ON PROVING FUTURE STABILITY OF COSMOLOGICAL SOLUTIONS WITH ACCELERATED EXPANSION

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ABSTRACT. In the late 90's, the standard perspective on how to model the universe changed dramatically; observational data concerning supernovae, obtained in 98–99, indicate that our universe is expanding at an accelerated rate. As a consequence, it is of interest to prove that cosmological solutions to Einstein's equations with accelerated expansion are future stable. That is the topic of the present contribution. The current standard models of the universe include different types of matter, but it turns out that many of the essential difficulties appear already in the vacuum setting. As a consequence, we here focus on giving a rough outline of how to prove future stability in the case of Einstein's vacuum equations with a positive cosmological constant. However, we also wish to give an overview of the stability results that have been obtained more generally, and to give an idea of how to arrive at the conclusion that the universe is expanding at an accelerated rate on the basis of observations.

1. Introduction

In the presence of a cosmological constant Λ, Einstein's equations read

$$G + \Lambda g = T, \tag{1}$$

where G is the Einstein tensor, T is the stress energy tensor, and g is the metric. In Einstein's heuristic derivation of his equations (based on an analogy with the Poisson equation), there is no reason to prefer a specific value of Λ. Even though certain non-zero values (ranges) have been preferred by various communities in the last hundred years, $\Lambda = 0$ has been the default choice for most of this period. However, due to the observational data collected in the late 90's, the situation has changed. Even though cosmologists need not necessarily include a cosmological constant in their models, a mechanism for inducing accelerated expansion is currently a standard ingredient when describing the universe. Due to the corresponding shift towards solutions to Einstein's equations with accelerated expansion, it is of interest to address the question of stability in that setting. In the present contribution, we wish to give an overview of results that have been obtained on this topic. However, our main purpose is to give a sketch of the proof of future stability in the case of Einstein's vacuum equations with a positive cosmological constant. It turns out that many of the essential difficulties arise already in this setting. Due to the central role played by the supernova observations in justifying the currently preferred models, we also devote one section to describing in what sense the observations lead to the conclusion that the universe is expanding at an accelerated rate.

The outline of this contribution is as follows. In Section 2, we describe how the observations of supernovae of type Ia can be used to limit the class of models consistent with observations. In Section 3, we then describe previous results that have been obtained on the topic of future stability in the case of accelerated expansion. Finally, most of the contribution is devoted to a description of how to prove future stability in the case of Einstein's vacuum equations with a positive cosmological constant. This is the subject of Section 4.

2. Observations

Background solutions. In order to be able to draw conclusions from the observations, it is necessary to first select a class of models in which the observations are to be interpreted. In cosmology, the starting point is always the assumption of spatial homogeneity and isotropy (i.e., *the cosmological principle*). However, it is also necessary to specify the matter content in order to be able to proceed. Currently, the preferred ingredients are ordinary matter (such as dust and radiation), dark matter and dark energy. However, in practice, it is common to model the matter by dust, a radiation fluid and a positive cosmological constant. Moreover, if one is interested in the early universe, the radiation dominates over the dust (due to the relevant scaling of the energy densities), so that one normally ignores the dust. Similarly, if one is interested in the late time behaviour, the dust dominates over the radiation, so that the radiation is normally ignored. The supernova observations are made in the late time regime, so that it is common to ignore the radiation. To summarize, the relevant metrics take the form

$$g = -dt^2 + a^2(t)\bar{g}_\Sigma, \tag{2}$$

on $M = I \times \Sigma$. Here I is an open interval, Σ is $\mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{S}^3$ (or a quotient thereof), and $\bar{g}_\Sigma$ is the standard metric on Σ. The matter is modelled by dust (a perfect fluid with vanishing pressure):

$$T = \rho_m dt \otimes dt. \tag{3}$$

The associated matter equation is obtained by requiring that T be divergence free with respect to the metric (2). The final ingredient of the model is a positive cosmological constant Λ, so that the relevant form of Einstein's equations is (1).

Drawing conclusions from the observations. The reason it is of interest to study supernovae of type Ia is that they are expected to be standard candles. What this means is that supernovae of type Ia have (approximately) a fixed peak luminosity (amount of energy they emit per unit time in the form of electromagnetic radiation when they are at their brightest), say L. Considering a supernova, it is natural to measure its radiant flux (the electromagnetic energy that crosses a unit area perpendicular to the line of sight per unit time) on earth, say F. Given these quantities, one can define the *luminosity distance* according to

$$d_L = \sqrt{\frac{L}{4\pi F}}.$$

Given the above class of background solutions, it turns out to be possible to derive an expression for d_L in terms of

- H_0 (the present value of the Hubble parameter),

- the redshift of the supernova, and
- two parameters describing the matter content, say $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$;

cf. [16, Chapter 5]. Here $\Omega_m = c_0\rho_m/H^2$, where ρ_m is the energy density associated with the dust, $H = \dot{a}/a$ is the Hubble parameter, and c_0 is an appropriately chosen constant. Moreover, $\Omega_\Lambda = c_1\Lambda/H^2$, where c_1 is a constant. The relevant values of c_0 and c_1 are to be found in [16, Chapter 5]; they depend on the speed of light and the gravitational constant, which we, for simplicity, have set equal to 1 here. Finally, $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ denote the present values of Ω_m and Ω_Λ. In fact, there is, for every $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, a function $F_{\Omega_{m,0},\Omega_{\Lambda,0}}$ such that

$$H_0 d_L = F_{\Omega_{m,0},\Omega_{\Lambda,0}}(z), \tag{4}$$

where z is the redshift of the emitting object. The observations can be used to determine $H_0 d_L$ and z. Combining this information with (4) yields a curve in the $\Omega_{m,0}, \Omega_{\Lambda,0}$-plane. However, the curves are different for different redshifts. Observing supernovae at different redshifts and using (4) then yields a limited region of the $\Omega_{m,0}, \Omega_{\Lambda,0}$-plane which is consistent with observations.

