SPECTRAL NOTIONS FOR CONFORMAL MAPS: A SURVEY

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ABSTRACT. The universal means spectrum of conformal mappings has been studied extensively in recent years. In some situations, sharp results are available, in others, only upper and lower estimates have been obtained so far. We review some of the classical results before discussing the recent work of Hedenmalm and Shimorin on estimates of the universal means spectrum near the origin. It is our ambition to explain how their method works and what its limitations are. We then move on to the recent study of the universal means spectrum of bounded functions near the point two conducted by Baranov and Hedenmalm. A number of open problems related to these topics are pointed out together with some auxilliary results which are interesting in their own right.

1. INTRODUCTION

Classes of conformal mappings. We say that a function $\varphi : \mathbb{D} \to \mathbb{C}$ which is univalent in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ belongs to the class $\mathcal{S}$ if it satisfies the requirements $\varphi(0) = 0$ and $\varphi'(0) = 1$. This means that the Taylor series of a function $\varphi \in \mathcal{S}$ is of the form

$$\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

We denote by $\mathcal{S}_b$ the class of bounded univalent functions in the disk (with the normalization $\varphi(0) = 0$ only). Finally, we shall also consider the subclass $\mathcal{S}_1$ of univalent functions φ in the unit disk with $\varphi(0) = 0$ and

$$\|\varphi\|_\infty = \sup_{z \in \mathbb{D}} |\varphi(z)| \le 1.$$

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A close relative of $\mathcal{S}$ is the class $\mathbf{\Sigma}$ of holomorphic functions $\psi : \mathbb{D}_e \to \mathbb{C}_\infty$ (we write $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere) which are univalent in the exterior disk

$$\mathbb{D}_e = \{z \in \mathbb{C}_\infty : |z| > 1\}$$

and have a power series expansion of the form

$$\psi(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}.$$

The classes $\mathcal{S}$, $\mathcal{S}_b$ and $\mathbf{\Sigma}$ have been the subjects of extensive studies; we suggest that the reader consult Peter Duren's book [12] for general background material.

The functions in $\mathcal{S}$ can be viewed as conformal mappings of the unit disk onto (normalized) simply connected domains in $\mathbb{C}$, while functions in $\mathcal{S}_b$ map onto bounded domains in $\mathbb{C}$. Similarly, functions in $\mathbf{\Sigma}$ map $\mathbb{D}_e$ onto $\mathbb{C}_\infty \setminus K$, where K is some (normalized) connected compact subset of $\mathbb{C}$ which does not divide the plane.

The simplest example of a function in $\mathcal{S}_b$ (and $\mathcal{S}_1$) is $\varphi(z) = z$, which maps the unit disk onto itself. Another, perhaps more interesting, function in the class $\mathcal{S}$ is the so-called *Kœbe function*

$$k(z) = \frac{z}{(1+z)^2}, \quad z \in \mathbb{D}. \tag{1.1}$$

The Kœbe function maps the unit disk onto the complement of the slit $[\frac{1}{4}, +\infty[$, an unbounded domain. Finally, we are provided with an example of a function in $\mathbf{\Sigma}$ by

$$l(z) = z + \frac{1}{z}, \quad z \in \mathbb{D}_e; \tag{1.2}$$

this function maps $\mathbb{D}_e$ onto the complement of the line segment $[-2, 2]$.

In many of the classical problems and theorems of *Geometric Function Theory*, the Kœbe function is extremal for the class $\mathcal{S}$. For instance, it is known that the image of *every* function in $\mathcal{S}$ contains a disk of radius 1/4. Hence the image of the Kœbe function is extremal for $\mathcal{S}$ in this respect. Moreover, for $\varphi \in \mathcal{S}$, we have the estimates

$$\frac{1-|z|}{(1+|z|)^3} \le |\varphi'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in \mathbb{D}, \tag{1.3}$$

which are sharp for the Kœbe function and its rotations.

Coefficient problems. The famous *Bieberbach conjecture* proved in 1985 by Louis de Branges asserts that if $\varphi \in \mathcal{S}$ and $\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then

$$|a_n| \le n, \quad n = 2, 3, 4, \ldots,$$

and that if equality holds for some n, then φ is the Kœbe function.

For $\psi \in \mathbf{\Sigma}$, there is a result similar to (1.3); namely, we have

$$\frac{|z|^2-1}{|z|^2} \le |\psi'(z)| \le \frac{|z|^2}{|z|^2-1}, \quad z \in \mathbb{D}_e. \tag{1.4}$$

Let us turn to the problem of estimating the coefficients of functions in $\mathbf{\Sigma}$, with expansion $\psi(z) = z + b_0 + \sum_n b_n z^{-n}$. Of course, we cannot say anything about b_0, since we may always introduce translations without disturbing the univalence of a function $\psi \in \mathbf{\Sigma}$. It follows immediately from Grönwall's celebrated *area theorem* (see [12], p.29) that $|b_1| \le 1$. The function in (1.2) shows that this result is sharp. One should note that

$$l(z) = \left[k(z^{-2})\right]^{-\frac{1}{2}} = z + \frac{1}{z},$$

where k is the Kœbe function. Next, in 1938, Max Schiffer proved that $|b_2| \le 2/3$, with equality for the function

$$\left[k(z^{-3})\right]^{-\frac{1}{3}} = z + \frac{2}{3}z^{-2} + \cdots .$$

Now, one might suspect that the inequality

$$|b_n| \le \frac{2}{n+1}$$

should hold for the remaining coefficients as well, but this turns out to be false already for the third coefficient. The correct sharp bound for the third coefficient is $|b_3| \le \frac{1}{2} + \mathrm{e}^{-6}$; this was obtained by Paul Garabedian and Schiffer in 1955 (see chapter 4.7 of [12]). The problem of finding optimal estimates for the coefficients of the class $\mathbf{\Sigma}$ is clearly a delicate matter. For instance, it seems that the extremal functions should map onto domains of a fractal nature; this is in contrast to the slit plane which is the image of the disk under the Kœbe function. To this day, no sharp bounds for $|b_n|$ are known for $n = 4, 5, 6, \ldots$.

It is also natural to consider coefficient problems for other classes of univalent functions in $\mathbb{D}$, for example, for bounded functions in $\mathcal{S}$ or for the class $\mathcal{S}_1$. As a matter of fact, the coefficient problem for $\mathcal{S}_1$ is closely related to that for $\mathbf{\Sigma}$ (Carleson and Jones [9]; see also section 8.1 of [12]). As we shall see below, the coefficient problems lead, in a natural way, to a study of the integral means of the derivative of functions in $\mathcal{S}$ and $\mathbf{\Sigma}$. At the same time, it is of independent interest to understand the behaviour of conformal maps *in the mean*. The remaining part of this paper is devoted to questions related to various spectral notions for conformal mappings.

