Delay-independent Stability of Cone-invariant Monotone Systems

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***Abstract*— Recent results in the literature have shown that particular classes of positive systems are insensitive to time-varying delays, giving the impression that the delay-insensitivity property stems from the fact that the system is positive. Nonetheless, it has been lately shown that a linear *cone-invariant* system is insensitive to time-varying delays, asserting that the property of delay-independence may stem from the fact that the system is cone-invariant rather than positive. In this paper, we provide additional evidence for this claim by analyzing the stability of cone-invariant *monotone* systems with bounded time-varying delays. We present a set of sufficient conditions for delay independent stability of discrete- and continuous-time cone-invariant monotone systems. For linear cone-invariant systems, we show that the stability conditions we have derived are also necessary.**

I. Introduction

Roughly speaking, a system is called *cone invariant* if a cone is an invariant set for the system. Such systems are encountered in a wide range of application areas, including epidemiology, multi-agent systems, and wireless networks [1]–[8]. Positive systems constitute a special case of cone-invariant systems, where the nonnegative orthant (which forms a cone) is an invariant set [9]–[13].

Physical systems are usually modeled based on the assumption that their evolution depends only on the current values of the state variables. However, in many cases, the system state may also be affected by previous values of the states. For example, delays are inherent in distributed systems due to communication and processing delays, forcing subsystems to act and update their internal states based on delayed information. For this reason, the study of stability and control of dynamical systems with delayed states is important and has attracted a lot of interest. It is well known that time delays limit the performance of closed-loop control systems and may even render an otherwise stable system unstable [14]. However, recent results have revealed that linear and particular classes of nonlinear positive systems are insensitive to bounded time-varying delays in the sense that a delayed system is asymptotically stable if the corresponding delay-free system is asymptotically stable [15]–[25].

Recent work by Tanaka *et al.* [26] and Shen and Zheng [27] show that the stability of continuous-time linear cone-invariant systems is insensitive to bounded time-varying delays. These results expand the class of systems that are insensitive to delays and lead to the conjecture that the insensitivity to time delays is due to cone-preservation, for which nonnegativity serves as a special case.

In this work, we provide additional evidence for this conjecture, by establishing delay-independent stability of *nonlinear* cone-invariant systems which are *monotone*. Monotone systems are those for which trajectories preserve a partial ordering on initial states. The theory of monotone dynamical systems is still an active field of research, see for example [28]–[32]. At the core of our paper, we derive a set of sufficient conditions for delay-independent stability of continuous- and discrete-time cone-invariant monotone systems with bounded time-varying delays. We demonstrate that for linear cone-invariant systems, the stability conditions we have developed are also necessary

The rest of the paper is organized as follows. In Section II, the notation and preliminaries needed for the development of our results are presented. The main results for discrete- and continuous-time monotone systems are presented in Sections III and IV, respectively. Finally, concluding remarks are given in Section V.

II. Notation and Preliminaries

A. Notation

Vectors are written in bold lower case letters and matrices in capital letters. We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of natural numbers and the set of natural numbers including zero, respectively. The non-negative orthant of the n-dimensional real space $\mathbb{R}^n$ is represented by $\mathbb{R}^n_+$. For a real interval $[a, b]$ and an open set $\mathcal{W} \subseteq \mathbb{R}^n$, $\mathcal{C}\big([a, b], \mathcal{W}\big)$ denotes the space of all real-valued continuous functions on $[a, b]$ taking values in $\mathcal{W}$. For a set $\mathcal{K} \subseteq \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{m\times n}$, by $A\mathcal{K}$ we mean that $A\mathcal{K} := \{A\boldsymbol{x} : \boldsymbol{x} \in \mathcal{K}\}$.

B. Preliminaries

Next, we review the key definitions and results necessary for developing the main results of this paper. We start with the definition of a *cone*.

Definition 1 *A set $\mathcal{K} \subseteq \mathbb{R}^n$ is called a cone if, for every $\boldsymbol{x} \in \mathcal{K}$ and $\theta \in \mathbb{R}_+$, we have $\theta\boldsymbol{x} \in \mathcal{K}$.*

A cone $\mathcal{K} \subseteq \mathbb{R}^n$ is said to be *solid* if its interior, denoted by **int** $\mathcal{K}$, is nonempty. It is called *pointed* if

$$\mathcal{K} \cap (-\mathcal{K}) = \{0\}.$$

Definition 2 *A cone $\mathcal{K} \subseteq \mathbb{R}^n$ is called a proper cone, if it is convex, closed, solid, and pointed.*

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A proper cone $\mathcal{K} \subseteq \mathbb{R}^n$ induces partial orderings $\leq_{\mathcal{K}}$ and $<_{\mathcal{K}}$ on $\mathbb{R}^n$ as follows:

$$\boldsymbol{x} \leq_{\mathcal{K}} \boldsymbol{y} \Longleftrightarrow \boldsymbol{y} - \boldsymbol{x} \in \mathcal{K},$$
$$\boldsymbol{x} <_{\mathcal{K}} \boldsymbol{y} \Longleftrightarrow \boldsymbol{y} - \boldsymbol{x} \in \textbf{int}\,\mathcal{K}.$$

When $\mathcal{K} = \mathbb{R}_+$, the partial orderings $\leq_{\mathcal{K}}$ and $<_{\mathcal{K}}$ are the usual ordering $\leq$ and $<$ on $\mathbb{R}$, respectively.

Definition 3 *Given a cone $\mathcal{K} \subseteq \mathbb{R}^n$, the set*

$$\mathcal{K}^\star = \left\{ \boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{x}^\top \boldsymbol{y} \geq 0 \text{ for all } \boldsymbol{x} \in \mathcal{K} \right\}$$

is called the dual cone of $\mathcal{K}$.

The following definition introduces *$\mathcal{K}$-positive* and *cross-positive* matrices.