On the basis of arguments of the above type, one is led to prefer models with a positive cosmological constant and Euclidean spatial geometry. It should of course be noted that there are other observational data supporting this conclusion. The above description is somewhat brief, and the reader interested in more details is referred to [16, Chapter 5].

3. Previous results

In the study of Einstein's equations with a positive cosmological constant, de Sitter space plays a prominent role due to its high degree of symmetry. Moreover, it expands both to the future and to the past. It is therefore natural to begin by proving that de Sitter space is stable (and not only future stable). In the case of $3+1$ dimensions, this was done in the work of Helmut Friedrich; cf. [3]. Later, he extended his results to include matter of Maxwell and Yang-Mills type; cf. [4]. The arguments used to prove the results are based on Friedrich's conformal field equations. Since the conformal field equations and their uses are described elsewhere in this volume, we shall not discuss this perspective further here. Even though the relevant ideas are very elegant, they do not seem to be well adapted to the problem of proving future stability in the presence of matter which does not have nice conformal invariance properties. Moreover, there seems to be an unnatural restriction on the dimension; even though the results of Friedrich have been extended to higher dimensions in the work of Michael Anderson, cf. [1], there is still the requirement that the spacetime dimension be even.

The current contribution is based on the ideas which were developed in [13] in the hope of obtaining more robust methods. The particular case considered in [13] was that of Einstein's equations coupled to a non-linear scalar field. There were two main reasons for considering this particular case. First of all, Einstein's equation with a positive cosmological constant are included as a special case. Since that case had already been dealt with using the conformal methods of Friedrich, it was, however, of interest to consider something more general. From a physics point of view, non-linear scalar fields are a natural class of matter models. The reason for this is that even though the observations indicate the the universe is expanding at an accelerated rate, various mechanisms are conceivable. A positive cosmological constant is one possibility, but one can also use a non-linear scalar field to explain the accelerated expansion. In the end, the methods developed in [13] turn out to be quite robust (cf. the examples of generalizations given below), and since the essential ideas are most easily explained in the case of Einstein's vacuum equations with a positive cosmological constant, we shall do so in Section 4.

3.1. **The Einstein–non–linear scalar field system.** Before proceeding, let us write down the Einstein–non–linear scalar field system. It is given by

$$G = T, \tag{5}$$

$$\nabla^{\alpha}\nabla_{\alpha}\phi - V' \circ \phi = 0, \tag{6}$$

where T is the stress energy tensor associated with the non–linear scalar field:

$$T_{\alpha\beta} = \nabla_{\alpha}\phi\nabla_{\beta}\phi - \left[\frac{1}{2}\nabla^{\gamma}\phi\nabla_{\gamma}\phi + V \circ \phi\right] g_{\alpha\beta}.$$

In the above expressions, V is a smooth function from $\mathbb{R}$ to itself, referred to as the *potential*. In order to be able to proceed, it is necessary to make assumptions concerning the potential. Various choices are of interest (cf. [11, 12] for a discussion), but we shall here mainly be interested in the following cases.

Potentials with a positive non-degenerate minimum at the origin. Potentials of this type are characterized by the conditions that $V(0) > 0$, $V'(0) = 0$ and $V''(0) > 0$. They were studied, e.g., in [13]. Note that by demanding that $\phi = 0$, one obtains Einstein's vacuum equations with a positive comological constant. In order to have something with which to compare, it is useful to write down a model solution. One simple example is given by

$$\phi = 0, \tag{7}$$

$$g = -dt^2 + e^{2Ht}\bar{g}, \tag{8}$$

where g and ϕ are defined on $\mathbb{R} \times \mathbb{T}^n$, $\bar{g}$ is the standard metric on $\mathbb{T}^n$, and

$$H = \left(\frac{2V_0}{n(n-1)}\right)^{1/2}. \tag{9}$$

For future reference, it is also of interest to introduce the terminology

$$\chi = V''(0)/H^2. \tag{10}$$

In particular, solutions typically exhibit exponential expansion in this setting. As is clear from the above, we do not require the spacetime dimension to be 4. The reason for this is that some of the results hold for all spatial dimensions $n \geq 3$, in contrast with the arguments based on conformal methods.

Exponential potentials. Potentials of this type are characterized by the conditions that

$$V(\phi) = V_0 e^{-\lambda\phi}, \tag{11}$$

where $V_0 > 0$ and λ are constants, and

$$0 < \lambda < 2(n-1)^{-1/2}, \tag{12}$$

where n is the space dimension. They were studied, e.g., in [7, 15]. In this case, a natural model solution is given by

$$g = -dt^2 + t^{2p}\bar{g} \tag{13}$$

$$\phi = \frac{2}{\lambda}\ln t - \frac{1}{\lambda}c_0, \tag{14}$$

on $\mathbb{R}_+ \times \mathbb{T}^n$, where $\mathbb{R}_+ = (0, \infty)$, $p > 1$ is a constant and

$$\lambda = \frac{2}{[(n-1)p]^{1/2}}, \tag{15}$$

$$c_0 = \ln\left[\frac{(n-1)(np-1)p}{2V_0}\right]. \tag{16}$$

Note that one obtains power law expansion, and that the restriction (12) has been chosen so that the scale factor $a(t)$ of the spatially homogeneous model solution equals t^p, with $p > 1$; in particular, $1/a$ is integrable.

