Regularizations of Differential-Algebraic Equations Revisited*

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Abstract

The present paper deals with quasilinear differential-algebraic equations with index 2. These equations are approximated by regularization methods. Such methods lead to singularly perturbed differential-algebraic equations. Using a geometric theory of singular perturbations convergence of the solutions of the regularized problems towards that of the index 2 problem is proved. The limits of the present theory are discussed and directions of future research are proposed.

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1 Introduction

In the present paper we consider quasilinear ordinary differential equations

$$A(x,\cdot)\dot{x} = g(x,\cdot). \tag{1.1}$$

Here, $\dot{} = d/dt$ denotes total differentiation with respect to t. If the coefficient matrix $A(x,t) \in \mathcal{B}(I\!R^m)$ is nonsingular everywhere, (1.1) is equivalent to an explicit ordinary differential equation. We are interested in the case when this matrix is singular for all (x,t) of the domain of definition of the matrix function A. More specifically, we even assume that the nullspace $\mathcal{N}(A(x,t))$ of $A(x,t)$ does not depend

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on (x, t). Those equations are called differential-algebraic equations (dae's). Such problems arise naturally in a variety of applications, e. g., electrical networks, constraint mechanical systems of rigid bodies, chemical reaction kinetics, in the method of lines applied to systems of partial differential equations of mixed type, and many more. This is why great interest has been devoted to the analysis, geometry and numerical treatment of dae's in recent years.

The present paper deals with so-called regularization methods for the approximation of index 2 dae's and their convergence behaviour.

For a long time, dae's (1.1) were considered to be more or less implicitly written ordinary differential equations (subject to algebraic constraints) such that their numerical solution should cause no trouble. In [5], Gear introduced the application of the backward differentiation formulae for solving dae's. After that a number of standard codes were written and incorporated in other programs, e. g., for solving electrical network equations. After repeated reports about the failure of such codes when applied to certain classes of problems, in the early eighties, an intensive study of dae's and their analytical properties began. As a result classes of dae's could be distinguished where usual ode methods were expected to behave as in the case of explicit ordinary differential equations, while other classes of equations could be identified where ode methods were likely to fail. With the background of linear constant coefficient dae's [4] where the index of a matrix pencil allows to distinguish between these classes, different notions of an index were introduced aiming at a classification, roughly speaking, how far a given dae differs from an explicit ordinary differential equation: Ordinary differential equations (i. e. $A(x, t)$ in (1.1) is nonsingular) are characterized by the index 0; the index 1 describes equations that could probably be solved by (possibly appropriately modified) ode methods. Higher index equations are those with an index greater than 1. Analytically, the algebraic relations contained in (1.1) cause the latter ones to include differentiation problems such that finite difference methods become unstable. Geometrically, dae's should induce a smooth vector field on a certain manifold [24]. In case of index 1 equations this manifold is completely given by the algebraic relations of (1.1), while in the higher index case this manifold is only a submanifold of that given by the algebraic relations. This submanifold is not easily available. An ode method can only be expected to work well if it approximates not only the vector field but also the submanifold.

An alternative to a straightforward application of discretization methods in higher index problems is the use of index reduction methods. Their essence is the differentiation of the algebraic relations of (1.1) until an index 1 equation is obtained (or even an explicit ordinary differential equation), which is *analytically* equivalent to the originally given dae. Such an approach has well-known drawbacks, e. g., drift-off, change of the stability properties, and computational complexity. Another method consists in small perturbations of (1.1) (measured by a small positive parameter ε). Such an approach is usually called *regularization of the dae*. A motivation for doing so (and how to insert ε) comes from different applications as well as from theoretical insight into the analytical properties of dae's. Here, (1.1) appears to be the reduced problem of a singularly perturbed ordinary differential equation or a singularly perturbed index 1 dae. From a practical point of view, for numerically solving the latter problems a number of codes which work well and efficiently at least if the singular perturbation parameter ε is not too small is available.

In the present paper we will study the problem of convergence of the regularized solutions towards that of (1.1) for index 2 equations. We will show that the nature of this question is intimately connected with the geometric structure of (1.1). Having adopted this point of view we are able to extend recent results about regularizations of index 2 dae's [13, 14]. Moreover, connecting our method of proof (which is analytical although we use geometric results) with recent results about the geometric nature of dae's by Reich [22] and Rabier/Rheinboldt [21, 20] leads to the open problem of generalizing the geometric singular perturbation theory by Fenichel [3]. The (hopefully positive) answer will lead to a complete geometric treatment of the convergence problem which, in our opinion, is the essence of singular perturbations in dae's.

In order to be more precise, let Q denote a projection of $\mathbb{R}^m$ onto the nullspace $\mathcal{N}(A(x,t))$ and $P = I - Q$. We consider the parametrization

$$A(x,t)\dot{x} = g(x + \varepsilon P\dot{x}, t + \varepsilon). \tag{1.2}$$

Using the usual trick of inserting the formal differential equation $\dot{t} = 1$ we can restrict our considerations to the autonomous case only. (1.2) is a special parametrization introduced by März [17] for linear systems. Convergence results for (1.2) in nonlinear semiexplicit index 2 dae's in Hessenberg form are proved in [13]. Other types of parametrizations are closely related such that methods similar to those used in the present paper can be applied (see e. g. [14, 2]).

As a special case of (1.2) we will consider semiexplicit systems first, i. e., $A(x,t) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$.

The outline of the paper is as follows: In Section 2 we consider a semiexlicit index 2 system and derive the underlying singular perturbation problem. In Section 3 we recall the necessary notations and the convergence result of a geometric singular perturbation theory developed in [3]. As an application we are now in a position to present the main convergence result for the semiexplicit system of Section 2 in Section 4. Section 5 contains the proof of Lemma 4.1, which lies at the heart of our approach. In Section 5 we show the same convergence result for the original problem (1.1) - (1.2). Concluding we ask the question of how to generalize Fenichels results, which is under investigation now.

In order to avoid technicalities we assume throughout the paper that all functions involved are defined for all values of their arguments and that they are sufficiently smooth. The theory can easily be made local. Moreover, the necessary smoothness of the functions can be derived from inspecting the used theorems.

2 Regularizing Semiexplicit Index 2 Dae's

Consider the system

$$\dot{y} = f(y, z), \tag{2.1a}$$
$$0 = g(y, z). \tag{2.1b}$$

Let subscripts denote partial derivatives, e. g., $f_y(y, z) := \frac{\partial}{\partial y} f(y, z)$ etc.
Assumption S2.

(i) $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ are sufficiently smooth.

(ii) $g_z(y, z)$ has constant rank.

(iii) $Q_D(y, z)$ is a projection of $\mathbb{R}^n$ onto the nullspace $\mathcal{N}(g_z(y, z))$ such that

$$\Delta(y, z) := g_z(y, z) + g_y(y, z) f_z(y, z) Q_D(y, z)$$

is nonsingular for all (y, z).

Remark 2.1 *(0) Posing Assumption (S2) globally is only for convenience. It will be sufficient to have it in a neighbourhood of a solution of (2.1) over a finite t-interval.*

(i) (S2 i, ii) together with the results of [7] imply that Q_D can be chosen as smooth as g_z.

(ii) Theorem A.13 of [10] shows that the nonsingularity of Δ is independent of the special choice of Q_D.

(iii) The case $Q_D(y, z) \equiv I$ represents Hessenberg systems whereas $Q_D(y, z) \equiv 0$ identifies index 1 dae's.

Assume from now on that Q_D is smooth. Frequently we will use the following lemma.

Lemma 2.2 *For all (y, z), $(p, q) \in \mathbb{R}^m \times \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n$ it holds:*

(i) $g_{yz}(y, z)(p, Q_D(y, z)\alpha) \in \mathcal{R}(g_z(y, z))$.

(ii) $g_{zz}(y, z)(q, Q_D(y, z)\alpha) \in \mathcal{R}(g_z(y, z))$.

The second derivative was written as a bilinear operator here.

Proof. For all (y, z), we have $g_z(y, z)Q_D(y, z) \equiv 0$. Consequently, $\phi(t) := g_z(y + tp, z)Q_D(y + tp, z) \equiv 0$ such that also $\phi'(t) \equiv 0$. We obtain

$$\phi'(0) = g_{yz}(y, z)(p, Q_D(y, z)\cdot) + g_z(y, z)(Q_{D,y}(y, z)p, \cdot)$$

and hence (i). (ii) is proved analogously. □

We want to formulate now that (S2, iii) is the index 2 condition for (2.1). Since there are a number of different notions of an index (e. g. the differentiation index [6], the index of tractability [18, 19], the perturbation index [11], the geometric index [22, 20, 8]), we should be careful about such a statement. Fortunately, under our assumptions they coincide [9]. So we will use these notions interchangably in order to make it most convenient.

