Asymptotic and Exponential Stability of a General Class of Continuous-Time Power Control Laws in Wireless Networks

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Abstract— **This paper develops a comprehensive stability analysis framework for continuous-time power control algorithms in wireless networks under bounded time-varying communication delays. Our first set of results establish global asymptotic stability of power control laws involving two-sided scalable interference functions, and include earlier work on standard interference functions as a special case. We then consider contractive interference functions and demonstrate that the associated continuous-time power control laws always have unique fixed points, which are exponentially stable even in the presence of bounded heterogeneous time-varying delays. For this class of interference functions, we derive an explicit bound on the decay rate that allows us to quantify the impact of delays on the convergence time of the algorithm. Numerical simulations illustrate our theoretical results.**

I. Introduction

The seminal work by Foschini and Miljanic [1] on continuous- and discrete-time power control triggered off numerous publications for both continuous-time power control (e.g., [2]–[5]) and discrete-time power control (e.g., [6]–[8]). An elegant axiomatic framework for studying a general class of discrete-time power control, the so-called *standard interference functions*, was proposed by Yates [6]. This framework includes the discrete-time Foschini-Miljanic (FM) algorithm [1] and several nonlinear power control algorithms as special cases. This framework was later extended by Sung and Leung [7] to a more general class, the so-called *two-sided scalable interference functions*, to allow for simple and powerful analysis of certain opportunistic power control laws. While these frameworks are general and guarantee synchronous and asynchronous convergence of discrete-time power control algorithms in their class, the existence of fixed points has to be established separately, and there is no information about the convergence rate of the algorithms. Recently, a new framework, called *contractive interference functions*, was proposed by Feyzmahdavian *et al.* [8] that guarantees contractivity of the interference functions and hence unique fixed points, as well as linear convergence rates for discrete-time synchronous and asynchronous iterations.

While communication delays are inevitably omnipresent in networks, they have not been considered for continuous-time power control until recently. Charalambous *et al.* [2] using the multivariate Nyquist criterion showed that the continuous-time FM algorithm is asymptotically stable for arbitrary constant delays, while Zappavinga *et al.* [4] using theory for positive systems proved that the FM algorithm is asymptotically stable even in the presence of bounded time-varying communication delays and topology changes. The continuous-time counterpart of the standard interference functions firstly appeared in [9] and it was later analysed in [3] using Lyapunov-Razumikhin functions.

In this paper, we consider general classes of continuous-time distributed power control algorithms, which are continuous-time versions of those proposed in [7] and [8]. We analyze the continuous-time two-sided scalable interference functions (for which standard interference functions constitute a special case) and contractive interference functions. Firstly, we prove that if there exists a feasible steady state power vector, power-control laws involving two-sided scalable interference functions are asymptotically stable for arbitrary bounded heterogeneous time-varying delays. Next, for the contractive interference functions, we prove that associated continuous-time power control algorithms converge exponentially to the unique fixed points, even in the presence of bounded communication delays. For this class of interference functions, we derive an explicit bound on the decay rate, thus quantifying the impact of delays on the convergence time of the algorithm. The validity of our theoretical results is demonstrated via an illustrative example for the continuous-time version of the Utility-Based Power Control (UBPC) algorithm proposed by Xiao *et al.* [10].

The remainder of the paper is organized as follows. In Section II, we introduce the notation that will be used throughout the paper and review some required preliminaries that are useful for the development of the results in this paper. Section III gives a description of the problem under consideration, while the main results of this paper are stated in Sections IV and V. An illustrative example is presented in Section VI, justifying the validity of our results. Finally, concluding remarks are given in Section VII.

II. Notation and Preliminaries

A. Notation

Vectors are written in bold lower case letters and matrices in capital letters. We have $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{N}_0$ for the set of real numbers, natural numbers, and the set of natural numbers including zero, respectively. The non-negative orthant of the n-dimensional real space $\mathbb{R}^n$ is represented by $\mathbb{R}^n_+$. The i^{th} component of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is denoted by x_i, and the notation $\boldsymbol{x} \geq \boldsymbol{y}$ implies that $x_i \geq y_i$ for all components i. For a matrix $A \in \mathbb{R}^{n \times n}$, a_{ij} denotes the entry in row i and column j. The spectral radius of a matrix A is the largest

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magnitude of the eigenvalues of A and is denoted by $\rho(A)$. Given a vector $\boldsymbol{v} > \mathbf{0}$, the weighted l_∞ norm is defined by

$$\|\boldsymbol{x}\|_\infty^{\boldsymbol{v}} = \max_{1\le i\le n} \frac{|x_i|}{v_i}.$$

The function $\text{sgn}(x)$ is the *signum* function defined by $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = 0$. For a given $\tau_{\max} > 0$, $\mathcal{C}$ denotes the space of continuous functions mapping the interval $[-\tau_{\max}, 0]$ into $\mathbb{R}^n$. We also denote

$$\mathcal{C}_{++} = \left\{\varphi(\cdot) \in \mathcal{C} \mid \varphi(\theta) > 0,\ \theta \in [-\tau_{\max}, 0]\right\}.$$

B. Preliminaries

Next, we review the key definitions and results necessary for developing the main results of this paper. The following definition introduce *contraction* mappings.