3.2. **Prototype results.** In order to have something with which to compare, let us give a rough formulation of a prototype result in the case of a potential with a positive non-degenerate minimum at the origin. Before stating the result, note that if $\gamma(t) = [t, \bar{\gamma}(t)]$ is a causal curve in $\mathbb{R} \times \mathbb{T}^n$ with respect to (8), then

$$-1 + e^{2Ht}|\dot{\bar{\gamma}}(t)|^2 \leq 0.$$

In particular

$$d[\bar{\gamma}(0), \bar{\gamma}(t)] \leq \frac{1}{H}$$

for $t \geq 0$, where d denotes the standard topological metric on $\mathbb{T}^n$. In particular, there is a bound on how far an observer can travel in the spatial directions. As a consequence, it is possible to obtain results which only involve local assumptions in space, but yield global conclusions in time. To be more specific, let us assume the following:

- We are given initial data to the Einstein–non–linear scalar field system $(\Sigma, \bar{g}, \bar{k}, \bar{\phi}_0, \bar{\phi}_1)$; here Σ is an n-dimensional manifold; $\bar{g}$ and $\bar{k}$ are a Riemannian metric and a symmetric covariant 2-tensor field on Σ respectively; and $\bar{\phi}_0$, $\bar{\phi}_1$ are smooth functions on Σ. Moreover, these objects satisfy the relevant constraint equations.
- On a sufficiently large ball, say $B_{4r_0}(p)$, the initial data (expressed with respect to suitable coordinates) are close enough to those of the model solution defined by (7) and (8).

Let (M, g, ϕ) be the maximal globally hyperbolic development of the initial data and $i : \Sigma \to M$ be the corresponding embedding. Then we obtain the following conclusions:

- The causal geodesics in (M, g) which start in $i[B_{r_0}(p)]$ are future complete.
- There is a region U, containing $J^+\{i[B_{r_0}(p)]\}$, in which it is possible to write down detailed asymptotics for the metric, second fundamental form, and the scalar field.

Clearly, the above is a rough formulation. Readers interested in a mathematically precise statement are referred to [13, Theorem 2, pp. 131–132]. Finally, let us point out that there are results of this type for general spatial dimensions $n \geq 3$.

In the case of an exponential potential, it is still true that causal curves can only travel a finite distance in the spatial directions (given the condition (12)). As a consequence, there is a similar result in that case; cf. [15, Theorem 2, pp. 160–161].

Stability of spatially homogeneous solutions. Combining results of the above type with an analysis of spatially homogeneous solutions and Cauchy stability, it is possible to derive stability results for spatially locally homogeneous solutions. To give an example of results of this type, let us state a slightly reformulated version of [13, Theorem 4, pp. 134–135]:

Theorem 1. *Let V be a smooth function such that $V(0) = V_0 > 0$, $V'(0) = 0$ and $V''(0) > 0$. Let H, $\chi > 0$ be defined by (9) and (10) respectively, let M be a connected and simply connected 3-dimensional manifold and let (M, g, k) be initial data to Einstein's equations with a positive cosmological constant $\Lambda = 3H^2$. Assume, furthermore, that one of the following conditions are satisfied:*

- *M is a unimodular Lie group different from $\mathrm{SU}(2)$ and g and k are left invariant under the action of this group.*
- *$M = \mathbb{H}^3$, where $\mathbb{H}^3$ is 3-dimensional hyperbolic space, and the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^3$.*
- *$M = \mathbb{H}^2 \times \mathbb{R}$ and the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^2 \times \mathbb{R}$.*

Assume finally that $\mathrm{tr}_g k > 0$. Let Γ be a cocompact subgroup of M in the case that M is a unimodular Lie group and a cocompact subgroup of the isometry group otherwise. Let Σ be the compact quotient. Then (Σ, g, k) are initial data. Make a choice of Sobolev norms $\|\cdot\|_{H^l}$ on tensorfields on Σ. Then there is an $\epsilon > 0$ such that if $(\Sigma, \rho, \kappa, \phi_0, \phi_1)$ are initial data for (5) and (6) satisfying

$$\|\rho - g\|_{H^4} + \|k - \kappa\|_{H^3} + \|\phi_0\|_{H^4} + \|\phi_1\|_{H^3} \leq \epsilon,$$

then the maximal globally hyperbolic development corresponding to $(\Sigma, \rho, \kappa, \phi_1, \phi_0)$ is future causally geodesically complete and there are expansions of the form given in the statement of [13, Theorem 2, pp. 131–132] *to the future.*

Remark 1. The restriction that the spatial dimension be 3 is due to the fact that the proof is based on a $3+1$-dimensional result concerning the future asymptotics of spatially homogeneous solutions. Given sufficient information concerning the future asymptotics of a spatially homogeneous solution in higher dimensions, we would obtain an analogous stability result concerning that solution.

Remark 2. The reader interested in an explanation for the additional restrictions imposed (that M not be isomorphic to $\mathrm{SU}(2)$ etc.) is referred to [13].