Lemma 2.3 *Under (S2), (2.1) has index 2.*

Proof. Let us differentiate (2.1b) and incorporate (2.1a):

$$0 = g_y(y, z)f(y, z) + g_z(y, z)\dot{z}. \tag{2.2}$$

The new system (2.1a), (2.2) has index 1. In order to show this we use the matrix criterion of [10]. (2.1a), (2.2) has index 1 iff its linearization has it. Dropping the arguments (y, z) the linearization of (2.1a), (2.2) is

$$A\begin{pmatrix} \dot{\bar{y}} \\ \dot{\bar{z}} \end{pmatrix} + B\begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix} = 0$$

with

$$\begin{aligned} A &= \begin{pmatrix} I & 0 \\ 0 & -g_z \end{pmatrix}, \\ B &= \begin{pmatrix} -f_y & -f_z \\ g_{yy}(f, \cdot) + g_y f_y + g_{yz}(\dot{z}, \cdot) & g_{yz}(f, \cdot) + g_y f_z + g_{zz}(\dot{z}, \cdot) \end{pmatrix}. \end{aligned}$$

According to Theorem A.13 of [10] the matrix pencil (A, B) is regular with index 1 iff $S \cap \mathcal{N}(A) = \{0\}$, where $S = \{x | Bx \in \mathcal{R}(A)\}$. Let now $x = (y, z) \in S \cap \mathcal{N}(A)$. Then $y = 0$ and $g_z z = 0$, i. e., $z = Q_D z$. Since $(y, z) \in S$, we obtain

$$g_{yz}(f, Q_D z) + g_y f_z Q_D z + g_{zz}(\dot{z}, Q_D z) \in \mathcal{R}(g_z).$$

By Lemma 2.2, $g_y f_z Q_D z \in \mathcal{R}(g_z)$ and hence also $\Delta Q_D z \in \mathcal{R}(g_z)$. Since $Q_D^2 = Q_D$, there is an $\alpha \in I\!R^n$ such that $Q_D z = Q_D \Delta^{-1} g_z \alpha$. On the other hand, $\Delta^{-1} g_z = \Delta^{-1} g_z(I - Q_D) = \Delta^{-1}(g_z + g_y f_z Q_D)(I - Q_D) = I - Q_D$ such that $Q_D z = 0$ and hence $z = 0$. □

A straightforward application of (1.2) to (2.1) gives

$$\dot{y} = f(y + \varepsilon\dot{y}, z), \tag{2.3a}$$
$$0 = g(y + \varepsilon\dot{y}, z). \tag{2.3b}$$

Instead, we consider the simpler version

$$\begin{aligned} \dot{y} &= f(y,z), && \text{(2.4a)} \\ 0 &= g(y+\varepsilon\dot{y}, z). && \text{(2.4b)} \end{aligned}$$

Obviously, convergence results for (2.4) carry over to (2.3) immediately. In [13] we proved convergence in terms of asymptotic expansions if (2.1) has index 2 Hessenberg form, i. e. g does not depend on z and, therefore, $Q_D = I$. Three essential steps were used:

- Index reduction of (2.1) and (2.4) by means of differentiation of the algebraic constraints.
- The resulting problem from (2.4) can be solved for the derivatives $\dot{y}$, $\dot{z}$ leading to a singularly perturbed system of ordinary differential equations whose reduced problem is equivalent to the index reduced form of (2.1).
- Now, Tikhonov's theorem applies and asymptotic expansions can be established.

Under Assumption (S2) we will follow the same lines. In an attempt to shorten the formulae we dot not write the arguments (y,z) of the different functions unless they are *not* (y,z). Differentiation of (2.4b) gives

$$0 = g_y(y+\varepsilon f, z)(I+\varepsilon f_y)f + \varepsilon g_y(y+\varepsilon f, z)f_z\dot{z} + g_z(y+\varepsilon f, z)\dot{z}. \tag{2.5}$$

From the linear case [12, 14] we learn that the singularly perturbed components of (y,z) in (2.4a), (2.5) should be Q_Dz. Therefore, we try to resolve (2.5) for $Q_D\dot{z}$ and $(I-Q_D)\dot{z}$, respectively. Note that $Q_D = Q_D(y,z)$! Let

$$z_1 := \varepsilon Q_D\dot{z}, \quad z_2 := (I-Q_D)\dot{z}.$$

Then (2.5) is equivalent to the system of equations

$$H_\varepsilon \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -g_y(y+\varepsilon f, z)(I+\varepsilon f_y)f \\ 0 \end{pmatrix} \tag{2.6}$$

with

$$H_\varepsilon = \begin{pmatrix} \left[\frac{1}{\varepsilon}g_z(y+\varepsilon f, z) + g_y(y+\varepsilon f, z)f_z\right]Q_D & g_z(y+\varepsilon f, z) + \varepsilon g_y(y+\varepsilon f, z)f_z \\ I-Q_D & Q_D \end{pmatrix}. \tag{2.7}$$

Lemma 2.4 *The limit of H_ε, for $\varepsilon \to 0$, exists and is equal to*

$$H_0 = \begin{pmatrix} g_{yz}(f, Q_D\cdot) + g_yf_zQ_D & g_z \\ I-Q_D & Q_D \end{pmatrix}. \tag{2.8}$$

Moreover, H_0 is nonsingular and its inverse is given by

$$H_0^{-1} = \begin{pmatrix} Q_D\Delta^{-1} & I - Q_D \\ (I-Q_D)\Delta^{-1} - \Delta^{-1}g_{yz}(f, Q_D\Delta^{-1}\cdot) & Q_D \end{pmatrix}. \tag{2.9}$$

Proof. Since $g_z Q_D \equiv 0$, we have

$$\begin{aligned} \frac{1}{\varepsilon} g_z(y+\varepsilon f, z) Q_D &= \frac{1}{\varepsilon}\left(g_z(y+\varepsilon f, z) - g_z\right) Q_D \\ &= g_{yz}(f, Q_D\cdot) + O(\varepsilon) \end{aligned}$$

such that (2.8) is true.

In order to give more insight into the procedure we will show first that H_0 is nonsingular and then, using the same methods, establish (2.9). Let

$$H_0 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then we have immediately $(I-Q_D)z_1 = 0$ and $Q_D z_2 = 0$ such that

$$g_{yz}(f, Q_D z_1) + g_y f_z Q_D z_1 + g_z(I-Q_D)z_2 = 0. \tag{2.10}$$

This is equivalent to

$$(g_z + g_y f_z Q_D)(I-Q_D)z_2 = -g_{yz}(f, Q_D z_1) - g_y f_z Q_D z_1$$

which yields

$$(I-Q_D)z_2 = -\Delta^{-1}\left\{g_{yz}(f, Q_D z_1) + g_y f_z Q_D z_1\right\}. \tag{2.11}$$

Inserting (2.11) into (2.10) we obtain

$$(I - g_z\Delta^{-1})\left\{g_{yz}(f, Q_D z_1) + g_y f_z Q_D z_1\right\} = 0.$$

Furthermore it holds that $g_z\Delta^{-1}g_y f_z Q_D = g_z\Delta^{-1}(g_z + g_y f_z Q_D)Q_D = g_z Q_D = 0$ and $g_y f_z Q_D = \Delta Q_D$ such that

$$Q_D z_1 = -\Delta^{-1}(I - g_z\Delta^{-1})g_{yz}(f, Q_D z_1).$$

By Lemma 2.2 there is an $\alpha \in \mathbb{R}^n$ such that $g_{yz}(f, Q_D z_1) = g_z\alpha$. But then $g_z\Delta^{-1}g_z\alpha = g_z\Delta^{-1}(g_z + g_y f_z Q_D)(I-Q_D)\alpha = g_z\alpha$, hence,

$$Q_D z_1 = 0.$$

Summing up, $z_1 = 0$ and, by (2.11) and $Q_D z_2 = 0$, also $z_2 = 0$ such that H_0 is nonsingular.

In order to derive a representation of H_0^{-1} let now

$$H_0 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{2.12}$$

The computation is similar to that used above but now with inhomogeneous right-hand side. The second equation of (2.12) gives immediately

$$(I - Q_D)z_1 = (I - Q_D)\beta, \quad Q_D z_2 = Q_D \beta. \tag{2.13}$$

Inserting these expressions into the first equation of (2.12) yields

$$g_{yz}(f, Q_D z_1) + g_y f_z Q_D z_1 + g_z (I - Q_D) z_2 = \alpha. \tag{2.14}$$

Now multiply this equation by $\Delta^{-1}(I - g_z \Delta^{-1})$. We obtain as above

$$Q_D z_1 = \Delta^{-1}(I - g_z \Delta^{-1})\alpha.$$

Since $\Delta^{-1} g_z = \Delta^{-1}(g_z + g_y f_z Q_D)(I - Q_D) = I - Q_D$ we get $\Delta^{-1}(I - g_z \Delta^{-1}) = Q_D \Delta^{-1}$ and, finally, $Q_D z_1 = Q_D \Delta^{-1}\alpha$ such that

$$z_1 = Q_D \Delta^{-1}\alpha + (I - Q_D)\beta. \tag{2.15}$$

In order to obtain an expression for z_2 note that $g_z = g_z(I - Q_D) = \Delta(I - Q_D)$ such that (2.14), (2.15) lead to

$$\begin{aligned}(I - Q_D)z_2 &= \Delta^{-1}\{\alpha - g_{yz}(f, Q_D z_1) - g_y f_z Q_D z_1\} \\ &= \Delta^{-1}\{\alpha - g_{yz}(f, Q_D \Delta^{-1}\alpha) - g_y f_z Q_D \Delta^{-1}\alpha\} \\ &= (I - Q_D)\Delta^{-1}\alpha - \Delta^{-1} g_{yz}(f, Q_D \Delta^{-1}\alpha). \end{aligned} \tag{2.16}$$

Note that $\Delta^{-1} g_{yz}(f, Q_D \Delta^{-1}\alpha) = \Delta^{-1} g_z \gamma = (I - Q_D)\gamma$ such that $\Delta^{-1} g_{yz}(f, Q_D \cdot) \in \mathcal{R}(I - Q_D)$. Now (2.13), (2.15), and (2.16) yield the representation of H_0^{-1}. □

Corollary 2.5 *For sufficiently small $\varepsilon > 0$ and $(y, z) \in K \subset \mathbb{R}^m \times \mathbb{R}^n$ with compact K, (2.5) is equaivalent to the system*

$$\varepsilon Q_D \dot{z} = Q_D h_1(y, z, \varepsilon), \quad (I - Q_D)\dot{z} = (I - Q_D) h_2(y, z, \varepsilon) \tag{2.17}$$

with some smooth functions h_1 and h_2.

Proof. $H_\varepsilon = H_0 - (H_0 - H_\varepsilon) = (I - (H_0 - H_\varepsilon)H_0^{-1})H_0$ such that $H_\varepsilon^{-1} = H_0^{-1}(I - (H_0 - H_\varepsilon)H_0^{-1})^{-1}$. (2.6) and Lemma 2.4 yield the assertion by Banach's lemma. □

Remark 2.6 *(i) We emphasize that Q_D depends on y and z in (2.17).*

(ii) If (2.1) is index 2 Hessenberg, $Q_D \equiv I$ and (2.17) consists of one singularly perturbed equation such that (2.1a), (2.17) has the usual form covered by Tikhonov's theorem. In fact, this property lies at the heart of the proofs in [13].

