral of the first term on the right hand side. Integrating by parts, it is given by (disregarding the factor $-1/2$)

$$
[t^{-1/2}e^{P_{\circ\pm}}(Q - Q_{\text{hom}})_{\circ\pm}]_{t_a}^{t_b} - \int_{t_a}^{t_b} \partial_t (t^{-1/2} e^{P_{\circ\pm}})(Q - Q_{\text{hom}})_{\circ\pm} dt = O(t_a^{-1/2-a}).
$$

Turning to the integral of the second term on the right hand side of (150), it can (after integration by parts) be written

$$
\begin{aligned}
& - [t^{1/2}(\partial_\mp P)_{\circ\pm} e^{P_{\circ\pm}}(Q - Q_{\text{hom}})_{\circ\pm}]_{t_a}^{t_b} + \int_{t_a}^{t_b} \partial_t [t^{1/2} e^{P_{\circ\pm}}](\partial_\mp P)_{\circ\pm}(Q - Q_{\text{hom}})_{\circ\pm} dt \\
& + \int_{t_a}^{t_b} t^{1/2} e^{P_{\circ\pm}}(\partial_\pm \partial_\mp P)_{\circ\pm}(Q - Q_{\text{hom}})_{\circ\pm} dt = O(t_a^{-1/2-a}),
\end{aligned}
$$

where we have used the fact that $\partial_\pm \partial_\mp P = O(t^{-2})$; cf. (26) and Lemma 53. Adding up, and letting t_b tend to infinity, we conclude that

$$
t_a^{1/2}[e^P \partial_\mp Q - e^{P_{\text{hom}}}\partial_\mp Q_{\text{hom}}]_{\circ\pm}(t_a, \theta) = O(t_a^{1/2-a_{\text{s}}-\delta_{\text{s}}}) + O(t_a^{-1/2-a}).
$$

As a consequence of this equality, it can be argued that

$$
\|e^P(\partial_\mp Q - \partial_\mp Q_{\text{hom}})\|_{C^0} \leq C t^{-1-a} + C t^{-a_{\text{s}}-\delta_{\text{s}}}. \tag{151}
$$

Let us turn to the derivatives of P. To begin with,

$$
\begin{aligned}
\partial_\pm [t^{1/2}\partial_\mp (P - P_{\text{hom}})] = & - \frac{1}{2} t^{-1/2}\partial_\pm (P - P_{\text{hom}}) + t^{1/2}(e^{2P}\partial_\pm Q \partial_\mp Q - e^{2P_{\text{hom}}}\partial_\pm Q_{\text{hom}}\partial_\mp Q_{\text{hom}}) \\
& - \frac{e^{P+\lambda/2}K^2}{2t^3} + \frac{e^{P_{\text{hom}}+\lambda_{\text{hom}}/2}K^2}{2t^3} + t^{1/2}\frac{\alpha_t}{2\alpha}\partial_\mp P - t^{1/2}\frac{\partial_t \alpha_{\text{hom}}}{2\alpha_{\text{hom}}}\partial_\mp P_{\text{hom}}.
\end{aligned}
\tag{152}
$$

The second term on the right hand side can be written

$$\begin{aligned}&t^{1/2}e^{2P}\partial_{\mp}Q\partial_{\pm}(Q-Q_{\mathrm{hom}})+t^{1/2}e^{2P}\partial_{\mp}(Q-Q_{\mathrm{hom}})\partial_{\pm}Q_{\mathrm{hom}}\\&+t^{1/2}(e^{2(P-P_{\mathrm{hom}})}-1)e^{2P_{\mathrm{hom}}}\partial_{\mp}Q_{\mathrm{hom}}\partial_{\pm}Q_{\mathrm{hom}}\\=&t^{1/2}e^{2P}\partial_{\mp}Q\partial_{\pm}(Q-Q_{\mathrm{hom}})+O(t^{-1/2-a_{\mathrm{s}}-r_{\infty}})+O(t^{-3/2-a-2r_{\infty}}).\end{aligned}$$

By arguments similar to ones given above, the last four terms on the right hand of (152) can be combined to give

$$O(t^{-3/2-a-\gamma})+O(t^{-1/2-a_{\mathrm{s}}-\gamma}).$$

Adding up,

$$\begin{aligned}\partial_t[t^{1/2}\partial_{\mp}(P-P_{\mathrm{hom}})]_{\circ\pm}=&-\frac{1}{2}t^{-1/2}[\partial_{\pm}(P-P_{\mathrm{hom}})]_{\circ\pm}+t^{1/2}[e^{2P}\partial_{\mp}Q\partial_{\pm}(Q-Q_{\mathrm{hom}})]_{\circ\pm}\\&+O(t^{-3/2-a})+O(t^{-1/2-a_{\mathrm{s}}-\delta_{\mathrm{s}}}).\end{aligned}$$

This equality is quite similar to (150), and arguments similar to ones given above yield

$$\|P_t-\partial_t P_{\mathrm{hom}}\|_{C^0}+\|P_\theta\|_{C^0}\le Ct^{-1-a}+Ct^{-a_{\mathrm{s}}-\delta_{\mathrm{s}}}.$$

Combining this estimate with (151), we obtain an improvement of (148); we can replace a_{s} with $\min\{a+1,a_{\mathrm{s}}+\delta_{\mathrm{s}}\}$. Iterating this improvement a finite number of times, we conclude that a_{s} can be chosen to equal $a+1$. The lemma follows. □

11 Characterising the pseudo-homogeneous solutions in terms of a lower bound on α

Finally, we are in a position to prove that only pseudo-homogeneous solutions are such that there is a positive lower bound on α.

