ORIGINS AND DEVELOPMENT OF THE CAUCHY PROBLEM IN GENERAL RELATIVITY

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Abstract. The seminal work of Yvonne Choquet-Bruhat published in 1952 demonstrates that it is possible to formulate Einstein's equations as an initial value problem. The purpose of this article is to describe the background to and impact of this achievement, as well as the result itself.

In some respects, the idea of viewing the field equations of general relativity as a system of evolution equations goes back to Einstein himself; in an argument justifying that gravitational waves propagate at the speed of light, Einstein used a special choice of coordinates to derive a system of wave equations for the linear perturbations on a Minkowski background. Over the following decades, Hilbert, de Donder, Lanczos, Darmois and many others worked to put Einstein's ideas on a more solid footing. In fact, the issue of local uniqueness (giving a rigorous justification for the statement that the speed of propagation of the gravitational field is bounded by that of light) was already settled in 1937 by the work of Stellmacher. However, the first person to demonstrate both local existence and uniqueness in a setting in which the notion of finite speed of propagation makes sense was Yvonne Choquet-Bruhat. In this sense, her work lays the foundation for the formulation of Einstein's equations as an initial value problem.

Following a description of the results of Choquet-Bruhat, we discuss the development of three research topics that have their origin in her work. The first one is local existence. One reason for addressing it is that it is at the heart of the original paper. Moreover, it is still an active and important research field, connected to the problem of characterizing the asymptotic behaviour of solutions that blow up in finite time. As a second topic, we turn to the questions of global uniqueness and strong cosmic censorship. These questions are of fundamental importance to anyone interested in justifying that the Cauchy problem makes sense globally. They are also closely related to the issue of singularities in general relativity. Finally, we discuss the topic of stability of solutions to Einstein's equations. This is not only an important and active area of research, it is also one that only became meaningful thanks to the work of Yvonne Choquet-Bruhat.

1. Introduction

Writing a review article on the work of Yvonne Choquet-Bruhat on the Cauchy problem [78], including the historical background and the impact it has had, is a difficult task. This is partly due to the fact that the roots go back to the inception of general relativity, and that the consequences stretch to the present. However, it is mainly due to the broad impact of the perspective developed by Choquet-Bruhat. In fact, the research areas that have their origin in the initial value formulation of Einstein's equations are so numerous and diverse that it is not possible for a single author to do them justice. We consequently hope that the reader is able to forgive the omissions which are due to the ignorance of the author.

From one perspective, the results of [78] can be considered to be abstract and mathematical in nature; they concern the local existence and uniqueness of solutions to Einstein's equations without providing any information concerning the global behaviour. However, the problems considered in [78] have their origin in the question of determinism in general relativity, as well as in the question of the speed of propagation of the gravitational field. Clearly, these are fundamental issues in any physical theory, and they played an important role in Einstein's thinking even prior to the formulation of general relativity in [72, 73]. It is interesting to trace the developments that led to the work of Choquet-Bruhat, starting with Einstein's papers concerning the speed of propagation of the gravitational field in [74, 75]; progressing via Hilbert's observations concerning the problem of uniqueness in [95]; de Donder's [68] and Lanczos' [109] introduction of the coordinates which lie at the heart of Choquet-Bruhat's argument; the remarkable insights of Darmois, stated in [66], concerning the problem of solving Einstein's equations given data on a spacelike hypersurface; the overview of the state of the field given in [67]; Stellmacher's resolution of the problem of local uniqueness in [161]; to Lichnerowicz's statement (in [111]) of the problem that Yvonne Choquet-Bruhat addresses in [78], and his discussion of the constraint equations in [112]. Describing this sequence of results is interesting since it illustrates how the small steps taken by each author in the end lead to insights, statements and questions which go a significant distance beyond the original ideas and perspectives. The questions and doubts that appear along the way also illustrate the strength of the current formulation of the initial value problem (as well as the painstaking effort required to arrive at it). For these reasons, we devote Section 2 to a discussion of the historical background to [78].

Turning to the impact of the work of Choquet-Bruhat, the results of [78] demonstrate the possibility of solving Einstein's equations, given initial data. Needless to say, this observation has far reaching consequences, since it opens up the possibility of using a wide variety of numerical and analytical techniques in order to study solutions. In the present article, we limit ourselves to a discussion of mathematical methods, since we do not have the competence required to describe, e.g., the numerical perspective (see, however, the contribution of Ulrich Sperhake to the present volume). Following a description of the results and arguments contained in [78], cf. Section 3, we thus turn to an overview of three fields of research that can be studied using mathematical methods, and that have their origin in the work of Choquet-Bruhat.

Local existence. A large part of [78] is devoted to proving local existence of solutions to the Cauchy problem that arises when expressing Einstein's equations with respect to the isothermal coordinates introduced by de Donder. The corresponding subject has a long history in its own right, and it is a history that it is at least partly necessary to understand in order to appreciate the contribution of Choquet-Bruhat. Moreover, the subject of local existence is still an active field of research. Partly for these reasons, we discuss this topic at some length in Section 4. However, the question of local existence is also related to the study of blow up phenomena more generally. The reason for this is that local existence results typically come with a so-called continuation criterion. A continuation criterion is a statement of the form: either the solution exists globally, or a certain norm of the solution becomes unbounded as the maximal time of existence is approached. Continuation criteria are of central importance in several contexts. First of all, they provide information concerning the behaviour of the solution close to the time at which it blows up (though, needless to say, it is necessary to keep in mind that the blow up could be caused by a bad choice of gauge, as opposed to representing a physically relevant singularity). Moreover, they are a basic first step in the problem of analyzing the global behaviour of solutions. In particular, they are needed in stability proofs. Another reason for taking an interest in this topic is related to the following question: to what extent can our universe or an isolated system be approximated by the highly idealized solutions normally used? To take one example, the universe is normally said to be almost spatially homogeneous and isotropic. However, in order to fit this statement with Einstein's equations, the notion of proximity has to be strong enough that the initial value problem is well posed with respect to the corresponding norm. If the notion is not strong enough, the proximity to spatial homogeneity and isotropy is due to some sophisticated non-linear phenomenon, as opposed to the consequence of a stability result. From the mathematical point of view, the natural perspective from which to address this issue is by trying to prove local existence of solutions in as low a regularity as possible.

Global uniqueness, strong cosmic censorship. Even though the contributions of Stellmacher and Choquet-Bruhat settle the question of local uniqueness of solutions to Einstein's vacuum equations, the question of global uniqueness remains. Again, this may seem to be a mathematical question of limited practical insterest. However, if it is not possible to associate a unique global development with initial data, there would be no contradiction in several authors obtaining several different solutions (corresponding to the same initial data) with completely different geometric properties. Solving Einstein's equations by means of the Cauchy problem would then not be of much use, and speaking of any type of "properties of the solution" corresponding to a given initial data set would, a priori, be meaningless. Interestingly, a very important step in the direction towards a global uniqueness result was taken by Yvonne Choquet-Bruhat, this time in collaboration with Robert Geroch. The result they prove in [24] is that there is a unique maximal globally hyperbolic development corresponding to a given initial data set. For the sake of brevity, we shall refer to this development as the maximal Cauchy development. Due to the importance of the result, we spend some time discussing the contents of [24] in Section 5. Unfortunately, there are examples of initial data for which the maximal Cauchy development has inequivalent maximal extensions; we describe some examples in Subsection 5.2. Since the examples are very special, one is led to the so-called strong cosmic censorship conjecture, one version of which states that for generic initial data, the maximal Cauchy development is inextendible. We formulate and discuss this conjecture, as well as a related conjecture concerning curvature blow up, in Subsection 5.2. We end the section by an overview of results that have obtained concerning strong cosmic censorship.

Stability. One problem of central importance in general relativity is to verify the stability of the highly symmetric solutions that are normally used to model the universe or isolated systems. We turn to this problem in Section 6. For natural reasons, the first spacetime one would like to prove stability of is Minkowski space. We give an overview of some of the results that have been obtained concerning this topic, and give a rough idea of the methods of proof used in the different approaches. As a next step, it would be of interest to prove stability of the Kerr family. So far, this has turned out to be a very difficult problem. However, there is an extensive literature on the topic of linear equations on black hole backgrounds, written with the aim of serving as a basis for a future stability proof. We give a brief overview of the corresponding results. Turning to cosmological spacetimes, we discuss some of the stability results that have been obtained in Subsection 6.3.

Omissions. The three fields of research discussed in Sections 4–6 only give a very limited idea of the impact of the work of Yvonne Choquet-Bruhat. For instance, we say nothing about the constraint equations (cf. [98, 13, 29] and references cited therein), the Penrose inequality (cf. [124] and references cited therein), numerical relativity (cf. [22], Ulrich Sperhake's contribution to the present volume, as well as references cited therein) etc. The reason for these omissions is related to the limited ability of the author to do the corresponding subjects justice.

2. Causality, gravitational waves and the notion of uniqueness of solutions to Einstein's equations

The purpose of the present section is to give a rough historical overview of some of the ideas that preceded [78]. Interestingly, questions related to the work of Choquet-Bruhat appeared as early as 1916, in Einstein's study of the question of the speed of propagation of gravitational waves.

Speed of propagation of gravitational waves. In the early days of general relativity, the answers to many fundamental questions were unclear. One such question was: does the gravitational field propagate at the speed of light? Already in 1916, Einstein wrote a paper to address this issue, see [74]. Note, however, that for reasons associated with shortcomings in the presentation in [74], he returned to this topic in his 1918 paper [75]. Considering a situation in which the metric is close to that of Minkowski space, he, in practice, studied the linearized problem. Using a special choice of coordinates (on a linearized level), he derived a wave equation for the perturbation, a result he used to justify the statement that the gravitational field propagates at the speed of light. This line of reasoning is also to be found in, for example, the presentations of Weyl and Eddington; cf. [177, pp. 213–216] and [71, pp. 128–131]. For a modern presentation of this material, see, e.g., [178, pp. 74–76]. The arguments of Einstein give an indication that, in some respects, the field equations of general relativity are a system of wave equations, and that the natural problem to pose on the mathematical level is the initial value problem. Nevertheless, the role of the choice of coordinates was not entirely clear at the time. In fact, Eddington [71, pp. 130–131] made the following comments concerning Einstein's justification (following the presentation of Einstein's argument):

> The statement that in the relativity theory gravitational waves are propagated with the speed of light has, I believe, been based entirely on the foregoing investigation; but it will be seen that it is only true in a very conventional sense. If coordinates are chosen so as to satisfy a certain condition which has no very clear geometrical importance, the speed is that of light; if the coordinates are slightly different the speed is altogether different from that of light. The

> result stands or falls by the choice of coordinates and, so far as can be judged, the coordinates here used were purposely introduced in order to obtain the simplification which results from representing the propagation as occurring with the speed of light. The argument thus follows a vicious circle.

The role of the coordinate freedom. That the diffeomorphism invariance of the equations makes it more complicated to assign a meaning to the notion of determinism (and to prove that it holds) was clear to Einstein already prior to his publication of the field equations. These complications are also discussed by Hilbert in [95]. In particular, Hilbert constructs two smooth coordinate systems on $\mathbb{R}^4$ that coincide for $t \leq 0$, but are different for $t > 0$. Considering a special solution to the Einstein-Maxwell system, he concludes that the corresponding coordinate representations of the solution coincide for $t \leq 0$, but not for $t > 0$. In particular, note that the two coordinate representations (and *all* their derivatives) coincide on the $t = 0$ hypersurface, even though they do not coincide in a neighbourhood of the $t = 0$ hypersurface. From this point of view, it is thus not possible to uniquely determine the solution on the basis of information of what it and its partial derivatives are at present.