(iii) *The other limiting case can be obtained if (2.1) has index 1 such that $Q_D \equiv 0$. Hence, (2.17) does not have a singular perturbation part. In turn, this means that, for index 1 systems, (2.4) is a regular perturbation of (2.1) as we should expect.*

For the classical Tikhonov theory of singular perturbations, the structure of (2.17) is a barrier which cannot be overcome by classical tools. Two observations, however, which suggest that geometric considerations could be the way out, can be made in (2.17).

Let $\bar{y}(t)$ and $\bar{z}(t)$ be two continuously differentiable functions. We will consider (2.17) on $K = \{(\bar{y}(t), \bar{z}(t)) | t \in [t_0, T]\}$. Denote $Q(t) := Q_D(\bar{y}(t), \bar{z}(t))$. Then we obtain, for $z_1 = Qz$ and $z_2 = (I - Q)z$, the system

$$\varepsilon \dot{z}_1 = \varepsilon \dot{Q}(z_1 + z_2) + Qh_1(y, z_1 + z_2, \varepsilon), \tag{2.18a}$$
$$\dot{z}_2 = -\dot{Q}(z_1 + z_2) + (I - Q)h_2(y, z_1 + z_2, \varepsilon). \tag{2.18b}$$

Proposition 2.7 *Every solution of (2.18) has the property: $(I-Q)z_1(t) = Qz_2(t) = 0$ for all $t \in [t_0, T]$ iff there exists a $t_1 \in [t_0, T]$ such that $(I-Q)z_1(t_1) = Qz_2(t_1) = 0$.*

Proof. Multiply (2.18b) by Q. Thus, $Q\dot{z}_2 = -Q\dot{Q}(z_1 + z_2)$. The multiplication rule for the derivative yields

$$\frac{d}{dt}(Qz_2) - \dot{Q}z_2 = -\dot{Q}(z_1 + z_2) + \dot{Q}Q(z_1 + z_2),$$

i. e.

$$\frac{d}{dt}(Qz_2) = \dot{Q}(I - Q)z_1 + \dot{Q}Qz_2. \tag{2.19}$$

Analogously, multiplication of (2.18a) by $I - Q$ yields

$$\frac{d}{dt}((I - Q)z_1) = -\dot{Q}(I - Q)z_1 + \dot{Q}z_2. \tag{2.20}$$

(2.19), (2.20) is a homogeneous linear system of ordinary differential equations with respect to Qz_2 and $(I - Q)z_1$. Thus, the proposition follows. □

Proposition 2.7 allows us to reduce the dimension of (2.17). It shows that it is sufficient to consider the singularly perturbed system (2.1a), (2.17) with the independent components $y, Qz, (I - Q)z$ on the subspaces $\mathbb{R}^m$, $\mathcal{R}(Q)$, and $\mathcal{R}(I - Q)$, respectively.

Proposition 2.8 *Let Q be as above and $e_1, \ldots, e_k \in \mathbb{R}^n$ be a basis of $\mathcal{R}(Q(t_0))$. If $E \in C^1([t_0, T], \mathcal{B}(\mathbb{R}^k, \mathbb{R}^n))$ solves the initial value problem*

$$\dot{E} = \dot{Q}E, \quad E(t_0) = (e_1, \ldots, e_k),$$

then $E(t)$ is a basis of $\mathcal{R}(Q(t))$ for all $t \in [t_0, T]$.

Proof. Since $E(t_0)$ has full rank k, $E(t)$ is so. Therefore, it is sufficient to show that $(I-Q)E \equiv 0$. But this can be done similar to the proof of Proposition 2.7. □

Let now E_α and E_β be constructed according to Proposition 2.8 such that $E_\alpha(t)$ is a basis of $\mathcal{R}(Q(t))$ and $E_\beta(t)$ is a basis of $\mathcal{R}(I-Q(t))$. Then let $z_1(t) = E_\alpha(t)\alpha(t)$, $z_2(t) = E_\beta(t)\beta(t)$ with $\alpha(t) \in I\!R^k$ and $\beta(t) \in I\!R^{n-k}$ in (2.18). Define now

$$\begin{aligned} F(y,\alpha,\beta,t,\varepsilon) &:= (E_\alpha^T E_\alpha)^{-1} E_\alpha^T \times \\ &\quad \left[\varepsilon \dot{Q}(E_\alpha \alpha + E_\beta \beta) + Q h_1(y, E_\alpha \alpha + E_\beta \beta, \varepsilon) - \varepsilon \dot{Q} E_\alpha \alpha\right] \\ G(y,\alpha,\beta,t,\varepsilon) &:= (E_\beta^T E_\beta)^{-1} E_\beta^T \times \\ &\quad \left[-\dot{Q}(E_\alpha \alpha + E_\beta \beta) + (I-Q) h_2(y, E_\alpha \alpha + E_\beta \beta, \varepsilon) + \dot{Q} E_\beta \beta\right] \end{aligned}$$

such that (2.18) is equivalent to

$$\varepsilon \dot{\alpha} = F(y,\alpha,\beta,t,\varepsilon) \tag{2.21a}$$
$$\dot{\beta} = G(y,\alpha,\beta,t,\varepsilon). \tag{2.21b}$$

Proposition 2.9 *Let $\bar{y}(t)$, $\bar{z}(t)$ be a solution of the reduced equation (2.1) (or, equivalently, of the reduced equation of (2.1a), (2.17)). Then*

$$F_\alpha(\bar{y}, \bar{\alpha}, \bar{\beta}, \cdot, 0) = -I, \tag{2.22}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are defined by $Q(t)\bar{z}(t) = E_\alpha(t)\bar{\alpha}(t)$ and $(I - Q(t))\bar{z}(t) = E_\beta(t)\bar{\beta}(t)$, respectively.

Proof. Because of our smoothness assumptions, the limiting process $\varepsilon \to 0$ and the differentiation can be interchanged. Thus, the right-hand side of (2.17) for $\varepsilon = 0$ is given by

$$\begin{aligned} \begin{pmatrix} Q_D h_1(y,z,0) \\ (I-Q_D) h_2(y,z,0) \end{pmatrix} &= H_0^{-1} \begin{pmatrix} -g_y f \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -Q_D \Delta^{-1} g_y f \\ -(I-Q_D)\Delta^{-1} g_y f + \Delta^{-1} g_{yz}(f, Q_D \Delta^{-1} g_y f) \end{pmatrix}. \end{aligned}$$

Hence,

$$F(\bar{y}, \bar{\alpha}, \bar{\beta}, \cdot, 0) = -(E_\alpha^T E_\alpha)^{-1} E_\alpha^T Q Q_D \Delta^{-1} g_y f$$

and

$$F_\alpha(\bar{y}, \bar{\alpha}, \bar{\beta}, \cdot, 0) = -(E_\alpha^T E_\alpha)^{-1} E_\alpha^T Q \frac{\partial}{\partial z}(Q_D \Delta^{-1} g_y f) E_\alpha.$$

Since $E_\alpha = Q E_\alpha$, it is convenient to consider the expression

$$\begin{aligned} \Psi &= Q \frac{\partial}{\partial z}(Q_D \Delta^{-1} g_y f) Q \\ &= Q[-\Delta^{-1} \frac{\partial}{\partial z} \Delta \Delta^{-1} g_y f + \Delta^{-1} \frac{\partial}{\partial z} g_y f + \Delta^{-1} g_y f_z] Q + Q Q_{D,z}(\Delta^{-1} g_y f, Q\cdot). \end{aligned} \tag{2.23}$$

The third term of (2.23) is

$$Q\Delta^{-1}g_y f_z Q = Q\Delta^{-1}(g_z + g_y f_z Q)Q = Q. \tag{2.24}$$

Here we have used $Q(t) = Q_D(\bar{y}(t), \bar{z}(t))$. The second term can be transformed as follows (cf. Lemma 2.2):

$$\begin{aligned} Q\Delta^{-1}\frac{\partial}{\partial z}g_y(Q\xi)f &= Q\Delta^{-1}g_{yz}(f, Q\xi) \\ &= Q\Delta^{-1}g_z\zeta \\ &= Q\Delta^{-1}(g_z + g_y f_z Q)(I - Q)\zeta \\ &= 0. \end{aligned} \tag{2.25}$$

The first term of (2.23) yields

$$\begin{aligned} Q\Delta^{-1}\frac{\partial}{\partial z}\Delta(Q\xi)\Delta^{-1}g_y f &= Q\Delta^{-1}[g_{zz}(\Delta^{-1}g_y f, Q\xi) + g_{yz}(f_z Q_D\Delta^{-1}g_y f, Q\xi) + \\ &\quad g_y f_{zz}(Q_D\Delta^{-1}g_y f, Q\xi) + g_y f_z Q_{D,z}(\Delta^{-1}g_y f, Q\xi)]. \end{aligned} \tag{2.26}$$

Similarly to (2.25), $Q\Delta^{-1}g_{zz}(\Delta^{-1}g_y f, Q\xi) = 0$ and $Q\Delta^{-1}g_{yz}(f_z Q_D\Delta^{-1}g_y f, Q\xi) = 0$. Since $\bar{y}(t)$, $\bar{z}(t)$ fulfil (2.1b), we obtain by differentiation $0 = g_y(\bar{y}(t), \bar{z}(t))\dot{\bar{y}}(t) + g_z(\bar{y}(t), \bar{z}(t))\dot{\bar{z}}(t)$, i. e. $g_y f \in \mathcal{R}(g_z)$. This yields

$$Q_D\Delta^{-1}g_y f = 0. \tag{2.27}$$

Since $Q_D^2 = Q_D$, we have $Q_{D,z} = Q_{D,z}Q_D + Q_D Q_{D,z}$. Inserting this identity into the last term of (2.26) we obtain by (2.27):

$$\begin{aligned} Q\Delta^{-1}g_y f_z Q_{D,z}(\Delta^{-1}g_y f, Q\xi) &= Q\Delta^{-1}g_y f_z Q_D Q_{D,z}(\Delta^{-1}g_y f, Q\xi) \\ &= Q\Delta^{-1}(g_z + g_y f_z Q_D)Q_D Q_{D,z}(\Delta^{-1}g_y f, Q\xi) \\ &= Q Q_{D,z}(\Delta^{-1}g_y f, Q\xi). \end{aligned}$$