3.3. **Previous results.** We are now in a position to describe results that have been obtained in the past. To begin with, there are the results by Friedrich and Anderson [1, 3, 4] which we have already discussed (see also Friedrich's contribution to the present volume). In the case of a potential with a positive non-degenerate minimum, we have desribed the main results in Subsection 3.2. When the potential has special properties, the conformal methods of Friedrich also apply to this case;

cf. [5]. Results analogous to those described in Subsection 3.2 hold in the case of an exponential potential; cf. [15, Theorem 2, pp. 160–161] and [15, Theorem 3, p. 162]. However, it is of interest to note that stability results in this setting have also been obtained in [7] (prior to the appearance of [15]); cf., in particular, [7, Theorem 1, pp. 2-3]. The results of [7] are based on a combination of Kaluza-Klein reduction and appealing to the stability of higher dimensional de Sitter spaces (cf. [1]). In other words, the methods are very different from those used in [15]. However, there is an associated restriction; the methods of [7] only apply for a discrete set of λ-values (cf. (11) and [7, (6), p. 2]), whereas [15] applies for all λ of the form (15) with $p > 1$.

The Einstein–Maxwell–non–linear scalar field setting. It is of interest to prove results of the type described in Subsection 3.2 in the case of the Einstein–Maxwell–non–linear scalar field system. In the case of a potential with a positive non-degenerate local minimum, this is done in [19]. The case of an exponential potential is considered in [9].

The Einstein–Euler system. The study of the Einstein–Euler system with a positive cosmological constant was initiated in [17]. In this paper, the authors study the irrotational case under the assumption that the equation of state takes the form $p = c\rho$, where $0 < c < 1/3$; here p is the pressure and ρ is the energy density. In particular, dust (corresponding to $p = 0$) and radiation ($p = \rho/3$) are excluded. In the case of $\mathbb{T}^3$ spatial topology, the authors prove future global non-linear stability of spatially homogeneous and isotropic solutions. In particular, it is of interest to note that no shocks form in the evolution. The authors do not prove results similar to those described in Subsection 3.2, but by combining ideas from [17] and [13], it should be possible to do so. The method of proof used in [17] is partly based on [13] (as far as dealing with the metric components is concerned). However, the analysis of the matter requires new ideas. In the irrotational case, the matter is described by one scalar function Φ, but the equation for Φ has a different symbol than the equations for the metric components. Moreover, it is a non-trivial issue to verify that the symbol does not degenerate in the course of the evolution.

It is of interest to ask if similar results can be obtained in the general case. That the answer to this question is yes is demonstrated in [18]. In the general setting, the relevant equations are a system of wave equations for the metric components, coupled to a system of first order equations for the matter fields. Nevertheless, it turns out to be possible to construct suitable energy currents in order to deal with the matter fields.

As mentioned earlier, dust and radiation fluids are excluded in the results described above. However, such fluids are important in the solutions physicists use to model the universe. Interestingly, the case of a radiation fluid is suited to a treatment using conformal methods. In fact, future stability of the FLRW models has been demonstrated in the case of Einstein's equations with a positive cosmological constant, coupled to a radiation fluid; cf. [8]. Finally, the case of dust and a positive cosmological constant has been treated in [6]. Note that of all the matter models discussed above, dust is the most relevant; the matter content is expected to behave as dust in the expanding direction.

The Einstein–Vlasov–non–linear scalar field system. Results analogous to those described in Subsection 3.2 have also been obtained in the case of the Einstein–Vlasov–non–linear scalar field system; cf. [16]. The relevant type of potential to which the results apply is one with a positive non–degenerate local minimum. In particular, the Einstein–Vlasov system with a positive comological constant is contained as a special case. Moreover, it is possible to use Vlasov matter to approximate dust and radiation fluids; cf. [16, Chapter 28]. As a consequence, future stability of models consistent with observations is contained as a special case of the results of [16]. A separate topic which is discussed in [16] is that of the restrictions on the topology of the universe imposed by observations. Given the current preference for spatially flat model solutions, the constructions provided in [16] indicate that there are no restrictions. Since we have discussed this topic at length elsewhere, we shall not do so here, but rather refer the reader interested in more details to [16, Section 7.9].

Stability of spatially inhomogeneous solutions, the cosmic no–hair conjecture. The above results yield future stability of large classes of spatially homogeneous solutions. However, it is also of interest to consider inhomogeneous solutions. In the case of the Einstein–Vlasov equations with a positive cosmological constant, it turns out to be possible to analyze the future asymptotics under the assumption of surface symmetry and under the assumption of $\mathbb{T}^3$-Gowdy type symmetry; cf. [20] and [2]. Moreover, [2] contains a proof of future stability of the $\mathbb{T}^3$-Gowdy symmetric solutions in the class of all solutions. The analogous problem in the case of surface symmetry is considered in [10]. One reason results of this type are interesting is that the relevant symmetry classes are such that both significant anisotropies and significant spatial inhomogeneities are allowed. However, due to the results of [20, 2], these anisotropies and spatial inhomogeneities vanish asymptotically from the point of view of observers. In fact, the solutions appear de Sitter like to late time observers. Moreover, the stability results indicate that this is not just a feature of the symmetric solutions, since it persists under perturbations. In particular, the evolution associated with Einstein's equations is such that the solutions tend to homogenize and isotropize from the point of view of the observers. This is clearly a desirable feature, given that the currently preferred models of the universe are spatially homogeneous and isotropic. Finally, let us point out that the expectation that solutions to Einstein's equations with a positive cosmological constant should appear de Sitter like to late time observers goes under the name of the *cosmic no–hair conjecture*; cf., e.g., [2] for a precise formulation of the conjecture.