Definition 4 *Let $A \in \mathbb{R}^{n\times n}$ and let $\mathcal{K} \subseteq \mathbb{R}^n$ be a proper cone. The square matrix A is said to be cross-positive on $\mathcal{K}$ if for any $\boldsymbol{x} \in \mathcal{K}$ and any $\boldsymbol{y} \in \mathcal{K}^\star$ with $\boldsymbol{y}^\top \boldsymbol{x} = 0$, we have $\boldsymbol{y}^\top A \boldsymbol{x} \geq 0$. It is called $\mathcal{K}$-positive if $A\mathcal{K} \subseteq \mathcal{K}$.*

Note that Metzler and non-negative matrices are cross-positive and $\mathcal{K}$-positive on the positive orthant $\mathcal{K} = \mathbb{R}^n_+$, respectively. We now define *cooperative* vector fields.

Definition 5 *A vector field $\boldsymbol{f} : \mathcal{K} \to \mathbb{R}^n$ which is continuously differentiable on the proper cone $\mathcal{K} \subseteq \mathbb{R}^n$ is said to be cooperative with respect to $\mathcal{K}$ if the Jacobian matrix $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{a})$ is cross-positive on $\mathcal{K}$ for all $\boldsymbol{a} \in \mathcal{K}$.*

The next definition introduces *sub-homogeneous vector fields*.

Definition 6 *A vector field $\boldsymbol{f} : \mathcal{K} \to \mathbb{R}^n$ is said to be sub-homogeneous of degree $\alpha > 0$ with respect to $\mathcal{K}$ if*

$$\boldsymbol{f}(\lambda \boldsymbol{x}) \leq_{\mathcal{K}} \lambda^\alpha \boldsymbol{f}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathcal{K},\ \forall \lambda \geq 1.$$

Finally, we define *order-preserving vector fields*.

Definition 7 *A vector field $\boldsymbol{g} : \mathcal{K} \to \mathbb{R}^n$ is called order-preserving on $\mathcal{K}$ if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$ such that $\boldsymbol{x} \leq_{\mathcal{K}} \boldsymbol{y}$, it holds that $\boldsymbol{g}(\boldsymbol{x}) \leq_{\mathcal{K}} \boldsymbol{g}(\boldsymbol{y})$.*

III. Discrete-Time Monotone Systems

A. Problem Statement

Consider the discrete-time nonlinear dynamical system

$$\Sigma : \begin{cases} \boldsymbol{x}(t+1) &= \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{g}\big(\boldsymbol{x}(t-\tau(t))\big), t \in \mathbb{N}_0, \\ \boldsymbol{x}(t) &= \boldsymbol{\varphi}(t), \qquad\qquad t \in \{-\tau_{\max}, \ldots, 0\}, \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state variable, $f, g : \mathcal{K} \to \mathbb{R}^n$ are continuous vector fields on the proper cone $\mathcal{K} \subseteq \mathbb{R}^n$, $\boldsymbol{\varphi} : \{-\tau_{\max}, \ldots, 0\} \to \mathbb{R}^n$ is the vector sequence specifying the initial state of the system, and $\tau(t)$ represents the time-varying delay which is bounded by a nonnegative constant $\tau_{\max}$; this is stated in the following assumption.

Assumption 1 *The delay $\tau : \mathbb{N}_0 \to \mathbb{N}_0$ satisfies*

$$0 \leq \tau(t) \leq \tau_{\max}, \quad t \in \mathbb{N}_0.$$

The time-delay system Σ given by (1) is said to be *monotone* if ordered initial states lead to ordered subsequent states. More precisely, let $\boldsymbol{x}(t, \boldsymbol{\varphi})$ denote the solution of Σ starting from the initial state $\boldsymbol{\varphi}(t)$. Then, Σ is monotone if

$$\boldsymbol{\varphi}(t) \leq_{\mathcal{K}} \boldsymbol{\varphi}'(t), \quad \forall t \in \{-\tau_{\max}, \ldots, 0\},$$

implies that

$$\boldsymbol{x}(t, \boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}'), \quad \forall t \in \mathbb{N}.$$

The following result provides a sufficient condition for monotonicity of Σ.

Proposition 1 *If $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$, then the time-delay system Σ given by (1) is monotone in $\mathcal{K}$.*

Proof: See Appendix A.

System Σ given by (1) is called *cone-invariant* with respect to a proper cone $\mathcal{K}$ if its state trajectory starting from any initial state $\boldsymbol{\varphi} \in \mathcal{K}$ will always remain in $\mathcal{K}$, that is $\boldsymbol{x}(t, \boldsymbol{\varphi}) \in \mathcal{K}$ for all $t \in \mathbb{N}_0$. When Σ is cone-invariant with respect to the positive orthant ($\mathcal{K} = \mathbb{R}^n_+$), it is called *positive*. We now provide a necessary and sufficient condition for cone-preservity of monotone systems of the form (1).

Proposition 2 *Assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$. Then, the monotone system Σ given by (1) is cone-invariant with respect to $\mathcal{K}$ if and only if*

$$\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \in \mathcal{K}. \tag{2}$$

Proof: See Appendix B.

Note that, according to Proposition 2, if the monotone system (1) has an equilibrium at the origin, *i.e.*,

$$\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0},$$

then it is cone-invariant.

B. Main Results

The following theorem states a sufficient condition for *local* asymptotic stability of cone-invariant monotone systems with bounded time-varying delays.

Theorem 1 *For the time-delay system Σ given by (1), suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$, and that (2) holds. Assume also that there exists $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ such that*

$$\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) - \boldsymbol{v} \in -\textbf{int}\,\mathcal{K}. \tag{3}$$

If $\boldsymbol{x}^\star$ is the only equilibrium point of the cone-invariant monotone system (1) in $\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{x} \leq_{\mathcal{K}} \boldsymbol{v}$, then for all bounded time-varying delays, $\boldsymbol{x}^\star$ is asymptotically stable with respect to initial conditions satisfying

$$\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{\varphi}(t) \leq_{\mathcal{K}} \boldsymbol{v}, \quad t \in \{-\tau_{\max}, \ldots, 0\}. \tag{4}$$

Proof: See Appendix C.

