Index Reduction and Regularization for Euler-Lagrange Equations of Constraint Mechanical Systems

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Abstract

The equations of motion of multibody systems with holonomic constraints are of index 3 and therefore not directly solvable by standard ODE or DAE methods. Until now, a number of regularization methods have been proposed to treat these systems [1, 2, 3, 4], but as they are based on different points of view they have never been compared with respect to their physical properties, invariance under diffeormorphisms, and convergence behaviour. In this paper we try to relate these methods to each other.

1 Introduction

In this paper the equations of motion of mechanical systems modelled by multibody systems are considered. These are systems of rigid or elastic bodies connected by force elements such as springs and damp ers or joints. As the former lead to known forces the latter result in algebraic constraint equations and lead to unknown constraint forces.

Since during the last decade the need for automatic multibody simulators in industry (simulation of robots, vehicles, mechanisms) increased, much interesthas been spent in the numerical treatment of differential-algebraic equations describing the motion:

$$M(q)\ddot{q} = f(q,\dot{q},t) + G^T(q)\lambda \qquad (1.1a)$$
$$0 = g(q) \qquad (1.1b)$$

with the symmetric positive definite mass matrix M, the generalized coordinates q, the generalized velocities $\dot{q}$, the forces f (applied forces as well as Coriolis, centrifugal and gravitational forces). g are holonomic constraints[1], the constraint matrix $G = \frac{\partial g}{\partial q}$ is supposed to have full rank. The term $G^T\lambda$ is the constraint force, λ the Lagrange multiplier. It is well-known that the index of (1.1) is 3.

Higher index systems, i.e. equations with an index greater than or equal to 2, are known to include differentiation problems: The solution of an index k system depends on derivatives of the right-hand side up to order $k-1$. This leads to the definition of the perturbation index in [5]. This index definition is based on analyzing the dependence of the solution on perturbations of the right-hand side of the DAE. Roughly speaking, the system has index k if its solution depends on derivatives of the perturbation up to order $k-1$. Perturbations occur during the numerical integration e.g. from discretization, truncation and roundoff errors. This is the reason why the numerical treatment of higher index systems causes numerical difficulties when treated by standard ODE methods.

Therefore some sort of index reduction process is advisable. One of such possibilities is the reduction by differentiation of the constraints. Twofold differentiation of (1.1b) leads to an index 1 system which can be numerically integrated by well-known numerical methods (e.g. BDF, RK or extrapolation methods, for an overview see [6, 5, 7]). Unfortunately, the numerical solution of the index-reduced system does not fulfil the original constraints (1.1b) and, furthermore, because of error propagation, a drift-off from these constraints occurs.

Another difficulty with index reduction by differentiation is that the stability behaviour can completely be changed by this process. For the case of linear constant coefficient DAEs it has been shown in [8] that additional zero-eigenvalues with Jordan-blocks of order $k-1$ are introduced for index k systems, which lead to instability. Inructive examples of the nonlinear case can be found in [9, 10].

Moreover, the computational complexity of these differentiated systems grows rapidly.

An survey of alternative approaches (e. g. [11, 12, 13, 14]) to treat these systems with index

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[1]The methods can be easily extended to the case of nonholonomic constraints.

reduction by differentiation and using some sort of projection to fulfil the original constraints is given in [15]. In that paper a classification for projection methods into coordinate projection methods and derivative projection methods is given. Those methods are not considered here.

Regularization methods can be considered as another way of index reduction. In the following section we describe various regularization methods for multibody system equations. More precisely, these are methods proposed by Baumgarte [1], Lötstedt [2], Knorrenschild [3], and Hanke [16]. Another well-known method for parameterizing higher index systems is the *pencil regularization* [17, 18]. This method does not fit completely into the former methods since it yields ordinary differential equations immediately. Therefore, it will not be considered here, but see [4] for this topic. More details on the topic can be found in [19].

2 Regularization Methods

In this section various methods which can be interpreted as regularization methods are described.

The oldest method that can be interpreted as a regularization was introduced by Baumgarte [1]. Motivated from control theory he introduced stabilizing control terms into the twofold differentiated constraints (1.1b), i.e. he chose a linear combination of different derivatives of (1.1b)

$$\ddot{g} + 2\alpha\dot{g} + \beta^2 g = 0.$$

α and β are chosen such that $g \equiv 0$ is a stable solution of this equation: $\alpha > 0$. Often $\beta = \alpha$ is selected, which corresponds to the aperiodic limit case. This yields

$$M\ddot{q} = f + G^T\lambda \tag{2.1a}$$

$$0 = G\ddot{q} + G_q(\dot{q},\dot{q}) + 2\alpha G\dot{q} + \beta^2 g. \tag{2.1b}$$

One difficulty with the Baumgarte approach is how to choose the parameters α, β. This question was discussed in [20], but has not sufficiently cleared until now. Choosing α, β too large results in stiff ODEs and a great amount of computing time. The smaller they become, the smaller the stabilizing effect. Note that these systems are not stiff in the sense that the stepsize is restricted by stability requirements, but by accuracy requirements such that stiff integrators are not the way out.