Thus, the last term of (2.23) and (2.26) cancel out. Summarizing we obtain $\Psi = Q$. Again using $E_\alpha = QE_\alpha$ we compute finally

$$F_\alpha(\bar{y}, \bar{\alpha}, \bar{\beta}, \cdot, 0) = -(E_\alpha^T E_\alpha)^{-1}E_\alpha^T Q^2 E_\alpha = -I.$$

□

The propositions 2.7 – 2.9 give us "almost" all ingredients for applying the classical Tikhonov theory. However, the transformation of (2.17) to get (2.18) and, consequently, the results basing on it are only valid at the solution of the reduced problem (2.1) such that important assumptions of Tikhonov's theorem are not fulfilled. Moreover, the difference between $Q(t)$ and Q_D is rather artificial. On the other hand, the construction reminds of the construction of a tangent space on a given manifold $M \subseteq \mathbb{R}^m \times \mathbb{R}^n$ at $x \in M$ such that $\begin{pmatrix} I & 0 \\ 0 & I - Q_D \end{pmatrix}$ appears to be a projection of the tangent space $T_x(\mathbb{R}^m \times \mathbb{R}^n)$ onto T_xM, where $M = \{(y,z) | g_y(y,z)f(y,z) \in \mathcal{R}(g_z(y,z))\}$. Later on we will show that Assumption (S2) assures that M is a differentiable submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. Therefore, we expect to overcome the conceptual difficulties when working with (2.18) if we consider the singular perturbation problem within a differential geometric context. The latter is very natural as it is shown in a number of papers by various authors [22, 23, 24, 21, 20]. Before we can do so in Section 4 we will shortly recall some facts of a geometric singular perturbation theory developed by Fenichel [3].

3 A Geometric Singular Perturbation Theory

The content of the present section is a short overview and a specialization of results in [3]. Although we are mainly interested in the local theory (and the theorem below is one indeed), in order to simplify notation, we assume again that all functions involved are defined for all values of their arguments and that they are sufficiently smooth.

For a motivation, consider a singularly perturbed system of ordinary differential equations in the (standard) form

$$\begin{aligned} \dot{y} &= F(y,z,\varepsilon), && (3.1a) \\ \varepsilon\dot{z} &= G(y,z,\varepsilon). && (3.1b) \end{aligned}$$

Let both functions be defined for all $(y,z) \in M := \mathbb{R}^m \times \mathbb{R}^n$. For $\varepsilon = 0$, we obtain the reduced system

$$\begin{aligned} \dot{y} &= F(y,z,0), && (3.2a) \\ 0 &= G(y,z,0). && (3.2b) \end{aligned}$$

Together with (3.1) we consider the system in the fast time scale $\tau = t/\varepsilon$. Denote $\cdot' := \frac{d}{d\tau}$. Then, for $\varepsilon \neq 0$, (3.1) is equivalent to

$$\begin{aligned} y' &= \varepsilon F(y,z,\varepsilon), && (3.3a) \\ z' &= G(y,z,\varepsilon). && (3.3b) \end{aligned}$$

The reduced system is now

$$\begin{aligned} y' &= 0, && (3.4a) \\ z' &= G(y,z,0). && (3.4b) \end{aligned}$$

Every solution of (3.2) must belong to the set $\mathcal{E} := \{(y,z) | G(y,z,0) = 0\}$. The point is here that $\mathcal{E}$ entirely consists of equilibrium points of (3.4). This fact will be used in order to define singular perturbation problems in a geometric context. In parallel to (3.1) and (3.3), (3.4) it is convenient to consider the equivalent systems in which ε is introduced as a dummy variable in the phase space. These systems are

$$\begin{aligned} \dot{y} &= F(y, z, \varepsilon), \\ \varepsilon\dot{z} &= G(y, z, \varepsilon), \\ \dot{\varepsilon} &= 0, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} y' &= \varepsilon F(y, z, \varepsilon), \\ z' &= G(y, z, \varepsilon), \\ \varepsilon' &= 0. \end{aligned} \tag{3.6}$$

For precise definitions of the differential geometric notions used below see any textbook on differential geometry, e. g. [1, 16]. Let M be a C^{r+1} l-dimensional manifold, $1 \leq r < \infty$. Let X be a C^r vector field on M, i. e., a mapping $X : M \to TM$ with $X(x) \in T_xM$. Here, T_xM denotes the tangent space of M at x and $TM := \bigcup_{x\in M} T_xM$ the tangent bundle. For any differentiable function ϕ defined on M near x, $X\phi(x)$ is defined by $D\phi(x)X(x)$ where D denotes the derivative. A flow $\Psi_X : (t, x) \mapsto x(t)$ is now uniquely defined by requiring that

$$\frac{d}{dt}\phi(x(t)) = X\phi(x(t))$$

for all differentiable functions ϕ, and $\Psi_X(0, \cdot) = I$. Let U be an open set in M, and let $\phi : U \to V \subseteq I\!R^l$ be a C^{r+1} diffeomorphism. Then $\xi = \phi(x)$ defines a local coordinate system. The flow of X satisfies

$$\dot{\xi} = X\phi(\phi^{-1}(\xi)) \tag{3.7}$$

in ξ-coordinates. The variational equation of (3.7) along an orbit $\xi(t)$ is

$$\begin{aligned} \dot{\delta\xi} = \; & D^2\phi(\phi^{-1}(\xi(t)))\langle D\phi^{-1}(\xi(t))\delta\xi, X(\phi^{-1}(\xi(t)))\rangle + \\ & D\phi(\phi^{-1}(\xi(t)))DX(\phi^{-1}(\xi(t)))D\phi^{-1}(\xi(t))\delta\xi. \end{aligned}$$

If $x = \phi^{-1}(\xi)$ is an equilibrium point of the flow of X, i. e. $X(x) = 0$, then we obtain

$$\dot{\delta\xi} = D\phi(x)DX(x)D\phi(x)^{-1}\delta\xi. \tag{3.8}$$

The coefficient matrix of (3.8) transforms by similarity and, hence, defines a linear map $TX(x) : T_xM \to T_xM$.

Let now X^ε be a family of vector fields on M parametrized by $\varepsilon \in [0, \varepsilon_0)$ such that $X^\varepsilon(x)$ is a C^r function of both variables (x, ε). Let $\mathcal{E}$ be a C^r submanifold of

M consisting entirely of equilibrium points of X^0. We will study the flow of X^ε near $\mathcal{E}$ for small nonzero ε. Let $\xi = \phi(x)$ be a C^{r+1} local coordinate system in M. In ξ-coordinates the flow of X^ε satisfies

$$\xi' = X^\varepsilon \phi(\phi^{-1}(\xi)) \tag{3.9}$$

subject to the condition

$$X^0 \phi(\phi^{-1}(\xi)) = 0 \text{ for } \xi \in \phi(\mathcal{E}). \tag{3.10}$$

(3.9) with (3.10) resembles (3.3). After rescaling $t = \varepsilon\tau$, (3.9) takes the form

$$\varepsilon\dot{\xi} = X^\varepsilon \phi(\phi^{-1}(\xi)) \tag{3.11}$$

which is the analogue of (3.1). Let $\mathbf{0}$ denote the zero vector field on $[0, \varepsilon_0)$. The vector fields $X^\varepsilon \times \mathbf{0}$ and $\varepsilon^{-1}X^\varepsilon \times \mathbf{0}$ are defined on $M \times [0, \varepsilon_0)$ and $M \times (0, \varepsilon_0)$, respectively. In local coordinates the flows of $X^\varepsilon \times \mathbf{0}$ and $\varepsilon^{-1}X^\varepsilon \times \mathbf{0}$ satisfy

$$\begin{array}{rcl} \xi' &=& X^\varepsilon(\phi(\phi^{-1}(\xi)) \\ \varepsilon' &=& 0 \end{array} \tag{3.12}$$

and

$$\begin{array}{rcl} \varepsilon\dot{\xi} &=& X^\varepsilon \phi(\phi^{-1}(\xi)) \\ \dot{\varepsilon} &=& 0, \end{array} \tag{3.13}$$

which are just the analogues of (3.5) and (3.6), respectively.

Let m the dimension of $\mathcal{E}$, and let n be the codimension of $\mathcal{E}$ in M. Let x be any point in $\mathcal{E}$. According to (3.8) we have a linear map

$$TX^0(x) : T_xM \to T_xM.$$

Since X^0 vanishes identically on $\mathcal{E}$, $T_X\mathcal{E}$ is in the kernel of $TX^0(x)$. The subspace $T_x\mathcal{E}$ is invariant under $TX^0(x)$, and so $TX^0(x)$ induces a linear map

$$QX^0(x) : T_xM/T_x\mathcal{E} \to T_xM/T_x\mathcal{E}$$

on the quotient space. $TX^0(x)$ has the eigenvalue 0 at least m-fold. The eigenvalues of $QX^0(x)$ are just the remaining ones of $TX^0(x)$.

Assumption (A). For the spectrum of $QX^0(x)$, it holds

$$\sigma(QX^0(x)) \subseteq \mathbb{C}^- := \{\lambda \in \mathbb{C} | \Re(\lambda) < 0\} \text{ for all } x \in \mathcal{E}.$$

In his original paper, Fenichel considers also hyperbolic splittings of the spectrum of $QX^0(x)$. As we have learnt from Proposition 2.9, Assumption (A) should be expected to be valid in our case.