One way to approach the objections of Eddington is to argue that gravitational waves propagate at the speed of light without appealing to a specific choice of coordinates. This perspective was taken by Vessiot in [173], a paper in which it is argued that the desired statement follows from the observation that discontinuities in the derivatives of the metric of order strictly higher than one are only allowed along null hypersurfaces. Note, however, that since [173] appeared before [71], Vessiot's work should not be thought of as an answer to Eddington's comments.

In a related development, de Donder observed that the coordinate choice of Einstein is the linearized version of the requirement that

$$g^{\sigma\tau}(\partial_\alpha g_{\sigma\tau} - 2\partial_\tau g_{\alpha\sigma}) = 0; \tag{1}$$

cf. [68, p. 40], in particular [68, $(117)_1$, p. 40]. Moreover, he noted that for coordinates satisfying (1), Einstein's vacuum equations take the form

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = L_{\mu\nu},$$

where $L_{\mu\nu}$ depends on the metric components and on their first derivatives, but not on their second derivatives; cf. [68, pp. 40–41]. This form of the equations makes it very plausible that the gravitational field propagates at the speed of light. However, the objections of Eddington apply equally well to de Donder's choice of coordinates. Interestingly, the same coordinates were also introduced one year later in the work of Lanczos [109]. Moreover, the slightly different perspective taken in [109] later led Darmois to the observation that (1) corresponds to the requirement that the coordinates satisfy the scalar wave equation with respect to the metric under consideration; cf. [67, pp. 16–17].

On the notion of initial data. In the truly remarkable paper [66], Georges Darmois poses the problem of studying Einstein's vacuum equations in the neighbourhood of a hypersurface. In his discussion of it, he addresses the freedom in choosing lapse and shift and the propagation of the constraints. Moreover, he identifies the induced metric and second fundamental form as natural initial data, and he points out that the initial data are limited by the requirement that the constraint equations be satisfied. An actual formulation of the constraint equations, based on the use of Gaussian coordinates, can be found in [67, pp. 8–9].

Combining the ideas. A general discussion, synthesizing the above observations, is to be found in [67]. In particular, in [67, Chapter 2], Darmois considers the question of recovering all the derivatives of the metric on a hypersurface, given the metric on the hypersurface, as well as its first normal derivative; cf. the above description of Vessiot's work [173]. On the basis of this analysis, he argues that characteristic hypersurfaces play a special role in the process of solving the equations (in that one cannot recover all the derivatives), leading to the above mentioned conclusion of Vessiot. One particular consequence of Darmois' analysis is that given the metric and its first normal derivative on a spacelike hypersurface, all the derivatives of the metric are determined on the hypersurface. Needless to say, this yields a local uniqueness result in the real analytic setting; cf. Subsection 4.1 below. Moreover, he notes that there is a linear homogeneous system of equations for the components of the Ricci tensor corresponding to the constraints. On the basis of this observation, he argues that it is not only necessary, but also sufficient, that the constraints be satisfied in order for there to be a real analytic solution to Einstein's equations (recall, however, the limitations associated with the real analytic setting illustrated by Hilbert's example described above). Darmois then goes on, in [67, Chapter 3], to address the question of why the coordinates used by Einstein are successful in demonstrating that the gravitational fields propagate at the speed of light. In order to do this in the general, non-perturbative regime, he first refers to the coordinate choice of de Donder [68] mentioned above. Darmois calls these coordinates *isothermal*, though the corresponding gauge choice is sometimes also referred to as harmonic, wave-coordinates or de Donder gauge. The reason for introducing this terminology is Darmois' observation that isothermal coordinates satisfy the wave equation associated with the metric: a function u solving the Laplace equation in the Euclidean setting can be thought of as corresponding to a static solution to the heat equation, and the surfaces of constant u are thus isothermal; thinking of the wave equation associated with the metric as the analogue of the Laplace equation, surfaces on which an isothermal coordinate is constant are thus 'isothermal' with respect to that coordinate. In addition, the property that they satisfy the scalar wave equation justifies, in Darmois' opinion, the naturalness of the coordinates, or, to use Darmois' own words (cf. [67, p. 18]):

> Nous avons donc la solution complète du problème posé, et nous voyons bien que ce n'est pas hasard si notre système de coordonnées présente sous une forme simple la propagation des potentiels. C'est parce qu'il est lié, de la manière la plus nette, à cette propagation elle-mème.

A geometric uniqueness proof. In spite of Darmois' optimistic assessment quoted above, a fundamental question remains. Given a solution to, say, Einstein's vacuum equations, there are two notions of causality. First, there is the causality associated with the metric, and, second, there is the notion of domain of dependence associated with solving Einstein's equations considered as a partial differential equation (PDE) (keeping the complications associated with the diffeomorphism invariance in mind). It is then of interest to know if these two notions coincide. This question was addressed by Stellmacher in [161] (on the suggestion of Friedrichs), as a part of his dissertation. Note that it cannot be addressed in the real analytic setting, since real analytic functions have the unique continuation property; cf. Subsection 4.1 below.

Stellmacher's argument is based on the use of isothermal coordinates. In fact, given two solutions to the Einstein-Maxwell equations (whose initial data coincide on a spacelike hypersurface), Stellmacher constructs isothermal coordinates (or 'de Dondersche Koordinatensysteme', as Stellmacher puts it) such that the PDE techniques of Friedrichs and Lewy can be applied. The conclusion is then that the solutions coincide up to a coordinate transformation. An important observation, which follows from Stellmacher's work, is that not only is it possible to construct isothermal coordinates, there is also a certain amount of free data that can be specified on a spacelike hypersurface.

Note, in particular, that Stellmacher's work constitutes a justification of the statment that the gravitational field propagates at a speed bounded by that of light. Moreover, the argument is such that Eddington's objections do not apply.

Rough formulation of the initial value problem. In the first chapter of [111], Lichnerowicz recapitulates much of the progress made. In a section entitled 'The exterior problem' [111, pp. 14–17], he states the initial value problem as that of finding the solution to Einstein's equations on the basis of the metric and its first derivatives on a hypersuface. He then solves the problem in the real analytic setting for spacelike hypersurfaces (using arguments going back to Darmois), and notes the importance of the constraints. Acknowledging the results of Stellmacher, he goes on to point out that it would be extremely important to similarly generalize the existence result to the non-real analytic setting; cf. [111, p. 17]. This is of course the problem solved by Yvonne Choquet-Bruhat.

Concluding remarks. Needless to say, the above presentation is incomplete. A more extensive description of the same material (from a somewhat different perspective) is given in [160].

3. The result of Yvonne Choquet-Bruhat

The seminal paper of Yvonne Choquet-Bruhat [78] represents the resolution of the problem posed by Lichnerowicz in [111, p. 17]. In other words, not only does local uniqueness hold in the class of C^k-functions (k times continuously differentiable functions) for k large enough (as demonstrated by Stellmacher [161]); *given initial data, there is a unique local solution.* As a consequence, [78] puts the Cauchy problem in general relativity on a solid footing in the C^k-setting. It is then natural to ask: why is the specific regularity class of importance? Why is it not sufficient to consider the case of real analytic functions? Since this is a somewhat technical topic, we discuss it separately in Subsection 4.1.

Turning to a more detailed description of the paper, a large part of the difficulty in obtaining the desired result lies in proving local existence of solutions to Einstein's equations in the prescribed regularity. Moreover, in order for the PDE methods to apply, it is necessary to use coordinates with respect to which the equations become hyperbolic (Choquet-Bruhat uses the isothermal coordinates introduced by de Donder [68]). Finally, it is necessary to connect the problem of solving the reduced equations (which only a posteriori correspond to Einstein's equations with respect to a specific choice of coordinates) with the constraint equations and the problem of solving Einstein's equations. It is important to note that the last step leads to problems that do not appear in the study of uniqueness.

The PDE aspect of the problem. The PDE that needs to be dealt with when solving Einstein's equations is of the form

$$A^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0, \tag{2}$$

where the $A^{\lambda\nu}$ depend on the unknowns W_s, and f_s depends on the unknowns and their first derivatives. In addition, the $A^{\lambda\nu}$ are assumed to satisfy the algebraic conditions that $A^{00} < 0$ and that A^{ij}, $i, j = 1, 2, 3$, are the components of a positive definite matrix (Choquet-Bruhat's conventions are somewhat different, but her requirements are effectively the same); needless to say, more conditions need to be imposed, but we omit them for the sake of brevity. In reality, Choquet-Bruhat studies the more general case when the $A^{\lambda\nu}$ are allowed to depend on the first derivatives of the unknowns W_s. The problem of solving (2) for given initial data is quite complicated and occupies some 74 pages of the paper. The first step involves considering linear equations. In particular, Choquet-Bruhat demonstrates that solutions to the relevant type of linear equations solve a system of integral equations; cf. [78, (20.2), p. 173].

Turning to the non-linear problem, it would be desirable to set up an iteration of the following form: given a function W_s', solve (2) with appropriate initial data and with $A^{\lambda\nu}$ and f_s replaced by the corresponding functions (of the coordinates only) obtained by composing $A^{\lambda\nu}$ and f_s with W_s' and its first derivatives. This yields a function W_s, which can be considered to be the image of W_s' under a map, say ϕ. Solving the Cauchy problem then turns into the problem of finding a fixed point of the map ϕ. In order to prove the existence of a fixed point, it is necessary to prove that ϕ has certain properties. In particular, there has to be an appropriate function space which is preserved by the map. In [78], a C^k solution is sought, and consequently, the relevant function space should consist of C^k functions. In particular, the map thus has to preserve the C^k class. Unfortunately, the above perspective leads to a loss of derivatives, so that ϕ cannot be used as desired. In order to overcome this difficulty, it is necessary to proceed somewhat differently. Differentiating (2) k times leads to an equation of the form

$$A^{\lambda\mu}\frac{\partial^2 U_S}{\partial x^\lambda \partial x^\mu} + B_S^{T\lambda}\frac{\partial U_T}{\partial x^\lambda} + F_S = 0, \tag{3}$$

where U_S collects all derivatives of W_s of order k, and $A^{\lambda\nu}$, $B_S^{T\lambda}$ and F_S depend on at most one, two and k derivatives of W_s respectively; cf. [78, p. 183]. In particular, assuming W_s to be $k+2$-times continuously differentiable, the coefficients $A^{\lambda\nu}$ and $B_S^{T\lambda}$ are at least $k+1$ and k times differentiable respectively. For a suitable choice of k, the idea is then to consider the integral equations corresponding to (3); to define a map similar to ϕ based on these equations (cf. the above); and to prove that this map has a fixed point. Unfortunately, it turns out that this perspective can only be applied if the $A^{\lambda\nu}$ do not depend on the first derivatives of W_s. However, by first differentiating the equation (2) once, and then proceeding as above, the more complicated case can also be handled. Needless to say, the above is a very rough sketch of the argument. The reader interested in more details is referred to [78].

The main PDE result is summarized in [78, pp. 218–219]. It is of interest to record the regularity requirements on the initial data: at the $t = 0$ hypersurface, W_s and $\partial_t W_s$ are assumed to be five and four times continuously differentiable respectively. On the other hand, the solution obtained is only four times continuously differentiable. In other words, the initial data induced on a $t = t_0$ hypersurface, with $t_0 \neq 0$, do not necessarily have the same degree of regularity as the original initial data. We discuss this topic further below.