Definition 1 *A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a* c-contraction *if there exists a constant $c \in [0, 1)$ such that*

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \le c\,\|\boldsymbol{x} - \boldsymbol{y}\|, \quad \forall x, y \in \mathbb{R}^n,$$

where $\|\cdot\|$ is some norm on $\mathbb{R}^n$.

An attractive feature of contraction mappings is that they always have a unique fixed point.

Proposition 1 ([11, Chapter 3]) *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a c-contraction, then f has a unique fixed point, i.e.,*

$$\exists\ \boldsymbol{x}^\star \in \mathbb{R}^n \quad \text{such that} \quad \boldsymbol{f}(\boldsymbol{x}^\star) = \boldsymbol{x}^\star.$$

III. Problem Statement

We consider a wireless system consisting of n mobile users communicating over the same frequency channel. The continuous-time power control algorithm, firstly considered in its general form by [9], is given by

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = k_i\big(-p_i(t) + I_i(\boldsymbol{p}(t))\big), \quad i = 1, \dots, n. \quad (1)$$

Here, $p_i(t)$ is the transmitted power of user i at time t, $I_i : \mathbb{R}_+^n \mapsto \mathbb{R}_+$ is the *interference function* modeling the interference and noise experienced by the intended receiver of user i, and k_i is a positive constant. The well-known continuous-time Foschini–Miljanic (FM) algorithm [1], for example, considers the *linear* interference function given by

$$I_i(\boldsymbol{p}) = \gamma_i \frac{\sum_{j\ne i} g_{ij}p_j + \eta_i}{g_{ii}}, \quad i = 1, \dots, n, \qquad (2)$$

where g_{ij} is the channel gain on the link between user j and the receiver of user i, γ_i is the target Signal-to-Interference-and-Noise Ratio (SINR) of user i, and η_i is the background noise at the receiver of user i.

In practice, there will always be a signalling delay associated with transmitting the perceived interference at the transmitter to the receiver, so that it can adjust the power according to the power control law. Consequently, a realistic analysis of the continuous-time power control algorithm must consider heterogeneous time-varying delays. More precisely, the continuous-time power control algorithm (1), when the time delay is introduced becomes

$$\begin{aligned} \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= k_i\big(-p_i(t) + I_i(\boldsymbol{p}^{d_i}(t))\big), \quad t \ge 0, \\ p_i(t) &= \varphi_i(t), \quad t \in [-\tau_{\max}, 0],\ i = 1, \dots, n. \end{aligned} \qquad (3)$$

Here, $\boldsymbol{\varphi}(\cdot) = [\varphi_1(\cdot), \dots, \varphi_n(\cdot)]^T$ is a continuous vector valued function specifying the initial condition of the system, and

$$\boldsymbol{p}^{d_i}(t) = [p_1(t - \tau_1^i(t)), \dots, p_n(t - \tau_n^i(t))]^T.$$

The delays $\tau_j^i(t)$ are assumed to be time-varying continuous functions with respect to t and satisfy

$$0 \le \tau_j^i(t) \le \tau_{\max}, \quad \forall i, j,\ t \ge 0,$$

where $\tau_{\max}$ is a positive constant providing an upper bound on the maximum allowable delay. Moreover, no restriction on the derivative of $\tau_j^i(t)$ is imposed.

To put our work in context, note that the discrete-time analog of (3) consists of iterations of the type

$$p_i(t+1) = I_i\big(\boldsymbol{p}^{d_i}(t)\big), \quad t \in \mathbb{N}_0. \qquad (4)$$

Such iterations have been studied under three frameworks; standard interference functions [6], two-sided scalable interference functions [7], and contractive interference functions [8]. Specifically, it was shown that if $\boldsymbol{I}(\boldsymbol{p})$ is a standard or two-sided scalable interference function, and a power vector $\boldsymbol{p}^\star$ satisfying $\boldsymbol{p}^\star = \boldsymbol{I}(\boldsymbol{p}^\star)$ exists, then the iterates generated by (4) converge asymptotically to $\boldsymbol{p}^\star$ [7, Theorem 12]. If $\boldsymbol{I}(\boldsymbol{p})$ is contractive, then the iterates (4) converges to a unique fixed point at a linear rate [8, Theorem 9], i.e., the distance between the iterates and the fixed point decays exponentially.