Integral means spectra. We define the *integral mean* of a function $\varphi \in \mathcal{S}$ for a real number t by

$$M_t[\varphi'](r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi'(\mathrm{e}^{i\theta})|^t \mathrm{d}\theta, \qquad 0 < r < 1,$$

and, more generally, for complex τ by setting

$$M_\tau[\varphi'](r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |[\varphi'(r\mathrm{e}^{i\theta})]^\tau| \mathrm{d}\theta, \qquad 0 < r < 1.$$

For functions ψ in $\mathbf{\Sigma}$, we consider the quantity

$$M_\tau[\psi'](r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |[\psi'(r\mathrm{e}^{i\theta})]^\tau| \mathrm{d}\theta, \quad \text{for} \quad 1 < r < +\infty.$$

The expression $[\varphi']^\tau$ is defined is terms of the complex logarithm. As the derivative φ' is never zero in $\mathbb{D}$, we may select a unique single-valued branch of $\log \varphi'(z)$ by requiring that $\log \varphi'(0) = 0$. For real t, the integral means enable us to measure the expansion and contraction associated with a given mapping, and by considering complex parameters, we also take rotation into account.

The classical properties of the class $\mathcal{S}$ ensure that for each τ there exist non-negative numbers β such that

$$M_\tau[\varphi'](r) = \mathcal{O}\left(\frac{1}{(1-r)^\beta}\right), \quad \text{as} \quad r \to 1^-. \tag{1.5}$$

For a given τ, we take $\beta_\varphi(\tau)$ to be the infimum of all non-negative β such that (1.5) holds. The function β_φ is usually called the *integral means spectrum* of φ. We note that equivalently, $\beta_\varphi(\tau)$ may be expressed as a limsup:

$$\beta_\varphi(\tau) = \limsup_{r\to 1} \frac{\log M_\tau[\varphi'](r)}{\log \frac{1}{1-r}}.$$

The *universal integral means spectra* for the classes $\mathcal{S}$ and $\mathcal{S}_b$ are defined by

$$\mathrm{B}_{\mathcal{S}}(\tau) = \sup_{\varphi\in\mathcal{S}} \beta_\varphi(\tau) \quad \text{and} \quad \mathrm{B}_{\mathcal{S}_b}(\tau) = \sup_{\varphi\in\mathcal{S}_b} \beta_\varphi(\tau).$$

It follows from the Hölder inequality that β_φ is a convex function for each $\varphi \in \mathcal{S}$, and hence we see that all the universal integral means spectra are convex functions of their respective arguments. Of course, we can define the integral means spectrum and the universal integral means spectrum for functions $\psi \in \mathbf{\Sigma}$ in a similar manner. In this case, we define $\beta_\psi(\tau)$ as the infimum of all β such that

$$M_\tau[\psi'](r) = \mathcal{O}\left(\frac{1}{(r-1)^\beta}\right), \quad \text{as} \quad r \to 1^+. \tag{1.6}$$

The universal integral means spectrum $\mathrm{B}_{\mathbf{\Sigma}}$ is then defined in the analogous fashion. It is possible to define universal integral means spectra in a slightly different manner, by interchanging the two limit operations:

$$\mathrm{B}_{\mathcal{S}}^{\infty}(\tau)=\limsup_{r\to 1}\sup_{\varphi\in\mathcal{S}}\frac{\log M_\tau[\varphi'](r)}{\log\frac{1}{1-r}}.$$

Similarly, we obtain spectral functions $\mathrm{B}_{\mathcal{S}_1}^{\infty}(\tau)$ and $\mathrm{B}_{\mathbf{\Sigma}}^{\infty}(\tau)$ (it is easy to see that $\mathrm{B}_{\mathcal{S}_b}^{\infty}(\tau)\equiv\mathrm{B}_{\mathcal{S}}^{\infty}(\tau)$, so we do not get anything new for the class $\mathcal{S}_b$). We call these *uniform universal integral means spectra*. Clearly, we have

$$\mathrm{B}_{\mathcal{S}_b}(\tau)\le\mathrm{B}_{\mathcal{S}_1}^{\infty}(\tau),\quad \mathrm{B}_{\mathbf{\Sigma}}(\tau)\le\mathrm{B}_{\mathbf{\Sigma}}^{\infty}(\tau),\quad \mathrm{B}_{\mathcal{S}}(\tau)\le\mathrm{B}_{\mathcal{S}}^{\infty}(\tau),$$

and one believes that all the stated inequalities are indeed identities. Most of the known estimates of integral means spectra actually apply to the uniform integral means spectra as well. In the sequel, we will mention uniform universal integral means spectra only when they are actually needed.

It turns out that it is a difficult problem to determine the universal integral means spectra. Sharp results are available only for certain values of real t; in general, we only have more or less refined estimates. The main objective of this paper is to present some recent methods to estimate the universal integral means spectrum. We will also point out some open problems that arise in connection with the study of integral means spectra of conformal mappings.

Integral means spectra and weighted Bergman spaces. We can also define $\beta_\varphi(t)$ in terms of *weighted Bergman spaces*. We let $\mathrm{d}A(z)$ denote the usual normalized area measure in the plane, that is,

$$\mathrm{d}A(z)=\frac{dxdy}{\pi},\quad z=x+\mathrm{i}y.$$

We also introduce the probability measure on $\mathbb{D}$ given by

$$\mathrm{d}A_\alpha(z)=(\alpha+1)(1-|z|^2)^\alpha\mathrm{d}A(z),\qquad -1<\alpha<+\infty.$$

For $-1<\alpha<+\infty$, the *Bergman space* $A^2_\alpha(\mathbb{D})$ then consists of functions f that are holomorphic in the unit disk and satisfy

$$\|f\|_\alpha^2=\int_{\mathbb{D}}|f(z)|^2\mathrm{d}A_\alpha(z)<+\infty. \tag{1.7}$$

In the case $\alpha=0$, we usually write $A^2(\mathbb{D})$ instead of $A^2_0(\mathbb{D})$. We then see that

$$\beta_\varphi(\tau)=\inf\left\{\alpha+1:[\varphi']^{\tau/2}\in A^2_\alpha(\mathbb{D})\right\}. \tag{1.8}$$

Similarly, we can define the spectrum for the class $\mathbf{\Sigma}$ in terms of Bergman spaces. In this case, we consider the Bergman spaces $A^2_\alpha(\mathbb{D}_e)$ of functions f that are holomorphic in $\mathbb{D}_e$ and satisfy

$$\|f\|^2_{\alpha,e} = \int_{\mathbb{D}_e} |f(z)|^2(1-|z|^{-2})^\alpha \frac{\mathrm{d}A(z)}{|z|^4} < +\infty, \tag{1.9}$$

and then we have

$$\beta_\psi(\tau) = \inf\big\{\alpha + 1 : [\psi']^{\tau/2} \in A^2_\alpha(\mathbb{D}_e)\big\}. \tag{1.10}$$

Regarding the uniform universal integral means spectra, we mention that $\alpha > \mathrm{B}^\infty_{\mathcal{S}}(\tau) - 1$ means that

$$\sup_{\varphi\in\mathcal{S}} \big\|[\varphi']^{\tau/2}\big\|_\alpha < +\infty;$$

analogous statements apply to the other classes $\mathcal{S}_1$ and $\mathbf{\Sigma}$.

