# HILBERT SPACES OF DIRICHLET SERIES

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A Dirichlet series is a function of the type $\sum_{n\in\mathbb{N}} a_n n^{-s}$. A natural space of such functions, first considered by H. Hedenmalm, P. Lindqvist and K. Seip in 1997, is the Dirichlet-Hardy space

$$\mathscr{H}^2 = \left\{ \sum_{n\in\mathbb{N}} a_n n^{-s} : (a_n) \in \ell^2 \right\}.$$

By the Cauhcy-Schwarz inequality, the functions in this space are analytic for $\operatorname{Re} s > 1/2$. A basic feature of this space is that the reproducing kernels are translates of the Riemann zeta function. Another indication that the space $\mathscr{H}^2$ is interesting follows from the observation by H. Bohr in 1913 that there is a natural correspondence between Dirichlet series and power series in infinitely many variables. This gives a natural identification of $\mathscr{H}^2$ with the Hardy space on the infinite dimensional torus, $H^2(\mathbb{T}^\infty)$.

A result of importance, previously known by analytic number theorists and rediscovered by Hedenmalm, Lindqvist and Seip, says that functions in the Dirichlet-Hardy space are locally in $L^2$ on the abscissa $\Re s = 1/2$. Later, it was shown by J.-F. Olsen and K. Seip that the Dirichlet-Hardy space and the classical Hardy space of the half-plane $\Re s > 1/2$ have the same bounded interpolating sequences.

We discuss these two results, and their counter-parts for other Hilbert spaces of Dirichlet series, or equivalently, spaces of power series in infinitely many variables. Examples include the Bergman and Dirichlet spaces on the infinite dimensonal torus, as well as the Drury-Arveson space on the infinite dimensional unit ball.