Solving Einstein's equations. In [78, pp. 219–224], Choquet-Bruhat describes how to construct solutions to Einstein's vacuum equations, given initial data. One way to write the vacuum equations is

$$S^{\lambda\mu} = 0,$$

where

$$S^{\lambda\mu} = R^{\lambda\mu} - \frac{1}{2} R g^{\lambda\mu},$$

$R^{\lambda\mu}$ are the components of the Ricci tensor of the metric g and R is the scalar curvature. Referring to the metric and its first derivatives on a spacelike hypersurface as the "initial data" for the equation, Choquet-Bruhat notes that the initial data have to satisfy the constraint equations. In order to be able to use isothermal coordinates, the initial data are also assumed to be such that

$$F^{\mu} := \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} g^{\lambda\mu})}{\partial x^{\mu}} = 0$$

for $t = 0$; cf. [78, (2.2), p. 220]; this can in fact be assumed without loss of generality.

The next step is to solve the hyperbolic system of equations Einstein's equations reduce to when *assuming* the coordinates to be isothermal. In particular, one is, a priori, not solving Einstein's equations. Due to the properties of isothermal coordinates, the problem of solving the relevant equations can be handled using the methods developed in [78, Chapters I–III]. Next, Choquet-Bruhat argues that since the F^{μ} are zero initially, and since the constraint equations are satisfied initially, the first order derivatives of the F^{μ} are zero initially. Finally, using the Bianchi identities, she demonstrates that the F^{μ} satisfy a homogeneous wave equation. By uniqueness, it then follows that the F^{μ} are zero, so that the coordinates are isothermal and the metric satisfies Einstein's vacuum equations. In short, this line of reasoning yields local *existence* of solutions. To end the paper, there is a proof of uniqueness on [78, pp. 223-224]. The final sentence of the paper is the statement of the main theorem, namely that there is a unique solution corresponding to initial data.

Comparing the formulation of the results with more recent versions, it is of interest to note that the definition of the initial data is not geometric in nature (the metric and its first derivatives are referred to as initial data, not the induced metric and second fundamental form). Moreover, the statement that there is a unique solution should be understood as saying that there is a unique local solution. Global issues are not addressed.

4. Regularity notions, local existence, and continuation criteria

As noted above, one central improvement obtained in [78] in comparison with earlier results is that real analyticity of the initial data is not required. In Subsection 4.1, we justify the importance of this improvement by defining real analyticity and arguing that the corresponding function class is inappropriate in the study of general relativity. Even though the class of functions used by Choquet-Bruhat does not suffer from the same deficiencies as the real analytic class, it is problematic if one is interested in taking the step from local to global considerations; cf. Subsection 4.2. The reason for this is closely related to the notion of a continuation criterion, which we discuss briefly in Subsection 4.3. We then return to the topic of proving local existence of solutions with a view towards obtaining results that might be useful in studying global questions; this is the subject of Subsection 4.4. In particular, for those readers unfamiliar with Sobolev spaces, we motivate why they are natural in the study of equations of the type (2). We end the section by a giving a description of some developments in the subject of local existence that have taken place since [78]. However, it should be said that the selection is not intended to be complete or to provide the correct reference concerning who was the first to obtain a result of a given kind; our ambition is only to give a rough idea of some of the developments.

4.1. **Real analyticity.** Let U be an open connected subset of $\mathbb{R}^d$, where $d \in \mathbb{N}$. A function $f : U \to \mathbb{R}$ is said to be *real analytic* in U if

- f has continuous derivatives of all orders in U, and if,
- for each $x_0 \in U$, there is an $R > 0$ such that the Taylor series expansion of f around x_0 converges and equals f in $B_R(x_0)$.

Here $B_R(x_0)$ denotes the open ball of radius R centered at x_0, and we tacitly assume that $B_R(x_0) \subset U$. It is very important to note that real analytic functions have the so-called *unique continuation property*. There are various ways of defining this notion, depending on the context, but for the purposes of the present discussion, we take it to mean the following: *Let f and g be two real analytic functions on U, all of whose derivatives agree at one point of U. Then $f = g$ on all of U.* In order to prove the statement, let A denote the set of points $x \in U$ such that $\partial^\alpha f(x) = \partial^\alpha g(x)$ for all multiindices α. By assumption, A is non-empty. Since f and g are continuously differentiable to all orders, A is closed. Finally, due to the defining property of real analytic functions, A is open. However, since U is connected, we know that any non-empty, open and closed subset of U equals U. As a consequence, $A = U$ and $f = g$ on all of U. One particular consequence of the unique continuation property is that if $f = g$ on an open subset of U, then $f = g$ on all of U.

The reason it is of importance to note that real analytic functions have the unique continuation property is that there is a tension between this property and the notion of finite speed of propagation (which is, of course, central in both special and general relativity). In order to illustrate this tension in the context of the initial value problem, consider

$$\Box u = 0, \tag{4}$$

$$u(0,x) = f(x), \tag{5}$$

$$u_t(0,x) = g(x), \tag{6}$$

where $\Box$ is the ordinary wave operator on $d+1$-dimensional Minkowski space. In order to discuss the propagation of information, we want the set up to be flexible enough that we can modify the initial data on some ball $B_R(x_0)$ in the initial hypersurface, and to study the influence of that change. Letting u_1 be the solution to the wave equation corresponding to the modified initial data, we then expect u to equal u_1 in the complement of the union of the causal future and the causal past of $\{0\} \times B_R(x_0)$. However, then u and u_1 would coincide on an open set. If they were real analytic, they would thus have to coincide everywhere. In fact, the real analytic class is so rigid that it is not even possible to make a non-trivial modification of the initial data on a ball $B_R(x_0)$ while keeping the data the same outside the ball. For this reason, it is not meaningful to speak of finite speed of propagation or to discuss the notion of causality in the real analytic class of functions. As a consequence, it is clear that working with real analytic functions is not compatible with special or general relativity.

In the context of the above discussion, it is of interest to note that solutions to the constraint equations are often obtained using the conformal method. This method involves solving a non-linear elliptic equation for a scalar function; cf., e.g., [97, pp. 2250–2251]. As a consequence, there is a problem in changing the initial data locally when using this method. Could it then be that the constraint equations are such that no local changes are allowed? That the answer to this question is no follows from the results obtained in [47]. In this paper, Corvino demonstrates that, given asymptotically flat initial data and an asymptotically flat end, it is possible to modify the data so that they are identically Schwarzschild outside a ball (in the specified asymptotically flat end) and so that they are identical to the original initial data inside a smaller ball. Results of a similar flavour are obtained in [28, 30, 48]. Moreover, an interesting recent related result is [23]. In this paper, the authors demonstrate that there are solutions to the constraint equations which are identical to Minkowski initial data outside of a cone. As a consequence of the existence of these data, it is possible to construct initial data for N bodies that do not interact for some finite amount of time. To conclude, there is no problem of principle in modifying solutions to Einstein's constraint equations locally.

4.2. **The space of k times continuously differentiable functions.** In the result of Choquet-Bruhat, the initial data for the metric and its first derivative are assumed to be five and four times continuously differentiable respectively. The corresponding class of functions is flexible enough that the notions of causality and finite speed of propagation make sense. As a consequence, the paper of Choquet-Bruhat is the first providing a proof of local existence and uniqueness of solutions to Einstein's vacuum equations in a setting in which the general theory of relativity is meaningful. On the other hand, one drawback of the result is that the solution obtained is only four times continuously differentiable, as noted above. In particular, the initial data induced on a hypersurface different from the original one need not have the same regularity as the original initial data. As far as local considerations are concerned, this is not an important problem. However, if one is interested in global issues, it is a problem. Ideally, one would like to have

- a norm such that if the solution blows up at some finite time T_+, then the norm of the initial data on constant-t hypersurfaces blows up as $t \to T_+$,
- a local existence theory which is powerful enough to guarantee that the norm remains bounded on hypersurfaces close to the intial one (i.e., locally).

The criterion corresponding to the norm being bounded is often referred to as a *continuation criterion*. The reason for this is that the above statement can be reformulated in the following way: as long as the continuation criterion is fulfilled, the solution can be continued. The use of the above information in the study of global questions is that, first of all, it makes it sufficient to control the norm corresponding to the continuation criterion in order to prove that the solution does not blow up in finite time. Moreover, if the continuation criterion is good enough, it might give useful information concerning the nature of the blow up, if it occurs. The second point above ensures that controlling the norm is at least possible locally.

In the case of [78], keeping control of the $C^5 \times C^4$ norm of the initial data is sufficient to guarantee that the solution can be continued (though, strictly speaking, [78] does not contain the statement that the size of the time period on which the solution exists only depends on the norm of the initial data, a statement that would be needed in order to obtain the desired conclusion). On the other hand, the local theory is not sufficiently strong to demonstrate that the required norm remains bounded even locally. However, this is not a deficiency of the arguments of Choquet-Bruhat; it is a property of hyperbolic equations. In fact, considering the initial value problem (4)–(6) with $C^{k+1} \times C^k$ initial data, the solution is typically not C^{k+1} (if the spatial dimension is 2 or greater); cf., e.g., [158, Theorem 1.1, p. 6]. This is an indication that C^k-spaces are not appropriate in the study of the Cauchy problem for hyperbolic equations.

The above discussion is a bit brief, and we shall therefore devote the next two subsections to giving examples of continuation criteria and describing function spaces that are better suited in the study of Einstein's equations.

4.3. **The notion of a continuation criterion.** In order to give an example of a continuation criterion, it is useful to begin with a simple setting. Let us therefore consider an autonomous system of ordinary differential equations. The corresponding initial value problem consists of the equations

$$\frac{dx}{dt} = f \circ x, \tag{7}$$

$$x(0) = x_0, \tag{8}$$

where f is, say, a smooth function from $\mathbb{R}^d$ to itself, $d \in \mathbb{N}$ and $x_0 \in \mathbb{R}^d$. By standard theory, there is a unique solution to (7) and (8). Moreover, there is a natural notion of a maximal interval of existence (take the union of all the intervals of existence corresponding to all solutions to (7) and (8); by uniqueness, the corresponding solutions coincide on the intersection of the existence intervals). However, none of these observations yield any conclusions concerning the global behaviour. Does the solution blow up in finite time? What are the asymptotics? In order to answer the first question, it is of interest to note that there is a continuation criterion for equations of the form (7). If x is a solution to (7) and (8), and $I = (t_-, t_+)$ is the corresponding maximal interval of existence, then either $t_+ = \infty$ or

$$\lim_{t \to t_+ -} |x(t)| = \infty.$$

There is a similar statement concerning t_-. In order to illustrate the use of this continuation criterion, let us consider a specific example.

Example. Consider the equation

$$\ddot{y} + y^k = 0,$$

where $k \geq 1$ is an odd integer. Introducing $x_1 = y$, $x_2 = \dot{y}$ and $x = (x_1, x_2)$, it is clear that x satisfies an autonomous system of equations. Moreover,

$$E = \frac{1}{2} x_2^2 + \frac{1}{k+1} x_1^{k+1}$$

is a conserved quantity. As a consequence, $|x|$ cannot blow up in finite time, and the continuation criterion implies that the solution exists globally.