The aim of this paper is to develop a stability analysis framework for the continuous-time power control algorithm (3) that is equally comprehensive as the theory for its discrete-time counterpart. Our main objectives are therefore to (i) study the asymptotic stability of the continuous-time power control algorithm with heterogeneous time-varying delays described by (3) when the interference function $\boldsymbol{I}(\boldsymbol{p})$ is two-sided scalable; and to (ii) analyse the exponential stability of (3) when the interference function is contractive and determine how the convergence rate depends on the magnitude of the time delays.

IV. Standard and Two-sided Scalable Interference Functions

As observed by Yates [6], many interference functions share important properties that allow them to be analyzed in a common framework. This observation led to the definition of *standard interference functions*.

Definition 2 *A function $I : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is called a standard interference function, if for all $\boldsymbol{p} \ge \mathbf{0}$ the following properties are satisfied.*

- *Positivity:* $\boldsymbol{I}(\boldsymbol{p}) > \mathbf{0}$,

- *Monotonicity: If* $\boldsymbol{p} \geq \boldsymbol{p}'$*, then* $\boldsymbol{I}(\boldsymbol{p}) \geq \boldsymbol{I}(\boldsymbol{p}')$*,*
- *Scalability: For all* $\alpha > 1$*,* $\alpha \boldsymbol{I}(\boldsymbol{p}) > \boldsymbol{I}(\alpha \boldsymbol{p})$*.*

Sung and Leung [7] introduced the following generalized class of interference functions, which are useful for analyzing certain classes of opportunistic power control laws.

Definition 3 *A function* $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *is called a two-sided scalable interference function, if for all* $\boldsymbol{p} \geq \boldsymbol{0}$*,* $\boldsymbol{I}(\boldsymbol{p})$ *satisfies*

- *Positivity:* $\boldsymbol{I}(\boldsymbol{p}) > \boldsymbol{0}$*,*
- *Two-sided scalability: For all* $\alpha > 1$*,*

$$\frac{1}{\alpha}\boldsymbol{p} \leq \boldsymbol{p}' \leq \alpha \boldsymbol{p} \Rightarrow \frac{1}{\alpha}\boldsymbol{I}(\boldsymbol{p}) < \boldsymbol{I}(\boldsymbol{p}') < \alpha \boldsymbol{I}(\boldsymbol{p}).$$

Even though the monotonicity and scalability conditions of standard interference functions have been replaced by the two-sided scalability condition, one can show that every standard interference function is also two-sided scalable [7, proposition 4]. However, the following example shows that the converse is, in general, not true. Consider the function

$$I(p) = \begin{cases} \sqrt{p}, & 0 < p \leq 1, \\ \dfrac{1}{\sqrt{p}}, & p > 1, \end{cases}$$

which is two-sided scalable. However, monotonicity property does not hold and, hence, the function is not standard.

The main properties of two-sided scalable interference functions can be summarized as follows.

Proposition 2 ([7], [12]) *Let* $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *be a two-sided scalable interference function. Then,*

(a) $\boldsymbol{I}(\boldsymbol{p})$ *is continuous at* $\boldsymbol{p}$ *for all* $\boldsymbol{p} > \boldsymbol{0}$*.*
(b) *If* $\boldsymbol{I}(\boldsymbol{p})$ *has a fixed point, then that fixed point is unique.*
(c) *If* $\boldsymbol{p} \neq \boldsymbol{p}'$*, then* $d\big(\boldsymbol{I}(\boldsymbol{p}), \boldsymbol{I}(\boldsymbol{p}')\big) < d(\boldsymbol{p}, \boldsymbol{p}')$*, where*

$$d(\boldsymbol{p}, \boldsymbol{p}') = \max_{i=1,\dots,n} \left\{ \max\left\{ \frac{p_i}{p'_i}, \frac{p'_i}{p_i} \right\} \right\}. \tag{5}$$

We begin by studying the asymptotic stability of the continuous-time power control algorithm (3) with $\boldsymbol{I}(\boldsymbol{p})$ being a two-sided scalable interference function. Thanks to k_i being a positive constant and $I_i(\cdot)$ being positive, differential equation (3) defines a positive system[1] [14, Chapter 3]. Therefore, the physical constraint that the power should be nonnegative ($p_i(t) \geq 0$) is automatically fulfilled. The following theorem is our first key result, which shows that if a two-sided scalable interference function $\boldsymbol{I}(\boldsymbol{p})$ has a fixed point $\boldsymbol{p}^\star > \boldsymbol{0}$, then the solution to (3) converges asymptotically to $\boldsymbol{p}^\star$ for any initial condition $\boldsymbol{\varphi}(\cdot) \in \mathcal{C}_{++}$.