Stability condition (3) does not include any information on the magnitude of delays, so it ensures *delay-independent* stability. This type of stability conditions is useful in practice, since the delays may not be easy to model precisely.

We will now show that under the stability condition that we have derived in Theorem 1, cone-invariant monotone systems whose vector fields are sub-homogeneous of degree smaller than or equal to one are *globally* asymptotically stable. Sub-homogeneous monotone systems constitute an important and useful class of nonlinear cone-invariant systems. Established models of many physical phenomena fall within this class. For example, biochemical reaction networks and most power control algorithms in wireless networks can be analyzed as sub-homogeneous monotone systems [33]–[35]. This class of monotone systems includes homogeneous monotone systems as a special case.

Theorem 2 *Assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving and sub-homogeneous of degree $\alpha \in (0,1]$ with respect to $\mathcal{K}$. Furthermore, assume that $\boldsymbol{x}^\star$ is the only equilibrium of* (1) *in $\mathcal{K}$. If there exists a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ such that*

$$\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) - \boldsymbol{v} \in -\textbf{int}\,\mathcal{K}, \tag{5}$$

then the sub-homogeneous cone-invariant monotone system (1) *is globally asymptotically stable for any bounded time-varying delays.*

Proof: See Appendix D.

Note that a linear cone-invariant system is a special case of sub-homogeneous cone-invariant monotone systems, where $\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$ and $\boldsymbol{g}(\boldsymbol{x}) = B\boldsymbol{x}$ with $\mathcal{K}$-positive matrices A and B. In this case, the global delay-independent stability condition (5) reduces to $(A+B)\boldsymbol{v} <_{\mathcal{K}} \boldsymbol{v}$.

C. Extensions

Our results can be easily extended to monotone systems with heterogeneous delays of the form:

$$x_i(t+1) = f_i(\boldsymbol{x}(t)) + g_i\big(x_1(t-\tau_1^i(t)), \ldots, x_n(t-\tau_n^i(t))\big).$$

Here, $i \in \{1,\ldots,n\}$, $\boldsymbol{x}(t) = (x_1(t),\ldots,x_n(t)) \in \mathbb{R}^n$, and $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ldots,f_n(\boldsymbol{x}))$ and $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}),\ldots,g_n(\boldsymbol{x}))$ are order-preserving on a proper cone $\mathcal{K}$. If the delays satisfy

$$0 \le \tau_i^j(t) \le \tau_{\max}, \quad \forall i,j \in \{1,\ldots,n\},$$

then the stability condition (3) ensures that the cone-invariant monotone system with heterogeneous time-varying delays is locally asymptotically stable.

IV. Continuous-Time Monotone Systems

A. Problem Statement

Next, we consider the continuous-time analog of (1):

$$\mathcal{G}: \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{g}\big(\boldsymbol{x}(t-\tau(t))\big), \quad t \ge 0, \\ \boldsymbol{x}(t) &= \boldsymbol{\varphi}(t), \qquad\qquad\qquad\qquad t \in [-\tau_{\max}, 0], \end{cases} \tag{6}$$

where $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ are continuously differentiable vector fields on the proper cone $\mathcal{K} \subseteq \mathbb{R}^n$. The time-varying delay $\tau(t)$ is continuous with respect to time and satisfies

$$0 \le \tau(t) \le \tau_{\max}, \quad \forall t \ge 0.$$

The time-delay system $\mathcal{G}$ given by (6) is called *monotone* if for any initial conditions $\boldsymbol{\varphi}(t), \boldsymbol{\varphi}'(t) \in \mathcal{C}\big([-\tau_{\max},0],\mathcal{K}\big)$, $\boldsymbol{\varphi}(t) \le_{\mathcal{K}} \boldsymbol{\varphi}'(t)$ for all $t \in [-\tau_{\max},0]$ implies that

$$\boldsymbol{x}(t,\boldsymbol{\varphi}) \le_{\mathcal{K}} \boldsymbol{x}(t,\boldsymbol{\varphi}'), \quad \forall t \ge 0.$$

Monotonicity of (6) is readily verified using the next result.

Proposition 3 **[28, Theorem 5.1.1]** *Suppose that $\boldsymbol{f}$ is cooperative with respect to the cone $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$. Then, the time-delay system $\mathcal{G}$ given by* (6) *is monotone in $\mathcal{K}$.*

System $\mathcal{G}$ is said to be *cone-invariant* with respect to a cone $\mathcal{K}$ if for any initial condition $\boldsymbol{\varphi}(t) \in \mathcal{C}\big([-\tau_{\max},0],\mathcal{K}\big)$, the corresponding state trajectory will never leave $\mathcal{K}$. The following result provides a necessary and sufficient condition for cone-preservity of $\mathcal{G}$.

Proposition 4 *Suppose that $\boldsymbol{f}$ is cooperative with respect to the cone $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$. Then, the monotone system* (6) *is cone-invariant with respect to $\mathcal{K}$ if and only if*

$$\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \in \mathcal{K}. \tag{7}$$

Proof: See Appendix E.

B. Main Results

We now provide a test for the local asymptotic stability of cone-invariant monotone systems of the form (6) with bounded time-varying delays.

Theorem 3 *For the time-delay system* (6)*, suppose that $\boldsymbol{f}$ is cooperative with respect to the cone $\mathcal{K}$, $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$, and* (7) *holds. Suppose also that there exist a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ such that*

$$\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) \in -\textbf{int}\,\mathcal{K}. \tag{8}$$

If $\boldsymbol{x}^\star$ is the only equilibrium point of the cone-invariant monotone system (6) *such that $\boldsymbol{0} \le_{\mathcal{K}} \boldsymbol{x}^\star \le_{\mathcal{K}} \boldsymbol{v}$, then for any initial conditions satisfying*

$$\boldsymbol{0} \le_{\mathcal{K}} \boldsymbol{\varphi}(t) \le_{\mathcal{K}} \boldsymbol{v}, \quad t \in [-\tau_{\max}, 0], \tag{9}$$

$\boldsymbol{x}^\star$ is asymptotically stable for all bounded time-varying delays.