4. Sketch of proof of future stability, vacuum setting

Let us now sketch the proof of stability of the solution

$$g = -dt^2 + e^{2Ht}\bar{g}$$

to Einstein's vacuum equations with a cosmological constant $\Lambda = n(n-1)H^2/2$; here $\bar{g}$ is the standard flat metric on $\mathbb{T}^n$, $H > 0$ is a constant, and the metric is defined on $\mathbb{R}\times\mathbb{T}^n$. In order to obtain a hyperbolic system of equations to which the appropriate analysis tools can be applied, it is necessary to make a gauge choice. There are of course many ways of doing so, but we shall here use gauge source functions. The idea underlying this perspective is the following. Note, first of all, that the Ricci tensor can be written

$$R_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)} + g^{\alpha\beta}g^{\gamma\delta}[\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}]. \tag{17}$$

In this equation,

$$\begin{aligned}\Gamma_{\alpha\gamma\beta} =& \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}),\\ \Gamma_\nu =& g^{\alpha\beta}\Gamma_{\alpha\nu\beta},\\ \nabla_\mu\Gamma_\nu =& \partial_\mu\Gamma_\nu - \Gamma^\alpha_{\mu\nu}\Gamma_\alpha.\end{aligned}$$

Moreover, a parenthesis denotes symmetrization. In other words,

$$\nabla_{(\mu}\Gamma_{\nu)} = \frac{1}{2}(\nabla_\mu\Gamma_\nu + \nabla_\nu\Gamma_\mu).$$

The equation we wish to solve is (1) with $T = 0$. This equation can be reformulated to

$$R_{\mu\nu} = \frac{2}{n-1}\Lambda g_{\mu\nu}. \tag{18}$$

Considering (17), it is clear that if the second term on the right hand side were absent, then (18) would be a system of hyperbolic partial differential equation for the metric components. The idea of using gauge source functions is then the following:

- Replace the Γ_ν appearing in (17) by some other functions, say F_ν; we shall refer to the functions F_ν as the *gauge source functions*. Assuming the F_ν only to depend on the spacetime coordinates and the metric components (but not on their derivatives), the corresponding modified Ricci tensor is a hyperbolic differential operator acting on the components of the metric, so that (18) is a hyperbolic system of equations.
- Let $\hat{R}_{\mu\nu}$ denote the object obtained when replacing Γ_ν by F_ν in (17). In other words,
$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{(\mu}\mathcal{D}_{\nu)},$$
where
$$\mathcal{D}_\mu = F_\mu - \Gamma_\mu.$$
- Due to the assumptions concerning F_ν, solving the equation
$$\hat{R}_{\mu\nu} = \frac{2}{n-1}\Lambda g_{\mu\nu} \tag{19}$$
locally is a matter of standard PDE theory.
- The equation (19) and the Bianchi identities imply that $\mathcal{D}_\nu$ satisfies a homogeneous wave equation.
- Setting up the initial data for (19) correctly (appropriate requirements are that $\mathcal{D}_\nu$ should vanish initially and that the constraint equations should be satisfied initially), it can be verified that the initial data for $\mathcal{D}_\nu$ vanish; note that $\mathcal{D}_\mu$ could vanish initially without the normal derivative of $\mathcal{D}_\mu$ vanishing initially. Since $\mathcal{D}_\nu$ satisfies a homogeneous wave equation, we conclude that $\mathcal{D}_\nu = 0$ whenever the solution to (19) is defined. As a consequence, we obtain a solution to (18).

The above description is a bit brief. The reader interested in a somewhat longer explanation (of an overview character) is referred to [16, Chapter 2]. Readers interested in the technical details are referred to [14, Chapter 14].

Global considerations. As far as local considerations are concerned, the particular choice of gauge source functions is not important. However, we are interested in proving future global existence, and in that context, the choice is important. There are of course many possibilities, but the most naive possibility would be to choose the gauge source functions to be the contracted Christoffel symbols of the background. Even so, there are, however, two different choices; we could choose equality with indices upstairs or with indices downstairs. It turns out to be convenient to choose equality with indices downstairs; i.e.,

$$F_\nu = nHg_{0\nu}. \tag{20}$$

The reason this choice is convenient is that it gives rise to a damping term in the equations. The question is then: are the equations (19), given the choice (20) of gauge source functions, appropriate for proving future stability? In order to develop some intuition concerning this question, it is useful to consider the equations that arise when the terms in the equations containing two or more factors that vanish on the background have been removed. Considering, for example, the 00-component of (19), it reads

$$-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{00} + \frac{1}{2}nH\partial_0 g_{00} + nH^2 - Hg^{ij}\partial_0 g_{ij} + 2Hg^{ij}\partial_i g_{j0} + \Delta_{A,00} = nH^2 g_{00},$$

where $\Delta_{A,00}$ consists of terms that are quadratic in expressions that vanish on the background; cf. [13, pp. 154–157]. Clearly, it would be preferable to have a simpler equation. Moreover, if we replace $g^{\alpha\beta}$ by the background metric in the first term, replace g_{ij}, g^{ij} and g_{j0} by their background values in the fourth and fifth terms, and if we remove $\Delta_{A,00}$, then the resulting equation reads

$$\frac{1}{2}\partial_0^2 g_{00} - \frac{1}{2}e^{-2Ht}\Delta g_{00} + \frac{1}{2}nH\partial_0 g_{00} - nH^2(g_{00}+1) = 0,$$

where Δ denotes the standard Laplacian on $\mathbb{T}^n$. This equation can be rewritten

$$\partial_0^2(g_{00}+1) - e^{-2Ht}\Delta(g_{00}+1) + nH\partial_0(g_{00}+1) - 2nH^2(g_{00}+1) = 0.$$