Growth models for continua. One way to approach the problem of estimating integral means spectra from below is to grow compact continua (which do not divide the plane) in some predictable manner. It is expected that the continua should exhibit fractal behavior in order to be at least close to extremal. The Lœwner evolution equation is a natural way to grow such continua. It is rather natural to try a probabilistic approach. If we pick Brownian motion on the unit circle as driving function, we get SLE (see, for instance, [28]). Another discrete probabilistic evolution was suggested by Beliaev and Smirnov [4]. Yet another model is known as DLA (diffusion limited aggregation), introduced by Witten and Sander [33]. A related deterministic model was studied by Carleson and Makarov [11].

Outline of the paper. We first discuss general results comparing the various integral means spectra. We then mention the various methods that give estimates from above and from below of the integral means spectra, with a focus on estimates from above. One of the methods is based on Bloch-type properties of $\log\varphi'$ for $\varphi\in\mathcal{S}$, and we mention interesting open problems in the context. We also look at the Carleson-Jones-Makarov method to estimate $\mathrm{B}_b(t)$ for t close to 2.

2. Sharp results

The class $\mathcal{S}$. Let us begin by reviewing the known results on the universal integral means spectrum for the class $\mathcal{S}$ for real values. Using the classical pointwise estimates for $\varphi\in\mathcal{S}$ given in (1.3) and the fact that $\varphi(z)=z$ belongs to $\mathcal{S}$, we immediately obtain the *trivial bounds* for the universal integral means spectrum:

$$0 \le \mathrm{B}_{\mathcal{S}}(t) \le \max\{3t, -t\}, \quad t\in\mathbb{R}.$$

While the pointwise estimates (1.3) are sharp for the Kœbe function $k(z) = z/(1-z)^2$ and its rotations $k_\lambda(z) = z/(1-\lambda z)^2$, $|\lambda| = 1$, one cannot truly expect that they should lead to sharp estimates for the integral means of functions in $\mathcal{S}$. The exact values of $\mathrm{B}_{\mathcal{S}}$ have been computed for large positive t by J. Feng and Thomas MacGregor (see [13]).

Theorem 2.1. *We have*

$$\mathrm{B}_{\mathcal{S}}(t) = 3t - 1 \quad \textit{for} \quad \frac{2}{5} \le t < \infty. \tag{2.1}$$

We refer the reader to [27], chapter 8, for details.

The universal integral means spectrum for large negative values of t has been studied by Lennart Carleson and Nikolai Makarov.

Theorem 2.2. *There exists a constant $t_0 < 0$ such that*

$$\mathrm{B}_{\mathcal{S}}(t) = -t - 1 \quad \textit{for} \quad -\infty < t \le t_0. \tag{2.2}$$

In their paper [10], Carleson and Makarov derive this theorem from certain results concerning counting the number of disks with large harmonic measure. Their paper also contains some interesting remarks on the properties one might expect of the functions corresponding to extremal growth.

One of the most interesting open problems in the theory of conformal mappings is to determine the optimal t_0. It is only known that $t_0 \le -2$. The statement that in fact we may take $t_0 = -2$, or equivalently, that $\mathrm{B}_{\mathcal{S}}(-2) = 1$, is usually referred to as *Brennan's conjecture*.

The classes $\mathcal{S}_b$ and $\mathbf{\Sigma}$. As we remarked before, the classes $\mathcal{S}_b$ and $\mathbf{\Sigma}$ are closely related. In fact, we shall show the following.

Theorem 2.3. *We have*

$$\mathrm{B}_{\mathcal{S}_b}(\tau) = \mathrm{B}_{\mathbf{\Sigma}}(\tau) \quad \textit{for} \quad \tau \in \mathbb{C}. \tag{2.3}$$

Proof. Our argument is inspired by Carleson and Jones [9]. First, let α' be such that $\alpha' < \mathrm{B}_{\mathcal{S}_b}(\tau)$. This means that there exists some $\varphi \in \mathcal{S}_b$ such that that $(\varphi')^{\tau/2}$ fails to be in $A^2_{\alpha'}(\mathbb{D})$. An *inversion* produces a function

$$\psi(z) = \frac{\varphi'(0)}{\varphi(1/z)}, \qquad z \in \mathbb{D}_e,$$

which is in $\mathbf{\Sigma}$, and a change of variables shows that

$$\int_{\mathbb{D}_e} |[\psi'(z)]^\tau| \, (1-|z|^{-2})^\alpha \frac{\mathrm{d}A(z)}{|z|^4} = |[\varphi'(0)]^\tau| \int_{\mathbb{D}} \left| \left[\frac{w^2 \varphi'(w)}{[\varphi(w)]^2} \right]^\tau \right| \mathrm{d}A_\alpha(z).$$

In Section 4 of [1], an estimate of $|\log(\varphi/z)|$ was obtained for $\varphi \in \mathcal{S}_b$, which shows that the contribution of $w/\varphi(w)$ to the integral on the right hand side is modest. In particular, the right hand side integral diverges for $\alpha = \alpha'$, so that $(\psi')^{\tau/2}$ is not in $A^2_{\alpha'}(\mathbb{D}_e)$. As a consequence, we get that $\alpha' \leq \mathrm{B}_{\boldsymbol{\Sigma}}(\tau)$. We conclude that

$$B_{\mathcal{S}_b}(\tau) \leq B_{\boldsymbol{\Sigma}}(\tau).$$

To get the reverse inequality, we proceed as follows. Suppose $\alpha' < \mathrm{B}_{\boldsymbol{\Sigma}}(\tau)$, so that there exists a $\psi \in \boldsymbol{\Sigma}$ so that $(\psi')^{\tau/2}$ is not in $A^2_{\alpha'}(\mathbb{D}_e)$. We choose $\Omega \subset \mathbb{D}_e$ to be a bounded simply connected domain with C^∞-smooth boundary such that

$$\int_\Omega \left|[\psi'(z)]^\tau\right| (1 - |z|^{-2})^{\alpha'} \frac{\mathrm{d}A(z)}{|z|^4} = +\infty; \tag{2.4}$$

we may convince ourselves about the existence of such an Ω by the following argument. First, we construct a bounded simply connected $\Omega_0 \subset \mathbb{D}_e$ with C^∞-smooth boundary such that $\partial\Omega$ contains an arc of $\mathbb{T} = \partial\mathbb{D}$ of length $> \pi$. The final Ω is chosen among the various rotations of Ω_0. It is an important observation that φ maps Ω onto a bounded region in $\mathbb{C}$.