Theorem 69. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that there is an $\alpha_0>0$ such that $\alpha\ge\alpha_0$ for all $(t,\theta)\in(t_0,\infty)\times\mathbb{S}^1$. Then the solution is pseudo-homogeneous.*

Proof. Since we have already proved that the conclusion holds in case $B=0$, cf. Theorem 59, let us assume $B\neq 0$. Due to Lemma 16, we know that there is a $C>0$ such that

$$(t/t_a)^r\hat{\mathcal{H}}(t)\ge C\hat{\mathcal{H}}(t_a) \tag{153}$$

holds for $t\ge t_a\ge t_1$; cf. (49). Here $\hat{\mathcal{H}}$ is defined in (48) and r can be chosen to equal $2(r_\infty+1)$. Moreover, $0<\rho<2\min\{\gamma,r_\infty\}$. On the other hand, due to Lemmas 67 and 68, we know that if we fix $\eta<6+2\min\{\gamma,r_\infty\}$, then there is a constant such that

$$\hat{\mathcal{H}}(t)\le Ct^{-\eta}$$

for all $t\ge t_1$. Combining this estimate with (153), we conclude that $\hat{\mathcal{H}}(t)=0$ for all $t\ge t_1$. Thus $P=P_{\mathrm{hom}}$ and $\lambda=\lambda_{\mathrm{hom}}$. Since Q and Q_{hom} converge to the same number and $Q_t=\partial_t Q_{\mathrm{hom}}$ for $t\ge t_1$, we obtain $Q=Q_{\mathrm{hom}}$. Finally, since $\alpha_t/\alpha=\partial_t\alpha_{\mathrm{hom}}/\alpha_{\mathrm{hom}}$ for large t and since α and α_{hom} converge to the same function, we conclude that $\alpha=\alpha_{\mathrm{hom}}$. The theorem follows. □

12 Proof of the main theorem

In the present section, we prove that $\langle\alpha^{-1/2}\rangle\to\infty$ for solutions that are not pseudo-homogeneous. Due to the monotonicity of $\langle\alpha^{-1/2}\rangle$, it is sufficient to prove that the assumption that $\langle\alpha^{-1/2}\rangle$ is bounded leads to a contradiction. The idea of how to achieve this is to prove that a bound on $\langle\alpha^{-1/2}\rangle$ implies that $t^{-1}\hat{H}$ is bounded to the future. Appealing to Lemma 26, we are then allowed to conclude that α has a positive lower bound. Theorem 69 then implies that the solution is pseudo-homogeneous, contrary to the assumption. In order to prove the desired bound on $t^{-1}\hat{H}$, it is natural to consider the energy

$$H_a = \int_{\mathbb{S}^1} \left\{ \alpha^{-1/2} \left[\left(P_t + \frac{1}{t} \right)^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) \right] + \frac{3}{t^2}\alpha^{-1/2} + \frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{7/2}} \right\} d\theta. \tag{154}$$

It can then be computed that

$$\frac{dH_a}{dt} = -\frac{2}{t}\int_{\mathbb{S}^1} \alpha^{-1/2}[(P_t+t^{-1})^2+\alpha e^{2P}Q_\theta^2]d\theta - \frac{6}{t^3}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta - \frac{5}{2t}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{7/2}}d\theta. \tag{155}$$

In order to prove decay, it would be desirable to trade, e.g.,

$$-\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}(P_t+t^{-1})^2 d\theta$$

against

$$-\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{1/2}P_\theta^2 d\theta.$$

One way of doing so is by introducing corrections, as in the case of $\mathbb{T}^3$-Gowdy. That is the topic of the first subsection.

12.1 Corrections

Define

$$\Gamma_1 = \frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t + \frac{1}{t}\right)(P - \langle P\rangle)d\theta.$$

Lemma 70. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Then*

$$\begin{aligned}\frac{d\Gamma_1}{dt} =& -\frac{2}{t}\Gamma_1 - \frac{1}{t}\int_{\mathbb{S}^1}\left[\alpha^{1/2}P_\theta^2 - \alpha^{-1/2}\left(P_t+\frac{1}{t}\right)^2\right]d\theta \\ &+ \frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}(Q_t^2-\alpha Q_\theta^2)(P-\langle P\rangle)d\theta - \frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)d\theta\left(\langle P_t\rangle + \frac{1}{t}\right).\end{aligned} \tag{156}$$

Remark 71. It is sometimes convenient to rewrite the last term as

$$-\frac{1}{t^2}A\left(\langle P_t\rangle + \frac{1}{t}\right) - \frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}Q_t(Q-\langle Q\rangle)d\theta\left(\langle P_t\rangle+\frac{1}{t}\right) - \frac{1}{t^2}B\langle Q\rangle\left(\langle P_t\rangle + \frac{1}{t}\right).$$