Theorem 1 *Suppose a two-sided scalable interference function* $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *has a fixed point* $\boldsymbol{p}^\star > \boldsymbol{0}$*. Then, the continuous-time power control algorithm* (3) *is asymptotically stable for any initial condition* $\boldsymbol{\varphi}(\cdot) \in \mathcal{C}_{++}$*, and for any proportionality constant,* $k_i > 0$*.*

[1] A dynamical system is said to be *positive* if every trajectory of the system starting from nonnegative initial conditions remains forever in the positive orthant [13]–[20].

Proof: See Appendix A. ∎

Remark 1 *The delay independence of continuous-time power control algorithms with standard interference functions was previously considered in [3, Theorem 4], using a Lyapunov-Razumikhin approach. Since every standard interference function is also two-sided scalable, Theorem 1 recovers the delay independence of standard interference functions as a special case.*

V. Contractive Interference Functions

Two-sided scalable interference functions do not necessarily have fixed points in the positive orthant (consider for example $I(p) = p + 1$), and the existence of fixed points has to be verified separately. Furthermore, no guarantees about the convergence rate of (3) are given. In [8], contractive interference functions, a slight modification of the standard interference functions, were introduced to allow for a more powerful analysis of distributed power control algorithms.

Definition 4 *A function* $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *is said to be a c-contractive interference function if it, for all* $\boldsymbol{p} \geq \boldsymbol{0}$*, satisfies the following conditions*

- *Positivity:* $\boldsymbol{I}(\boldsymbol{p}) > \boldsymbol{0}$*,*
- *Monotonicity: If* $\boldsymbol{p} \geq \boldsymbol{p}'$*, then* $\boldsymbol{I}(\boldsymbol{p}) \geq \boldsymbol{I}(\boldsymbol{p}')$*,*
- *Contractivity: There exists a constant* $c \in [0, 1)$*, and a vector* $\boldsymbol{v} > \boldsymbol{0}$ *such that for all* $\epsilon > 0$*,*

$$\boldsymbol{I}(\boldsymbol{p} + \epsilon \boldsymbol{v}) \leq \boldsymbol{I}(\boldsymbol{p}) + c\epsilon \boldsymbol{v}.$$

Contrary to the result for standard and two-sided scalable interference functions, contractive interference functions always define contraction mappings in the weighted l_∞ norm, hence $\boldsymbol{I}(\boldsymbol{p}) = \boldsymbol{p}$ has a unique solution in $\mathbb{R}^n_+$:

Proposition 3 ([8]) *If* $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *is a c-contractive interference function, then it has a unique fixed point* $\boldsymbol{p}^\star \in \mathbb{R}^n_+$*, and*

$$\|\boldsymbol{I}(\boldsymbol{p}) - \boldsymbol{I}(\boldsymbol{p}')\|^{\boldsymbol{v}}_\infty \leq c\|\boldsymbol{p} - \boldsymbol{p}'\|^{\boldsymbol{v}}_\infty, \quad \forall \boldsymbol{p}, \boldsymbol{p}' \in \mathbb{R}^n,$$

where $c \in [0, 1)$*.*

In [8], it was shown that the linear interference function (2), as well as many practical interference functions from the literature are contractive. In particular, all the examples provided in Yate's original paper [6] were shown to be contractive.

Throughout this section we will use the following concept of *exponential stability*.

Definition 5 *The solution* $\boldsymbol{p}(t) = \boldsymbol{p}^\star$ *of* (3) *is said to be globally exponentially stable if there exist positive reals* α *and* β *such that for every initial function* $\boldsymbol{\varphi}(\cdot)$*, the solution* $\boldsymbol{p}(t)$ *of* (3) *satisfies*

$$\|\boldsymbol{p}(t) - \boldsymbol{p}^\star\| \leq \beta e^{-\alpha t} \left(\sup_{-\tau_{\max} \leq s \leq 0} \|\boldsymbol{\varphi}(s) - \boldsymbol{p}^\star\| \right), \quad \forall t \geq 0.$$

Clearly, exponential stability implies asymptotic stability.

We will now show that if the interference function is c-contractive, then the continuous-time power control law described by (3) converges exponentially to the unique fixed point. Moreover, an explicit bound on the convergence rate of (3) is provided that allows us to quantify the impact of the magnitude of the time delays on the convergence rate. Note that contractive interference functions are also positive and hence, the physical constraint that the power should be nonnegative ($p_i(t) \geq 0$) is fulfilled.