There exists a C^∞-smooth conformal mapping ϕ which maps $\mathbb{D}$ onto Ω. The map $\varphi(z) = \psi \circ \phi(z)$ is then bounded and univalent, so that $\tilde{\varphi}(z) = \psi \circ \phi(z) - \psi \circ \phi(0)$ belongs to $\mathcal{S}_b$. Exploiting the fact that ϕ' is non-zero throughout the closed unit disk and performing the obvious change of variables, we obtain the following chain of inequalities:

$$\begin{aligned}
\int_{\mathbb{D}} \left|[\tilde{\varphi}'(z)]^\tau\right| (1 - |z|^2)^\alpha \mathrm{d}A(z) &= \int_{\mathbb{D}} \left|[\varphi'(z)]^\tau\right| (1 - |z|^2)^\alpha \mathrm{d}A(z) \\
&= \int_{\mathbb{D}} \left|[\phi'(z)]^\tau \, [\psi' \circ \phi(z)]^\tau\right| (1 - |z|^2)^\alpha \mathrm{d}A(z) \\
&= \int_\Omega \left|[\psi'(w)]^\tau\right| (1 - |\phi^{-1}(w)|^2)^\alpha \left|\{[\phi^{-1}]'(w)\}^{2-\tau}\right| \mathrm{d}A(w) \\
&\geq \delta \int_\Omega \left|[\psi'(w)]^\tau\right| (|w|^2 - 1)^\alpha \mathrm{d}A(w),
\end{aligned} \tag{2.5}$$

for some small but positive δ. In view of (2.4), the bottom integral diverges for $\alpha = \alpha'$, and hence $[\tilde{\varphi}']^{\tau/2} \notin A^2_{\alpha'}(\mathbb{D})$. In particular, $\alpha' \leq \mathrm{B}_{\boldsymbol{\Sigma}}(\tau)$, so that

$$\mathrm{B}_{\boldsymbol{\Sigma}}(\tau) \leq \mathrm{B}_{\mathcal{S}_b}(\tau).$$

The assertion of the theorem is now immediate. □

The exact values of the universal integral means spectrum $\mathrm{B}_{\mathcal{S}_b}$ are known for large t.

Theorem 2.4. *We have*

$$\mathrm{B}_{\mathcal{S}_b}(t) = t - 1, \quad \textit{for} \quad t \geq 2. \tag{2.6}$$

The universal integral means spectrum near the point two has been studied by Peter Jones and Makarov, who obtained the following result in [22].

Theorem 2.5. *For real t, the following estimate holds:*

$$\mathrm{B}_{\mathcal{S}_b}(2 - t) = 1 - t + \mathcal{O}(t^2), \quad \textit{as} \quad t \to 0. \tag{2.7}$$

In the introduction, we mentioned that the integral means of derivatives are related to certain coefficient problems. More precisely, $B_{\mathcal{S}_1}(1)$ determines the slowest possible rate of decay of the Taylor coefficients of functions in $\mathcal{S}_1$ and, similarily, $B_{\mathbf{\Sigma}}(1)$ determines the rate of decay of the Laurent series coefficients of functions in $\mathbf{\Sigma}$. Let us see why this is so.

By applying the Cauchy estimate

$$n|a_n| \leq \frac{1}{2\pi r^{n-1}} \int_{|z|=r} |\varphi'(re^{i\theta})| \mathrm{d}\theta$$

with $r = 1 - \frac{1}{n}$ to the coefficients a_n of a function $\varphi \in \mathcal{S}_1$, we immediately get an upper estimate in terms of the integral means:

$$|a_n| \leq \frac{C}{n} M_1[\varphi'] \left(1 - \frac{1}{n}\right), \quad n = 2, 3, 4, \ldots,$$

for some positive constant C. The corresponding estimate for the coefficients b_n of a function in $\psi \in \mathbf{\Sigma}$ reads

$$|b_n| \leq \frac{C}{n} M_1[\psi'] \left(1 + \frac{1}{n}\right), \quad n = 2, 3, 4, \ldots.$$

In their paper [9], Carleson and Jones show that these estimates are essentially sharp. To be precise, let us put

$$B_n = \sup_{\psi \in \mathbf{\Sigma}} |b_n|;$$

Carleson and Jones prove that there exists a positive constant c_0 such that

$$B_n \geq \frac{c_0}{n} \sup_{\psi \in \mathbf{\Sigma}} M_1[\psi'] \left(1 + \frac{1}{n}\right);$$

and a similar result is deduced for functions in $\mathcal{S}_1$. As we interpret these inequalities in terms of the uniform universal integral means spectrum, we see that the uniform bound for the Laurent coefficients of functions in $\mathbf{\Sigma}$ decay at a rate proportional to

$$n^{-1+\mathrm{B}^{\infty}_{\mathbf{\Sigma}}(1)+\varepsilon}$$

for each fixed $\varepsilon > 0$, as n tends to infinity, and the corresponding statement holds for functions in $\mathcal{S}_1$. This fact provides us with some additional motivation to study the spectra $\mathrm{B}_{\mathcal{S}_b}$ and $\mathrm{B}_{\boldsymbol{\Sigma}}$ (as well as their uniform counterparts). The value of $\mathrm{B}_{\boldsymbol{\Sigma}}(1)$ is also important for a problem of Littlewood on the extremal behavior of polynomials (see [3]).

From the convexity of $\mathrm{B}_{\mathcal{S}_b}$, we obtain the trivial estimate

$$\mathrm{B}_{\mathcal{S}_b}(1) \leq \frac{1}{2},$$

which has been improved by James Clunie, Christian Pommerenke and others. For example, Arkadi Grinshpan and Pommerenke (see [15]) have shown that $\mathrm{B}_{\mathcal{S}_b}(1) \leq 0.4884$. Recently, Hedenmalm and Shimorin [19] obtained $\mathrm{B}_{\mathcal{S}_b}(1) \leq \mathrm{B}^{\infty}_{\boldsymbol{\Sigma}}(1) \leq 0.46$. Carleson and Jones have conjectured that $\mathrm{B}_{\mathcal{S}_b}(1) = \mathrm{B}^{\infty}_{\mathcal{S}_1}(1) = \mathrm{B}^{\infty}_{\boldsymbol{\Sigma}}(1) = 1/4$.