Note that there are exponentially growing spatially homogeneous solutions to this equation. On the other hand, we would like $g_{00}+1$ to converge to zero (preferably exponentially). To conclude, to choose the equation (19) does not seem to be a good idea. However, there is still a freedom in modifying the equations. In particular, adding multiples of $\mathcal{D}_\nu$ to the equations is allowed. It turns out to be convenient to consider

$$\hat{R}_{\mu\nu} - \frac{2}{n-1}\Lambda g_{\mu\nu} + M_{\mu\nu} = 0,$$

where

$$M_{00} = -2Hg^{0\lambda}\mathcal{D}_\lambda, \quad M_{0i} = 2H\mathcal{D}_i, \quad M_{ij} = 0.$$

With this choice, it turns out that the equations can be written

$$-g^{\alpha\beta}\partial_\alpha\partial_\beta u + (n+2)H\partial_0 u + 2nH^2 u + \Delta_{00} = 0 \tag{21}$$

$$-g^{\alpha\beta}\partial_\alpha\partial_\beta g_{0m} + nH\partial_0 g_{0m} + 2(n-2)H^2 g_{0m} - 2Hg^{ij}\Gamma_{imj} + \Delta_{0m} = 0 \tag{22}$$

$$-g^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + nH\partial_0 h_{ij} + \Delta_{ij} = 0, \tag{23}$$

where $\Delta_{\mu\nu}$ are quadratic in terms that vanish on the background, $u = g_{00} + 1$ and $h_{ij} = e^{-2Ht} g_{ij}$; cf. [13, Lemma 14, p. 171]. Replacing $g^{\alpha\beta}$ with the corresponding object for the background, and removing Δ_{00}, the equation (21) becomes

$$\partial_0^2 u - e^{-2Ht}\Delta u + (n+2)H\partial_0 u + 2nH^2 u = 0.$$

It can quite easily be demonstrated that solutions to this equation decay to zero exponentially. Modifying (23) similarly, one concludes that h_{ij} converges exponentially. Finally, modifying (22) similarly, and assuming h_{ij} to converge at the appropriate rate, one can conclude that g_{0m} converges exponentially. On this naive level, the choice (21)–(23) thus seems to be appropriate.

Bootstrap argument. The essence of the proof of future stability is a bootstrap argument; assuming that certain bootstrap assumptions are fulfilled (concerning u, g_{0m} and h_{ij}) on a time interval, say $[0, T]$, the idea is to improve the bootstrap assumptions on this time interval (given that the initial data are close enough to those of the background). If it is possible to improve the bootstrap assumptions, it follows that they hold for the entire future. In particular, the argument yields future global existence. Moreover, the knowledge concerning the asymptotics can later on be improved in order to obtain detailed information concerning the behaviour of solutions.

The bootstrap assumptions can be divided into two groups. The first group consists of assumptions concerning the metric $g_{\mu\nu}$ in the supremum norm. To begin with, it is important to make sure that $g_{\mu\nu}$ are the components of a Lorentz metric in the course of the evolution. However, it is also natural to make assumptions that are adapted to the expected asymptotic behaviour. Judging by the background solution, u should be small and g_{ij} should expand as e^{2Ht}. Moreover, g_{0m} should be small. In the end, it turns out that g_{0m} converges, but that only follows by a rather complicated argument carried out a posteriori. Moreover, it is not natural to assume that g_{0m} is bounded (as part of the bootstrap assumptions). The reason for this is that there is a natural scaling associated with the number of downstairs spatial indices in the expressions that have to be estimated; it is natural to associate a factor of e^{Ht} with each downstairs spatial index. The natural way to state that g_{0m} is small is to say that $e^{-Ht} g_{0m}$ is small; in practice this corresponds to saying that $g^{ij} g_{0i} g_{0j}$ is small, or that the one form field $g(\partial_t, \cdot)$ is small relative to the Riemannian metric induced on the constant t hypersurfaces by g. To summarize, the first group of bootstrap assumptions consists of the requirements that u be small, that h_{ij} remain equivalent to the Kronecker delta δ_{ij}, and that $e^{-Ht} g_{0m}$ be small (in fact, we require $e^{-Ht} g_{0m}$ not only to be small, but also to be exponentially decaying).

The second group of bootstrap assumptions consists of the requirements that certain energies associated with u, g_{0m} and h_{ij} are small. The relevant energies for u and g_{0m} are given by

$$E_{\mathrm{l},k} = \frac{1}{2} \sum_{|\alpha| \le k} \int_{\mathbb{T}^n} \{(\partial^\alpha \partial_t u)^2 + g^{ij} \partial^\alpha \partial_i u \partial^\alpha \partial_j u + H^2 (\partial^\alpha u)^2\} dx, \tag{24}$$

$$E_{\mathrm{s},k} = \frac{1}{2} \sum_{|\alpha| \le k} \sum_i \int_{\mathbb{T}^n} [(\partial^\alpha \partial_t g_{0i})^2 + g^{lm} \partial^\alpha \partial_l g_{0i} \partial^\alpha \partial_m g_{0i} + H^2 (\partial^\alpha g_{0i})^2] dx. \tag{25}$$

The energy for h_{ij} is somewhat different, since h_{ij} is expected to converge. Note that in order for these energies to make sense it is necessary to make assumptions concerning the metric; this is one of the reasons for dividing the bootstrap assumptions into two groups. The second group of bootstrap assumptions concerning u and g_{0m} then consists of the requirement that

$$e^{2aHt}E_{\mathrm{l},k} + e^{-2Ht+2aHt}E_{\mathrm{s},k} \leq \epsilon^2 \tag{26}$$

for some small parameter ϵ, where $a > 0$ is an appropriately chosen parameter. The assumption concerning h_{ij} is somewhat more technical, but it is similar in character.