Theorem 2 *Suppose $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is c-contractive. Then, the unique fixed point $\boldsymbol{p}^\star$ of the continuous-time power control algorithm* (3) *is exponentially stable for any initial condition $\boldsymbol{\varphi}(\cdot) \in \mathcal{C}_{++}$ and for any proportionality constant, $k_i > 0$. In particular, for all $t \geq 0$,*

$$\|\boldsymbol{p}(t) - \boldsymbol{p}^\star\|^{\boldsymbol{v}}_\infty \leq e^{-k_{\min}(1-\bar{c})t} \left(\sup_{-\tau_{\max} \leq s \leq 0} \|\boldsymbol{\varphi}(s) - \boldsymbol{p}^\star\|^{\boldsymbol{v}}_\infty \right),$$

where $k_{\min} = \min_{1 \leq i \leq n} k_i$, and $\bar{c}$ is the unique positive solution of the equation

$$\bar{c} = c\, e^{k_{\min}(1-\bar{c})\tau_{\max}}. \tag{6}$$

Proof: See Appendix B. ∎

Although Equation (6) does not admit an explicit solution, its solution $\bar{c}$ is always unique and satisfies

$$c \leq \bar{c} < 1.$$

Moreover, $\bar{c}$ is monotonically increasing with $\tau_{\max}$ and approaches one as $\tau_{\max}$ tends to infinity. Hence, while the power control law remains exponentially stable for arbitrary delays and proportionality constant, the convergence rate deteriorates with increasing delays

VI. Numerical Examples

We consider a continuous-time implementation of the Utility-Based Power Control (UBPC) algorithm from [10], whose associated interference function is

$$I^u_i(\mathbf{p}) = \left(\frac{\sum_{j \neq i} g_{ij} p_j + \eta_r}{g_{ii}} \right) f_i^{-1} \left(\alpha_i \frac{\sum_{j \neq i} g_{ij} p_j + \eta_r}{g_{ii}} \right),$$

where $i = 1, \ldots, n$, and α_i is a price coefficient. Here, $f_i(SIR_i) = U_i'(SIR_i)$ in the concave part of U_i where U_i is a utility function of user i. In their paper, Xiao *et al.* use a sigmoidal utility function

$$U_i(SIR_i) = \frac{1}{1 + e^{-a_i(SIR_i - b_i)}}, \tag{7}$$

where

$$b_i = \gamma_i - a_i^{-1} \ln(a_i \gamma_i - 1)\ . \tag{8}$$

Let us define $M^b = [m^b_{ij}]$ to be

$$m^b_{ij} = \begin{cases} b_i \frac{g_{ij}}{g_{ii}}, & j \neq i, \\ 0, & j = i. \end{cases}$$

In [8], it is shown that if $c = \rho(M^b) < 1$, then I^u is a c-contractive interference function. We will next show that Theorem 2 provides an explicit bound on the decay rate of UBPC algorithm that allows us to analytically quantify the impact of delays on the convergence rate of the algorithm.

Consider a wireless network with 4 users, characterized by matrix (9),

$$M = \begin{bmatrix} 0 & 0.3558 & 0.0354 & 0.1737 \\ 0.0522 & 0 & 0.0012 & 0.0229 \\ 0.2169 & 1.2433 & 0 & 0.5503 \\ 1.1938 & 0.8062 & 0.0206 & 0 \end{bmatrix}. \tag{9}$$

The SINR threshold and the thermal noise for each node is set to $\gamma_i = 1.5$ and $\eta_i = 0.04$ mWatts, respectively. The parameter α_i is set to 5000. We assume that four users in the system use sigmoidal utility function with $a_i = 1.33$, and b_i is found according to (8). The spectral radius of the network is $\rho(M^b) = 0.88297$, hence $c = 0.88297$. The initial power $p_i(0)$ for all users is set to 1 mWatt. A simulation of the network characterized by the matrix (9) shows that SINRs converge to the desired SINR and power levels to the minimal power vector; see Figure 1.

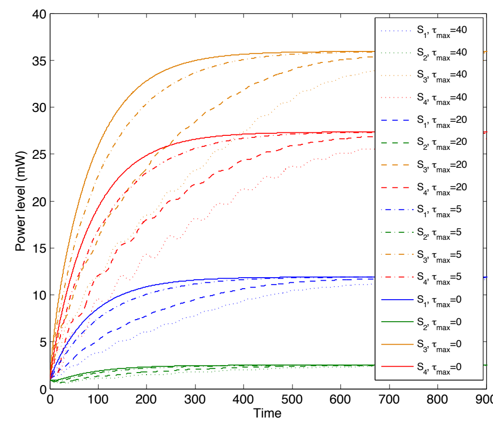

Fig. 1. Convergence of the UBPC algorithm with desired SINR $\gamma_i = 1.5$, thermal noise $\eta = 0.04\ mWatts$ and proportionality gain $k_i = 0.1$ for all users. The solid, dashed, dot-dashed and dotted lines illustrate the convergence of the algorithm in the presence of delays with maximum delay 0, 5, 20 and 40, respectively.