The next approach considered here was introduced by Lötstedt [21, 22, 2] for constrained mechanical systems and, in a more general form, in [2] for index-2-systems. He introduces additional springs and dampers in order to fulfil the algebraic constraint resulting in

$$M\ddot{q} = f + G^T(Kg + DGv).$$

The spring and damper parameters K, D are chosen such that no oscillations occur.

In [22] he motivates his approach as a penalty technique applied to Hamilton's principle. Lötstedt's method can be regarded as a combination of Baumgarte's method with singular perturbations: In a first step, the algebraic constraint is replaced by

$$0 = \dot{g} + \kappa g \quad \Longleftrightarrow \quad 0 = G\dot{q} + \kappa g$$

such that $g \equiv 0$ is a stable solution again (i.e. $\kappa > 0$). Regularizing the resulting index 2 system as in [2] yields:

$$\varepsilon\lambda = -(G\dot{q} + \kappa g). \tag{2.2}$$

Solving this for λ and substituting into the differential equation implies

$$M\ddot{q} = f - \frac{1}{\varepsilon}G^T(G\dot{q} + \kappa g).$$

A third approach was proposed by Knorrenschild [3]. As he gives regularizations reducing the index by one, we have to apply his procedure twice, yielding a two-stage process. In a first step (1.1b) is replaced by

$$0 = g(q + \varepsilon\dot{q}, t) \tag{2.3}$$

and, in a second step, the resulting index 2 system is parameterized in a similar manner, i.e. (2.3) is replaced by

$$0 = g(q + \mu\dot{q} + \varepsilon(\dot{q} + \mu\ddot{q}))$$

resulting in

$$M\ddot{q} = f + G^T\lambda \tag{2.4a}$$

$$0 = g(q + (\varepsilon + \mu)\dot{q} + \mu\varepsilon M^{-1}(f + G^T\lambda)) \tag{2.4b}$$

which is an index 1 system.

From an analytical point of view, higher index systems are ill-posed problems in the sense of Hadamard in some naturally given topologies [4]. The problem of solving $Tx = q, \quad T : X \to Y$ is well-posed if existence and uniqueness of solution are guaranteed and the solution depends continuously on the data. This last point is not fulfilled in case of higher index systems, since derivatives are involved. This observation motivates the approach of Hanke [16]. He introduces parameters in order to arrive at an index 1 problem and to obtain a regularization in the sense of Tikhonov. Applying this approach to constrained mechanical systems yields

$$\begin{aligned} \dot q &= v + \mu \dot v && \text{(2.5a)}\\ M\dot v &= f + G^T\lambda && \text{(2.5b)}\\ 0 &= g(q + \varepsilon \dot q + \mu\varepsilon M^{-1}(f + G^T\lambda)). && \text{(2.5c)} \end{aligned}$$

In the original paper $\varepsilon = \mu$ was selected, but for some reasons it makes sense to choose $\varepsilon \neq \mu$ (see Section 3 below).

3 Properties of the Different Parameterizations

In this section, the different approaches introduced above in order to stabilize the algebraic constraints in the index reduction process are discussed. We show that the forces tangential to the manifold are the same, the methods differ only in orthogonal forces. The methods proposed in [3, 4] differ from [1] only in higher order terms of the parameters ε, μ. All these methods are invariant under diffeomorphisms in the coordinate space and scalings of g, this is not the case for (2.2).

In the following the matrix product

$$P = M^{-1}G^T(GM^{-1}G^T)^{-1}G. \qquad (3.1)$$

plays an important role.

Lemma 3.1 *Let P be given by (3.1). Then*

1. *$I - P$ is a projection onto the nullspace of G.*
2. *P is an orthogonal projection with respect to the scalar product*

$$< x, y >_M := x^T M y.$$

3. *P is invariant with respect to coordinate changes, i.e. if $T : U_{\bar q} \to U_q$ is a diffeomorhism and $\bar G(\bar q) = \frac{\partial}{\partial \bar q} g(T(\bar q))$, then $P = M^{-1}\bar G^T(\bar G M^{-1}\bar G)^{-1}\bar G$.*
   P is also invariant under scalings of g.

The invariance of P with respect to coordinate changes is a remarkable property. It is well-known that the Lagrangian calculus is independent of the special choice of the generalized coordinates q. More precisely, the motion of a system is independent of diffeormorphisms of the generalized coordinates. P has this property, too, which leads to the invariance of the equations of motion described now.

Proposition 3.2 *For the unperturbed system as well as for all parameterizat ions it holds*

$$(I - M^{-1}G^T(GM^{-1}G^T)^{-1}G)\dot v = (I - M^{-1}G^T(GM^{-1}G^T)^{-1}G)M^{-1}f \qquad (3.2)$$

or, equivalently,

$$(I - P)\dot v = (I - P)M^{-1}f.$$

As a consequence of Lemma 3.1 we see that $I - P$ is a projection onto the tangent space of the manifold

$$\mathcal{M} = \{q | g(q) = 0\}.$$

That means that (3.2) describes the dynamics of the system restricted to the manifold. Only those components of the forces contribute to a change of absolute value of the velocity which are parallel to the manifold. In this sense, additional accelerations perpendicular to the constraint manifold are a consequence of the curvature of this manifold. This can be seen from the next proposition:

Proposition 3.3 *For the constrained mechanical system it holds that*

$$P\dot v = -M^{-1}G^T(GM^{-1}G^T)^{-1}G_q(\dot q, \dot q) \qquad (3.3)$$

For the parameterized systems, the analogous properties are summarized in the following proposition.