Enery estimates. The main step in improving the bootstrap assumptions consists of improving the estimates for the energies. In order to do so, we need to construct energies that are roughly of the form (24) and (25), but which are adapted to the equations (21)–(23). For appropriate choices of constants γ and δ, it turns out that basic energies of the form

$$\mathcal{E}_{\gamma,\delta}[v] = \frac{1}{2}\int_{\mathbb{T}^n}[-g^{00}(\partial_0 v)^2 + g^{ij}\partial_i v\partial_j v - 2\gamma H g^{00} v\partial_0 v + \delta H^2 v^2]dx \tag{27}$$

are appropriate. In fact, if v satisfies the equation

$$-g^{\alpha\beta}\partial_\alpha\partial_\beta v + \alpha H\partial_0 v + \beta H^2 v = F, \tag{28}$$

where $\alpha, \beta > 0$, then the constants γ and δ can be chosen so that $\mathcal{E}_{\gamma,\delta}[v]$ is equivalent to

$$\int_{\mathbb{T}^n}[(\partial_0 v)^2 + g^{ij}\partial_i v\partial_j v + H^2 v^2]dx.$$

Moreover,

$$\frac{d\mathcal{E}_{\gamma,\delta}}{dt} \leq -\eta H\mathcal{E}_{\gamma,\delta} + \int_{\mathbb{T}^n}\{(\partial_0 v + \gamma H v)F + \Delta_{E,\gamma,\delta}[v]\}dx,$$

where $\eta > 0$ is a constant, and $\Delta_{E,\gamma,\delta}[v]$ is an 'error term' which can be controlled (see below). In the process of improving the bootstrap assumptions, we need to take higher derivatives into account, and then we need to consider

$$\mathcal{E}_k = \sum_{|\alpha|\leq k}\mathcal{E}_{\gamma,\delta}[\partial^\alpha v].$$

However, up to a commutator term,

$$[g^{\alpha\beta}\partial_\alpha\partial_\beta, \partial^\gamma]v,$$

$\partial^\gamma v$ satisfies the same equation as v, so that the same energy estimates can be applied. In fact,

$$\begin{aligned}\frac{d\mathcal{E}_k}{dt} \leq & -\eta H\mathcal{E}_k + \sum_{|\alpha|\leq k}\int_{\mathbb{T}^n}\{(\partial_0\partial^\alpha v + \gamma H\partial^\alpha v)(\partial^\alpha F + [-g^{\mu\nu}\partial_\mu\partial_\nu, \partial^\alpha]v) \\ & + \Delta_{E,\gamma,\delta}[\partial^\alpha v]\}dx.\end{aligned} \tag{29}$$

Estimates. As mentioned above, the main step in improving the bootstrap assumptions consists of improving the estimates for the energies. The main tool in obtaining this improvement is (29). In order to be able to use this estimate, we need to estimate

$$\|\Delta_{E,\gamma,\delta}[\partial^\alpha v]\|_{L^1}, \quad \|[-g^{\mu\nu}\partial_\mu\partial_\nu, \partial^\alpha]v\|_{L^2}, \quad \|\partial^\alpha\Delta_{\mu\nu}\|_{L^2},$$

where $\Delta_{\mu\nu}$ is the 'error term' that appears in (21)–(23). In these expressions, v is one of u, g_{0m} or h_{ij}. It turns out that the terms that need to be estimated are quite involved; cf. [13, pp. 154–158] for details. Moreover, the different metric components (and derivatives thereof) that appear in the expressions have very different asymptotic behaviour; g_{00} should be expected to tend to -1 exponentially, g_{0m} should a priori be expected to tend to infinity exponentially, and g_{ij} should tend to infinity as e^{2Ht}. Due to the sheer number of terms in the expressions that need to be dealt with, it is extremely important to develop a systematic way of estimating them. There are several aspects to this problem. One aspect is the specific appearance of the terms; this is not something that much can be done about. Another aspect is the choice of bootstrap assumptions. As mentioned earlier, it is in the end possible to prove that u converges to zero exponentially, that g_{0m} converges to a (typically non-zero) limit exponentially to the future, and that h_{ij} converges exponentially. However, the estimates for the 'error terms' become quite complicated if one phrases the bootstrap assumptions in a way that naturally incorporates such asymptotics (moreover, it is very difficult to improve bootstrap assumptions for g_{0i} that involve a bound; i.e., no exponential growth). In the end, the choice of bootstrap assumptions is such as to make a systematic estimate of the 'error terms' as easy as possible. In particular, the choice is not motivated by a desire to obtain the correct asymptotics immediately from the bootstrap assumptions.