The theoretical upper bounds on the decay rate of the UBPC algorithm as obtained in Theorem 2 have been compared with the actual decay rate of the UBPC algorithm for the wireless network characterized by the matrix (9), see Figure 2. Since the delays are time-varying and usually smaller than the maximum, there is a gap between the theoretical and the actual decay rates that one observes in simulations. As we have already shown, when delays are introduced, $\bar{c}$ increases monotonically with $\tau_{\max}$, meaning that the decay rate decreases. For this network, we have numerically calculated $\bar{c}$ for the values of delays we consider in the simulations and we have $\bar{c}_5 = 0.9193$, $\bar{c}_{20} = 0.9588$ and $\bar{c}_{40} = 0.9752$.

VII. Conclusions

This paper developed a comprehensive stability analysis framework for continuous-time power control laws under bounded time-varying communication delays. The first set of

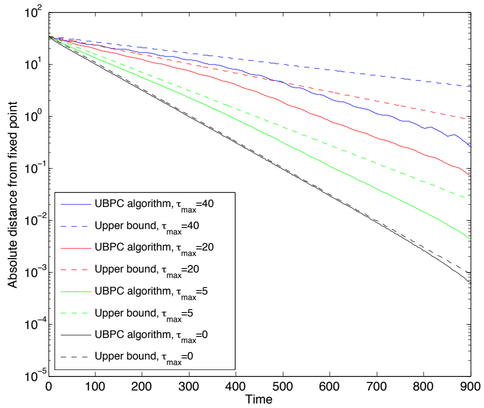

Fig. 2. Comparison of upper bound on the decay rate of the UBPC algorithm and the actual decay rate of the UBPC algorithm for the wireless network characterized by matrix (9), for $\tau_{\max} = 5, 20$ and 40. The dashed lines show the theoretical upper bound on the decay rate, while the solid lines show the actual decay rate of the UBPC algorithm.

results established global asymptotic stability of power control laws involving two-sided scalable interference functions. Next, we proved global exponential stability of power control laws involving contractive interference functions and derived explicit bounds on the decay rate which allows to, for the first time, quantify the impact of delays on the convergence rate of the transmit powers.

Appendix

A. Proof of Theorem 1

First note that if $\boldsymbol{p}^\star$ is a fixed point of the two-sided scalable interference function $\boldsymbol{I}(\boldsymbol{p})$, then $\boldsymbol{p}^\star = \boldsymbol{I}(\boldsymbol{p}^\star) > \boldsymbol{0}$. Since the initial condition $\boldsymbol{\varphi}(\cdot)$ is assumed to be positive ($\boldsymbol{\varphi}(\cdot) > \boldsymbol{0}$) and $\boldsymbol{I}(\boldsymbol{p}) > \boldsymbol{0}$ for all $\boldsymbol{p} \geq \boldsymbol{0}$, the solutions of the positive system (3) satisfy $\boldsymbol{p}(t) > \boldsymbol{0}$ for all $t \geq 0$ [14, Chapter 3]. This allows us to use the Lyapunov function

$$V(\boldsymbol{p}) = d\big(\boldsymbol{p}, \boldsymbol{p}^\star\big) - 1,$$

where $d(\cdot,\cdot)$ is defined as in (5). Let m be the index for which the maximum is achieved at time t. That is,

$$m = \arg\max_{1\leq i\leq n} \left\{ \max\left\{ \frac{p_i(t)}{p_i^\star}, \frac{p_i^\star}{p_i(t)} \right\} \right\}.$$

This implies that

$$V(\boldsymbol{p}) = \max\left\{ \frac{p_m(t)}{p_m^\star}, \frac{p_m^\star}{p_m(t)} \right\} - 1.$$

It is clear that $V(\boldsymbol{p}) > 0$ for all $\boldsymbol{p} \neq \boldsymbol{p}^\star$. The time-derivative of the candidate Lyapunov function along the trajectories of (3) is given by

$$\dot{V}(\boldsymbol{p}) = \begin{cases} \dfrac{\dot{p}_m(t)}{p_m^\star}, & \text{if } p_m(t) > p_m^\star, \\ 0, & \text{if } p_m(t) = p_m^\star, \\ -\dfrac{p_m^\star}{p_m^2(t)}\dot{p}_m(t), & \text{if } p_m(t) < p_m^\star. \end{cases}$$

a) If $p_m(t) > p_m^\star$, then

$$\begin{aligned}\dot{V}(\boldsymbol{p}) &= k_m \left(- \frac{p_m(t) + I_m\big(\boldsymbol{p}^{d_m}(t)\big)}{p_m^\star} \right) \\ &= k_m \left(- \frac{p_m(t)}{p_m^\star} + \frac{I_m\big(\boldsymbol{p}^{d_m}(t)\big)}{I_m(\boldsymbol{p}^\star)} \right),\end{aligned}$$