Before we are able to define the reduced vector field we must introduce a certain projection $\Pi^{\mathcal{E}}$ as follows. For each $x \in \mathcal{E}$, $T_x\mathcal{E}$ has a unique complement N_x in T_xM which is invariant under $TX^0(x)$, namely the linear hull of the algebraic eigenspaces corresponding to the nonvanishing eigenvalues of $TX^0(x)$. Let $\Pi^{\mathcal{E}}(x)$ now be the projection onto $T_x\mathcal{E}$ according to this decomposition. The reduced vector field X_R on $\mathcal{E}$ is now given by

$$X_R(x) := \Pi^{\mathcal{E}} \frac{\partial}{\partial \varepsilon} X^{\varepsilon}(x)\bigg|_{\varepsilon=0}. \tag{3.14}$$

Since TX^0 is C^{r-1}, $\Pi^{\mathcal{E}}$ is so, too, and hence also X_R.

Let Ψ be a flow on M. We say that $V \subseteq M$ is locally invariant under the flow Ψ if there is a neighbourhood U of V such that for all $p \in V$ and all $t \in I\!R$ it holds: If $t \geq 0$ ($t \leq 0$) and $\Psi([0,t],p) \subseteq U$ ($\Psi([t,0],p) \subseteq U$), then $\Psi([0,t],p) \subseteq V$ ($\Psi([t,0],p) \subseteq V$). For a locally invariant V let $\{S(p)|p \in V\}$ be a family of subsets of M parametrized by $p \in V$. We say that $\{S(p)|p \in V\}$ is locally positively invariant if $\Psi(t,S(p)) \subseteq S(\Psi(t,p))$ for all $p \in V$ and $t \geq 0$ such that $\Psi([0,t],p) \subseteq V$. Suppose that V is a locally invariant submanifold of M, and $\{S(p)|p \in V\}$ is a family of submanifolds of M parametrized by $p \in V$. We say that $\{S(p)|p \in V\}$ is a C^r family of manifolds if $\{(p,p')|p \in V, p' \in S(p)\}$ is a C^r submanifold of $M \times M$.

Let now M, X^{ε}, and $\mathcal{E}$ given as above with Assumption (A) being fulfilled. Let $K \subseteq \mathcal{E}$ be a compact subset. We call a manifold $\mathcal{C}$ a center manifold for $X^{\varepsilon} \times \mathbf{0}$ near K if

(i) $K \times \{0\} \subseteq \mathcal{C}$,

(ii) $\mathcal{C}$ is locally invariant under the flow of $X^{\varepsilon} \times \mathbf{0}$,

(iii) for all $x \in K$, $\mathcal{C}$ is tangent to $\mathcal{E} \times [0,\varepsilon_0)$ at $(x,0)$.

Let U be a locally invariant neighbouhood of $K \times \{0\}$ in M. We say that a family $\{\mathcal{F}^s(p)|p \in U\}$ is a C^{r_2} family of C^{r_1} stable manifolds for U near K if

(i) $\mathcal{F}^s(p)$ is a C^{r_1} manifold for each $p \in U$,

(ii) $p \in \mathcal{F}^s(p)$ for each $p \in U$,

(iii) $\mathcal{F}^s(p)$ and $\mathcal{F}^s(q)$ are disjoint or identical for each $p,q \in U$,

(iv) $\mathcal{F}^s(x,0)$ is tangent to the invariant subspace of $T_{(x,0)}(M \times [0,\varepsilon_0))$ associated with the nonvanishing eigenvalues of $T(X^{\varepsilon} \times \mathbf{0})(x,0)$ for each $x \in K$,

(v) $\{\mathcal{F}^s(p)|p \in U\}$ is a positively invariant C^{r_2} family of manifolds.

With these definitions the following theorem is true.

Theorem 3.1 *Let Assumption (A) be fulfilled. Then:*

(i) *There is a C^r center manifold for $X^\varepsilon \times \mathbf{0}$ near K. There is a neighbourhood V of K such that, for all $p \in V$, if $\overline{\Psi_{X^\varepsilon \times \mathbf{0}}((\mathbb{R}), p)} \subseteq V$, then $\overline{\Psi_{X^\varepsilon \times \mathbf{0}}((\mathbb{R}), p)} \subseteq \mathcal{C}$.*

(ii) *There is a C^{r-1} family $\mathcal{F}^s(p)$ of C^r stable manifolds for U near K. If $p \in M \times \{\varepsilon\}$, then $\mathcal{F}^s(p) \subseteq M \times \{\varepsilon\}$. Each manifold $\mathcal{F}^s(p)$ intersects $\mathcal{C}$ transversally in exactly one point.*

(iii) *Let $\sigma > 0$ such that, for all $x \in K$, the real parts of the eigenvalues of $QX^0(x)$ are less than $-\sigma$. Then there is a constant C such that if $p \in U$ and $q \in \mathcal{F}^s(p)$, then*

$$d(\Psi_{X^\varepsilon \times \mathbf{0}}(\tau, p), \Psi_{X^\varepsilon \times \mathbf{0}}(\tau, q)) \leq C e^{-\sigma\tau} d(p, q)$$

for all $\tau \geq 0$ such that $\Psi_{X^\varepsilon \times \mathbf{0}}([0, \tau], p) \subseteq U$. Moreover, if $\overline{\Psi_{X^\varepsilon \times \mathbf{0}}([0, \infty), p)} \subseteq U$, then

$$\mathcal{F}^s(p) = \{q \in U | d(\Psi_{X^\varepsilon \times \mathbf{0}}(\tau, p), \Psi_{X^\varepsilon \times \mathbf{0}}(\tau, q)) \leq C e^{-\sigma\tau} d(p, q) \text{ for all } \tau \geq 0\}.$$

(iv) *Define for $(x, \varepsilon) \in \mathcal{C}$*

$$X_{\mathcal{C}}(x, \varepsilon) := \begin{cases} \varepsilon^{-1} X^\varepsilon(x) \times \{0\} & \text{if } \varepsilon > 0, \\ X_R(x) \times \{0\} & \text{if } \varepsilon = 0. \end{cases}$$

Then $X_{\mathcal{C}}$ is a C^{r-1} vector field on $\mathcal{C}$ near $K \times \{0\}$.

This theorem is a specialized version of Theorem 9.1 in [3]. As an immediate consequence we have

Corollary 3.2 *Let the assumptions of Theorem 3.1 be fulfilled. Let $x \in \mathcal{E}$ be fixed. Let $K := \Psi_{X_{\mathcal{C}}}([0, T], (x, 0)) \subseteq \mathcal{C}$ for some $T > 0$. Then there exists a neighbourhood V of $(x, 0)$ such that, for all $q \in V$ with $\varepsilon > 0$, $\Psi_{\varepsilon^{-1} X^\varepsilon \times \mathbf{0}}(t, q) \in U$, $t \in [0, T]$, and there exist constants $C_1, C_2 > 0$ such that*

$$d(\Psi_{X_{\mathcal{C}}}(t, (x, 0)), \Psi_{X^\varepsilon \times \mathbf{0}}(t, q)) \leq C_1 e^{-\sigma t/\varepsilon} + C_2 \varepsilon.$$

Proof. Note first that $K \subseteq \mathcal{E}$. Let $q = (y, \varepsilon) \in U$. Then there exists exactly one $p (= (\bar{y}, \varepsilon)) \in \mathcal{C}$ such that $q \in \mathcal{F}^s(p)$. Then we have by definition of $X_{\mathcal{C}}$

$$\begin{aligned} & d(\Psi_{X_{\mathcal{C}}}(t, (x, 0)), \Psi_{X^\varepsilon \times \mathbf{0}}(t, q)) \\ & \quad \leq \; d(\Psi_{X_{\mathcal{C}}}(t, (x, 0)), \Psi_{X_{\mathcal{C}}}(t, p)) + d(\Psi_{X^\varepsilon \times \mathbf{0}}(t/\varepsilon, p), \Psi_{X^\varepsilon \times \mathbf{0}}(t/\varepsilon, q)) \\ & \quad \leq \; C_2 \varepsilon + C_1 e^{-\sigma t/\varepsilon} \end{aligned}$$

for q sufficiently close to $(x, 0)$. □

For more details we refer to the original paper [3]. Compare also [15]. If we consider initial value problems for (3.1), we have additional initial conditions

$$y(0) = \eta(\varepsilon), \quad z(0) = \zeta(\varepsilon). \tag{3.15}$$

Now $x \in \mathcal{E}$ represents the fact that $G(\eta(0), \zeta(0), 0) = 0$. Since, under our assumptions, G_z is nonsingular, this means nothing else than dropping the initial condition for z if we consider the reduced problem. The manifold $\mathcal{F}^s(\eta(0), \zeta(0), 0)$ represents just the set of initial values $(y(0), z(0))$ which converge after the rapid transient phase to the solution of the reduced problem determined by $(\eta(0), \zeta(0))$. In the simple case (3.15), $z(0)$ can be chosen arbitrarily while $y(0)$ is restricted to be $\eta(0)$.