Proposition 3.4

1. *For (2.1) (Baumgarte's method), it holds that*

$$\begin{aligned} P\dot v = M^{-1}G^T(GM^{-1}G^T)^{-1}G\ddot q &= \qquad (3.4)\\ -M^{-1}G^T(GM^{-1}G^T)^{-1}& \\ \times(G_q(\dot q, \dot q) + \alpha G\dot q + \beta g)& \qquad (3.5) \end{aligned}$$

2. *For (2.2) (Lötstedt's method), it holds that*

$$\begin{aligned} P\dot v &= M^{-1}G^T(GM^{1}G^T)^{-1}G\ddot q\\ &= -\underbrace{M^{-1}G^T(GM^{-1}G^T)^{-1}G}_{P} M^{-1}f\\ &\quad -\varepsilon^{-1}M^{-1}G^T(G\dot q + \kappa g) \end{aligned}$$

3. *For (2.4) (Knorrenschild's method), it holds that*

$$\begin{aligned} P\dot{v} &= M^{-1}G^T(GM^{-1}G^T)^{-1}G\ddot{q} \\ &= -M^{-1}G^T(GM^{-1}G^T)^{-1} \times \\ &\left(\frac{(\mu+\varepsilon)^2}{2\varepsilon\mu}G_q(\dot{q},\dot{q}) + \frac{\mu+\varepsilon}{\varepsilon\mu}G\dot{q} + \frac{1}{\varepsilon\mu}g\right) \\ &+R \end{aligned} \tag{3.6}$$

with

$$R = \mathcal{O}\left((\mu+\varepsilon)\|\dot{q}\|\|\ddot{q}\| + \mu\varepsilon\|\ddot{q}\|^2 + \frac{(\mu+\varepsilon)^2}{\varepsilon\mu}(\|\dot{q}\|^3 + \|\ddot{q}\|^3)\right).$$

4. *For (2.5) (Hanke's method), it holds that*

$$\begin{aligned} P\dot{v} &= M^{-1}G^T(GM^1G^T)^{-1}G\ddot{q} \\ &= -M^{-1}G^T(GM^{-1}G^T)^{-1} \\ &\quad \times\left(\frac{\varepsilon}{2\mu}G_q(\dot{q},\dot{q}) + \frac{1}{\mu}G\dot{q} + \frac{1}{\varepsilon\mu}g\right)(3.7)R \end{aligned} \tag{3.8}$$

with

$$R = \mathcal{O}\left(\varepsilon\|\dot{q}\|\|\ddot{q}\| + \varepsilon\mu\|\ddot{q}\|^2 + \frac{\varepsilon^2}{\mu}\|\dot{q}+\varepsilon\ddot{q}\|^3\right)$$

Obviously, Lötstedt's method is not independent with respect to coordinate changes as the others are.

A nice property of Baumgarte's approach is that $g = 0$ is a first integral in contrast to the others.

In the linearizations, Knorrenschild's and Hanke's method are almost identical. The close relationship between these methods and [1] can be seen by comparing (3.5) and (3.6),(3.8), respectively. They only differ by R, which is of higher order in ε, μ if $\varepsilon = \mathcal{O}(\mu), \mu = \mathcal{O}(\varepsilon)$.

Compared with (3.3) the parameters of (3.8) should satisfy $\varepsilon = 2\mu$. Comparing (3.3) and (3.6) would lead to $(\varepsilon+\mu)^2 = 2\varepsilon\mu$, which would yield $\varepsilon = \mu = 0$. Hence, in this case, a "canonical" choice of these parameters cannot be derived. In the other case damped oscillations are always be present in the solution.

4 Conclusion

Besides the physical interpretation, there remains the problem of convergence to be solved. It can be shown by the methods developed in [16, 4] that, under reasonable assumptions, the solutions of the regularized systems possess asymptotic expansions with respect to the (small) parameters ε, μ as long as $0 < c_1 \leq \frac{\varepsilon}{\mu} \leq c_2 < \infty$ (cf also [19]).

Although no attempt is made to compare these methods with respect to their numerical behaviour, all of them will lead to stiff equations causing a high amount of computing effort. As a way out of this trouble the asymptotic expansions can be used for extrapolation $\varepsilon \to 0$ [4]. The question how they can be used to give error estimates for the difference between the solutions of the perturbed and unperturbed systems is still under investigation.

Another way out is the integration of the index-reduced DAE together with some stabilizing projection [15]. This will not introduce extra stiffness into the equations. In this case the constraint equations are not fulfilled asymptotically, i.e. for $t \to \infty$, but

without time delay. However, we have made no attempt to compare these methods numerically.

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