Relationship between $\mathrm{B}_{\boldsymbol{\Sigma}}$ and $\mathrm{B}_{\mathcal{S}}$. We now point out an important relationship between the spectra of the classes $\mathcal{S}$ and $\boldsymbol{\Sigma}$ (and thus, between $\mathrm{B}_{\mathcal{S}}$ and $\mathrm{B}_{\mathcal{S}_b}$) found by Makarov (see [24]).

Theorem 2.6. *We have*

$$\mathrm{B}_{\mathcal{S}}(t) = \max\big\{\mathrm{B}_{\boldsymbol{\Sigma}}(t), 3t-1\big\}, \qquad t \in \mathbb{R}. \tag{2.8}$$

Hence, we would know $\mathrm{B}_{\mathcal{S}}$ if we could compute the universal integral means spectrum for functions in $\boldsymbol{\Sigma}$. For complex values τ, we do not know much about the values of the universal integral means spectra. In an unpublished manuscript, Ilia Binder extends Makarov's results to the complex setting (see [7]).

Theorem 2.7. *Suppose* $\operatorname{Re}\tau \leq 0$. *Then*

$$\mathrm{B}_{\mathcal{S}}(\tau) = \mathrm{B}_{\boldsymbol{\Sigma}}(\tau). \tag{2.9}$$

If, on the other hand, $\operatorname{Re}\tau > 0$, *then*

$$\mathrm{B}_{\mathcal{S}}(\tau) = \max\big\{\mathrm{B}_{\boldsymbol{\Sigma}}(\tau), |\tau| + 2\operatorname{Re}\tau - 1\big\}. \tag{2.10}$$

A heuristic argument suggesting that $\mathrm{B}^{\infty}_{\boldsymbol{\Sigma}} = \mathrm{B}_{\boldsymbol{\Sigma}}$. From the definitions, it is clear that $\mathrm{B}_{\boldsymbol{\Sigma}}(\tau) \leq \mathrm{B}^{\infty}_{\boldsymbol{\Sigma}}(\tau)$ holds for all complex τ. Next suppose that for some α, $-1 < \alpha < +\infty$, we have

$$\sup_{\psi \in \boldsymbol{\Sigma}} \big\|[\psi']^{\tau/2}\big\|_{\alpha,e} = +\infty.$$

We would then like to construct a single $\psi_0 \in \boldsymbol{\Sigma}$ such that

$$\big\|[\psi_0']^{\tau/2}\big\|_{\alpha',e} = +\infty$$

for each α' with $-1 < \alpha' < \alpha$. First, we find a sequence ψ_j in $\mathbf{\Sigma}$ such that

$$\big\|[\psi_j']^{\tau/2}\big\|_{\alpha,e} \geq m_j,$$

where m_j is a very rapidly increasing sequence (one could pick, for instance, $m_j = 2^{2^j}$). Let $K_j = \mathbb{C} \setminus \psi(\mathbb{D}_e)$ be the associated compact continua. We will tag on a linear segment to each K_j by the following procedure. Consider the convex hull $\widehat{K}_j$ of K_j, and pick a strictly convex boundary point w_j of $\widehat{K}_j$; then, clearly, $w_j \in K_j$. A line segment L_j emanating from w_j is chosen in a direction perpendicular to $\partial\widehat{K}_j$ if there exists a boundary tangent at the point; if there is no tangent, we have more freedom and just make sure that the angle to each of the two tangential directions at w_j is at least $\frac{1}{2}\pi$. The length of L_j should be allowed to increase with j, but at a rate much slower than that of m_j. Next, we realize that $K_j \cup L_j$ is a continuum which does not divide the plane; we also scale this new continuum so that it gets diameter equal to 1, and call it M_j. We may repeat the procedure of adding a line segment, and can thus assume that M_j has diameter 1 but two line segments emanating from it, of the same length, pointing in approximately the opposite directions. Unless we made a particularly unlucky choice of the two points where the two line segments were adjoined, the conformal maps from $\mathbb{D}_e$ onto $\mathbb{C}_\infty \setminus M_j$ which preserve the point at infinity will have essentially the same properties as ψ_j. The final step is to construct a continuum with all the essential geometric ingredients of all the K_j present simultaneously. We rescale each M_j to have diameter $1/j^2$, and place all the rescaled M_j's along the real line so that the line segments point left-right. We realign the rescaled M_j's slightly so that the line segments may be merged, and form their union. The union (we should add a limit point, too) – call it M – is a compact continuum, and it has all the ingredients of each K_j at once. It should be possible to show that the corresponding conformal map $\psi_1 : \mathbb{D}_e \to \mathbb{C}_\infty \setminus M$ which preserves ∞ fails to have belong to the slightly smaller weighted Bergman space $A^2_{\alpha'}(\mathbb{D}_e)$. We get the desired ψ_0 by normalization: $\psi_0(z) = \psi_1(z)/\psi_1'(\infty)$.

3. The universal integral means spectrum near the origin

Estimates from above. We have seen that the universal integral means spectrum is known for both large and small values of t. This section is devoted to estimates of $\mathrm{B}_{\mathcal{S}}$ for values of t near the origin. We begin by recalling some classical results in this direction.

An upper estimate for $B_{\mathcal{S}}$ was found by James Clunie and Pommerenke (see the book [27], chapter 8, for details). Their result is not sharp, but it covers all real values of t.

Theorem 3.1. *We have*

$$\mathrm{B}_{\mathcal{S}}(t) \le t - \frac{1}{2} + \left(4t^2 - t + \frac{1}{4}\right)^{1/2} \tag{3.1}$$

for $t \in \mathbb{R}$. *In particular, we get the asymptotic estimate*

$$\limsup_{t\to 0} t^{-2}\, \mathrm{B}_{\mathcal{S}}(t) \le 3.$$

In the proof, the elementary pointwise estimate

$$\left|\frac{\varphi''(z)}{\varphi'(z)} - \frac{2\bar{z}}{1-|z|^2}\right| \le \frac{4}{1-|z|^2}, \qquad z \in \mathbb{D}, \tag{3.2}$$

is used along with Hardy's identity to reduce to problem to the study of a certain ordinary differential inequality. More precisely, a decisive step in the proof is the estimate

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi'(r\mathrm{e}^{i\theta})|^t \left|\frac{\mathrm{e}^{i\theta}\varphi''(r\mathrm{e}^{i\theta})}{\varphi'(r\mathrm{e}^{i\theta})} - \frac{2r}{1-r^2}\right|^2 \mathrm{d}\theta \le \frac{16}{(1-r^2)^2} M_t[\varphi'](r). \tag{3.3}$$

Estimates of a similar type appear in the work of Hedenmalm and Shimorin, which will be discussed later in this paper. Their method, however, avoids the use of pointwise estimates and leads to better results.