The above example is a special case of a general principle: *if a differential equation has an associated conserved quantity strong enough to bound the norm involved in the continuation criterion, then solutions cannot blow up in finite time (so that they exist globally).* When it is applicable, this principle is clearly very powerful. It is obviously of interest to find conserved (or, possibly, monotone) quantities and to find a continuation criterion which involves as weak a norm as possible. The existence of conserved or monotone quantities often follows from the existence of a natural energy for the system of equations under consideration; we shall not discuss this issue further. The problem of finding a continuation criterion which involves as weak a norm as possible is strongly related with the problem of proving local existence in as low a regularity as possible. Note that in the case of Einstein's equations, a general result saying that solutions do not blow up in finite time is not to be expected. In certain circumstances, global existence is, however, to be expected (in the study of stability of Minkowski or de Sitter space, for example). Moreover, even if global existence is not obtained, the weaker the norm involved in the continuation criterion, the stronger the conclusions concerning the character of the blow up.

4.4. **Local existence.** Let us now turn to the developments concerning the issue of local existence, the goal being to find as good a continuation criterion as possible. A complete description would be very long and complicated, and it is not within our field of competence to do the subject full justice. For that reason, we shall only briefly describe some developments we consider to be important.

Sobolev spaces. In the work of Choquet-Bruhat, the function space used for the initial data is $C^5 \times C^4$. As mentioned above, this regularity class is not preserved by the evolution. It is consequently inappropriate. The real analytic class we have already discarded on the basis of incompatibility with the fundamental ideas of special and general relativity. Starting with smooth initial data, it can be demonstrated that there is a smooth local solution. However, the associated continuation criterion is much too strong. In order to arrive at a more reasonable class of functions, it is natural to turn to the wave equation (4). Associated with a solution to this equation (corresponding, say, to initial data that vanish outside a ball of a certain radius), there is an energy

$$E(t)=\frac{1}{2}\int_{\mathbb{R}^d}[|u_t(t,x)|^2+|\nabla u(t,x)|^2]dx. \tag{9}$$

Differentiating under the integral sign and integrating by parts leads to the conclusion that E is conserved. Sometimes it is of interest to control the function itself, and not only its derivatives (as in (9)). It is then natural to consider, e.g.,

$$\mathcal{E}(t)=\frac{1}{2}\int_{\mathbb{R}^d}[|u_t(t,x)|^2+|\nabla u(t,x)|^2+|u(t,x)|^2]dx. \tag{10}$$

If u is a solution to the wave equation, the quantity $\mathcal{E}$ is typically not conserved. However, $\mathcal{E}(t)\le e^{|t|}\mathcal{E}(0)$. In this sense, $\mathcal{E}(0)$ controls $\mathcal{E}(t)$ at later times. In particular, $\mathcal{E}$ cannot go from being finite to being infinite in finite time (in contrast with the $C^{k+1}\times C^k$-norm). If α is a d-multiindex (corresponding to the d spatial dimensions), then $\partial^\alpha u$ is also a solution of (4), since ∂^α and $\Box$ commute. As a consequence, it is natural to consider

$$\mathcal{E}_k(t)=\frac{1}{2}\sum_{|\alpha|\le k}\int_{\mathbb{R}^d}[|\partial^\alpha u_t(t,x)|^2+|\nabla\partial^\alpha u(t,x)|^2+|\partial^\alpha u(t,x)|^2]dx.$$

This quantity also has the property that $\mathcal{E}_k(t)\le e^{|t|}\mathcal{E}_k(0)$. Thus $\mathcal{E}_k(t)$ remains bounded if it is initially bounded. The above observations suggest that it is appropriate to make requirements of the following form: the H^{k+1}-norm of $u|_{t=0}$ is bounded, and the H^k-norm of $u_t|_{t=0}$ is bounded. Here, the H^k-norm of a function v on $\mathbb{R}^d$ is defined as follows:

$$\|v\|_{H^k}=\left(\sum_{|\alpha|\le k}\int_{\mathbb{R}^d}|\partial^\alpha v(x)|^2dx\right)^{1/2}.$$

An important question in this context is: what is the natural class of functions for which the H^k-norm is defined? Clearly, if v is k times continuously differentiable, and if it vanishes outside a ball of a fixed radius, then the H^k-norm is well defined. However, these requirements are too restrictive. The reason for this is partly that C^k-type regularity is not preserved by the evolution. However, the main reason is that it is of central importance for the corresponding space to be complete, a notion we now define.

Complete function spaces. Proofs of existence of solutions to non-linear equations often proceed by constructing a sequence of approximations. The goal is to prove that the sequence converges to a solution. On the other hand, the best one can hope to prove concerning the sequence, say u_n, is that $\|u_n-u_m\|$ converges to zero when n and m tend to infinity (where $\|\cdot\|$ is the relevant norm in the argument under consideration); a sequence u_n with this property is said to be a *Cauchy sequence* with respect to the norm $\|\cdot\|$. From this knowledge, one would like to draw the conclusion that there is a function u such that u_n converges to u (and then to prove that u is a solution to the equation under consideration). If the norm $\|\cdot\|$ is defined on a space X such that every Cauchy sequence converges in this sense, then the space X with the norm $\|\cdot\|$ is said to be *complete*. In order to obtain completeness in the case of the H^k-norm, it is natural to define the associated space, denoted $H^k(\mathbb{R}^d)$, to be the set of k-times weakly differentiable functions on $\mathbb{R}^d$ whose derivatives of order up to and including k are square integrable (we omit the technical definition of weak differentiability, and refer the interested reader to books on PDE; cf., e.g., [147, Definition 5.2, p. 36]). The spaces $H^k(\mathbb{R}^d)$ are referred to as *Sobolev spaces*, and they are complete.

Fractional Sobolev spaces. In certain circumstances, it is of interest to define Sobolev spaces for indices k that are not non-negative integers. One way of doing so is by observing that there is a characterization of Sobolev spaces in terms of the Fourier transform. In fact,

$$(2\pi)^{-d} \sum_{|\alpha| \leq k} \int_{\mathbb{R}^d} \xi^{2\alpha} |\hat{v}(\xi)|^2 d\xi = \|v\|_{H^k}^2,$$

where $\hat{v}$ denotes the Fourier transform of v. As a consequence, it can be argued that there are constants $C_{i,k} > 0$, $i = 1, 2$, such that

$$C_{1,k} \int_{\mathbb{R}^d} (1 + |\xi|^2)^k |\hat{v}(\xi)|^2 d\xi \leq \|v\|_{H^k}^2 \leq C_{2,k} \int_{\mathbb{R}^d} (1 + |\xi|^2)^k |\hat{v}(\xi)|^2 d\xi.$$

Using this inequality, there is an alternate definition of the Sobolev norm, namely

$$\|v\|_{H_s} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi \right)^{1/2},$$

which makes sense for all real numbers s. If k is a non-negative integer, then the H^k-norm is equivalent to the H_k-norm. However, the H_s-norm makes sense for all real numbers s (and a corresponding space can be defined such that it is complete). Let us now return to the topic of local existence.

Local existence results. Even prior to the work of Choquet-Bruhat, authors such as Friedrichs, Lewy, Hadamard, Schauder and many others made important contributions to the subject of solving the intial value problem for hyperbolic PDE's. Going forward in time to the early 50's, one very important reference is [110]. In this book, Leray obtains far reaching conclusions concerning local existence of solutions to quite general systems of equations. However, the results of [110] suffer from the same deficiency as that of Choquet-Bruhat; more regularity is assumed of the initial data than is obtained for the solution. In the 60's, results of the desired type were, nevertheless, obtained; cf. [69, 157]. In fact, the theorem stated on [157, p. 222] constitutes a local existence result for a second order quasi-linear equation. Moreover, the initial data induced on constant-t hypersurfaces different from the original one have the same degree of regularity as the original initial data. Sobolev undoubtedly obtained this result much earlier; in [69], Dionne refers to a paper of Sobolev's in the mid 50's, which, according to Dionne, deals with the case of second order equations. In the work of Dionne, cf. [69, Theorem 8, p. 8] and [69, Theorem 5, p. 9], similar results are obtained for more general classes of equations.

An important related question concerning the preservation of regularity is the following: given C^∞ (infinitely differentiable) initial data, does one obtain a C^∞ local solution? Since one obtains a local C^k solution for each k, there might, naively, not seem to be a problem. However, the size of the region on which the C^k solution is obtained might depend on k, and as k tends to infinity, the region might shrink to the initial hypersurface (so that, in the end, there is no region on which there is a C^∞ solution). It is quite interesting to note that the question of existence of smooth solutions was addressed as late as 1971; cf. [25].

Symmetric hyperbolic systems. A different perspective on the problem of solving Einstein's equations is to be found in [76, 96]. In [76, 96], the authors reformulate equations of the form (2) to symmetric hyperbolic systems; in particular, to first order systems. Once that step has been taken, the equations can be thought of as first order ODE's in a Banach space (and at that stage, the theory of semigroups can be applied). Refining the work of previous authors (such as Friedrichs, Lax, Kato and many others), they obtain local existence results in a lower degree of regularity. Moreover, the regularity they obtain for the solutions is of the form

$$C^0([0,T], H_{s+1}(\mathbb{R}^d, \mathbb{R}^N)) \cap C^1([0,T], H_s(\mathbb{R}^d, \mathbb{R}^N)), \tag{11}$$

given that the initial data are in

$$H_{s+1}(\mathbb{R}^d, \mathbb{R}^N) \times H_s(\mathbb{R}^d, \mathbb{R}^N),$$

where $s > d/2 + 1$; cf. [96, Theorem III, p. 282]. As a consequence, the initial data induced at later times have the same degree of regularity as the original initial data. However, it is interesting to note that the formulation of [96, Theorem III, p. 282] does not include an explicit statement of a continuation criterion, even though it is concievable that the authors could have derived one using their methods.

Continuation criterion. A very interesting proof of local existence of solutions to systems of conservation laws is to be found in [123, Chapter 2]. The proof is elementary in nature in that it simply consists of proving that a sequence of solutions to a sequence of linear equations converges to a solution of the non-linear equation. Moreover, it does not appeal to the semi-group theory used by Kato and his collaborators, but is simply based on energy estimates. Nevertheless, Majda demonstrates that the solution has regularity of the form (11). Finally, he proves a continuation criterion which essentially states that for any $s > d/2$ (where d is the spatial dimension), the maximal interval $[0, T)$ on which a solution with regularity (11) exists is either such that $T = \infty$ or such that

$$\limsup_{t \to T-} (\|u_t(t, \cdot)\|_\infty + \|\nabla u(t, \cdot)\|_\infty) = \infty.$$

In short: as long as the first derivatives do not blow up, the solution can be continued. Note that since the continuation criterion is independent of s, local existence of smooth solutions (given smooth initial data) is immediate.

Recent developments, the bounded L^2-curvature theorem. In the late 90's and early 00's, a significant amount of progress was made on the problem of decreasing the degree of regularity required of the initial data for quasi-linear wave equations; cf., e.g., [11, 12, 168, 104, 105, 155] and references cited therein (see also [174, 175] for related work in the case of constant mean curvature foliations). In particular, combining the results of [155] with the the counterexamples obtained in [1, 116, 117] yields the conclusion that for equations of the form

$$g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u = q^{\alpha\beta}(u)\partial_\alpha u \partial_\beta u \tag{12}$$

on $\mathbb{R}^{3+1}$,

- the initial value problem is locally well posed for initial data in $H_{s+1} \times H_s$ (where $s > 1$),
- there are equations of the form (12) for which the initial value problem is not locally well posed for initial data in $H^2 \times H^1$.