Proof. Compute

$$\begin{aligned}\frac{d\Gamma_1}{dt} =& -\frac{2}{t}\Gamma_1 + \frac{1}{t^2}\int_{\mathbb{S}^1}\partial_t\left[t\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)\right](P-\langle P\rangle)d\theta \\ &+\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)^2 d\theta - \frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)d\theta\left(\langle P_t\rangle+\frac{1}{t}\right).\end{aligned}$$

Due to (21), we obtain the conclusion of the lemma. □

Let us estimate the corrections in terms of H_a and the integral of $\alpha^{-1/2}$.

Lemma 72. *Consider a solution to (2)–(6) on* $(t_0,\infty)\times\mathbb{S}^1$ *with* $K\neq 0$*. Define* H_a *according to (154). Then*

$$|\Gamma_1|\leq\frac{1}{4t^2}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta\ tH_a. \tag{157}$$

Proof. Note, to begin with, that

$$\|P-\langle P\rangle\|_{C^0}\leq\frac{1}{2}\int_{\mathbb{S}^1}|P_\theta|d\theta\leq\frac{1}{2}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta\right)^{1/2}\left(\int_{\mathbb{S}^1}\alpha^{1/2}P_\theta^2d\theta\right)^{1/2}. \tag{158}$$

Thus

$$\begin{aligned}&\left|\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}(P_t+t^{-1})(P-\langle P\rangle)d\theta\right|\\ \leq&\frac{1}{t}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta\right)^{1/2}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}(P_t+t^{-1})^2d\theta\right)^{1/2}\|P-\langle P\rangle\|_{C^0}\\ \leq&\frac{1}{2t}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta\left(\int_{\mathbb{S}^1}\alpha^{-1/2}(P_t+t^{-1})^2d\theta\right)^{1/2}\left(\int_{\mathbb{S}^1}\alpha^{1/2}P_\theta^2d\theta\right)^{1/2}\\ \leq&\frac{1}{4t^2}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta\ t\int_{\mathbb{S}^1}\alpha^{-1/2}[(P_t+t^{-1})^2+\alpha P_\theta^2]d\theta.\end{aligned}$$

The lemma follows. □

Lemma 73. *Consider a solution to (2)–(6) on* $(t_0,\infty)\times\mathbb{S}^1$ *with* $K\neq 0$*. Define* H_a *according to (154). If* $\langle\alpha^{-1/2}\rangle$ *is bounded, there is a constant* C*, depending on the solution, such that*

$$|\Gamma_2|\leq\frac{C}{t}H_a$$

for all $t\geq t_1$*, where* Γ_2 *is defined in (121).*

Proof. Estimate

$$\left|\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}Q_t(Q-\langle Q\rangle)d\theta\right|\leq\frac{1}{t}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta\right)^{1/2}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}Q_t^2d\theta\right)^{1/2}\|e^P(Q-\langle Q\rangle)\|_{C^0}.$$

Due to (158), the assumed bound on $\langle\alpha^{-1/2}\rangle$ and the fact that H_a is bounded, there exists a constant C such that

$$\|P-\langle P\rangle\|_{C^0}\leq C$$

for all $t\geq t_1$. As a consequence,

$$\begin{aligned}\|e^P(Q-\langle Q\rangle)\|_{C^0}\leq&Ce^{\langle P\rangle}\|Q-\langle Q\rangle\|_{C^0}\leq Ce^{\langle P\rangle}\int_{\mathbb{S}^1}|Q_\theta|d\theta\\ \leq&Ce^{\langle P\rangle}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta\right)^{1/2}\left(\int_{\mathbb{S}^1}\alpha^{1/2}Q_\theta^2d\theta\right)^{1/2}\leq C\left(\int_{\mathbb{S}^1}\alpha^{1/2}e^{2P}Q_\theta^2d\theta\right)^{1/2}\end{aligned}$$

for all $t\geq t_1$. Adding up the above, we obtain

$$|\Gamma_2|\leq\frac{C}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}(Q_t^2+\alpha Q_\theta^2)d\theta\leq\frac{C}{t}H_a(t) \tag{159}$$

for all $t\geq t_1$. The lemma follows. □

12.2 Limiting value of H_a

In order to prove that tH_a is bounded, we proceed in two steps. First, we prove that $H_a \to 0$. Then we use the corrections defined in the previous subsection to prove that tH_a is bounded. In the present subsection, we focus on the first step. The argument is similar to one presented in [24]; we prove that

$$t^{-1}H_a \in L^1([t_1,\infty)). \tag{160}$$

Since we know that H_a decays and is bounded from below, we know that it converges to a real number ≥ 0. If this number is different from zero, we obtain a contradiction to (160).