Proof: See Appendix F.

Theorem 3 allows us to prove that when a cone-invariant sub-homogeneous monotone system of the form (6) has a unique equilibrium point in the proper cone $\mathcal{K}$, it is globally asymptotically stable under the stability condition (8).

Theorem 4 *Assume that $\boldsymbol{f}$ is cooperative with respect to the cone $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$. Furthermore, assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are sub-homogeneous of degree $\alpha > 0$ with respect to $\mathcal{K}$. If $\boldsymbol{x}^\star$ is the only equilibrium of* (6) *in $\mathcal{K}$, and there is a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ satisfying* (8)*, then the sub-homogeneous cone-invariant monotone system $\mathcal{G}$ is globally asymptotically stable for any bounded time-varying delays.*

Proof: The proof is similar to the one of Theorem 2.

C. A Special Case: Cone-invariant Linear Systems

We now discuss delay-independent stability of a special case of (6), namely linear systems on the form

$$\begin{cases} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{x}(t-\tau(t)), \qquad t \geq 0, \\ \boldsymbol{x}(t) &= \boldsymbol{\varphi}(t), \qquad\qquad\qquad t \in [-\tau_{\max}, 0]. \end{cases} \tag{10}$$

In terms of (6), $\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$ and $\boldsymbol{g}(\boldsymbol{x}) = B\boldsymbol{x}$. One can verify that if A is cross-positive and B is $\mathcal{K}$-positive, then (10) is a sub-homogeneous cone-invariant monotone system. Theorem 4 helps us to derive a necessary and sufficient condition for delay-independent stability of (10). Specifically, we note the following.

Corollary 1 *Consider the linear system* (10) *where A is cross-positive and B is $\mathcal{K}$-positive. Then, the following statements are equivalent.*

(a) *There exists a vector $\boldsymbol{v} \in$ **int** $\mathcal{K}$ such that*

$$(A+B)\boldsymbol{v} \in -\textbf{int}\,\mathcal{K}.$$

(b) *The cone-invariant linear system* (10) *is globally asymptotically stable for all bounded time-varying delays.*

Tanaka *et al.* [26] proved that the stability of continuous-time cone-invariant linear systems is insensitive to arbitrary *constant* time delays using a "DC-dominant" property. More recently, Shen and Zheng [27], by comparing the trajectory of the constant delay system and that of the time-varying delay system, proved that the stability of cone-invariant linear systems is insensitive to bounded time-varying delays. The latter result is equivalent to Corollary 1.

V. Conclusions

This paper has been concerned with delay-independent stability of a significant class of nonlinear (continuous- and discrete-time) cone-invariant systems. First, we have presented a set of conditions for establishing local asymptotic stability of cone-invariant monotone systems with bounded time-varying delays. Then, we have derived sufficient conditions for global delay-independent stability of sub-homogeneous cone-invariant monotone systems. Finally, for linear cone-invariant systems, we have shown that the stability conditions developed are also necessary.

Appendix

A. Proof of Proposition 1

Let $\boldsymbol{\varphi}(t)$ and $\boldsymbol{\varphi}'(t)$ be arbitrary initial conditions satisfying $\boldsymbol{\varphi}(t) \leq_{\mathcal{K}} \boldsymbol{\varphi}'(t)$ for all $t \in \{-\tau_{\max}, \dots, 0\}$. We show by induction that

$$\boldsymbol{x}(t,\boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(t,\boldsymbol{\varphi}') \tag{11}$$

holds for all $t \in \mathbb{N}$. Since

$$\boldsymbol{x}(0,\boldsymbol{\varphi}) = \boldsymbol{\varphi}(0) \leq_{\mathcal{K}} \boldsymbol{\varphi}'(0) = \boldsymbol{x}(0,\boldsymbol{\varphi}'),$$

the induction hypothesis is true for $t = 0$. Assume for induction that (11) holds for $t \in \{0, \dots, \hat{t}\}$ with $\hat{t} \in \mathbb{N}_0$. It is clear that $\boldsymbol{x}(\hat{t},\boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(\hat{t},\boldsymbol{\varphi}')$. Moreover, as

$$-\tau_{\max} \leq \hat{t} - \tau(\hat{t}) \leq \hat{t},$$

we have $\boldsymbol{x}\big(\hat{t}-\tau(\hat{t}),\boldsymbol{\varphi}\big) \leq_{\mathcal{K}} \boldsymbol{x}\big(\hat{t}-\tau(\hat{t}),\boldsymbol{\varphi}'\big)$ by the induction hypothesis. It now follows from order-preservity of $\boldsymbol{f}$ and $\boldsymbol{g}$ on $\mathcal{K}$ that

$$\begin{aligned} \boldsymbol{x}\big(\hat{t}+1,\boldsymbol{\varphi}\big) &= \boldsymbol{f}\big(\boldsymbol{x}(\hat{t},\boldsymbol{\varphi})\big) + \boldsymbol{g}\big(\boldsymbol{x}(\hat{t}-\tau(\hat{t})),\boldsymbol{\varphi})\big) \\ &\leq_{\mathcal{K}} \boldsymbol{f}\big(\boldsymbol{x}(\hat{t},\boldsymbol{\varphi}')\big) + \boldsymbol{g}\big(\boldsymbol{x}(\hat{t}-\tau(\hat{t})),\boldsymbol{\varphi}')\big) \\ &= \boldsymbol{x}\big(\hat{t}+1,\boldsymbol{\varphi}'\big). \end{aligned}$$

We conclude that (11) is true for all $t \in \mathbb{N}_0$. Hence, the time-delay system (1) is monotone.