In view of this fact, it is remarkable that Einstein's equations admit local solutions, given initial data in $H^2 \times H^1$. The corresponding result, referred to as the *bounded L^2-curvature theorem*, demonstrates that the non-linearity in Einstein's equations has a better structure than general equations of the form (12); cf. [107, 108, 163, 164, 165, 166, 167]. The statement of the main theorem is to be found on [108, p. 8], and it is concerned with maximal foliations of solutions to Einstein's vacuum equations in the asymptotically flat setting. The corresponding continuation criterion states that as long as the L^2-norm of the Ricci curvature and the L^2-norm of the gradient of the second fundamental form remain bounded, then the solution can be continued (strictly speaking, a condition concerning the volume radius also needs to be imposed; cf. [108, p. 8] for the details).

5. Global uniqueness, strong cosmic censorship

The work [78] of Yvonne Choquet-Bruhat constitutes a fundamental first step in putting the Cauchy problem in general relativity on a solid footing. However, it is in some respects incomplete. As a consequence of [78], we know that there is a local solution, given initial data. Moreover, we know that two local solutions are *locally* the same. On the other hand, we neither have information about global uniqueness, nor about global behaviour. To expect information about the global behaviour in general is too optimistic. However, if the general theory of relativity is supposed to be deterministic, there should be a global uniqueness result. A very important step in that direction was taken in [24]; in a sense, the full depth and importance of the work of Yvonne Choquet-Bruhat on the Cauchy problem only becomes apparent when taking both [78] and [24] into account. For that reason, we begin the present section with a description of the results of [24].

5.1. **Existence of a maximal globally hyperbolic development.** In order to obtain a uniqueness result, it is necessary to demand some type of maximality of the development. It turns out that demanding maximality in the class of all developments does not yield uniqueness (see below for a motivation of this statement). However, demanding maximality in the class of globally hyperbolic developments of the initial data does. In order to state the main result of [24], it is necessary to introduce the terminology used in [24]. First of all, the authors study Einstein's vacuum equations (even though this is not an essential restriction), and they define *an initial data set* to be a 3-dimensional manifold Σ on which a Riemannian metric $\bar{g}$ and a symmetric covariant 2-tensor field $\bar{k}$ are defined, where $\bar{g}$ and $\bar{k}$ satisfy the vacuum constraint equations. Already here, it is of interest to note the contrast with the formulation in [78]; in [78], initial data are taken to be the metric and the first derivative of the metric restricted to the initial hypersurface (clearly, a non-geometric formulation). In contrast, the definition of an initial data set in [24] is geometric in nature, and it is intrinsic to the initial hypersurface. In [24], the notion of a development of initial data is defined by the following:

- a manifold M with a Lorentz metric g satisfying Einstein's vacuum equations,
- an embedding $i : \Sigma \to M$ such that if κ is the second fundamental form induced on $S := i(\Sigma)$, then $i^*g = \bar{g}$ and $i^*\kappa = \bar{k}$ (where i^* denotes the pull-back),

- S is a Cauchy hypersurface in (M, g) (so that (M, g) is a globally hyperbolic manifold).

Again, in contrast with [78], the formulation of the notion of a development is geometric in nature. Whether one should include the third requirement (i.e., that S be a Cauchy hypersurface) is debatable. Sometimes this requirement is not included, and then a development in the sense of [24] is referred to as a *globally hyperbolic development*. Since we wish to contrast the case of globally hyperbolic developments with developments that are not globally hyperbolic, we here do not a priori assume developments to be globally hyperbolic, and state explicitly when they are.

Given the above definitions, the authors of [24] note that the local existence of solutions to Einstein's equations can be formulated as follows: given an initial data set, there is a corresponding globally hyperbolic development; cf. [24, Theorem 1, p. 331]. Strictly speaking, in order to take the step from [78] to this form of local existence, it is necessary to construct initial data appropriate for appealing to the result of [78], given an initial data set in the sense of [24]; to prove that the local solutions constructed in [78] can be patched together to a yield a globally hyperbolic development etc.

Another important ingredient needed in order to prove the main theorem of [24] is a local uniqueness result. It is not completely obvious what the corresponding statement should be. In the case of two solutions to (7) and (8), say, it is natural to compare them on the intersection of their intervals of existence, and to prove that they equal there. Given two globally hyperbolic developments of a given initial data set, it is less clear how to make the comparison.

In order to phrase a geometric local uniqueness result, the authors of [24] introduce the following terminology: if (M, g) and (M', g') (with corresponding embeddings i and i' respectively) are globally hyperbolic developments of the same initial data set $(\Sigma, \bar{g}, \bar{k})$, then (M, g) is said to be an *extension* of (M', g') if there is a map $\psi : M' \to M$ which is a diffeomorphism onto its image, and which is such that $\psi^* g = g'$ and $\psi \circ i' = i$. Given this definition, local uniqueness can be formulated as follows: any two globally hyperbolic developments (of the same initial data set) are the extensions of a common development; cf. [24, Theorem 2, p. 331]. Again, in order to prove this result, it is necessary to patch up the local coordinate changes constructed in [78] in order to produce the map ψ. A more recent presentation of the proofs of [24, Theorems 1–2, p. 331] is to be found in [147]; cf. [147, Theorem 14.2, p. 156] and [147, Theorem 14.3, p. 158].

Based on these two observations, the main result of [24] is [24, Theorem 3, p. 332]. Using our terminology, it can be phrased as follows:

Theorem 1. *Let $(\Sigma, \bar{g}, \bar{k})$ be an initial data set. Then there is a globally hyperbolic development of $(\Sigma, \bar{g}, \bar{k})$ which is an extension of every other globally hyperbolic development of $(\Sigma, \bar{g}, \bar{k})$. This development is unique (up to isometry).*

The relevant development is sometimes referred to as the *maximal globally hyperbolic development* or *maximal Cauchy development*. The proof can be divided into two steps. First, the authors prove that there is a globally hyperbolic development which is maximal in the sense that it cannot be extended; this step largely consists of an application of Zorn's lemma. The second step is more difficult, and consists of proving that given two globally hyperbolic developments, there is a common *extension*. The method of proof in the second step is largely Lorentz geometric in nature. The proof is quite intricate, and the presentation in [24] is a bit terse; [150, Chapter 23] contains a more detailed exposition (the proof in [150] is in the Einstein-Vlasov-non-linear scalar field setting, but, as mentioned, this does not have a significant effect on the argument). Let us also note that the problem of how to remove the use of Zorn's lemma in the proof has been addressed in [154].

Importance of the result. The main reason why Theorem 1 is so important is the following. Due to [78], there is a development, given an initial data set. However, if there is not a preferred development which is uniquely singled out by the initial data (and, possibly, some additional criterion), then the initial value problem does not make sense; cf. the introduction. Theorem 1 guarantees that in the class of globally hyperbolic developments, there is a preferred member: the maximal Cauchy development. In some respects, this restriction to globally hyperbolic developments is natural; if (M, g) is a development of an initial data set, it is not to be expected that the initial data control the behaviour of the solution in the complement of the domain of dependence of the initial hypersurface (and if the development is not globally hyperbolic, this complement is non-empty), so that uniqueness beyond the domain of dependence of the initial hypersurface is not to be expected.

Even though Theorem 1 is an important result, it does lead to a new question: are there initial data sets such that the corresponding maximal Cauchy development is extendible? Could there be many different extensions of the maximal Cauchy development? Unfortunately, the answer to both of these questions is yes, and this leads us to the strong cosmic censorship conjecture.

5.2. **Strong cosmic censorship.** That the maximal Cauchy development is sometimes extendible can be seen by a very simple example. Consider the hyperboloid in Minkowski space consisting of the set of future directed unit timelike vectors. This set is a spacelike hypersurface. Let $(\Sigma, \bar{g}, \bar{k})$ denote the corresponding initial data set. The corresponding maximal Cauchy development is the timelike future of the origin in Minkowski space, denoted $I^+(0)$. Clearly, this Lorentz manifold can be extended to all of Minkowski space. Moreover, by removing points from the complement of $I^+(0)$ in Minkowski space, inequivalent extensions of the maximal Cauchy development are obtained. However, these extensions are, in some respects, unnatural. One reason for this is that the extensions obtained by removing points are not maximal. It is of greater interest to know if there are inequivalent maximal extensions to the maximal Cauchy development (in case the maximal extension of the maximal Cauchy development is uniquely determined, the problem of determinism is resolved in a satisfactory manner). Moreover, it is more natural to limit one's attention to the following physically relevant situations: the isolated systems setting and the cosmological setting. Here, we take the isolated systems setting to correspond to asymptotically flat initial data, and we take the cosmological setting to correspond to initial data $(\Sigma, \bar{g}, \bar{k})$ such that Σ is a closed manifold. One could also consider the case of initial data sets that are asymptotically hyperboloidal (as in the example above). However, one should then keep in mind that extendibility is to be expected to the past, and that the region of interest is the future.

In order to develop some intuition concerning what can go wrong, it is instructive to discuss some examples. We begin by describing the so-called Taub-NUT spacetimes.

Taub-NUT. The Taub spacetimes are the maximal Cauchy developments of left invariant and locally rotationally symmetric vacuum initial data on SU(2); cf. [45] or [147] for details. It turns out that these spacetimes are past and future causally geodesically incomplete; cf. [115]. Moreover, considering the hypersurfaces of spatial homogeneity, they start from zero volume, increase to a maximal volume, and then shrink again to zero volume. In this respect, there is a big bang and a big crunch. However, the curvature remains bounded as one approaches the boundary of the spacetime, and it turns out that the maximal Cauchy developments are extendible. Considering a globally hyperbolic Taub region in a maximal extension, it can be seen that the boundary of the globally hyperbolic region consists of two null hypersurfaces, beyond which there are closed timelike curves. Proving that there are two inequivalent maximal extensions is difficult. However, this result was obtained in [45]. A somewhat more detailed argument, based on the ideas in [45], is to be found in [147]. It is of interest to note that the proof given in [45] is crucially dependent on the fact that the Cauchy horizon has two components.

Polarized $\mathbb{T}^3$*-Gowdy.* There is a geometric characterization of the polarized $\mathbb{T}^3$-Gowdy spacetimes. However, for the purposes of the present discussion, it is more convenient to define them to be metrics of the form

$$g = e^{-2U}[e^{2A}(-dt^2 + d\theta^2) + t^2 dy^2] + e^{2U} dx^2$$

on $M = (0, \infty) \times \mathbb{T}^3$, where $t \in (0, \infty)$ and $(\theta, x, y) \in \mathbb{T}^3$; cf. [45, (32), p. 1622]. In this expression, U and A are functions of t and θ only. Moreover, it is of interest to note that Einstein's vacuum equations imply that U satisfies a linear wave equation and that A can be obtained by integrating expressions in U; cf. [45, (33), p. 1622]. For this class of spacetimes, $t = 0$ is expected to correspond to a big bang singularity, and $t = \infty$ is expected to correspond to an expanding direction. In fact, one can prove that polarized $\mathbb{T}^3$-Gowdy spacetimes are future causally geodesically complete and past causally geodesically incomplete. On the other hand, by setting $U = A = 0$, one obtains the flat Kasner solution (which is simply a quotient of a part of Minkowski space). Moreover, the flat Kasner solution is extendible; it is the basic example extendibility in the class of polarized $\mathbb{T}^3$-Gowdy spacetimes. However, more interesting solutions can be obtained by using the fact that spatial variation in θ is allowed. In particular, solutions can be constructed with the properties that

- they can be extended beyond two disjoint intervals I_i, $i = 1, 2$, (in θ) in the $t = 0$ hypersurface,
- they cannot be extended through the complement of these intervals.