4 The Geometric Convergence Approach

Consider again the system

$$\begin{aligned} \dot{y} &= f(y,z) && (4.1a) \\ 0 &= g(y,z) && (4.1b) \end{aligned}$$

subject to Assumption (S2) together with the parametrized form

$$\begin{aligned} \dot{y} &= f(y,z) && (4.2a) \\ 0 &= g(y+\varepsilon f(y,z), z). && (4.2b) \end{aligned}$$

Reducing the index of both systems by 1 via differentiation yields now

$$\begin{aligned} \dot{y} &= f && (4.3a) \\ g_z \dot{z} &= -g_y f && (4.3b) \end{aligned}$$

and

$$\begin{aligned} \dot{y} &= f && (4.4a) \\ [g_z(y+\varepsilon f, z) + \varepsilon g_y(y+\varepsilon f, z) f_z]\dot{z} &= -g_y(y+\varepsilon f, z)(I+\varepsilon f_y)f, && (4.4b) \end{aligned}$$

respectively. Here we used again the convention to drop the arguments of the various functions if they are (y, z). The transformation $\tau = t/\varepsilon$ in (4.4) gives $(\cdot' \equiv d/d\tau)$

$$\begin{aligned} y' &= \varepsilon f && (4.5a) \\ [g_z(y+\varepsilon f, z) + \varepsilon g_y(y+\varepsilon f, z) f_z] z' &= -\varepsilon g_y(y+\varepsilon f, z)(I+\varepsilon f_y)f. && (4.5b) \end{aligned}$$

Lemma 4.1 *Let $M = \mathbb{R}^m \times \mathbb{R}^n$. Then it holds:*

(i) For $0 < \varepsilon < \varepsilon_0$, (4.5) induces a smooth vector field

$$X^\varepsilon : (y,z) \mapsto \begin{pmatrix} \varepsilon f \\ -\varepsilon[g_z(y+\varepsilon f, z) + \varepsilon g_y(y+\varepsilon f, z) f_z]^{-1} g_y(y+\varepsilon f, z)(I+\varepsilon f_y) f \end{pmatrix} \quad (4.6)$$

on M.

(ii) $X^\varepsilon \times \mathbf{0}$ *can be extended to a* C^1 *vector field on* $M \times [0, \varepsilon_0)$*, where*

$$X^0(y,z) = \begin{pmatrix} 0 \\ -Q_D\Delta^{-1}g_y f \end{pmatrix}. \tag{4.7}$$

(iii) The manifold $\mathcal{E}$ *of equilibrium points is given by*

$$\mathcal{E} = \{(y,z)|Q_D\Delta^{-1}g_y f = 0\}.$$

(iv) $QX^0(y,z)$ *has only the eigenvalue* -1 *for all* $(y,z) \in \mathcal{E}$*.*

(v) The reduced vector field X_R *according to (3.14) is given by*

$$X_R(y,z) = \begin{pmatrix} f \\ -(I-Q_s)\Delta^{-1}g_y f - \frac{\partial}{\partial y}(Q_D\Delta^{-1}g_y f)f \end{pmatrix}, \tag{4.8}$$

where $Q_s = Q_s(y,z)$ *denotes the projection*

$$Q_s = Q_D\Delta^{-1}\{g_y f_z + g_{yz}(f,\cdot) - g_{zz}(\Delta^{-1}g_y f,\cdot)\} \tag{4.9}$$

which projects $\mathbb{R}^n$ *onto the nullspace of* g_z*.*

We postpone the proof of Lemma 4.1 to the next section.

Remark 4.2 *(i) As the proof of Lemma 4.1 will show, the smoothness of* X^ε *is expected to be even higher than* C^1 *also at* $\varepsilon = 0$ *and to depend only on the smoothness of* f *and* g*.*

(ii) The manifold of equilibrium points $\mathcal{E}$ *is just the constraint manifold of the index 1 system (4.3). This follows easily by the results of [20] and*

$$Q_D\Delta^{-1}g_y f = 0 \text{ iff } g_y f \in \mathcal{R}(g_z).$$

The latter equivalence will be shown in Section 5.

(iii) The projection Q_s *plays a similar role as the so-called canonical projection introduced by Griepentrog and März [10, 18] for linear dae's. Moreover, if* g *is linear,* Q_s *reduces essentially to that projection. Therefore, we will call* Q_s *of (4.9) a canonical projection again.*

(iv) The reduced equation of (4.4) is just (4.3). (4.3b) is equivalent to $(I-Q_D)\dot{z} = -\Delta^{-1}g_y f$*. Since* $Q_sQ_D = Q_D$ *(cf. Lemma 4.1 (v)),*

$$(I-Q_s)\dot{z} = -(I-Q_s)\Delta^{-1}g_y f.$$

Compare this with the second equation of (4.8), which reads

$$\dot{z} = -(I-Q_s)\Delta^{-1}g_y f - \frac{\partial}{\partial y}(Q_D\Delta^{-1}g_y f)f. \tag{4.10}$$

In contrast to $(I-Q_s)\Delta^{-1}g_y f$*, the latter is a vector field over* $\mathcal{E}$*. The second term of the right-hand side of (4.10) takes into account changes in* $Q_s z$ *due to the curvature of* $\mathcal{E}$*.*

The convergence theorem for (4.2) is now an immediate consequence of Corollary 3.2 and Lemma 4.1.

Theorem 4.3 *Let $(y(t), z(t))$ be a solution of (4.1) for $t \in [0,T]$. Let $y_{0,\varepsilon}$, $z_{0,\varepsilon}$ be sufficiently close to $(y(0), z(0))$ such that $g(y_{0,\varepsilon} + \varepsilon f(y_{0,\varepsilon}, z_{0,\varepsilon}), z_{0,\varepsilon}) = 0$. Then there exists a unique solution $(y_\varepsilon(t), z_\varepsilon(t))$ of (4.2) on $[0,T]$ for ε sufficiently small, and it holds*

$$|(y_\varepsilon(t), z_\varepsilon(t)) - (y(t), z(t))| \le C_1 e^{-\sigma t/\varepsilon} + C_2\varepsilon$$

for some constants C_1, $C_2 > 0$, $0 < \sigma < 1$.

5 Proof of Lemma 4.1

The first assertion of Lemma 4.1 depends on the invertibility of the matrix in brackets on the left-hand side of (4.5b). Once this is assured, the smoothness of (4.6) follows from that of f and g. With the exception of a factor of ε, (4.5b) is equivalent to (2.5). Hence, we can use the same trick and represent the system using the matrix H_ε. By Lemma 2.4, for sufficiently small $\varepsilon > 0$, (4.6) is true and

$$z' = -\varepsilon \left(\frac{1}{\varepsilon} I, I\right) H_\varepsilon^{-1} \begin{pmatrix} g_y(y+\varepsilon f, z)(I + \varepsilon f_y) f \\ 0 \end{pmatrix}. \tag{5.1}$$

Taking the limit of (5.1) for $\varepsilon \to 0$ we obtain the expression

$$\begin{aligned} & -(I,0)H_0^{-1}\begin{pmatrix} g_y f \\ 0 \end{pmatrix} \\ = \; & -(I,0)\begin{pmatrix} Q_D\Delta^{-1} g_y f \\ (I - Q_D)\Delta^{-1} g_y f - \Delta^{-1} g_{yz}(f, Q_D\Delta^{-1} g_y f) \end{pmatrix} \\ = \; & -Q_D \Delta^{-1} g_y f. \end{aligned} \tag{5.2}$$

Thus, (4.6), (4.7) define a C^0 vector field on $M \times [0, \varepsilon_0)$. It is now a straightforward but tedious calculation to show that the derivative of (5.1) has a limit for $\varepsilon \to 0$ which is just the derivative of (5.2). We omit it. Assertion (iii) is now a consequence of (4.7).

The next step is the computation of $TX^0(x)$ for $x \in \mathcal{E}$. It holds

$$DX^0(x) = \begin{pmatrix} 0 & 0 \\ -\frac{\partial}{\partial y}(Q_D\Delta^{-1} g_y f) & -\frac{\partial}{\partial z}(Q_D\Delta^{-1} g_y f) \end{pmatrix} \tag{5.3}$$

subject to the condition $x = (y,z) \in \mathcal{E}$. The eigenvalues of (5.3) are 0 (m-fold) and the eigenvalues of $-\frac{\partial}{\partial z}(Q_D\Delta^{-1} g_y f)$. We compute

$$\begin{aligned} & -\frac{\partial}{\partial z}(Q_D\Delta^{-1} g_y f) \\ = \; & -Q_{D,z}(\Delta^{-1} g_y f, \cdot) + Q_D\Delta^{-1}\left(\frac{\partial}{\partial z}\Delta\right)\Delta^{-1} g_y f - Q_D\Delta^{-1} g_{yz}(f, \cdot) - Q_D\Delta^{-1} g_y f_z. \end{aligned} \tag{5.4}$$

Since $(y,z) \in \mathcal{E}$, $Q_D\Delta^{-1}g_y f = 0$. Moreover, since $\mathcal{R}(g_z\Delta^{-1}) = \mathcal{R}(g_z)$ and

$$g_z\Delta^{-1}g_z\Delta^{-1} = g_z\Delta^{-1}(g_z + g_y f_z Q_D)(I - Q_D)\Delta^{-1} = g_z\Delta^{-1},$$

$g_z\Delta^{-1}$ is a projection onto $\mathcal{R}(g_z)$. Taking into account that $Q_D\Delta^{-1} = \Delta^{-1}(I - g_z\Delta^{-1})$ we arrive at

$$(y,z) \in \mathcal{E} \text{ iff } g_y f \in \mathcal{R}(g_z). \tag{5.5}$$

Let now $\alpha \in I\!R^n$ such that $g_y f = g_z\alpha$. Then

$$\Delta^{-1}g_y f = \Delta^{-1}g_z\alpha = \Delta^{-1}(g_z + g_y f_z Q_D)(I - Q_D)\alpha = (I - Q_D)\alpha. \tag{5.6}$$

Moreover,

$$0 \equiv \frac{\partial}{\partial z}[Q_D(I - Q_D)] = Q_{D,z}(I - Q_D) - Q_D Q_{D,z}. \tag{5.7}$$

Furthermore,

$$\frac{\partial}{\partial z}\Delta = g_{zz} + g_{yz}(f_z Q_D\cdot,\cdot) + g_y f_{zz}(Q_D\cdot,\cdot) + g_y f_z Q_{D,z}$$

such that, by (5.6),

$$\begin{aligned}
&\left(\frac{\partial}{\partial z}\Delta\right)\Delta^{-1}g_y f \\
&= g_{zz}(\Delta^{-1}g_y f,\cdot) + g_{yz}(f_z Q_D(I - Q_D)\alpha,\cdot) + g_y f_{zz}(Q_D(I - Q_D)\alpha,\cdot) + \\
&\quad g_y f_z Q_{D,z}((I - Q_D)\alpha,\cdot) \\
&= g_{zz}(\Delta^{-1}g_y f,\cdot) + g_y f_z Q_{D,z}((I - Q_D)\alpha,\cdot).
\end{aligned}$$