B. Proof of Proposition 2

(i) Suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$, and that $\boldsymbol{f}(\boldsymbol{0})+\boldsymbol{g}(\boldsymbol{0}) \in \mathcal{K}$. We will prove that the time-delay system (1) is cone-invariant. Let $\boldsymbol{x}(t,\boldsymbol{\varphi}_0)$ be the solution to (1) with the initial condition $\boldsymbol{\varphi}_0(t) = \boldsymbol{0}$, $t \in \{-\tau_{\max}, \dots, 0\}$. Clearly,

$$\boldsymbol{x}\big(1,\boldsymbol{\varphi}_0\big) = \boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \in \mathcal{K}.$$

We will use induction to show that

$$\boldsymbol{x}\big(t,\boldsymbol{\varphi}_0\big) \in \mathcal{K}, \quad \forall t \in \mathbb{N}. \tag{12}$$

If (12) is true for all t up to some $\hat{t}$, then $\boldsymbol{x}\big(\hat{t},\boldsymbol{\varphi}_0\big) \in \mathcal{K}$, or, equivalently, $\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{x}\big(\hat{t},\boldsymbol{\varphi}_0\big)$. Also, since $\boldsymbol{\varphi}_0(t) \in \mathcal{K}$ for all $t \in \{-\tau_{\max}, \dots, 0\}$ and $\hat{t} - \tau(\hat{t}) \in [-\tau_{\max}, \hat{t}]$, by induction hypothesis, we have

$$\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{x}\big(\hat{t}-\tau(\hat{t}),\boldsymbol{\varphi}_0\big).$$

As $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$, it follows that

$$\begin{aligned} \boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) &\leq_{\mathcal{K}} \boldsymbol{f}\big(\boldsymbol{x}(\hat{t},\boldsymbol{\varphi}_0)\big) + \boldsymbol{g}\big(\boldsymbol{x}(\hat{t}-\tau(\hat{t})),\boldsymbol{\varphi}_0)\big) \\ &= \boldsymbol{x}\big(\hat{t}+1,\boldsymbol{\varphi}_0\big), \end{aligned}$$

implying that (12) holds for all $t \in \mathbb{N}$.

Now, let $\boldsymbol{\varphi}$ be an arbitrary initial state satisfying $\boldsymbol{\varphi}(t) \in \mathcal{K}$. It is clear that

$$\boldsymbol{0} = \boldsymbol{\varphi}_0(t) \leq_{\mathcal{K}} \boldsymbol{\varphi}(t), \;\; t \in \{-\tau_{\max}, \dots, 0\}.$$

According to Proposition 1, the time-delay system (1) is monotone. Thus, $\boldsymbol{x}\big(t,\boldsymbol{\varphi}_0\big) \leq_{\mathcal{K}} \boldsymbol{x}\big(t,\boldsymbol{\varphi}\big)$ for all $t \in \mathbb{N}$. It follows from (12) that $\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{x}\big(t,\boldsymbol{\varphi}\big)$, or, equivalently,

$$\boldsymbol{x}\big(t,\boldsymbol{\varphi}\big) \in \mathcal{K}, \quad \forall t \in \mathbb{N}.$$

Therefore, the monotone system (1) is cone-invariant.

(ii) Assume that the monotone system (1) is cone-invariant. If, for contradiction, $\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \notin \mathcal{K}$, then there exists $\boldsymbol{z} \in \mathcal{K}^\star$ such that $\boldsymbol{z}^\top\big(\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0})\big) < 0$. In this case, we have

$$\boldsymbol{z}^\top\boldsymbol{x}\big(1,\boldsymbol{\varphi}_0\big) = \boldsymbol{z}^\top\big(\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0})\big) < 0,$$

implying that $\boldsymbol{x}\big(1,\boldsymbol{\varphi}_0\big) \notin \mathcal{K}$. This contradicts the fact that the time-delay system (1) is cone-invariant.

C. Proof of Theorem 1

Before proving Theorem 1, we state the following lemma that is key to our argument.

Lemma 1 *Consider the following time-delay dynamical system with constant delays, closely related to system* (1)*:*

$$\Sigma' : \begin{cases} \boldsymbol{y}(t+1) &= \boldsymbol{f}\big(\boldsymbol{y}(t)\big) + \boldsymbol{g}\big(\boldsymbol{y}(t-\tau_{\max})\big), t \in \mathbb{N}_0, \\ \boldsymbol{y}(t) &= \boldsymbol{\varphi}(t), \qquad\qquad t \in \{-\tau_{\max}, \dots, 0\}. \end{cases} \tag{13}$$

Assume that $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *are order-preserving on* $\mathcal{K}$*. The following statements hold.*

(i) *If there exists a vector* $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ *such that*

$$\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) - \boldsymbol{v} \in -\textbf{int}\,\mathcal{K}, \tag{14}$$

then

$$\boldsymbol{y}(t+1, \boldsymbol{\varphi}_v) \le_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_v), \quad t \in \mathbb{N}_0, \tag{15}$$

$$\boldsymbol{x}(t, \boldsymbol{\varphi}_v) \le_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_v), \quad t \in \mathbb{N}_0, \tag{16}$$

where $\boldsymbol{\varphi}_v(t) = \boldsymbol{v}$ *for all* $t \in \{-\tau_{\max}, \dots, 0\}$*, and* $\boldsymbol{x}(t, \boldsymbol{\varphi}_v)$ *and* $\boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ *are solutions to* (1) *and* (13)*, respectively.*

(ii) *If* $\boldsymbol{f}(\mathbf{0}) + \boldsymbol{g}(\mathbf{0}) \in \mathcal{K}$*, then the solution* $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ *to* (13) *starting from* $\boldsymbol{\varphi}_0(t) = 0$, $t \in \{-\tau_{\max}, \dots, 0\}$*, satisfies*

$$\boldsymbol{y}(t, \boldsymbol{\varphi}_0) \le_{\mathcal{K}} \boldsymbol{y}(t+1, \boldsymbol{\varphi}_0), \quad t \in \mathbb{N}_0,$$
$$\boldsymbol{y}(t, \boldsymbol{\varphi}_0) \le_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}_0), \quad t \in \mathbb{N}_0,$$

where $\boldsymbol{x}(t, \boldsymbol{\varphi}_0)$ *is the solution to* (1)*.*

Proof:

(i) Let $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ be a vector satisfying (14). Since

$$\boldsymbol{y}\big(1, \boldsymbol{\varphi}_v\big) = \boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) \le_{\mathcal{K}} \boldsymbol{v} = \boldsymbol{y}\big(0, \boldsymbol{\varphi}_v\big),$$

Inequality (15) holds for $t = 0$. Assume that (15) is true for all t up to $\hat{t}$. It follows from the induction hypothesis that

$$\boldsymbol{y}(\hat{t}+1, \boldsymbol{\varphi}_v) \le_{\mathcal{K}} \boldsymbol{y}(\hat{t}, \boldsymbol{\varphi}_v),$$
$$\boldsymbol{y}(\hat{t} - \tau_{\max} + 1, \boldsymbol{\varphi}_v) \le_{\mathcal{K}} \boldsymbol{y}(\hat{t} - \tau_{\max}, \boldsymbol{\varphi}_v).$$

These inequalities together with the order-preservity of $\boldsymbol{f}$ and $\boldsymbol{g}$ imply that

$$\begin{aligned} \boldsymbol{y}\big(\hat{t}+2, \boldsymbol{\varphi}_v\big) &= \boldsymbol{f}\big(\boldsymbol{y}(\hat{t}+1, \boldsymbol{\varphi}_v)\big) + \boldsymbol{g}\big(\boldsymbol{y}(\hat{t} - \tau_{\max} + 1, \boldsymbol{\varphi}_v)\big) \\ &\le_{\mathcal{K}} \boldsymbol{f}\big(\boldsymbol{y}(\hat{t}, \boldsymbol{\varphi}_v)\big) + \boldsymbol{g}\big(\boldsymbol{y}(\hat{t} - \tau_{\max}, \boldsymbol{\varphi}_v)\big) \\ &= \boldsymbol{y}\big(\hat{t}+1, \boldsymbol{\varphi}_v\big). \end{aligned}$$

Therefore, (15) holds for all $t \in \mathbb{N}_0$.

By using induction, we now prove Inequality (16). The induction hypothesis is true for $t = 1$, since

$$\boldsymbol{x}\big(1, \boldsymbol{\varphi}_v\big) = \boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) = \boldsymbol{y}\big(1, \boldsymbol{\varphi}_v\big).$$

Assuming it is true for a given $t = \hat{t}$, we then have

$$\boldsymbol{x}(\hat{t}, \boldsymbol{\varphi}_v) \le_{\mathcal{K}} \boldsymbol{y}(\hat{t}, \boldsymbol{\varphi}_v),$$
$$\boldsymbol{x}(\hat{t} - \tau(\hat{t}), \boldsymbol{\varphi}_v) \le_{\mathcal{K}} \boldsymbol{y}(\hat{t} - \tau(\hat{t}), \boldsymbol{\varphi}_v).$$

As $\hat{t} - \tau_{\max} \le \hat{t} - \tau(\hat{t})$, it follows from (15) that

$$\boldsymbol{y}(\hat{t} - \tau(\hat{t}), \boldsymbol{\varphi}_v) \le_{\mathcal{K}} \boldsymbol{y}(\hat{t} - \tau_{\max}, \boldsymbol{\varphi}_v),$$

implying that

$$\boldsymbol{x}(\hat{t} - \tau(\hat{t}), \boldsymbol{\varphi}_v) \le_{\mathcal{K}} \boldsymbol{y}(\hat{t} - \tau_{\max}, \boldsymbol{\varphi}_v).$$

Since $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$, we obtain

$$\begin{aligned} \boldsymbol{x}\big(\hat{t}+1, \boldsymbol{\varphi}_v\big) &= \boldsymbol{f}\big(\boldsymbol{x}(\hat{t}, \boldsymbol{\varphi}_v)\big) + \boldsymbol{g}\big(\boldsymbol{x}(\hat{t} - \tau(\hat{t}), \boldsymbol{\varphi}_v)\big) \\ &\le_{\mathcal{K}} \boldsymbol{f}\big(\boldsymbol{y}(\hat{t}, \boldsymbol{\varphi}_v)\big) + \boldsymbol{g}\big(\boldsymbol{y}(\hat{t} - \tau_{\max}, \boldsymbol{\varphi}_v)\big) \\ &= \boldsymbol{y}\big(\hat{t}+1, \boldsymbol{\varphi}_v\big). \end{aligned}$$

The induction proof is complete.

(ii) The proof is similar to the one of part *(i)* and thus omitted. ∎

D. Proof of Theorem 2

Let $\boldsymbol{\varphi} \in \mathcal{K}$ be an arbitrary initial condition and let $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ be a vector satisfying (5). For all $\boldsymbol{z} \in \mathcal{K}^\star \backslash \{\mathbf{0}\}$, we have $\boldsymbol{z}^\top \boldsymbol{v} > 0$ [36, Proposition 3.1]. Thus, we can find sufficiently large $\gamma \ge 1$ such that $\boldsymbol{z}^\top \boldsymbol{\varphi}(t) \le \gamma \boldsymbol{z}^\top \boldsymbol{v}$, for $t \in \{-\tau_{\max}, \dots, 0\}$, which implies that

$$\boldsymbol{\varphi}(t) \le_{\mathcal{K}} \gamma \boldsymbol{v}. \tag{17}$$

Since $\boldsymbol{f}$ and $\boldsymbol{g}$ are sub-homogeneous of degree $\alpha \in (0, 1]$, we have

$$\begin{aligned} \boldsymbol{f}(\gamma \boldsymbol{v}) + \boldsymbol{g}(\gamma \boldsymbol{v}) &\le_{\mathcal{K}} \gamma^\alpha \big(\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v})\big) \\ &\le_{\mathcal{K}} \gamma^\alpha \boldsymbol{v} \\ &\le_{\mathcal{K}} \gamma \boldsymbol{v}, \end{aligned}$$

where the second inequality follows from (5). Therefore, the vector $\gamma \boldsymbol{v}$ also satisfies (5). It follows from Theorem 1 and (17) that $\lim_{t\to\infty} \boldsymbol{x}(t, \boldsymbol{\varphi}) = \boldsymbol{x}^\star$.