A solution with these properties can be extended, and the Cauchy horizon consists of two disjoint components. Moreover, in [45] it is demonstrated that there are two inequivalent maximal extensions of such a solution. Interestingly, this procedure can be generalized. Taking N disjoint intervals, with properties similar to the above, yields 2^{N-1} inequivalent maximal extensions. As a consequence, there is no bound to the number of inequivalent maximal extensions that might exist. The above description is a bit brief, and we refer the reader to [45] for a more detailed discussion.

Formulation of the conjecture. The above examples illustrate that there might be several inequivalent maximal extensions corresponding to a given initial data set. As a consequence, it might seem that determinism is lost. On the other hand, it is clear that the solutions that admit inequivalent maximal extensions are quite special. It is thus tempting to make the following conjecture.

Conjecture 1 (Strong cosmic censorship)**.** *For generic vacuum initial data sets* $(\Sigma, \bar{g}, \bar{k})$ *which are either*

- *asymptotically flat, or*
- *such that* Σ *is a closed manifold,*

the corresponding maximal Cauchy development is inextendible.

Remark 1. *Similar formulations can also be made in the presence of matter.*

Given the above discussion, this formulation (which is essentially the same as the one to be found in [44, Section 1.3]) is quite natural. However, it was preceded by slightly different perspectives; cf. [70] and [128]; see also [40] (which includes a discussion of weak cosmic censorship).

The formulation of the conjecture is problematic in many ways. First of all, the meaning of the words "generic" and "inextendible" is not completely clear. In the case of an ODE, we could take a set of initial data to be generic if the complement has zero measure. However, this notion becomes more problematic in the infinite dimensional setting. Another perspective would be to say that a set is generic if it is open and dense with respect to some suitable topology (there are also more technical notions in the same vein; dense G_δ set, for example). In the ideal situation, it is possible to prove that the set of initial data leading to extendible maximal Cauchy developments has positive codimension. Nevertheless, which definition of the term "generic" is most appropriate depends on the context. In the general formulation of the conjecture, the term is therefore left unspecified.

The word "inextendible" can also be interpreted in many ways. For (M', g') to be an extension of (M, g), it is clear that both M and M' should be connected manifolds of the same dimension. Moreover, there should be a map $\phi : M \to M'$ which is a diffeomorphism onto its image and which is such that $\phi^* g' = g$ (and such that the relevant matter fields are preserved similarly). Finally, it is clear that $\phi(M) \neq M'$ should hold. However, it is not obvious that these criteria are enough. One could, for example, also demand that Einstein's equations (together with possible matter equations) should hold. In addition to this, there is the question of regularity; what degree of regularity should we demand of the extension (M', g')? Is mere continuity enough? In what sense should we demand that the equations hold? Should we demand the existence of classical solutions, or is it enough to demand that (M', g') is a weak solution? Again, several suggestions exist. As a consequence, these issues need to be discussed in each individual case. However, there is an interesting connection to the question of curvature blow up, which we discuss next.

Curvature blow up. Due to the work of Hawking and Penrose, we know that the existence of singularities (in the sense of causal geodesic incompleteness) is generic; cf., e.g., [88, 89, 127, 90, 178, 126] and the contribution of David Garfinkle and José Senovilla to this volume. However, as the Taub-NUT and polarized $\mathbb{T}^3$-Gowdy examples illustrate, the fact that there is a singularity in the sense of causal geodesic incompleteness does not guarantee the existence of a singularity in the sense of curvature blow up. In analogy with the above, it is thus tempting to make the following conjecture.

Conjecture 2 (Curvature blow up)**.** *For generic vacuum initial data sets* $(\Sigma, \bar{g}, \bar{k})$ *which are either*

- *asymptotically flat, or*
- *such that* Σ *is a closed manifold,*

the Kretschmann scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ *is unbounded in the incomplete directions of causal geodesics.*

Again, one could of course formulate the same conjecture in the presence of matter. Conjecture 2 actually implies the strong cosmic censorship conjecture. However, it is then necessary to take extendibility to mean C^2-extendibility. That not all authors consider this to be the most natural degree of regularity is illustrated by [40, p. A26]. Nevertheless, Conjecture 2 is of great interest in its own right.

Next we turn to a brief discussion of some results that have been obtained in the past.

5.3. **Results, strong cosmic censorship.** The literature on the strong cosmic censorship conjecture is quite extensive. We shall therefore limit our discussion of the results to a few examples. Conjecture 1 has not yet been addressed in full generality. The only results that exist concern the analogous problem obtained by imposing symmetry conditions. The idea is thus to first specify a setting (cosmological or isolated systems setting, for example); then to specify a matter model (such as vacuum or a scalar field); and, finally, to specify a symmetry requirement (spherical symmetry, Bianchi type VIII spacetimes etc.). Given these choices, there is a statement analogous to Conjecture 1. From a logical point of view, the corresponding statement is unrelated to Conjecture 1 (since symmetric initial data are non-generic). However, the hope is that the resolution of the resulting problems might shed some light on how to proceed in the general case. Moreover, considering the special solutions described above (for which the maximal Cauchy development is extendible), it is of interest to prove that they are unstable in some larger class of solutions. In what follows, we discuss the isolated systems setting and the cosmological setting separately.

Isolated systems. In the case of asymptotically flat initial data, the most natural symmetry class is that of spherical symmetry. However, the only spherically symmetric vacuum solutions to Einstein's equations that are asymptotically flat are the Schwarzschild spacetimes. As a consequence, it is necessary to add some form of matter in order to obtain an interesting problem. This was done in the work of Christodoulou (cf. [32, 33, 34, 35, 36, 37, 38, 39]), who considered matter of scalar field type; cf. [40, p. A29] for a motivation of the choice. The sequence of papers stretches from a proof of global existence and dispersion for small data; a study of the formation of trapped surfaces; via a proof of the existence of naked singularities; to a proof of the fact that the naked singularities are non-generic. The exact notion of genericity used in the result is clarified in the statement of [39, Theorem 4.1, p. 216]. Roughly speaking, the result says that the set of initial data leading to naked singularities has positive codimension. The above description is very brief, and the readers interested in a more detailed discussion (which is still of an overview character) are referred to [40].

Another interesting sequence of results that have been obtained in the asymptotically flat setting are represented by [49, 50]. In these papers, Dafermos considers the Einstein-Maxwell-scalar field equations. The motivation for considering this particular matter model is the following. Ideally, it would be of interest to consider what happens in a symmetry class which allows the Kerr metrics as a special case. However, proving strong cosmic censorship in the axially symmetric setting currently seems too difficult. However, as is argued in [49], Maxwell's equations can work as a substitute. Interestingly, the results of [49] demonstrate that there is an open set of initial data such that the future boundary of the maximal Cauchy development contains a null component with the property that the metric can be extended continuously beyond it. However, the curvature blows up along the boundary, and the metric is C^1-inextendible. In this respect, the conclusion concerning strong cosmic censorship depends on the notion of extendibility used in the formulation. The results of [49] are based on the characteristic initial value problem. However, combining [49] with [50, 56], the extendibility of the maximal Cauchy development is obtained for a large set of asymptotically flat initial data sets in the relevant symmetry class. More recently, the results of [51] demonstrate that there is an open set of initial data for which the boundary of the maximal Cauchy development has no spacelike component at all (and that the metric is continuously extendible across the boundary).

So far, proving strong cosmic censorship in the isolated systems setting under less stringent symmetry conditions than spherical symmetry appears difficult. However, it is of interest to note that, recently, a large class of solutions has been constructed which conjecturally gives a general picture of what to expect concerning the interior of black holes, cf. [122].

The cosmological setting. In cosmology, the spatially homogeneous spacetimes are a natural class in which to start studying the issue of strong cosmic censorship. For a given time direction (future or past), it is to be expected that all causal geodesics are complete, or that all causal geodesics are incomplete. Since it is not possible to extend the maximal Cauchy development in directions in which the causal geodesics are complete, it is natural to focus on the incomplete directions (we refer to the corresponding asymptotic regions as singularities). In case there is matter present, there are typically curvature invariants that become unbounded as one approaches a singularity; cf. [132]. As a consequence, C^2-inextendibility follows under quite general circumstances. In the absence of matter, the situation is, however, more subtle. Considering, for example, the Bianchi class A spacetimes, the locally rotationally symmetric vacuum solutions are such that the maximal Cauchy development can be extended (in the case of Bianchi IX, there are even inequivalent maximal extensions, as noted above). This led Chruściel and Rendall to consider the question of strong cosmic censorship in the spatially homogeneous vacuum setting; cf. [46]. In particular, the authors prove strong cosmic censorship for Bianchi IX vacuum solutions, assuming the spatial topology is, for example, spherical. Interestingly, the authors do not approach the problem by analyzing the asymptotics of solutions in general. Instead, they start by assuming that, say, a Bianchi IX solution has a Cauchy horizon, and then prove that this implies that the solution has more symmetries than general Bianchi IX solutions. Nevertheless, the results of [46] leave the issue of curvature blow up unanswered. This led Rendall to return to this question in [133]. In this reference, Rendall demonstrates that for all the Bianchi class A spacetimes (except types VIII and IX), there are two possibilities:

- either the maximal Cauchy development is extendible (this occurs in the locally rotationally symmetric cases, as mentioned above), or
- the Kretschmann scalar blows up in the incomplete directions of causal geodesics.

The remaining classes (VIII and IX) exhibit more complicated behaviour in the direction of the singularity. Nevertheless, the same conclusion concerning Bianchi type VIII and IX vacuum spacetimes was obtained in [139]; cf. also [140] which contains more detailed information concerning the asymptotics. A full statement of the relevant result is to be found in [147, Theorem 24.12, p. 258]. Note, in particular, that the Taub spacetimes are exceptional in the Bianchi IX class.

Taking a step away from spatial homogeneity, it is natural to consider the case that there is a 2-dimensional group of isometries. In fact, given the above examples of inequivalent maximal extensions in the case of polarized Gowdy solutions, it is of interest to consider the issue of strong cosmic censorship in that setting. This was done in [43], in which the authors verify that for generic initial data in the relevant symmetry class, the curvature is unbounded in the incomplete directions of causal geodesics. Even though there is a large class of solutions for which the maximal Cauchy development has several inequivalent maximal extensions, this class is thus non-generic. One aspect which simplifies the analysis in the case of polarized $\mathbb{T}^3$-Gowdy vacuum spacetimes is the fact that the equation for one of the metric components is linear, and that the remaining components are obtained by integrating expressions in the solution to the linear equation. In the case of general $\mathbb{T}^3$-Gowdy vacuum spacetimes, one obtains a system of non-linear wave equations instead of a linear scalar equation. Nevertheless, it can be demonstrated that strong cosmic censorship holds in this case as well; cf. [141, 142, 143, 144, 145]. In fact, for a set of initial data which is open and dense in the C^∞-topology, detailed information concerning the asymptotics is obtained (both in the expanding direction and in the direction towards the singularity), including curvature blow up in the incomplete directions of causal geodesics. The above description is somewhat brief, and the interested reader is referred to [149] for a more extensive overview of the topic of strong cosmic censorship in the case of Gowdy spacetimes. Finally, let us note that there are related results in the presence of matter; cf. [53, 54, 55, 156].