Now (5.4) leads to

$$\begin{aligned}
&-\frac{\partial}{\partial z}(Q_D\Delta^{-1}g_y f) \\
&= -(I - Q_D\Delta^{-1}g_y f_z)Q_{D,z}((I - Q_D)\alpha,\cdot) + \\
&\quad Q_D\Delta^{-1}\{g_{zz}(\Delta^{-1}g_y f,\cdot) - g_{yz}(f,\cdot) - g_y f_z\}.
\end{aligned} \tag{5.8}$$

Since $Q_D\Delta^{-1}g_y f_z Q_D = Q_D\Delta^{-1}(g_z + g_y f_z Q_D)Q_D = Q_D$, we see by (5.7) that the first term of (5.8) vanishes. Defining

$$Q_s := Q_D\Delta^{-1}\{g_y f_z + g_{yz}(f,\cdot) - g_{zz}(\Delta^{-1}g_y f,\cdot)\}$$

we see that (5.8) is now

$$-\frac{\partial}{\partial z}(Q_D\Delta^{-1}g_y f) = -Q_s.$$

Obviously, $\mathcal{R}(Q_s) \subseteq \mathcal{R}(Q_D)$. Moreover, by Lemma 2.2

$$\begin{aligned}
g_{yz}(f, Q_D\alpha) &= g_z\beta \\
g_{zz}(\Delta^{-1}g_y f, Q_D\alpha) &= g_z\gamma
\end{aligned}$$

for some β, γ. Since $Q_D\Delta^{-1}(g_z + g_y f_z Q_D)(I - Q_D) = 0$, we obtain $Q_s Q_D = Q_D\Delta^{-1}g_y f_z Q_D = Q_D$. Similarly, $Q_s^2 = Q_s$. Therefore, Q_s is a projection onto $\mathcal{N}(g_z)$.

Since $T_{(y,z)}\mathcal{E} \subseteq \mathcal{N}(DX^0(y,z)$ and $\dim\, T_{(y,z)}\mathcal{E} = m+n-\dim\,\mathcal{N}(g_z)$, the remaining eigenvalues of $QX^0(y,z)$ must be -1 according to the dimension of $Q_s(y,z)$. Thus (iv) is proved.

It remains to show (4.8). First compute $\Pi^{\mathcal{E}}$. (5.3) has the eigenvalues 0 and -1 and there are no generalized eigenvectors to the eigenvalue 0. This means that $\mathcal{N}(DX^0(x)^2) = \mathcal{N}(DX^0(x))$. Denote for short $A = -\frac{\partial}{\partial y}(Q_D\Delta^{-1}g_y f)$. Then

$$DX^0(x)^2 = \begin{pmatrix} 0 & 0 \\ -QA & Q \end{pmatrix}.$$

This leads to $A\alpha - Q\beta = 0$ iff $(\alpha,\beta) \in \mathcal{N}(DX^0(x))$ iff $(\alpha,\beta) \in \mathcal{N}(DX^0(x)^2)$ iff $-QA\alpha + Q\beta = 0$. Hence, $A = QA$. The geometric eigenspace to the eigenvalue -1 is $L := \{(\alpha,\beta) | \alpha = 0, (I - Q_s)\beta = 0\}$. By a dimension argument, there are no generalized eigenvectors such that the geometric and the algebraic eigenspace (the latter being just the invariant subspace corresponding to the eigenvalue -1) coincide. Hence, $\Pi^{\mathcal{E}}$ is the projection onto $\mathcal{N}(\Pi^{\mathcal{E}}) = L$. This yields

$$\Pi^{\mathcal{E}} = \begin{pmatrix} I & 0 \\ A & I - Q_s \end{pmatrix}. \tag{5.9}$$

Indeed, because of $A = QA$, $\Pi^{\mathcal{E}}$ is a projection. Since $DX^0(x)\Pi^{\mathcal{E}} = 0$, $\mathcal{R}(\Pi^{\mathcal{E}}) \subseteq \mathcal{N}(DX^0(x))$. On the other hand, if $(\alpha,\beta) \in \mathcal{N}(DX^0(x))$, $A\alpha - Q\beta = 0$ and $\Pi^{\mathcal{E}}(\alpha,\beta) = (\alpha,\beta)$ such that $(\alpha,\beta) \in \mathcal{R}(\Pi^{\mathcal{E}})$. Consequently, $\mathcal{R}(\Pi^{\mathcal{E}}) = \mathcal{N}(DX^0(x))$. Moreover, since $\Pi^{\mathcal{E}}(\alpha,\beta) = 0$ for all $(\alpha,\beta) \in L$, a dimensional argument yields $\mathcal{N}(\Pi^{\mathcal{E}}) = L$.

According to (3.14) we must now compute

$$X^R(x) = \Pi^{\mathcal{E}} \frac{\partial}{\partial \varepsilon} X^\varepsilon(x)\Big|_{\varepsilon=0}$$

for $x \in \mathcal{E}$. Denote

$$Z^\varepsilon := -\varepsilon T_\varepsilon^{-1} g_y(y + \varepsilon f, z)(I + \varepsilon f_y) f \tag{5.10}$$

with

$$T_\varepsilon := g_z(y + \varepsilon f, z) + \varepsilon g_y(y + \varepsilon f, z) f_z.$$

Then (4.2) yields

$$X^R(x) = \begin{pmatrix} f \\ -\frac{\partial}{\partial y}(Q_D\Delta^{-1}g_y f) f + (I - Q_s)\, \frac{\partial}{\partial \varepsilon} Z^\varepsilon(x)\Big|_{\varepsilon=0} \end{pmatrix}. \tag{5.11}$$

Now,

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}Z^\varepsilon &= -T_\varepsilon^{-1}g_y(y+\varepsilon f,z)(I+\varepsilon f_y)f \\ &\quad +\varepsilon T_\varepsilon^{-1}\frac{\partial}{\partial\varepsilon}T_\varepsilon T_\varepsilon^{-1}g_y(y+\varepsilon f,z)(I+\varepsilon f_y)f \\ &\quad -\varepsilon T_\varepsilon^{-1}g_{yy}(y+\varepsilon f,z)((I+\varepsilon f_y)f,f) \\ &\quad -\varepsilon T_\varepsilon^{-1}g_y(y+\varepsilon f,z)f_yf. \qquad (5.12)\end{aligned}$$

Denote the terms on the right-hand side of (5.12) by t_1,t_2,t_3,t_4. Using the representation

$$T_\varepsilon^{-1}\alpha = \left(\frac{1}{\varepsilon}I,I\right)H_\varepsilon^{-1}\begin{pmatrix}\alpha\\0\end{pmatrix}$$

induced by Lemma 2.4 we obtain

$$\begin{aligned}(I-Q_s)t_4|_{\varepsilon=0} &= -(I-Q_s)(I,0)H_0^{-1}\begin{pmatrix}g_yf_yf\\0\end{pmatrix} \\ &= -(I-Q_s)Q_D\Delta^{-1}g_yf_yf \\ &= 0 \qquad (5.13)\end{aligned}$$

because of $Q_sQ_D=Q_D$, and

$$\begin{aligned}(I-Q_s)t_3|_{\varepsilon=0} &= -(I-Q_s)(I,0)H_0^{-1}\begin{pmatrix}g_{yy}(f,f)\\0\end{pmatrix} \\ &= -(I-Q_s)Q_D\Delta^{-1}g_{yy}(f,f) \\ &= 0. \qquad (5.14)\end{aligned}$$

Now we consider, for $\alpha\in I\!R^n$, the limit of $(I-Q_s)T_\varepsilon^{-1}\alpha$. By Lemma 2.4,

$$\begin{aligned}&(I-Q_s)T_\varepsilon^{-1}\alpha \\ &= (I-Q_s)\left(\frac{1}{\varepsilon}I,I\right)H_\varepsilon^{-1}\begin{pmatrix}\alpha\\0\end{pmatrix} \\ &= (I-Q_s)\left(\frac{1}{\varepsilon}I,I\right)\left(I-H_0^{-1}(H_0-H_\varepsilon)\right)^{-1}H_0^{-1}\begin{pmatrix}\alpha\\0\end{pmatrix} \\ &= (I-Q_s)\left(\frac{1}{\varepsilon}I,I\right)\times \\ &\quad \left\{\begin{pmatrix}I&0\\0&I\end{pmatrix}+\right. \\ &\quad \left.\begin{pmatrix}Q_D\Delta^{-1}M_1Q_D & Q_D\Delta^{-1}M_2\\ \{(I-Q_D)\Delta^{-1}-\Delta^{-1}g_{yz}(f,Q_D\Delta^{-1}\cdot)\}M_1Q_D & \{(I-Q_D)\Delta^{-1}-\Delta^{-1}g_{yz}(f,Q_D\Delta^{-1}\cdot)\}M_2\end{pmatrix}+O(\varepsilon^2)\right\}\times \\ &\quad \begin{pmatrix}Q_D\Delta^{-1}\alpha\\(I-Q_D)\Delta^{-1}\alpha-\Delta^{-1}g_{yz}(f,Q_D\Delta^{-1}\alpha)\end{pmatrix} \qquad (5.15)\end{aligned}$$

where M_1, $M_2=O(\varepsilon)$. Since $(I-Q_s)Q_D=0$, the limit $\varepsilon\to0$ of (5.15) can be taken and one obtains

$$(I-Q_s)T_\varepsilon^{-1}\alpha\to(I-Q_s)\left[\Delta^{-1}\alpha-\Delta^{-1}g_{yz}(f,Q_D\Delta^{-1}\alpha)\right].$$

With this expression,

$$\begin{aligned}(I-Q_s)t_1|_{\varepsilon=0} &= -(I-Q_s)\left[\Delta^{-1}g_y f - \Delta^{-1}g_{yz}(f, Q_D\Delta^{-1}g_y f)\right] \\ &= -(I-Q_s)\Delta^{-1}g_y f \qquad (5.16)\end{aligned}$$

since $\Delta^{-1}g_y f = (I-Q_D)\beta$ for some $\beta \in I\!R$ on $\mathcal{E}$. Applying (5.15) again we have

$$\begin{aligned}(I-Q_s)t_2|_{\varepsilon=0} &= (I-Q_s)[\Delta^{-1} - \Delta^{-1}g_{yz}(f, Q_D\Delta^{-1}\cdot)] \times \\ &\quad [g_{y,z}(f,\cdot) + g_y f_z](I,0)H_0^{-1}\begin{pmatrix} g_y f \\ 0 \end{pmatrix} \\ &= (I-Q_s)[\Delta^{-1} - \Delta^{-1}g_{yz}(f, Q_D\Delta^{-1}\cdot)] \times \\ &\quad [g_{y,z}(f,\cdot) + g_y f_z]Q_D\Delta^{-1}g_y f \\ &= 0 \qquad (5.17)\end{aligned}$$

on $\mathcal{E}$. (5.11), (5.13), (5.14), (5.16), (5.17) prove (v) of Lemma 4.1. Now the proof of Lemma 4.1 has been completed.