E. Proof of Proposition 4

(i) Let $\boldsymbol{\varphi}_0(t)$ be the initial condition satisfying $\boldsymbol{\varphi}_0(t) = \mathbf{0}$, $t \in [-\tau_{\max}, 0]$. Since $\boldsymbol{f}$ is cooperative with respect to $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$, it follows from Proposition 3 that system (6) is monotone. Thus, if $\boldsymbol{\varphi}_0(t) \le_{\mathcal{K}} \boldsymbol{\varphi}(t)$ for all $t \in [-\tau_{\max}, 0]$, then

$$\boldsymbol{x}(t, \boldsymbol{\varphi}_0) \le_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}), \quad \forall t \ge 0. \tag{18}$$

Let $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ be the solution to the following delayed differential equation with constant delays:

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \boldsymbol{f}\big(\boldsymbol{y}(t)\big) + \boldsymbol{g}\big(\boldsymbol{y}(t-\tau_{\max})\big), t \ge 0, \\ \boldsymbol{y}(t) = \boldsymbol{\varphi}(t), \qquad\qquad t \in [-\tau_{\max}, 0]. \end{cases} \tag{19}$$

According to [28, Corollary 5.2.2], $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ is non-decreasing, *i.e.*, $\mathbf{0} = \boldsymbol{\varphi}_0(0) \le_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_0)$, $\forall t \ge 0$. Moreover, by [37, Lemma 2], $\boldsymbol{y}(t, \boldsymbol{\varphi}_0) \le_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}_0)$ for all $t \ge 0$. Therefore, $\mathbf{0} \le_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi})$, which implies that $\boldsymbol{x}(t, \boldsymbol{\varphi}) \in \mathcal{K}$ for all $t \ge 0$.

Conversely, assume that (6) is cone-invariant. Suppose, for contradiction, that $\boldsymbol{f}(\mathbf{0}) + \boldsymbol{g}(\mathbf{0}) \notin \mathcal{K}$. Then, there is $\boldsymbol{z} \in \mathcal{K}^\star$ such that $\boldsymbol{z}^\top \big(\boldsymbol{f}(\mathbf{0}) + \boldsymbol{g}(\mathbf{0})\big) < 0$, which implies that

$$\boldsymbol{z}^\top \dot{\boldsymbol{x}}(0, \boldsymbol{\varphi}_0) = \boldsymbol{z}^\top \big(\boldsymbol{f}(\mathbf{0}) + \boldsymbol{g}(\mathbf{0})\big) < 0.$$

Hence, there exists sufficiently small $\delta > 0$ such that

$$\boldsymbol{z}^\top \boldsymbol{x}(t, \boldsymbol{\varphi}_0) < \boldsymbol{z}^\top \boldsymbol{x}(0, \boldsymbol{\varphi}_0) = 0, \quad \forall t \in (0, \delta).$$

Therefore, $\boldsymbol{x}(t) \notin \mathcal{K}$ for $t \in (0, \delta)$, which is a contradiction.

F. Proof of Theorem 3

Let $\boldsymbol{v}$ be a vector satisfying (8). Define $\boldsymbol{\varphi}_0(t) = \mathbf{0}$ and $\boldsymbol{\varphi}_v(t) = \boldsymbol{v}$, $t \in [-\tau_{\max}, 0]$. As $\boldsymbol{f}$ is cooperative and $\boldsymbol{g}$ is order-preserving, according to Proposition 3, the time-delay system (6) is monotone. Thus,

$$\boldsymbol{x}(t, \boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}_v), \quad \forall t \geq 0.$$

Let $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t, \boldsymbol{\psi}_v)$ be solutions to the time-delay system (19) starting from $\boldsymbol{\varphi}_w(t)$ and $\boldsymbol{\varphi}_v(t)$, respectively. According to [37, Lemma 2], $\boldsymbol{y}(t, \boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}_0)$ and $\boldsymbol{x}(t, \boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ for all $t \geq 0$, implying that

$$\boldsymbol{y}(t, \boldsymbol{\varphi}_w) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_v), \quad \forall t \geq 0. \tag{20}$$

Since $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ is non-decreasing and $\boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ is non-increasing for $t \geq 0$ [28, Corollary 5.2.2], we have

$$\mathbf{0} \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_w) \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{v}, \quad \forall t \geq 0.$$

Thus, both $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ are bounded and monotone. Therefore, by [28, Theorem 1.2.1], $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ converge to an equilibrium of (19) in $[\boldsymbol{w}, \boldsymbol{v}]$, which must be $\boldsymbol{x}^\star$, *i.e.*,

$$\lim_{t \to \infty} \boldsymbol{y}(t, \boldsymbol{\varphi}_0) = \lim_{t \to \infty} \boldsymbol{y}(t, \boldsymbol{\varphi}_v) = \boldsymbol{x}^\star. \tag{21}$$

It now follows from (20) and (21) that $\lim_{t \to \infty} \boldsymbol{x}(t, \boldsymbol{\varphi}) = \boldsymbol{x}^\star$.

REFERENCES

[1] R. Bhattacharya, J. Fung, A. Tiwari, and R. Murray, "Ellipsoidal cones and rendezvous of multiple agents," *IEEE Conference on Decision and Control (CDC)*, pp. 171–176, 2004.