In addition, estimates for special values of $B_{\mathcal{S}}(t)$ were found by Pommerenke (see [27]):

$$\mathrm{B}_{\mathcal{S}}(-1) \le 0.601$$

and by Daniel Bertilsson (see [5], [6]):

$$\mathrm{B}_{\mathcal{S}}(-2) \le 1.547.$$

Estimates from below. A considerable amount of work has been devoted to finding estimates for the universal integral means spectrum from below. There are both analytic and numerical results in this direction.

Makarov first proved that there exists a constant $c > 0$ such that $B_{\mathcal{S}}(t) \ge ct^2$ for t near the origin; this results was later improved by Steffen Rodhe (once again, see [27]).

Theorem 3.2. *There exists a map $\varphi \in \mathcal{S}$ with $\beta_\varphi(-1) > 0.109$ and $\beta_\varphi(t) \geq 0.117\, t^2$ for small t. Hence, we have*

$$B_{\mathcal{S}}(t) \geq 0.117\, t^2 \tag{3.4}$$

for small t.

An improved lower estimate (for positive t) was recently found by Ilgiz Kayumov (see [26]).

Theorem 3.3. *We have*

$$B_{\mathcal{S}}(t) \geq \frac{t^2}{5}, \qquad 0 < t \leq \frac{2}{5}. \tag{3.5}$$

On the numerical side, we should mention the experimental work of Philipp Kraetzer (see [23]). In view of his results, Kraetzer suggested that the equality

$$\mathrm{B}_{\mathcal{S}}(t) = \frac{t^2}{4}, \qquad -2 \leq t \leq 2, \tag{3.6}$$

might hold. This is sometimes called the *Kraetzer conjecture.*

In a recent paper (see [4]), Dmitry Beliaev and Stanislav Smirnov conduct a numerical study of the universal integral means spectrum $\mathrm{B}_{\boldsymbol{\Sigma}}$ based on the construction of a family of random fractals which they call *random conformal snowflakes.* Using this technique, Beliaev and Smirnov get the estimate $\mathrm{B}_{\boldsymbol{\Sigma}}(1) > 0.23\ldots$.

4. Recent progress near the origin

We shall now discuss in some detail the recent work of Serguei Shimorin and the first-named author on upper estimates of the universal integral means spectrum. It is our primary goal to explain the basic ideas contained in their papers. This means that we sometimes omit certain details and technical points; in such cases, we refer the reader to [31] and [19] for complete arguments.

Preliminaries. In the introduction, we mentioned that the integral means spectrum β_φ could be defined for $\varphi \in \mathcal{S}$ as

$$\beta_\varphi(\tau) = \inf\left\{\alpha + 1 : [\varphi']^{\tau/2} \in A^2_\alpha(\mathbb{D})\right\},$$

where $A^2_\alpha(\mathbb{D})$ is a weighted Bergman space on the unit disk. In fact, if we have

$$\left\|[\varphi']^{\tau/2}\right\|^2_\alpha < +\infty$$

for *every* $\varphi \in \mathcal{S}$, then $\mathrm{B}_{\mathcal{S}}(\tau) \leq \alpha + 1$. Moreover, if we have the stronger assertion

$$\sup_{\varphi\in\mathcal{S}} \left\|[\varphi']^{\tau/2}\right\|^2_\alpha < +\infty,$$

then $\mathrm{B}^{\infty}_{\mathcal{S}}(\tau) \le \alpha + 1$.

Hilbert spaces of functions on the bidisk. We now introduce a class of weighted Bergman spaces in the bidisk. For $-1 < \alpha, \beta < +\infty$ and $-\infty < \gamma < +\infty$, we consider the Hilbert space $L^2_{\alpha,\beta,\gamma}(\mathbb{D}^2)$ of measurable functions in the bidisk $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ that satisfy

$$\|f\|^2_{\alpha,\beta,\gamma} = \int_{\mathbb{D}^2} |f(z,w)|^2 |z-w|^{2\gamma} \mathrm{d}A_\alpha(z)\mathrm{d}A_\beta(w) < +\infty. \tag{4.1}$$

The Bergman space $A^2_{\alpha,\beta,\gamma}(\mathbb{D}^2)$ is the closed subspace of $L^2_{\alpha,\beta,\gamma}(\mathbb{D}^2)$ that consists of the holomorphic functions. The spaces $A^2_{\alpha,\beta,\gamma}(\mathbb{D}^2)$ were studied by Hedenmalm, Shimorin, and Sola [21], while the special case $\beta = 0$ was investigated earlier by Hedenmalm and Shimorin [19].

For a function $f \in A^2_{\alpha,\beta,\gamma}(\mathbb{D}^2)$, we define the diagonal restriction operation $\oslash$ by setting

$$\oslash[f](z) = f(z,z), \quad z \in \mathbb{D}.$$

Using reproducing kernel techniques, one sees that the image of the space $A^2_{\alpha,\beta,\gamma}(\mathbb{D}^2)$ under the diagonal restriction operator $\oslash$ may be identified (as a linear space) with the Bergman space $A^2_{\alpha+\beta+2\theta+2}(\mathbb{D})$. Moreover, we have the sharp norm inequality

$$\frac{1}{\sigma(\alpha,\beta,\gamma)} \|\oslash f\|^2_{\alpha+\beta+2\gamma+2} \le \|f\|^2_{\alpha,\beta,\gamma}, \qquad f \in A^2_{\alpha,\beta,\gamma}(\mathbb{D}^2). \tag{4.2}$$

where

$$\frac{1}{\sigma(\alpha,\beta,\gamma)} = \int_{\mathbb{D}^2} |z-w|^{2\gamma} \mathrm{d}A_\alpha(z)\mathrm{d}A_\beta(w). \tag{4.3}$$

This fact will be of crucial importance in subsequent sections.

Using the inequality (4.2) as a starting point, it is in fact possible to expand the norm of a function $f \in A^2_{\alpha,\beta,\gamma}(\mathbb{D}^2)$ in terms of the norms of its derivatives with respect to one of the variables in Bergman spaces on the unit disk:

$$\|f\|^2_{\alpha,\beta,\gamma} = \sum_{N=0}^{\infty} \frac{1}{\sigma(\alpha,\beta+N)} \left\| \sum_{k=0}^{N} a_{k,N} \boldsymbol{\partial}_z^{N-k} \oslash [\boldsymbol{\partial}_z^k f] \right\|^2_{\alpha+\beta+2\gamma+2N+2}. \tag{4.4}$$

Here, $a_{k,N}$ are certain coefficients, which depend on α, β, γ as well as on k and N. Also, $\boldsymbol{\partial}_z$ stands for the differentiation operator

$$\boldsymbol{\partial}_z = \frac{d}{dz}.$$

We omit the proof of these results, which rely on *reproducing kernel* methods, and require a considerable amount of computation. A full discussion of the elegant theory of reproducing kernels is beyond the scope of this paper; we suggest that the reader consult the book [29] as well as the paper [21].