Lemma 74. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that $\langle\alpha^{-1/2}\rangle$ is bounded. Then*

$$\lim_{t\to\infty} H_a(t) = 0.$$

Proof. As described prior to the statement of the lemma, we wish to prove that (160) holds. Due to (155), the only thing that remains to be proved is that

$$\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}(e^{2P}Q_t^2+\alpha P_\theta^2)d\theta \in L^1([t_1,\infty)).$$

However, that

$$\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}Q_t^2 d\theta$$

is integrable is an immediate consequence of the fact that

$$\frac{d(t^{-2}\hat{H})}{dt} = -\frac{2}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}(P_t^2+e^{2P}Q_t^2)d\theta - \frac{6}{t^3}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta - \frac{2}{t}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{7/2}}d\theta$$

and that $t^{-2}\hat{H}$ is bounded from below by zero. Turning to P, note that (156) implies that

$$\begin{aligned}\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{1/2}P_\theta^2 =& \frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)^2 d\theta - \frac{d\Gamma_1}{dt} - \frac{2}{t}\Gamma_1\\ &+\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}(Q_t^2-\alpha Q_\theta^2)(P-\langle P\rangle)d\theta\\ &-\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)d\theta\left(\langle P_t\rangle+\frac{1}{t}\right).\end{aligned} \tag{161}$$

Integrating this equality from t_1 to t_b and then letting $t_b\to\infty$, we see that the first three terms cause no problems; in the case of the first term, we can appeal to (155) and the lower bound on H_a; in the case of the second and third terms, it is sufficient to appeal to the bound $|\Gamma_1|\leq Ct^{-1}$ which holds under the assumptions of the lemma. The fourth term is integrable due to the information we have already derived concerning the Q-energy and the fact that $\|P-\langle P\rangle\|_{C^0}$ is bounded to the future under the assumptions of the lemma. What remains is thus to consider the last term. However, it can be estimated in absolute value by

$$\frac{1}{2\pi t}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta\right)^{1/2}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+t^{-1}\right)^2 d\theta\left(\int_{\mathbb{S}^1}\alpha^{1/2}d\theta\right)^{1/2},$$

a function we know to be integrable. □

12.3 Rate of decay for H_a

In the present subsection, we prove that tH_a is bounded, given a bound on $\langle\alpha^{-1/2}\rangle$. The main idea of the proof is to use the corrections Γ_1 and Γ_2. As a consequence of the result, we are able to prove the main theorem.

Lemma 75. *Consider a solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K\neq 0$. Assume that $\langle\alpha^{-1/2}\rangle$ is bounded. Then there is a constant C such that $tH_a(t)\leq C$ for all $t\geq t_1$.*

Proof. Define the energy

$$\mathcal{E}=H_a+\Gamma_1-\Gamma_2.$$

Due to Lemmas 72 and 73 and the assumptions of the lemma, we know that

$$|\Gamma_1|+|\Gamma_2|\leq Ct^{-1}H_a.$$

In particular, $\mathcal{E}$ and H_a are thus equivalent for large t. Compute

$$\begin{aligned}\frac{d\mathcal{E}}{dt}=&-\frac{1}{t}\mathcal{E}-\frac{1}{t}(\Gamma_1-\Gamma_2)-\frac{3}{t^3}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta-\frac{3}{2t}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{7/2}}d\theta\\&+\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}(Q_t^2-\alpha Q_\theta^2)(P-\langle P\rangle)d\theta-\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)d\theta\left(\langle P_t\rangle+\frac{1}{t}\right)\\&+\frac{1}{t^2}B\langle Q_t\rangle.\end{aligned}\tag{162}$$

Thus

$$\frac{d\mathcal{E}}{dt}\leq-\frac{1}{t}\mathcal{E}+\frac{C}{t^2}\mathcal{E}+\frac{C}{t}\mathcal{E}^{3/2}-\frac{1}{t}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)d\theta\left(\langle P_t\rangle+\frac{1}{t}\right)+\frac{1}{t^2}B\langle Q_t\rangle$$

for large t, where we have used (158), Lemmas 72 and 73 and the fact that $\mathcal{E}$ and H_a are equivalent for large t. Let $0<\rho<1$ and define $\mathcal{E}_\rho=t^\rho\mathcal{E}$. Then

$$\frac{d\mathcal{E}_\rho}{dt}\leq-\frac{1-\rho}{t}\mathcal{E}_\rho+\frac{C}{t^2}\mathcal{E}_\rho+\frac{C}{t}\mathcal{E}^{1/2}\mathcal{E}_\rho-\frac{1}{t^{1-\rho}}\int_{\mathbb{S}^1}\alpha^{-1/2}\left(P_t+\frac{1}{t}\right)d\theta\left(\langle P_t\rangle+\frac{1}{t}\right)+\frac{1}{t^{2-\rho}}B\langle Q_t\rangle\tag{163}$$

for large t. There is a $T_\rho>t_0$ such that for $t\geq T_\rho$, the sum of the first three terms is negative; note that $\mathcal{E}$ converges to zero. What we need to concern ourselves with is consequently the integral of the last two terms. Let us begin with the integral of the last term. It is given by

$$\int_{t_a}^{t_b}\frac{1}{t^{2-\rho}}B\langle Q_t\rangle dt=\left[\frac{1}{t^{2-\rho}}B\langle Q\rangle\right]_{t_a}^{t_b}+\int_{t_a}^{t_b}\frac{2-\rho}{t^{3-\rho}}B\langle Q\rangle dt.\tag{164}$$