6. Stability results

In most applications of general relativity, a few highly symmetric solutions to Einstein's equations play a central role; Schwarzschild and Kerr in the case of isolated systems, and spatially homogeneous and isotropic solutions in the case of cosmology. Clearly, these solutions are very important. However, they are highly idealized, and it is of importance to prove that they are robust. This observation naturally leads to the question of stability, which is meaningful thanks to the work of Yvonne Choquet-Bruhat [78]. In order to give a rough formulation of what we mean by stability, let us assume that $(\Sigma, \bar{g}, \bar{k})$ are initial data of the solution, say (M, g), we want to prove stability of (in case there is matter present, initial data for the matter fields should be added as well). A stability result would then be a statement of the form: given initial data on Σ close enough to $(\bar{g}, \bar{k})$, the corresponding solution is globally similar. The formulation is here intentionally vague. The exact meaning of the words "close" and "globally similar" depends on the context. In fact, there are several proofs of the stability of Minkowski space, but they are not equivalent; the assumptions and conclusions are quite different. For that reason, a precise definition of the ingredients of the statement has to be given in each individual result.

Due to the central role played by Minkowski space in general relativity, we begin the present section by discussing its stability; cf. Subsection 6.1. As a next step, it would be of interest to prove that the Kerr family is stable. Unfortunately, this problem seems to be too difficult at present (even though important and very interesting results on the problem of demonstrating the formation of trapped surfaces, given initial data far from containing trapped surfaces, have recently appeared; cf., e.g., [41, 106, 103, 2]). Nevertheless, significant progress has been made concerning associated linear problems, and we discuss this work briefly in Subsection 6.2. Finally, in Subsection 6.3, we turn to the question of stability in the cosmological setting.

6.1. **Minkowski space.** The problem of proving stability of Minkowski space has been approached from several different perspectives. To begin with, one could choose the metric and second fundamental form induced on

- the $t = 0$ hypersurface, or
- on the standard hyperboloid (consisting of the future directed unit timelike vectors)

as the initial data to be perturbed. To distinguish between these two cases, we below speak of asymptotically flat and asymptotically hyperboloidal initial data respectively. The exact meaning of these words depends on the context, and we refer to the references given for precise definitions. Note that in the case of asymptotically flat initial data, one can hope to prove stability of Minkowski space. In the case of asymptotically hyperboloidal initial data, one can only hope to prove stability to the future of the hyperboloid (it would typically be possible to go some distance into the past, but not far enough for the information obtained to be of interest). Let us begin by discussing the asymptotically hyperboloidal perspective.

The conformal approach of Helmut Friedrich. When taking the hyperboloidal perspective, it is important to note that Minkowski space can be be embedded into the Einstein cosmos; cf., e.g., [82, p. 23] for details. By rescaling the embedded Minkowski metric by the square of a conformal factor, say Ω, the resulting metric equals that of the Einstein cosmos. In particular, the rescaled metric can be extended to the boundary of the image of Minkowski space in the Einstein cosmos. Moreover, the boundary consists of two 3-dimensional manifolds $\mathscr{I}^{\pm}$ and three points i^0 and $i^{\pm}$. The sets $\mathscr{I}^{\pm}$ correspond to the future $(+)$ and past $(-)$ endpoints of null geodesics. They are therefore called *future and past null infinity* respectively. The points $i^{\pm}$ are similarly called *future and past timelike infinity*, and the point i^0 is referred to as *spacelike infinity*. Adding $\mathscr{I}^{\pm}$ to the image of

Minkowski space in the Einstein cosmos yields a smooth manifold with boundary. Moreover, the conformal factor Ω extends smoothly to the boundary in such a way that $\Omega > 0$ on the region corresponding to Minkowski space, $\Omega = 0$ on the boundary, but $d\Omega \neq 0$ on the boundary (these conditions correspond to part of the requirements defining an asymptotically simple spacetime; cf., e.g., [82, Definition 1.1, pp. 24-25]). Below, we refer to the boundary as the *conformal boundary.* The idea is then to try to extend Einstein's equations to include a conformal factor, and in such a way that the evolution can be extended beyond the conformal boundary. The point of this idea is that, in the conformal picture, the region to the future of the closure of the standard hyperboloid in Minkowski space is actually compact. If the extended equations (after suitable gauge fixing) are hyperbolic, the problem of proving stability is then reduced to Cauchy stability for a finite time interval (which is a standard result for hyperbolic PDE's). When taking this perspective, all the difficulty is thus in finding the right equations. This was done by Helmut Friedrich; cf. [79, 80, 82, 81]. As a consequence, future stability of Minkowski space is obtained. Moreover, the perspective yields very detailed information concerning the asymptotics. On the other hand, it is clearly necessary to make specific requirements of the asymptotically hyperboloidal initial data in order for the method to apply. Relating these requirements to requirements on asymptotically flat initial data turns out to be a subtle issue. We shall not discuss it here, but rather refer the interested reader to [83, 172, 84] and references cited therein. For a general overview of the perspective described above, the reader is referred to [82].

The Christodoulou-Klainerman approach. Let us now turn to the case of asymptotically flat initial data, starting with the methods developed by Christodoulou and Klainerman; cf. [42]. In [42], the authors do not use conformal rescalings, so that it is necessary to prove global existence of solutions to Einstein's vacuum equations for small data. The proof is based on energy estimates. Specifically, the so-called Bel-Robinson tensor plays a central role in the argument. Given a tensor field W with the symmetries of the Weyl tensor (we shall refer to such tensor fields as Weyl fields), the Bel-Robinson tensor Q is a quadratic expression in the Weyl field and its dual. Two important properties of the Bel-Robinson tensor are

- $\mathcal{E} = Q(T_1, T_2, T_3, T_4) \geq 0$, if T_i, $i = 1, \dots, 4$, are future directed timelike vectors (below, we refer to expressions such as $\mathcal{E}$ as Bel-Robinson energy densities),
- if W satisfies the Bianchi equations, then Q is divergence free.

In order to obtain suitable energies, it is necessary to integrate Bel-Robinson energy densities over appropriate spacelike hypersurfaces. Moreover, it is not sufficient to only consider the Bel-Robinson tensor associated with the Weyl tensor of the solution; it is necessary to control higher order derivatives. Since higher order derivatives are obtained by applying derivative operators (associated with certain vector fields) to the Weyl tensor, the main problem is that of constructing appropriate foliations and appropriate vector fields. In [42], the authors construct two foliations; one spacelike foliation consisting of maximal hypersurfaces (so that the mean curvature of the leaves vanishes) and one null foliation associated with a solution u to the eikonal equation. The energies are obtained by integrating appropriate Bel-Robinson energy densities over the leaves of the maximal foliation. The vector fields are obtained by constructing approximate Killing and conformal Killing vector fields. On the background (Minkowski space), there are of course many conformal Killing vector fields, but the perturbed solution typically has none. On the other hand, since the solution is close to Minkowski space, there are approximate conformal Killing fields. However, constructing them is a delicate issue. In the end, all of the above issues are tied together in the proof which consists of a bootstrap argument involving the foliations, the construction of the vector fields, the energies etc. The asymptotic information obtained as a result is less detailed than that of Friedrich, but stability (and not only future stability) is obtained, as well as detailed information concerning the asymptotics. Moreover, it is not clear that the results of Friedrich can be applied to initial data of the degree of generality considered in [42]; in order for this to be true, it would be necessary to relate asymptotically flat initial data with asymptotically hyperboloidal initial data (cf. the above discussion).

The methods of Christodoulou and Klainerman have turned out to be of use in many different contexts. In particular, Nina Zipser proved stability of Minkowski space in the Einstein-Maxwell setting; cf. [179, 180]. Moreover, Lydia Bieri proved stability of Minkowski space under weaker assumptions on the initial data; cf. [17, 18]. In particular, Bieri requires control of one less derivative of the initial data, and allows a worse decay rate (one less power of r) in the definition of asymptotic flatness. As a consequence, she obtains less detailed information concerning the asymptotic behaviour. On the other hand, since it is sufficient to improve weaker bootstrap assumptions in the argument, it turns out that some of the constructions of almost conformal Killing fields become unnecessary, which simplifies the proof. As a final comment, it is of interest to note that the results [42, 18, 180] only concern matter models with conformal invariance properties.

The Lindblad-Rodnianski approach. In [118, 119], Lindblad and Rodnianski develop a very different approach to proving stability of Minkowski space. In particular, they consider Einstein's equations with respect to the isothermal coordinates introduced by de Donder, just as Choquet-Bruhat did in [78]. However, as opposed to the work of Choquet-Bruhat, the isothermal coordinates are here used to settle global issues. A central role in the argument is played by the so-called weak null condition. In order to justify the terminology, it is natural to consider the equations

$$\Box u = -\, u_t^2, \tag{13}$$

$$\Box u = -\, u_t^2 + |\nabla u|^2 \tag{14}$$

in $3+1$-dimensions, where $\Box$ is the Minkowski space wave operator. Both (13) and (14) admit $u = 0$ as a solution, but the stability properties of this solution is different for the two equations. In fact, $u = 0$ is a stable solution of (14), but it is an unstable solution of (13); cf., e.g., [99] and [158, Chapter II]. The reason for this is that (14) satisfies the so-called null-condition, whereas (13) does not; cf. [158, Chapter II] for a definition of this notion. The null-condition can be defined for a wide class of equations, and there are quite general stability results for solutions to equations satisfying the null-condition; cf. [101, 31, 102]. Unfortunately, Einstein's equations, expressed using isothermal coordinates, do not satisfy the null-condition. However, the authors of [118, 119] devised a way for extracting an *asymptotic system* from the equations (it is obtained by neglecting derivatives that are tangential to the light cone as well as cubic terms (such terms can be expected to decay faster)), and the equations are said to satisfy the *weak null condition* if the asymptotic system admits global solutions. It turns out that Einstein's vacuum equations, when expressed with respect to isothermal coordinates, satisfy the weak null condition.

Just as in [42], the proof of stability consists of a bootstrap argument based on energy estimates. As opposed to [42], it is, however, sufficient to use the conformal Killing vector fields of Minkowski space in order to prove stability (which significantly simplifies the argument). The way the weak null condition manifests itself in the bootstrap argument is that the equations have a hierarchical structure; it is possible to first improve the bootstrap assumptions for some of the components of the metric; this information can then be fed into the equations for the remaining components; and the desired conclusion follows.

Clearly, the above discussion only gives a very rough idea of the argument. The reader interested in more details is referred to [118, 119]. There are two clear advantages of [118, 119]. First of all, the argument is substantially shorter than that of [42]. Moreover, the methods can be used to prove stability for matter models that do not have conformal invariance properties. On the other hand, the conclusions obtained concerning the solution are not as detailed as those of [42].