A multiplier estimate. We begin by reviewing the results of Shimorin's paper [31]. The *Schwarzian derivative* of a function $\varphi \in \mathcal{S}$ is given by the expression

$$S[\varphi](z) = \frac{\varphi'''(z)}{\varphi'(z)} - \frac{3}{2}\left[\frac{\varphi''(z)}{\varphi'(z)}\right]^2, \qquad z \in \mathbb{D}.$$

It is classical (see [12], p. 263) that the Schwarzian derivative admits the pointwise estimate

$$|S[\varphi](z)| \le \frac{6}{(1-|z|^2)^2}, \qquad z \in \mathbb{D},$$

which is sharp for the Kœbe function. From this it follows that, for a fixed $\varphi \in \mathcal{S}$, we can view the Schwarzian derivative $S[\varphi]$ as a multiplier between the weighted Bergman spaces $A^2_{\alpha-2}(\mathbb{D})$ and $A^2_{\alpha+2}(\mathbb{D})$, with the multiplier norm estimate

$$\big\|S[\varphi]\, g\big\|^2_{\alpha+2} \le 36\,\frac{\alpha+3}{\alpha-1}\,\|g\|^2_{\alpha-2}, \quad -1 < \alpha < +\infty, \tag{4.5}$$

for all $g \in A^2_{\alpha-2}(\mathbb{D})$. Shimorin was able to improve this "trivial" inequality by avoiding the use of pointwise estimates.

Theorem 4.1. *Fix $\varphi \in \mathcal{S}$ and a real parameter α, $1 < \alpha < +\infty$. We then have the estimate*

$$\big\|S[\varphi]\, g\big\|^2_{\alpha+2} \le 36\,\frac{\alpha+1}{\alpha-1}\,\|g\|^2_{\alpha-2}, \qquad g \in A^2_{\alpha-2}(\mathbb{D}). \tag{4.6}$$

The proof runs as follows. We introduce two functions F and G by putting

$$F(z,w) = \log\left[\frac{(\varphi(w)-\varphi(z))\,wz}{(w-z)\varphi(w)\varphi(z)}\right], \quad (z,w) \in \mathbb{D}^2,\ z \ne w, \tag{4.7}$$

and

$$G(z,w) = \frac{\partial^2 F}{\partial z \partial w}(z,w), \quad (z,w) \in \mathbb{D}^2,\ z \ne w. \tag{4.8}$$

The functions F and G can be extended holomorphically to the entire bidisk $\mathbb{D}^2$. It is easy to see that

$$\oslash[G](z) = \frac{1}{6}\, S[\varphi](z), \qquad z \in \mathbb{D}.$$

Using the Grunsky inequalities, or an invariant version of Grönwall's area theorem (see [12]), we obtain the estimate

$$\big\|z \mapsto G(z,w)\big\|_0^2 \le \frac{1}{(1-|w|^2)^2}, \qquad w \in \mathbb{D}. \tag{4.9}$$

The multiplier estimate (4.6) for $g \in A^2_{\alpha-2}(\mathbb{D})$ now follows from (4.9) and (4.2) applied to the holomorphic function

$$f(z,w) = 6\,G(z,w)g(w), \quad (z,w) \in \mathbb{D}^2.$$

Indeed, since $\sigma(0,\alpha,0) = 1$, we note that

$$\|S[\varphi]g\|^2_{\alpha+2} = \|\oslash f\|^2_{\alpha+2} \le \|f\|^2_{0,\alpha,0},$$

and that

$$\begin{aligned}\|f\|^2_{0,\alpha,0} &= 36\int_{\mathbb{D}^2} |G(z,w)g(w)|^2\,\mathrm{d}A(z)\mathrm{d}A_\alpha(w)\\ &= 36\int_{\mathbb{D}} \|z\mapsto G(z,w)\|_0^2\,|g(w)|^2\,\mathrm{d}A_\alpha(w)\\ &\le 36\int_{\mathbb{D}} |g(w)|^2\,\frac{\mathrm{d}A_\alpha(w)}{(1-|w|^2)^2} = 36\,\frac{\alpha+1}{\alpha-1}\,\|g\|^2_{\alpha-2}.\end{aligned}$$

Derivatives of powers of $[\varphi']^\lambda$ and derivatives of Bergman space functions. In what follows, we will apply Theorem 4.1 to functions g of the type

$$g(z) = [\varphi'(z)]^\lambda = \exp[\lambda f(z)], \quad \lambda \in \mathbb{C}, \tag{4.10}$$

where $f = \log\varphi'$. It is easy to compute derivatives of these functions; we have

$$g' = \lambda f' \exp(\lambda f) = \lambda f' g, \tag{4.11}$$

$$g'' = (\lambda f'' + \lambda^2 [f']^2)\,g, \tag{4.12}$$

and so on. We should mention that the linear combinations and products of derivatives of φ' one obtains in this fashion have rather interesting algebraic properties. This is mentioned in [19] and [20].

We need to be able to compare the norm of a function in $A^2_\alpha(\mathbb{D})$ with the norms of its successive derivatives in the appropriate weighted Bergman spaces. The precise statement is as follows.

Proposition 4.2. *Suppose $-1 < \alpha < \infty$, and fix a real parameter ν with $0 < \nu \le 1$. We then have, for $n = 1, 2, 3, \ldots$ and for $g \in A^2_\alpha(\mathbb{D})$,*

$$0 \le (\alpha+2)_{2n}\|g\|^2_\alpha - \|g^{(n)}\|^2_{\alpha+2n} = \mathcal{O}(\|g\|^2_{\alpha+\nu}).$$

Here, the standard Pochhammer notation is used:

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1).$$

One way to prove Proposition 4.2 is to consider the Taylor coefficients of the given function and the coefficients of its derivative and use the Parseval formula to express the norm in the weighted Bergman space.

Remark 4.3. It is an open problem to find a relation similar to that of Proposition 4.2 for the Bergman L^p spaces $A^p_\alpha(\mathbb{D})$, with $0 < p < +\infty$, that is, the space of functions g that are holomorphic in the unit disk and satisfy the norm boundedness condition

$$\|g\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |g(z)|^p \mathrm{d}A_\alpha(z) < +\infty. \tag{4.13}$$

It is known (see the first chapter of [18]) that for $1 \le p < +\infty$,

$$C_1(\alpha)\|f\|_{A^p_\alpha} \le \|f'\|^p_{A^p_{\alpha+2}} \le C_2(\alpha)\|f\|^p_{A^p_\alpha} \tag{4.14}$$

holds for $f \in A^p_\alpha(\mathbb{D})$ with $f(0) = 0$. Here, $C_1(\alpha)$ and $C_2(\alpha)$ are some positive constants. The optimal constants are not known here.