6 The Regularization of Quasilinear Dae's

While the main result is given by Theorem 4.3, we include this section for the sake of completeness. We are now in a position to derive an analogous result for the quasilinear dae

$$A(x)\dot{x} = g(x). \qquad (6.1)$$

Assumption (SL2).

(i) $A : I\!R^m \to \mathcal{B}(I\!R^m)$, $g : I\!R^m \to I\!R^m$ are sufficiently smooth.

(ii) The nullspace $\mathcal{N}(A(x))$ is independent of x.

(iii) The geometric index of (6.1) is 2.

Let Q denote a projection of $I\!R^m$ onto the nullspace of $A(x)$ and $P = I - Q$. The parametrization of (6.1) is now

$$A(x)\dot{x} = g(x + \varepsilon P\dot{x}). \qquad (6.2)$$

Lemma 6.1 *Let (SL2, i, ii) be true. Then there exists a nonsingular matrix $S \in \mathcal{B}(I\!R^m)$ such that*

$$PS = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \qquad (6.3)$$

Moreover, for every $x_0 \in I\!R^m$, there exist a neighbourhood U of x_0 and a smooth function $T : U \to \mathcal{B}(I\!R^M)$ such that $T(x)$ is nonsingular and

$$T^{-1}(x)A(x)S = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad (6.4)$$

for all $x \in U$.

Proof. Let $s_{k+1}, \ldots, s_m$ be a basis of $\mathcal{N}(A(x))$ and $s_1, \ldots, s_k$ a basis of $\mathcal{N}(Q)$. Hence, $s_1, \ldots, s_m$ is a basis of $I\!R^m$ and $S = (s_1, \ldots, s_m)$ is nonsingular. It is easy to see that (6.3) holds. Furthermore,

$$A(x)S = A(x)PS = \begin{pmatrix} A_1(x) \\ 0 \end{pmatrix},$$

where $A_1(x)$ has full rank k. $T(x)$ can now be constructed using e. g. the Gram-Schmidt procedure. Hence (6.4). □

Let now S be chosen according to Lemma 6.1 and

$$\overline{x} = \begin{pmatrix} y \\ z \end{pmatrix} = S^{-1}x. \tag{6.5}$$

By (6.4), (6.1) and (6.2) are locally equivalent to the semiexplicit systems

$$\begin{pmatrix} \dot{y} \\ 0 \end{pmatrix} = T^{-1}(y,z)g(y,z) \tag{6.6}$$

and

$$\begin{pmatrix} \dot{y} \\ 0 \end{pmatrix} = T^{-1}(y,z)g(y + \varepsilon\dot{y}, z), \tag{6.7}$$

respectively. This is nothing else but (2.3). It is easy to see that (SL2) implies now (S2) since the scaling by T^{-1} and the constant coordinate change (6.5) do not have any influence on the index. As an immediate consequence of Theorem 4.3 we obtain

Theorem 6.2 *Let (SL2) be fulfilled. Let $x(t)$ be a solution of (6.1) for $t \in [0,T]$. Assume that there exists a neighbourhood U of $\{x(t)|t \in [0,T]\}$ such that a smooth $T(x)$ with (6.4) can be chosen. Let $x_{0,\varepsilon}$ be a consistent initial value for (6.2) sufficiently close to $x(0)$. Then there exists a unique solution $x_\varepsilon(t)$ of (6.2) on $[0,T]$ for ε sufficiently small, and it holds*

$$|x_\varepsilon(t) - x(t)| \leq C_1 e^{-\sigma t/\varepsilon} + C_2\varepsilon$$

for some constants C_1, $C_2 > 0$, $0 < \sigma < 1$.

7 Conclusions

In the preceding sections we proved the convergence of a regularization method for quasilinear index 2 differential-algebraic equations

$$A(x)\dot{x} = g(x) \tag{7.1}$$

provided that the nullspace of $A(x)$ does not depend on x. Although our method of proof was purely analytical, the essential ingredient was a deep result of a differential geometric singular perturbation theory. As a consequence of our considerations in Section 2 it turned out that the classical Tikhonov theory is not sufficient to prove a convergence result unless the nullspace of $g_z(y,z)$ does not depend on (y,z). Therefore, a revision of the three steps of the proof within the context of a differential geometric approach seems to be appropriate:

I) The first step is an index reduction by differentiation of the algebraic constraints. For this, Reich [22] suggested a geometric theory considering a dae as a differential equation on a manifold. A deeper analysis of this idea is given in [20]. The first step of this index reduction process can be formulated as follows.

Consider the autonomous dae

$$F(x, \dot{x}) = 0, \tag{7.2}$$

where $x \in I\!\!R^m$. If $x : I \to I\!\!R^m$ is a (continuously differentiable) solution of (7.2), $(x(t), \dot{x}(t))$ must belong to the set $M_1 := \{(x,p) \in I\!\!R^m \times I\!\!R^m | F(x,p) = 0\}$. Assume that M_1 is a differentiable manifold. Denote the projection onto the first component by $\pi : M \to I\!\!R^m$. If $W_1 := \pi(M_1)$ is a submanifold of $I\!\!R^m$ (a so-called constraint manifold), the tangent space $T_x W_1$ for every $x \in W_1$ is well-defined. Hence, $\dot{x}(t) \in T_{x(t)} W_1$. Therefore, not only $(x(t), \dot{x}(t)) \in M_1$ has to hold, but additionally

$$(x(t), \dot{x}(t)) \in M_1 \cap TW_1 \tag{7.3}$$

with the tangent bundle TW_1 of W_1. Instead of (7.2) a relation like (7.3) is now called a dae. More precisely, for a given differentiable manifold M, a relation $(x, \dot{x}) \in M$ is called a dae. If M is the preimage of 0 of some smooth mapping $F : I\!\!R^m \times I\!\!R^m \to I\!\!R^m$, the dae reads $F(x, \dot{x}) = 0$. (7.2) and (7.3) can be stated in the following way:

(7.2) Find $x : I \to I\!\!R^m$ continuously such that $x(t), \dot{x}(t)) \in M_1$.

(7.3) Find $x : I \to I\!\!R^m$ continuously such that $x(t), \dot{x}(t)) \in M_1 \cap TW_1$.

Obviously, if $M_2 := M_1 \cap TW_1$ is a manifold again, this process can be repeated. It will stop if, for some integer j, $M_{j+1} = M_j$. The least index j with this property is called the geometric index of (7.2). Obviously, if the dae $(x, \dot{x}) \in M_1$ has the geometric index j, then the dae $(x, \dot{x}) \in M_2$ has the geometric index $j-1$. In this way the successive computation of M_i describes an index reduction process. Analytically, this is just index reduction by differentiation. One can show that if $M_{j+1} = M_j$, $\pi\big|_{M_j} : M_j \to W_j$ is (locally) a diffeomorphism and, therefore, there is a smooth function ϕ defined on $V \subseteq W_j$ such that $\left(\pi\big|_{M_j}\right)^{-1} : V \subseteq W_j \to U \subseteq M_j$ has the representation $\left(\pi\big|_{M_j}\right)^{-1}(x) = (x, \phi(x))$. Since $M_j = M_{j+1} = M_j \cap TW_j$, ϕ is a vector field on W_j. In other words, $(x, \dot{x}) \in M_j$ is equivalent to $\dot{x} = \phi(x)$, $x \in M_j$, i. e. a differential equation on the manifold M_j.

II) Let now (7.2) be an index 2 dae. For small parameters $\varepsilon \neq 0$ consider dae's

$$F_\varepsilon(x, \dot{x}) = 0. \tag{7.4}$$

Assume that (7.4) has index 1. (7.2) describes a differential equation on the manifold M_2 whereas (7.4) is a differential equation on $M_1(\varepsilon)$. Simulating the process of Section 2 we have to consider

$$(x, \dot{x}) \in M_2 \tag{7.5}$$

and

$$(x, \dot{x}) \in M_1(\varepsilon), \tag{7.6}$$

respectively, where the latter should be a singular perturbation of the first one. By (I), vector fields on M_2 and $M_1(\varepsilon)$, respectively, are defined.

III) Unfortunately, the theory of Section 3 does not apply to (7.5), (7.6) since $M_1(\varepsilon) \not\supseteq M_2$ in general. Therefore, the following questions must be answered in order to give a complete geometric characterization of the regularization in dae's:

1. Which relations exist between M_2 and $M_1(\varepsilon)$?
2. Can Fenichel's theorem be generalized to this situation?

Answers even for the special case (7.1) with the nullspace of $A(x)$ depending on x would be a considerable progress.

Investigations about both questions are just under progress. I believe that the answers will give new insights into what will happen when regularizing differential-algebraic equations.

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