In order to estimate the right hand side, we need an estimate of $B\langle Q\rangle$. Note, to this end, that

$$A=\int_{\mathbb{S}^1}t\alpha^{-1/2}\left(P_t+\frac{1}{t}-e^{2P}Q_t(Q-\langle Q\rangle)\right)d\theta-B\langle Q\rangle.$$

Under the assumptions of the lemma, the first term on the right hand side is bounded in absolute value by Ct. Thus there is a constant C such that

$$|B\langle Q\rangle|\leq Ct\tag{165}$$

for all $t\geq t_1$. As a consequence, both of the terms on the right hand side of (164) are bounded. Let us turn to the second to last term on the right hand side of (163). It can be written

$$-\frac{A}{t^{2-\rho}}\left(\langle P_t\rangle+\frac{1}{t}\right)-\frac{1}{t^{1-\rho}}\int_{\mathbb{S}^1}\alpha^{-1/2}e^{2P}Q_t(Q-\langle Q\rangle)d\theta\left(\langle P_t\rangle+\frac{1}{t}\right)-\frac{1}{t^{2-\rho}}B\langle Q\rangle\left(\langle P_t\rangle+\frac{1}{t}\right);\tag{166}$$

cf. Remark 71. Note that the second term can be estimated by $Ct^{-1}\mathcal{E}^{1/2}\mathcal{E}_\rho$ and can therefore be absorbed by the first term on the right hand side of (163); cf. the above argument. Turning to the integral of the first term on the right hand side of (166), we have

$$-\int_{t_a}^{t_b} \frac{A}{t^{2-\rho}}\left(\langle P_t\rangle + \frac{1}{t}\right)dt = -\left[\frac{A}{t^{2-\rho}}\left(\langle P\rangle + \ln t\right)\right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \frac{(2-\rho)A}{t^{3-\rho}}\left(\langle P\rangle + \ln t\right)dt.$$

Due to Corollary 20, both terms appearing on the right hand side are bounded. Note that this is still the case if $\rho = 1$. Finally, note that

$$\int_{t_1}^{t_b}\left|\frac{1}{t^{2-\rho}}B\langle Q\rangle\left(\langle P_t\rangle + \frac{1}{t}\right)\right|dt \leq C\int_{t_1}^{t_b} t^{-1+\rho}\left(|\langle P_t\rangle| + \frac{1}{t}\right)dt \leq C_\rho,$$

where we have used Corollary 22. Adding up the above, we conclude that there is a constant C_ρ such that $\mathcal{E}_\rho(t) \leq C_\rho$ for all $t \geq t_1$. As a consequence, there is, for every $0 < \rho < 1$, a constant C_ρ such that $t^\rho H_a(t) \leq C_\rho$ for all $t \geq t_1$. Returning to the proof of (165), we conclude that for every $\eta > 1/2$, there is a constant C_η such that

$$|B\langle Q\rangle| \leq C_\eta t^\eta \tag{167}$$

for all $t \geq t_1$. Going through the above estimates with $\rho = 1$ and using the improved estimate (167), we conclude that

$$t\mathcal{E}(t) \leq C + C\int_{t_1}^{t} s^{-1}|B\langle Q\rangle\langle P_t\rangle|ds.$$

However, appealing to (167) and Corollary 22, we know that the right hand side is bounded. Thus $\mathcal{E}(t) \leq Ct^{-1}$ for all $t \geq t_1$. In particular, $tH_a(t) \leq C$ for all $t \geq t_1$. □

Finally, we are in a position to prove the main theorem, Theorem 3.

Theorem 3. Due to Lemma 75, we know that $tH_a(t) \leq C$ for $t \geq t_1$. We would like to appeal to Lemma 26. However, in order to be able to do so, we need to demonstrate that $t^{-1}\hat{H}(t) \leq C$ for $t \geq t_1$. However, most of the terms appearing in $t^{-1}\hat{H}$ appear also in tH_a. All that remains is to estimate

$$\int_{\mathbb{S}^1} \alpha^{-1/2}P_t^2 d\theta \leq 2\int_{\mathbb{S}^1} \alpha^{-1/2}\left[\left(P_t + \frac{1}{t}\right)^2 + \frac{1}{t^2}\right]d\theta \leq Ct^{-1},$$

where the last step is a consequence of the bound on H_a mentioned above and the assumption that $\langle\alpha^{-1/2}\rangle$ is bounded. Due to the assumptions, we know that α cannot converge to zero uniformly. Adding up the above observations, we are therefore allowed to appeal to Lemma 26 in order to conclude that α has a positive lower bound. Consequently, the solution is pseudo-homogeneous; cf. Theorem 69. □

13 The polarised case

In the present section, we prove Proposition 1. As a first step, we prove that there is an open set of initial data such that H_a decays as $1/t$. One particular consequence of this result is that suitably small perturbations of pseudo-homogeneous solutions have energy decay of this type; cf. Proposition 8. Combining this observation with the fact that $\langle\alpha^{-1/2}\rangle \to \infty$ for non-pseudo-homogeneous solutions, we are then able to derive the desired conclusions; cf. Lemma 79.