where we used the fact that $p_m^\star = I_m(\boldsymbol{p}^\star)$ to get the second equality. Since $d\big(\boldsymbol{p}(t), \boldsymbol{p}^\star\big) = \frac{p_m(t)}{p_m^\star}$, and

$$\frac{I_m\big(\boldsymbol{p}^{d_m}(t)\big)}{I_m(\boldsymbol{p}^\star)} \leq d\big(\boldsymbol{I}(\boldsymbol{p}^{d_m}(t)), \boldsymbol{I}(\boldsymbol{p}^\star)\big),$$

we have

$$\dot{V}(\boldsymbol{p}) \leq k_m \big(-d(\boldsymbol{p}(t), \boldsymbol{p}^\star) + d(\boldsymbol{I}(\boldsymbol{p}^{d_m}(t)), \boldsymbol{I}(\boldsymbol{p}^\star)) \big). \tag{10}$$

On the other hand, by Proposition 2(c), we obtain

$$\begin{aligned} & d\big(\boldsymbol{I}(\boldsymbol{p}^{d_m}(t)), \boldsymbol{I}(\boldsymbol{p}^\star)\big) \\ &\quad < d\big(\boldsymbol{p}^{d_m}(t), \boldsymbol{p}^\star\big) \\ &\quad = \max_{1\leq i\leq n} \left\{ \max\left\{ \frac{p_i(t-\tau_i^m(t))}{p_i^\star}, \frac{p_i^\star}{p_i(t-\tau_i^m(t))} \right\} \right\} \\ &\quad \leq \max_{1\leq i\leq n} d\big(\boldsymbol{p}(t-\tau_i^m(t)), \boldsymbol{p}^\star\big) \\ &\quad \leq \max_{t-\tau_{\max}\leq s\leq t} d\big(\boldsymbol{p}(s), \boldsymbol{p}^\star\big), \end{aligned} \tag{11}$$

where the last inequality follows from the fact that $\tau_i^m(t) \leq \tau_{\max}$. Combining (10) and (11) gives

$$\begin{aligned} \dot{V}(\boldsymbol{p}) &< k_m \big(-d\big(\boldsymbol{p}(t), \boldsymbol{p}^\star\big) + \max_{t-\tau_{\max}\leq s\leq t} d(\boldsymbol{p}(s), \boldsymbol{p}^\star) \big) \\ &= k_m \big(-V\big(\boldsymbol{p}(t)\big) + \max_{t-\tau_{\max}\leq s\leq t} V\big(\boldsymbol{p}(s)\big) \big). \end{aligned} \tag{12}$$

b) Similarly, if $p_m(t) < p_m^\star$, we have

$$\begin{aligned} \dot{V}(\boldsymbol{p}) &\leq k_m \frac{I_m\big(\boldsymbol{p}(t)\big)}{p_m(t)} \big(-d(\boldsymbol{p}(t), \boldsymbol{p}^\star) + d(\boldsymbol{I}(\boldsymbol{p}^{d_m}(t)), \boldsymbol{I}(\boldsymbol{p}^\star)) \big) \\ &< k_m \frac{I_m\big(\boldsymbol{p}(t)\big)}{p_m(t)} \big(-V(\boldsymbol{p}(t)) + \max_{t-\tau_{\max}\leq s\leq t} V(\boldsymbol{p}(s)) \big). \end{aligned} \tag{13}$$

In order to address the asymptotic stability of the continuous-time power control algorithm (3), we now apply the Invariance-Principle theorem for delay differential equations [21, Corollary 1], [22, Corollary 3.3.2]. In the light of this theorem, we are interested in the case

$$\max_{t-\tau_{\max}\leq s\leq t} V\big(\boldsymbol{p}(s)\big) = V\big(\boldsymbol{p}(t)\big).$$

It follows from (12) and (13) that while the above condition holds, $\dot{V}(\boldsymbol{p}) < 0$ for all $\boldsymbol{p} \neq \boldsymbol{p}^\star$. Therefore, $\boldsymbol{p} = \boldsymbol{p}^\star$ is globally asymptotically stable.