6.2. **Linear equations on black hole backgrounds.** After proving the stability of Minkowski space, the natural next step is to consider the Schwarzschild and Kerr spacetimes. In fact, for reasons related to physical intuition and uniqueness results characterizing the Kerr family, the Kerr solutions are expected to be the natural end states of gravitational collapse to a black hole. In order to justify this picture, it is, at the very least, necessary to prove that the Kerr family is stable. In other words, to prove that perturbed Kerr data give rise to a maximal Cauchy development which asymptotes to a member of the Kerr family. Needless to say, giving precise definitions of what it means for initial data to be "close" to a member of the Kerr family, and what it means for a solution to "asymptote" to a member of this family is a part of the problem. So far, proving stability of Kerr has turned out to be too difficult; there are no results. As a consequence, researchers have focused on associated linear problems. The most natural linear equations to consider, in view of the above observations, are the linearized Einstein equations on a Kerr background. However, due to the degree of complication of this system, it turns out to be inappropriate to start by trying to analyze the asymptotics of its solutions. A simpler problem would be to consider the ordinary wave equation on Kerr. As this also turns out to be quite complicated, researchers interested in proving stability of Kerr started by considering the linear wave equation on a Schwarzschild background; cf., e.g., [100, 10, 19, 20, 21, 57, 58]. There are various ways to approach the problem, but the method of using vector fields and energies (as in [42]) is currently the most prominent one. In order to describe one way of constructing energies, let ϕ be a solution to the wave equation

$$\Box_g \phi = 0,$$

where g is a Schwarzschild metric. Then the associated stress energy tensor,

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi,$$

is divergence free. If X is a vector field, contracting the stress energy tensor with X yields a current

$$J^X_\mu = T_{\mu\nu}X^\nu$$

with divergence

$$\nabla^\mu J^X_\mu = {}^{(X)}\pi_{\mu\nu}T^{\mu\nu},$$

where

$${}^{(X)}\pi_{\mu\nu} = \frac{1}{2}(\mathcal{L}_X g)_{\mu\nu}.$$

Note, in particular, that if X is a Killing field, then the associated current is divergence free. Given the above constructions, it is of course natural to appeal to the divergence theorem. The corresponding equality can be used to relate the boundary terms and the bulk term. Given a priori information concerning the bulk and boundary terms (for instance that the bulk term vanishes (in case X is a Killing field, this is true), that one of the boundary terms is controlled by the initial data, that the bulk or boundary terms have suitable signs etc.), the equality can be used to control the bulk term or some of the boundary terms. In Schwarzschild, it is natural to start with the ∂_t vectorfield, since it, in particular, is Killing. In this case, the bulk term vanishes, and (for a natural choice of region) there are three boundary terms. One boundary term corresponds to an energy at the initial hypersurface, another corresponds to an energy at a later time, and the final boundary term corresponds to the part of the horizon in between them. In particular, the equality leads to a bound on the energy at later times in terms of the initial energy. Away from the horizon, this leads to a bound on the solution. However, near the horizon, the energy degenerates, and it has to be replaced by something else. Fortunately, it turns out that a vector field capturing the red shift effect can be used to control the solution near the horizon. Another complication that appears in Schwarzschild is associated with the presence of trapped null geodesics. However, it again turns out that this problem can be addressed using similar methods. The results concerning Schwarzschild represent a progression from bounds on the solution in the early works, to decay in the later contributions. The exact decay statements are somewhat technical and are strongly dependent on the geometry.

Turning to the case of Kerr, an additional complication which arises is the fact that the ∂_t vector field becomes spacelike in the ergoregion. Nevertheless, the questions of boundedness and decay have also been analyzed for solutions to the wave equation on Kerr backgrounds; cf., e.g., [59, 60, 61, 169, 5] for the case of $|a| \ll M$ and [62] for the case $|a| < M$. The literature on linear wave equations on black hole backgrounds is vast, and we have only given a few references. The reader interested in an introduction to the subject which is fairly up to date is referred to [60].

Recently, Dafermos, Holzegel and Rodnianski have announced that linear stability of Schwarzschild holds; cf. [52]. Due to this result, proving stability of black hole spacetimes now seems to be within reach.

6.3. **Stability, cosmological setting.** The model solutions in the cosmological setting are spatially homogeneous and isotropic; i.e., they satisfy the cosmological principle. However, as opposed to the standard black hole spacetimes, they nomally include matter. For a long time, it was taken for granted that it is only meaningful to include matter models satisfying the strong and dominant energy conditions, and that the cosmological constant is zero. Due to the observations of supernovae of type Ia carried out in '98-'99, cf. [138, 129], this perspective has changed. In fact, most current models of the universe include a mechanism which induces accelerated expansion; cf. [134, 135, 136, 137] and references cited therein for some of the possibilities. Historically, the main focus was thus on the case of a vanishing cosmological constant, but recently, the case of a positive cosmological constant has attracted more attention. We here discuss both cases. This is partly due to the fact that the cosmological constant has come and gone several times in the history of general relativity. However, it is mainly due to the following point of view: since the conclusion that there is accelerated expansion is based on fitting a very limited class of models to the observations, it is of interest to develop a feeling for more general solutions (with and without a positive cosmological constant).

Cosmology, the case of non-accelerated expansion. To our knowledge, the first stability result concerning a cosmological solution without accelerated expansion is [6]. In [6], the authors prove future stability of the Milne model, which is a vacuum solution of the form

$$g_M = -dt^2 + t^2 \bar{g}_H,$$

where $\bar{g}_H$ is a hyperbolic metric on a closed manifold. The Milne model can be thought of as a quotient of the timelike future of the origin in Minkowski space. The stability proof includes the construction of a future global CMC foliation; it guarantees future causal geodesic completeness; and it contains a demonstration of the fact that, after an appropriate rescaling, the metric and second fundamental form of the CMC leaves converge to those of the background. In this respect, the Milne model can be considered to be an attractor of the Einstein flow. The argument is based on the use of energies associated with the Bel-Robinson tensor. However, in a later work, cf. [7], the authors generalize the results to higher dimensions and more general situations using a rougher type of energy estimates.

It is very important to note that the results of [6, 7] fit into a much bigger context. In fact, there is a conjecture relating the future asymptotic behaviour of solutions to Einstein's vacuum equations in the cosmological setting with the geometrization of 3-manifolds. Moreover, one fundamental aspect of the conjecture is that the parts of the manifold corresponding to the hyperbolic pieces in the geometric decomposition should dominate asymptotically (in the sense that the fraction of the volume contained in the non-hyperbolic regions should tend to zero). The first step in verifying this conjecture is of course to verify that it holds for the Milne model itself. This is achieved in [6, 7]. Since it would be out of place to discuss the conjecture mentioned above in detail here, we refer the interested reader to [3, 77, 130, 131] and references cited therein for more information.

One important rule of thumb that indicates that the study of the stability properties of the Milne model (as opposed to some other spacetime) is the most natural first step is the following: spatial hypersurfaces with hyperbolic geometry maximize the expansion (which leads to maximal decay of perturbations and makes it easier to prove stability). This rule of thumb should of course be taken with a large grain of salt. However, an additional indication of its use is given by [26, 27]. In these papers, the authors demonstrate future stability in the $U(1)$ symmetric setting, provided the 2-dimensional hypersurface in which spatial variation is allowed is a higher genus surface.

Cosmology, the case of accelerated expansion. The simplest setting in which to consider cosmological solutions with accelerated expansion is Einstein's vacuum equations with a positive cosmological constant. Moreover, the model solution in this case is de Sitter space. That de Sitter space is stable was demonstrated in [80]. Similarly to the proof of the stability of Minkowski space to the future of a standard hyperboloid, the method of proof is based on the conformal field equations developed by Helmut Friedrich. Later on, the results were extended to include matter of Maxwell and Yang-Mills type; cf. [81]. While Friedrich only considers $3+1$-dimensional spacetimes, Michael Anderson proves stability of de Sitter space in arbitrary even spacetime dimensions in [4]. On the other hand, the methods used by Anderson are similar in spirit to those of Friedrich. Interestingly, it turns out that the results of Anderson are useful in the proof of stability of solutions to the Einstein-non-linear scalar field equations in $3+1$-dimensions in the case of an exponential potential; cf. [91]. In this case, the proof is based on a combination of Anderson's result and Kaluza-Klein reduction techniques. More recently, the conformal perspective has been used to address the stability of cosmological solutions to Einstein's equations with a positive cosmological constant and a radiation fluid; cf. [120]. Moreover, Friedrich proved stability of solutions to the Einstein-non-linear scalar field system (cf. [85]), assuming that a specific relation between the cosmological constant and the mass holds.

In spite of the successes of the conformal perspective, it does not seem suited to settle the issue of stability for all the matter models of physical interest. The problem of proving stability of de Sitter space was, for this reason, revisited in [146]. In this paper, the author demonstrates stability of de Sitter space in all spacetime dimensions. Moreover, the stability of solutions to the Einstein-non-linear scalar field system is demonstrated under more general circumstances than those considered in [85]. The method used in [146] is based on expressing the equations with respect to coordinates similar to the isothermal coordinates of de Donder. However, instead of demanding that the contracted Christoffel symbols vanish, they are here required to equal prescribed functions of the metric components and of the coordinates. The prescribed functions are referred to as *gauge source functions*, and the idea of introducing this additional freedom goes back to [86]. Unfortunately, it is not sufficient to only introduce appropriate gauge source functions. It is also necessary to add additional terms (that vanish when the gauge source functions equal the contracted Christoffel symbols) to the equations in order to obtain a system with good properties. In order to analyze the behaviour of solutions to the resulting system, energy methods can be used. However, it is of interest to note that in order to close the bootstrap argument, it is necessary to consider the energy associated with the different metric components separately. The reason for this is that it is possible to derive a system of differential inequalities for the energies associated with the different components, and it is necessary to improve the bootstrap assumptions concerning the different energies in a hierarchical fashion.

Following [146], several results using similar methods have been obtained. For instance, the case of the Einstein-non-linear scalar field with an exponential potential is considered in [148] (under more general circumstances than those considered in [91]). The Einstein-Maxwell-non-linear scalar field case is considered in [162, 121]. Turning to perfect fluids, there is a sequence of papers on this topic; cf.

[151, 159, 87]. Finally, the problem of proving stability in the case of the Einstein-Vlasov equations with a positive cosmological constant is discussed in [150]; cf. also [9, 125, 170]. For each of the matter models coupled to Einstein's equations, there are, needless to say, additional complications that need to be dealt with. However, we shall refrain from saying anything about the details here, and simply refer the reader interested in more information to the above references.

Cosmology, the direction of the singularity. All of the above results concern the future stability of cosmological solutions. It would also be of interest to prove stability in the direction of the singularity. However, the behaviour in that direction can in general be expected to be quite complicated. In particular, it is expected to be oscillatory, as can already be seen in the spatially homogeneous setting; cf., e.g., [139, 140, 176, 92, 93, 113, 14, 114]. The latter results are also consistent with a general picture concerning how the behaviour close to the singularity should be. This picture goes back to Belinskii, Khalatnikov and Lifschitz (cf., e.g., [15, 16]), and has later been developed further; cf., e.g., [64, 65, 94, 171] and references cited therein. However, for special types of matter models, the behaviour can be expected to be less complicated. In particular, the presence of a scalar field or a stiff fluid is expected to suppress the oscillations. The resulting behaviour is for that reason referred to as quiescent. In the spatially homogeneous setting, this expectation can be confirmed in some cases; cf. [140]. Moreover, using so-called Fuchsian techniques, large classes of solutions with quiescent singularities can be constructed; cf. [8, 63]. Even though the results obtained in [8, 63] are important, they suffer from two deficiencies. First of all, the solutions are real analytic. Moreover, the construction is based on prescribing the asymptotic behaviour. In that respect, the results correspond to specifying initial data at the singularity. It would be preferable to start with initial data on an ordinary Cauchy hypersurface, and to prove stability of the quiescent behaviour. Recently, this question has been addressed in [152, 153]. In fact, [153] constitutes the first stability result in the direction of the singularity.

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