Lemma 76. *Consider a polarised solution to (2)–(6) on* $(t_0, \infty) \times \mathbb{S}^1$ *with* $K \neq 0$. *Assume that, for some* $t_a \geq t_1$,

$$t_a H_a(t_a) \leq 6, \quad t_a\mathcal{E}(t_a) \leq 1, \quad \frac{8|A|[\langle\alpha^{1/2}\rangle(t_a)]^{1/2}}{\sqrt{2\pi t_a}} \leq 1, \tag{168}$$

where $\mathcal{E} = H_a + \Gamma_1$. Then $tH_a(t) \leq 6$ for all $t \geq t_a$. Moreover,

$$\frac{1}{t^2}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta, \quad \frac{1}{t}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{5/2}}d\theta \in L^1([t_1,\infty)).$$

Remark 77. Pseudo-homogeneous solutions are such that the left hand sides in (168) converge to zero as $t_a \to \infty$.

Remark 78. Non pseudo-homogeneous solutions which satisfy the conditions have the property that α converges to zero uniformly; cf. Lemma 26 and the proof of Theorem 3.

Proof. Let $\gamma \in (0,1)$. Due to (157), we know that as long as

$$tH_a(t) \leq 12\gamma, \tag{169}$$

then

$$(1-\gamma)H_a \leq \mathcal{E} \leq (1+\gamma)H_a. \tag{170}$$

Due to (162), we know that

$$\frac{d\mathcal{E}}{dt} = -\frac{1}{t}\mathcal{E} - \frac{1}{t}\Gamma_1 - \frac{A}{t^2}\left(\langle P_t\rangle + \frac{1}{t}\right) - \frac{3}{t^3}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta - \frac{3}{2t}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{7/2}}d\theta. \tag{171}$$

As long as

$$tH_a(t) \leq 12, \tag{172}$$

the second and the fourth terms on the right hand side of (171) add up to something non-positive, so that

$$\frac{d\mathcal{E}}{dt} \leq -\frac{1}{t}\mathcal{E} - \frac{A}{t^2}\left(\langle P_t\rangle + \frac{1}{t}\right),$$

and

$$\frac{d}{dt}(t\mathcal{E}) \leq -\frac{A}{t}\left(\langle P_t\rangle + \frac{1}{t}\right).$$

Assuming (172) to hold on the interval $[t_a, t_b]$, we obtain

$$\begin{aligned} t\mathcal{E}(t) \leq & t_a\mathcal{E}(t_a) + |A|[\langle\alpha^{1/2}\rangle(t_a)]^{1/2}\int_{t_a}^{t}\frac{1}{s\sqrt{2\pi}}H_a^{1/2}(s)ds \\ \leq & t_a\mathcal{E}(t_a) + \frac{4|A|[\langle\alpha^{1/2}\rangle(t_a)]^{1/2}}{\sqrt{2\pi}}\int_{t_a}^{t}\frac{1}{s^{3/2}}ds \leq t_a\mathcal{E}(t_a) + \frac{8|A|[\langle\alpha^{1/2}\rangle(t_a)]^{1/2}}{\sqrt{2\pi t_a}} \end{aligned} \tag{173}$$

for all $t \in [t_a, t_b]$. By assumption, (168) holds. Note that, as a consequence,

$$t\mathcal{E}(t) \leq 2 \tag{174}$$

on the interval $[t_a, t_b]$, assuming (172) to hold there. Let $\mathcal{A}$ be the set of $s \in [t_a, \infty)$ such that

$$tH_a(t) \leq 6 \tag{175}$$

and (174) hold for $t \in [t_a, s]$. Clearly, $\mathcal{A}$ is non-empty and connected. Moreover, it is closed by definition. In order to prove openness, let $\tau \in \mathcal{A}$. Due to (175), we know that (169) holds with $\gamma = 1/2$. Thus (170) implies that

$$\tau H_a(\tau) \leq 2\tau\mathcal{E}(\tau) \leq 4.$$

Consequently, (175) clearly holds in an open neighbourhood of τ. We are thus allowed to appeal to (173) in $[t_a, \tau+\epsilon]$ for some $\epsilon > 0$. As a consequence, we can extend (174) to $[t_a, \tau+\epsilon]$. Thus $\mathcal{A} = [t_a, \infty)$.

Returning to (171), note that

$$\begin{aligned}\frac{d(t\mathcal{E})}{dt} =& -\Gamma_1 - \frac{A}{t}\left(\langle P_t\rangle + \frac{1}{t}\right) - \frac{3}{t^2}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta - \frac{3}{2t}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{5/2}}d\theta\\ \leq& -\frac{A}{t}\left(\langle P_t\rangle + \frac{1}{t}\right) - \frac{3}{2t^2}\int_{\mathbb{S}^1}\alpha^{-1/2}d\theta - \frac{3}{2t}\int_{\mathbb{S}^1}\frac{\alpha^{-1/2}e^{P+\lambda/2}K^2}{t^{5/2}}d\theta,\end{aligned}$$

where we used the fact that (157) and (175) hold. Since we know that the first term on the far right hand side is integrable, we conclude that the second and third terms are integrable. □

Note that one particular consequence of the last lemma is that $t^{-2}\langle\alpha^{-1/2}\rangle \in L^1([t_1,\infty))$. It is of interest to use this assumption as a starting point for further analysis.