[2] W. M. Haddad, V. Chellaboina, and Q. Hui, *Nonnegative and Compartmental Dynamical Systems*. Princeton, 2010.

[3] A. Rantzer, "Distributed control of positive systems," *IEEE Conference on Decision and Control (CDC)*, pp. 6608–6611, 2011.

[4] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson, "Stability and performance of continuous-time power control in wireless networks," *IEEE Transactions on Automatic Control*, vol. 59, no. 8, pp. 2012–2023, 2014.

[5] M. Valcher and P. Misra, "On the stabilizability and consensus of positive homogeneous multi-agent dynamical systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1936–1941, 2014.

[6] C. Grussler and A. Rantzer, "Modified balanced truncation preserving ellipsoidal cone-invariance," *IEEE Conference on Decision and Control (CDC)*, pp. 2365–2370, 2014.

[7] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson, "Asymptotic and exponential stability of general classes of continuous-time power control laws in wireless networks," *IEEE Conference on Decision and Control (CDC)*, pp. 49–54, 2013.

[8] V. Jonsson, A. Rantzer, and R. M. Murray, "A scalable formulation for engineering combination therapies for evolutionary dynamics of disease," *IEEE American Control Conference (ACC)*, 2014.

[9] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. John Wiley and Sons, New York, 2000.

[10] M. Colombino and R. S. Smith, "Convex characterization of robust stability analysis and control synthesis for positive linear systems," *IEEE Conference on Decision and Control*, pp. 4379–4384, 2014.

[11] Y. Ebihara, D. Peaucelle, and D. Arzelier, "LMI approach to linear positive system analysis and synthesis," *Systems & Control Letters*, vol. 63, pp. 50–56, 2014.

[12] J. Shen and J. Lam, "l_∞/L_∞-gain analysis for positive linear systems with unbounded time-varying delays," *IEEE Transactions on Automatic Control*, vol. 60, no. 3, pp. 857–862, 2015.

[13] A. Rantzer, "Scalable control of positive systems," *European Journal of Control*, 2015.

[14] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*. Springer, 2014.

[15] W. Haddad and V. Chellaboina, "Stability theory for non-negative and compartmental dynamical systems with time delay," *Systems & Control Letters*, vol. 51, no. 5, pp. 355–361, 2004.

[16] X. Liu, W. Yu, and L. Wang, "Stability analysis of positive systems with bounded time-varying delays," *IEEE Transactions on Circuits and Systems II*, vol. 56, no. 7, pp. 600–604, 2009.

[17] M. Ait Rami, "Stability analysis and synthesis for linear positive systems with time-varying delays," *International Symposium on Positive Systems (POSTA)*, pp. 205–216, 2009.

[18] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson, "On the rate of convergence of continuous-time linear positive systems with heterogeneous time-varying delays," *European Control Conference (ECC)*, pp. 3372–3377, 2013.

[19] X. Liu, W. Yu, and L. Wang, "Stability analysis for continuous-time positive systems with time-varying delays," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 1024–1028, 2010.

[20] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson, "Asymptotic stability and decay rates of positive linear systems with unbounded delays," *IEEE Conference on Decision and Control (CDC)*, pp. 1423–1428, 2013.

[21] O. Mason and M. Verwoerd, "Observations on the stability of cooperative systems," *Systems & Control Letters*, pp. 461–467, 2009.

[22] V. Bokharaie, O. Mason, and M. Verwoerd, "D-stability and delay-independent stability of homogeneous cooperative systems," *IEEE Transactions on Automatic Control*, pp. 2882–2885, 2010.

[23] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson, "Exponential stability of homogeneous positive systems of degree one with time-varying delays," *IEEE Transactions on Automatic Control*, vol. 59, pp. 1594–1599, 2014.

[24] V. S. Bokharaie and O. Mason, "On delay-independent stability of a class of nonlinear positive time-delay systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1974–1977, 2014.

[25] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson, "Asymptotic stability and decay rates of homogeneous positive systems with bounded and unbounded delays," *SIAM Journal on Control and Optimization*, vol. 52, no. 4, pp. 2623–2650, 2014.

[26] T. Tanaka, C. Langbort, and V. Ugrinovskii, "DC-dominant property of cone-preserving transfer functions," *Systems & Control Letters*, vol. 62, no. 8, pp. 699–707, 2013.

[27] J. Shen and W. X. Zheng, "Stability analysis of linear delay systems with cone invariance," *Automatica*, vol. 53, pp. 30–36, 2015.

[28] H. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. AMS, 1995.

[29] D. Angeli and E. Sontag, "Monotone control systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 10, pp. 1684–1698, 2003.

[30] S. Dashkovskiy, B. Ruffer, and F. Wirth, "Discrete-time monotone systems: criteria for global asymptotic stability and applications," *MTNS Conference*, pp. 89–97, 2006.

[31] A. Rantzer and B. Bernhardsson, "Control of convex-monotone systems," *IEEE Conference on Decision and Control (CDC)*, pp. 2378–2383, 2014.

[32] E. Lovisari, G. Como, and K. Savla, "Stability of monotone dynamical flow networks," *IEEE Conference on Decision and Control (CDC)*, pp. 2384–2389, 2014.

[33] J. D. Murray, *Mathematical Biology I: An Introduction*. Springer, New York, 2002.

[34] R. Yates, "A framework for uplink power control in cellular radio systems," *IEEE Journal on Selected Areas in Communications*, vol. 13, no. 7, pp. 1341–1347, 1995.

[35] H. R. Feyzmahdavian, M. Johansson, and T. Charalambous, "Contractive interference functions and rates of convergence of distributed power control laws," *IEEE Transactions on Wireless Communications*, vol. 11, no. 12, pp. 4494–4502, 2012.

[36] M. W. Hirsch and H. Smith, *Monotone dynamical systems*, 2005.

[37] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson, "Subhomogeneous positive monotone systems are insensitive to heterogeneous time-varying delays," *International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, 2014.