System of differential inequalities. Once the terms $\Delta_{\mu\nu}$ etc. have been estimated and the relevant energies have been defined, it is possible to derive a system of differential inequalities for the energies. The system one obtains is the following (assuming that the bootstrap assumptions hold):

$$\frac{d\hat{H}_{\mathrm{l},k}}{dt} \leq -4aH\hat{H}_{\mathrm{l},k} + CH\epsilon e^{-aHt}\hat{H}_k^{1/2}\hat{H}_{\mathrm{l},k}^{1/2}, \tag{30}$$

$$\frac{d\hat{H}_{\mathrm{s},k}}{dt} \leq -4aH\hat{H}_{\mathrm{s},k} + CH\hat{H}_{\mathrm{m},k}^{1/2}\hat{H}_{\mathrm{s},k}^{1/2} + CH\epsilon e^{-aHt}\hat{H}_k^{1/2}\hat{H}_{\mathrm{s},k}^{1/2}, \tag{31}$$

$$\frac{d\hat{H}_{\mathrm{m},k}}{dt} \leq He^{-aHt}\hat{H}_{\mathrm{m},k} + CH\epsilon e^{-aHt}\hat{H}_k^{1/2}\hat{H}_{\mathrm{m},k}^{1/2}. \tag{32}$$

In these inequalities, a and ϵ are the same parameters that appear in the bootstrap assumption (26). In order to define the energies $\hat{H}_{\mathrm{l},k}$ and $\hat{H}_{\mathrm{s},k}$, one proceeds as follows. Consider (21) and (22). These equations are of the form (28). As a consequence, it is possible to associate appropriate γ's and δ's with them and to construct energies $\mathcal{E}_k$ as described above. The $\hat{H}_{\mathrm{l},k}$ and $\hat{H}_{\mathrm{s},k}$ are then appropriately rescaled versions of these energies. In particular, if γ_{l} and δ_{l} are the constants associated with (21) according to the above scheme, then

$$\hat{H}_{\mathrm{l},k} = H^{-2}e^{2aHt}\sum_{|\alpha|\leq k}\mathcal{E}_{\gamma_{\mathrm{l}},\delta_{\mathrm{l}}}[\partial^\alpha u].$$

Similarly,

$$\hat{H}_{\mathrm{s},k} = H^{-2}e^{-2Ht+2aHt}\sum_i\sum_{|\alpha|\leq k}\mathcal{E}_{\gamma_{\mathrm{s}},\delta_{\mathrm{s}}}[\partial^\alpha u_i].$$

The definition of the energy $\hat{H}_{\mathrm{m},k}$ is slightly different, but similar. Finally,

$$\hat{H}_k = \hat{H}_{\mathrm{l},k} + \hat{H}_{\mathrm{s},k} + \hat{H}_{\mathrm{m},k}.$$

In order to develop some intuition for the differential inequalities, it is useful to compare them with the equations (21)–(23). Considering (21), this equation should yield exponential decay (up to some 'error terms'). This intuition should be compared with (30); the first term on the right hand side corresponds to the expected exponential decay. However, there is an error term, corresponding to the second term on the right hand side. Consider (22). If it were not for the second to last term on the left hand side of this equation, we would expect the same structure in the differential inequality for $\hat{H}_{\mathrm{s},k}$. However, the term $-2Hg^{ij}\Gamma_{imj}$ does make a difference. In fact, it gives rise to the second term on the right hand side of (31). Note that this is the most problematic term in the system of differential inequalities; if we were not to distinguish between the different components of the energy, this term would read $CH\hat{H}_k$, and it would make it impossible to improve the bootstrap assumptions. Finally, consider (23). This equation is such that one does not expect convergence to zero of h_{ij}. However, one does expect the energy to converge to a non-zero number. This expectation fits well with (32).

Proving future stability; improving the bootstrap assumptions. The main step in proving future global non-linear stability consists of improving the bootstrap assumptions. Moreover, the main bootstrap assumption is essentially equivalent to

$$\hat{H}_{k_0}(t) \leq \epsilon^2 \tag{33}$$

on some time interval, say $[0, T]$; here k_0 is an integer strictly larger than $n/2 + 1$. In other words, this is the estimate we need to improve. In order for this to be possible, we clearly have to have a strict inequality at $t = 0$. In fact, we shall assume

$$\hat{H}_{k_0}(0) \leq c_0^2 \epsilon^2$$

for some constant $c_0 \in (0, 1)$. The tool relevant for improving the bootstrap assumptions is the system of differential inequalities (30)–(32). The most naive way to proceed would be to add the differential inequalities in order to obtain a differential inequality for $\hat{H}_k$. However, the resulting inequality would be of the form

$$\frac{d\hat{H}_k}{dt} \leq C\hat{H}_k,$$

where $C \geq 0$ is an unknown constant. Clearly, this is not useful in improving the bootstrap assumptions. On the other hand, combining the differential inequality for $\hat{H}_{l,k}$ with the bootstrap assumption yields

$$\frac{d\hat{H}_{\mathrm{l},k_0}}{dt} \leq CH\epsilon^3 e^{-aHt}.$$

This inequality can be integrated to

$$\hat{H}_{\mathrm{l},k_0}(t) \leq \hat{H}_{\mathrm{l},k_0}(0) + Ca^{-1}\epsilon^3 \leq c_0^2\epsilon^2 + Ca^{-1}\epsilon^3.$$

Clearly, we can thus assume $\hat{H}_{\mathrm{l},k_0}$ to be as small a factor as we wish times ϵ^2 (by assuming c_0 and ϵ to be small enough). Similarly, it is possible to improve the estimates for $\hat{H}_{\mathrm{m},k_0}$. Finally, once these improvements have been obtained, it is possible to use (31) in order to improve the estimate for $\hat{H}_{\mathrm{s},k_0}$. Adding up arguments of the above type yields an improvement of the bootstrap assumptions, and as a consequence future global existence follows. Moreover, using the fact that the bootstrap assumptions hold for the entire future, it is possible to derive more detailed asymptotics. However, we omit the details.

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