Lemma 79. *Consider a non-pseudo-homogeneous polarised solution to (2)–(6) on $(t_0,\infty)\times\mathbb{S}^1$ with $K \neq 0$. Assume that $t^{-2}\langle\alpha^{-1/2}\rangle \in L^1([t_1,\infty))$. Then there is a constant C such that $tH_a(t) \leq C$ for all $t \geq t_1$. In particular, α converges to zero uniformly. Moreover, $t^{-1}\langle\alpha^{-1/2}\rangle \to 0$, and*

$$\lim_{t\to\infty}\|P(t,\cdot) - \langle P(t,\cdot)\rangle\|_{C^0} = 0.$$

Finally,

$$\lim_{t\to\infty}\left\|\frac{P(t,\cdot)}{\ln t} + 1\right\|_{C^0} = 0 \tag{176}$$

and there is a time sequence $t_k \to \infty$ such that

$$\lim_{k\to\infty}\left\|\frac{\lambda(t_k,\cdot)}{\ln t_k} - 5\right\|_{C^0} = 0. \tag{177}$$

Proof. Let $f(t) = t^{-1}\langle\alpha^{-1/2}\rangle$. By assumption, $t^{-1}f \in L^1([t_1,\infty))$. Moreover,

$$\partial_t f = -t^{-1}f + t^{-1}\langle\partial_t\alpha^{-1/2}\rangle \geq -t^{-1}f.$$

Thus Lemma 10 applies, and we conclude that $t^{-1}\langle\alpha^{-1/2}\rangle \to 0$. Combining this observation with (157), we conclude that for every $\epsilon > 0$, there is a $T > t_0$ such that $|\Gamma_1| \leq \epsilon H_a$ for all $t \geq T$. Consequently, for t large enough, $\mathcal{E}$ and H_a are equivalent. Compute, using (171) and (157), that

$$\begin{aligned}\frac{d}{dt}(t\mathcal{E}+1) \leq& -\Gamma_1 - \frac{A}{t}\left(\langle P_t\rangle + \frac{1}{t}\right) \leq \frac{C}{t^2}\langle\alpha^{-1/2}\rangle(t\mathcal{E}+1) + \frac{C}{t}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}(P_t+t^{-1})^2d\theta\right)^{1/2}\\ \leq& C(t^{-3/2} + t^{-2}\langle\alpha^{-1/2}\rangle)(t\mathcal{E}+1).\end{aligned}$$

As a consequence, $t\mathcal{E}$, and thus tH_a, is bounded. By earlier observations, we thus know that α converges to zero uniformly. Combining the fact that $t^{-1}\langle\alpha^{-1/2}\rangle$ converges to zero with the bound on $tH_a(t)$, we conclude that the spatial variation of P converges to zero; cf. (158).

Let us turn to the limit of $P/\ln t$. Note, to begin with, that $\|\lambda - \langle\lambda\rangle\|_{C^0}$ is bounded; cf. (75). Since the spatial variation of P tends to zero, there is thus a constant $C > 0$ such that

$$\partial_t\langle\alpha^{-1/2}\rangle \geq C\langle\alpha^{-1/2}\rangle\frac{e^{\langle P\rangle+\langle\lambda\rangle/2}K^2}{t^{5/2}}.$$

In particular, we thus have

$$\int_{t_1}^{\tau}\frac{e^{\langle P\rangle+\langle\lambda\rangle/2}K^2}{t^{5/2}}dt \leq C\ln\frac{\langle\alpha^{-1/2}(\tau,\cdot)\rangle}{\langle\alpha^{-1/2}(t_1,\cdot)\rangle} \leq C\ln\tau \tag{178}$$

for all $\tau \geq t_1$. Let us compute

$$\partial_t\left(\frac{1}{\langle\alpha^{-1/2}\rangle}\int_{\mathbb{S}^1}\alpha^{-1/2}Pd\theta\right) = -\frac{2\pi}{t} + \frac{A}{t\langle\alpha^{-1/2}\rangle} + \int_{\mathbb{S}^1}\partial_t\left(\frac{\alpha^{-1/2}}{\langle\alpha^{-1/2}\rangle}\right)(P - \langle P\rangle)d\theta; \tag{179}$$

note that

$$\int_{\mathbb{S}^1} \partial_t \left(\frac{\alpha^{-1/2}}{\langle \alpha^{-1/2} \rangle} \right) \langle P \rangle d\theta = 0.$$

Note that the first term on the right hand side of (179) dominates over the second. What we need to analyse is thus the third term. It can be estimated by

$$\int_{\mathbb{S}^1} \left| \partial_t \left(\frac{\alpha^{-1/2}}{\langle \alpha^{-1/2} \rangle} \right) \right| d\theta \| P - \langle P \rangle \|_{C^0}.$$