B. Proof of Theorem 2

We use the candidate Lyapunov function

$$V(\boldsymbol{p}) = \|\boldsymbol{p} - \boldsymbol{p}^\star\|_\infty^{\boldsymbol{v}} = \max_{1\leq i\leq n} \frac{1}{v_i} |p_i - p_i^\star|.$$

Let m be the index for which the maximum is achieved at time t. Calculating the time derivative of $V(\boldsymbol{p})$ along the solutions of (3), we get

$$\begin{aligned}\dot{V}(\boldsymbol{p}) &= \frac{1}{v_m}\operatorname{sgn}(p_m(t)-p_m^\star)\dot{p}_m(t)\\ &= \frac{1}{v_m}\operatorname{sgn}\big(p_m(t)-p_m^\star\big)k_m\big(-p_m(t)+I_m(\boldsymbol{p}^{d_m}(t))\big)\\ &= k_m\big(-\frac{1}{v_m}\operatorname{sgn}(p_m(t)-p_m^\star)(p_m(t)-p_m^\star)\big)\\ &\quad + k_m\big(\frac{1}{v_m}\operatorname{sgn}(p_m(t)-p_m^\star)(I_m(\boldsymbol{p}^{d_m}(t))-I_m(\boldsymbol{p}^\star))\big)\\ &\le k_m\Bigg(-\frac{1}{v_m}\big|p_m(t)-p_m^\star\big| + \\ &\qquad \frac{1}{v_m}\big|I_m(\boldsymbol{p}^{d_m}(t))-I_m(\boldsymbol{p}^\star)\big|\Bigg). \quad (14)\end{aligned}$$

where we used the fact that $p_m^\star = I_m(\boldsymbol{p}^\star)$ to obtain the third equality. The interference function $\boldsymbol{I}(\boldsymbol{p})$ is c-contractive. Thus,

$$\begin{aligned}\frac{1}{v_m}\big|I_m\big(\boldsymbol{p}^{d_m}(t)\big)-I_m(\boldsymbol{p}^\star)\big| &\le \big\|\boldsymbol{I}(\boldsymbol{p}^{d_m}(t))-\boldsymbol{I}(\boldsymbol{p}^\star)\big\|_\infty^{\boldsymbol{v}}\\ &\le c\|\boldsymbol{p}^{d_m}(t)-\boldsymbol{p}^\star\|_\infty^{\boldsymbol{v}}, \quad (15)\end{aligned}$$

where we used the definition of weighted l_∞ norm to get the first inequality, and Proposition 3 to obtain the second inequality. Substituting (15) into (14) yields

$$\begin{aligned}\dot{V}(\boldsymbol{p}) &\le k_m\left(-\|\boldsymbol{p}(t)-\boldsymbol{p}^\star\|_\infty^{\boldsymbol{v}} + c\|\boldsymbol{p}^{d_m}(t)-\boldsymbol{p}^\star\|_\infty^{\boldsymbol{v}}\right)\\ &= k_m\left(-\|\boldsymbol{p}(t)-\boldsymbol{p}^\star\|_\infty^{\boldsymbol{v}} + c\max_{1\le i\le n}\frac{1}{v_i}\big|p_i(t-\tau_i^m(t))-p_i^\star\big|\right)\\ &\le k_m\left(-\|\boldsymbol{p}(t)-\boldsymbol{p}^\star\|_\infty^{\boldsymbol{v}} + c\max_{1\le i\le n}\|\boldsymbol{p}(t-\tau_i^m(t))-\boldsymbol{p}^\star\|_\infty^{\boldsymbol{v}}\right)\\ &= k_m\left(-V\big(\boldsymbol{p}(t)\big)+c\max_{1\le i\le n}V\big(\boldsymbol{p}(t-\tau_i^m(t))\big)\right)\\ &\le k_m\left(-V\big(\boldsymbol{p}(t)\big)+c\max_{t-\tau_{\max}\le s\le t}V\big(\boldsymbol{p}(s)\big)\right).\end{aligned}$$

Since $V(\boldsymbol{p})\ge 0$, by Halanay Inequality [23, pp. 389–390], we obtain

$$V\big(\boldsymbol{p}(t)\big) \le \left(\sup_{-\tau_{\max}\le s\le 0} V\big(\boldsymbol{p}(s)\big)\right)e^{-\eta_m t}, \quad t\ge 0,$$

where η_m is the unique positive solution of the equation

$$\eta_m = k_m(1-ce^{\eta_m\tau_{\max}}).$$

Let η_i be the positive solution of the equation

$$\eta_i = k_i(1-ce^{\eta_i\tau_{\max}}),$$

for $i=1,\ldots,n$. Since η_i, as a function of $k_i > 0$, is monotonically increasing, we have $\eta\le\eta_i$ for all i where

$$\eta = k_{\min}(1-\ ce^{\eta\tau_{\max}}).$$

This implies that $\eta\le\eta_m$, and hence $e^{-\eta_m t}\le e^{-\eta t}$ for $t\ge 0$. Therefore,

$$V\big(\boldsymbol{p}(t)\big) \le \left(\sup_{-\tau_{\max}\le s\le 0} V\big(\boldsymbol{p}(s)\big)\right)e^{-\eta t}.$$

Letting $\bar{c} = 1-\frac{\eta}{k_{\min}}$, the result follows immediately.

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