Due to (105), the first factor of this expression can be estimated by

$$-\int_{\mathbb{S}^1} \frac{1}{2} \frac{\alpha_t}{\alpha} \frac{\alpha^{-1/2}}{\langle \alpha^{-1/2} \rangle} d\theta + \int_{\mathbb{S}^1} \frac{1}{\langle \alpha^{-1/2} \rangle} \left\langle -\frac{1}{2} \alpha^{-1/2} \frac{\alpha_t}{\alpha} \right\rangle \frac{\alpha^{-1/2}}{\langle \alpha^{-1/2} \rangle} d\theta \leq C \frac{e^{\langle P \rangle + \langle \lambda \rangle / 2} K^2}{t^{5/2}} \tag{180}$$

for all $t \geq t_1$. Let $\epsilon > 0$. Then there is a $t_a \geq t_1$ such that $\| P - \langle P \rangle \|_{C^0} \leq \epsilon$ for all $t \geq t_a$. Dividing the interval of integration into the pieces $[t_1, t_a]$ and $[t_a, t]$ yields

$$\int_{t_1}^{\tau} \int_{\mathbb{S}^1} \left| \partial_t \left(\frac{\alpha^{-1/2}}{\langle \alpha^{-1/2} \rangle} \right) \right| d\theta \| P - \langle P \rangle \|_{C^0} dt \leq C_\epsilon + C\epsilon \int_{t_a}^{\tau} \frac{e^{\langle P \rangle + \langle \lambda \rangle / 2} K^2}{t^{5/2}} dt \leq C_\epsilon + C\epsilon \ln \tau,$$

where we used (180) in the first step and (178) in the second. There is a similar estimate for the integral of the second term on the right hand side of (179). Integrating (179), we thus have

$$\left| \frac{1}{\langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} \alpha^{-1/2} P d\theta + 2\pi \ln \tau \right| \leq C_\epsilon + C\epsilon \ln \tau.$$

In particular,

$$\lim_{t \to \infty} \left| \frac{1}{2\pi \langle \alpha^{-1/2} \rangle \ln t} \int_{\mathbb{S}^1} \alpha^{-1/2} P d\theta + 1 \right| = 0.$$

On the other hand,

$$\frac{1}{2\pi \langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} \alpha^{-1/2} P d\theta - \langle P \rangle = \frac{1}{2\pi \langle \alpha^{-1/2} \rangle} \int_{\mathbb{S}^1} \alpha^{-1/2} (P - \langle P \rangle) d\theta \to 0$$

as $t \to \infty$. Since $\| P - \langle P \rangle \|_{C^0} \to 0$ as $t \to \infty$, we obtain (176).

In order to prove the statement concerning λ, let us fix $\epsilon > 0$. Assume that there is a T such that

$$\frac{\lambda(t, \theta)}{\ln t} \geq 5 + \epsilon$$

for all $(t, \theta) \in [T, \infty) \times \mathbb{S}^1$. Due to (176), we are allowed to assume that T is such that $P(t, \theta) \geq -\ln t - \epsilon \ln t / 4$ for all $(t, \theta) \in [T, \infty) \times \mathbb{S}^1$. Combining this information with (4), we conclude that

$$\frac{\alpha_t}{\alpha} \leq -t^{-1+\epsilon/4} K^2.$$

Integrating this inequality from T to t and exponentiating the result, we obtain

$$\alpha(t, \theta) \leq \exp\left(-\frac{4}{\epsilon} t^{\epsilon/4} K^2 + \frac{4}{\epsilon} T^{\epsilon/4} K^2 \right) \alpha(T, \theta).$$

Raising this equality to $-1/2$ and averaging over the result, we obtain a lower bound on $\langle \alpha^{-1/2} \rangle$ which is inconsistent with the fact that $t^{-1} \langle \alpha^{-1/2} \rangle$ converges to zero. Consequently, the above assumption leads to a contradiction. Assume that there is a T such that

$$\frac{\lambda(t, \theta)}{\ln t} \leq 5 - \epsilon$$

for all $(t, \theta) \in [T, \infty) \times \mathbb{S}^1$. By an argument similar to the one given above, this assumption leads to the conclusion that α is bounded from below by a positive constant, something we know to be false. Adding up the above information, we conclude that there is a time sequence $t_k \to \infty$ and a sequence of $\theta_k \in \mathbb{S}^1$ such that

$$5 - \epsilon \leq \frac{\lambda(t_k, \theta_k)}{\ln t_k} \leq 5 + \epsilon.$$

Since the spatial variation of λ is bounded by, say, C, we conclude that

$$5 - \epsilon - \frac{C}{\ln t_k} \leq \frac{\lambda(t_k, \theta)}{\ln t_k} \leq 5 + \epsilon + \frac{C}{\ln t_k}$$

for all $\theta \in \mathbb{S}^1$. The lemma follows. □

Finally, we are in a position to prove Proposition 1.

Proposition 1. Combining Proposition 8 and Lemma 76, we conclude that for $\epsilon > 0$ small enough, the conditions of Lemma 79 are met. Moreover the conclusions of this lemma yield the desired result. □

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