Estimate of $\mathrm{B}_{\mathcal{S}}(-1)$. We are now ready to implement our results and recover Shimorin's estimates. Next, we put $\lambda = -\frac{1}{2}$ in (4.12), and see that

$$\boldsymbol{\partial}^2_z\big\{[\varphi'(z)]^{-1/2}\big\} = -\frac{1}{2}S[\varphi](z)\,[\varphi'(z)]^{-1/2}.$$

Using this together with Proposition 4.2, we get the following chain of inequalities:

$$\begin{aligned}\big\|[\varphi']^{-1/2}\big\|^2_{\alpha-2} &= \frac{1}{(\alpha)_4}\,\big\|\boldsymbol{\partial}^2_z\{[\varphi']^{-1/2}\}\big\|^2_{\alpha+2} + \mathcal{O}\big(\big\|[\varphi']^{-1/2}\big\|^2_{\alpha-2+\nu}\big)\\ &= \frac{1}{4(\alpha)_4}\big\|[\varphi']^{-1/2}\,S[\varphi]\big\|^2_{\alpha+2} + \mathcal{O}\big(\big\|[\varphi']^{-1/2}\big\|^2_{\alpha-2+\nu}\big)\\ &\le \frac{9\,(\alpha+1)}{(\alpha-1)(\alpha)_4}\|(\varphi')^{-1/2}\|^2_{\alpha-2} + \mathcal{O}\big(\big\|[\varphi']^{-1/2}\big\|^2_{\alpha-2+\nu}\big).\end{aligned} \tag{4.15}$$

We may now rewrite this as

$$\left(1 - \frac{9(\alpha+1)}{(\alpha-1)(\alpha)_4}\right)\big\|[\varphi']^{-1/2}\big\|^2_{\alpha-2} \le \mathcal{O}\big(\big\|[\varphi')^{-1/2}\big\|^2_{\alpha-2+\nu}\big). \tag{4.16}$$

Suppose that α is chosen so that

$$\frac{9(\alpha+1)}{(\alpha-1)(\alpha)_4} < 1, \tag{4.17}$$

and suppose moreover that we may find a ν, $0 < \nu \le 1$, such that

$$\sup_{\varphi \in \mathcal{S}} \big\| [\varphi']^{-1/2} \big\|^2_{\alpha-2+\nu} < +\infty,$$

due, for instance, to some a priori estimates of distortion type. We then obtain that

$$\sup_{\varphi \in \mathcal{S}} \big\| [\varphi']^{-1/2} \big\|^2_{\alpha-2} < +\infty.$$

It is immediate that $\mathrm{B}_{\mathcal{S}}(-1) \le \mathrm{B}^{\infty}_{\mathcal{S}}(-1) \le \alpha - 1$. The critical condition is (4.17), as it is usually easy to obtain a reasonable a priori estimate. An implementation yields the estimate $\mathrm{B}_{\mathcal{S}}(-1) \le 0.4196$.

4.1. **Area-type inequalities and norm expansions.** We proceed with a discussion of the methods developed in the paper [19].

Our starting point is an inequality which follows from a classical result of Prawitz (see [19] for details).

Theorem 4.4. *Let $\varphi \in \mathcal{S}$ and $0 < \theta \le 1$. We then have*

$$\int_{\mathbb{D}} \left| \varphi'(z) \left[\frac{z}{\varphi(z)} \right]^{\theta+1} - 1 \right|^2 \frac{\mathrm{d}A(z)}{|z|^{2\theta+2}} \le \frac{1}{\theta}.$$

If we are to follow the path trodden by Shimorin, our next objective is to transform the inequality of Theorem 4.4 into a multiplier inequality for functions in $A^2_\alpha(\mathbb{D})$. First, we introduce a second variable into our inequality. Actually, in the integral of Theorem 4.4, two points are present, the origin 0 and the point z. The origin may be moved to another (free) point w in $\mathbb{D}$ via a suitable Mœbius automorphism of $\mathbb{D}$, while z is kept as integration variable. After these changes, the inequality of Prawitz takes the following appearance:

$$\int_{\mathbb{D}} \big| \Phi_\theta(z,w) + L_\theta(z,w) \big|^2 \frac{\mathrm{d}A(z)}{|z-w|^{2\theta}} \le \frac{1}{\theta} \frac{1}{(1-|w|^2)^{2\theta}}, \tag{4.18}$$

where $w \in \mathbb{D}$ and $0 < \theta \le 1$. The function Φ_θ is given by

$$\Phi_\theta(z,w) = \frac{1}{z-w} \left\{ \frac{\varphi'(z)}{\varphi'(w)} \left(\frac{\varphi'(w)(z-w)}{\varphi(z)-\varphi(w)} \right)^{\theta+1} - 1 \right\}, \tag{4.19}$$

while L_θ is defined to be

$$L_\theta(z,w) = \frac{1}{z-w} \left\{ 1 - \left(\frac{1-|w|^2}{1-\overline{w}z} \right)^{1-\theta} \right\}; \tag{4.20}$$

it is assumed that $(z,w)\in\mathbb{D}^2$ and $z\neq w$. The function Φ_θ extends holomorphically across the diagonal; the diagonal values are

$$\Phi_\theta(z,z)=\frac{1-\theta}{2}\,\frac{\varphi''(z)}{\varphi'(z)},\quad z\in\mathbb{D}. \tag{4.21}$$

A similar smooth extension is possible for L_θ.

The next step is now to multiply both sides of (4.18) by a function $g\in A^2_{\alpha-2\theta}(\mathbb{D})$ and integrate with respect to the measure $\mathrm{d}A_\alpha$ in the variable w. This yields

$$\int_{\mathbb{D}^2}\big|g(w)\Phi_\theta(z,w)+g(w)L_\theta(z,w)\big|^2|z-w|^{-2\theta}\mathrm{d}A(z)\mathrm{d}A_\alpha(w)\\ \le\frac{\alpha+1}{\theta(\alpha-2\theta+1)}\,\|g\|^2_{\alpha-2\theta},$$

and we interpret the left-hand side as the norm of the function

$$g(w)\Phi_\theta(z,w)+g(w)L_\theta(z,w)$$

in the space $L^2_{0,\alpha,-\theta}(\mathbb{D}^2)$. Thus, we find that

$$\big\|g(w)\Phi_\theta(z,w)+g(w)L_\theta(z,w)\big\|^2_{0,\alpha,-\theta}\le\frac{\alpha+1}{\theta(\alpha-2\theta+1)}\,\|g\|^2_{\alpha-2\theta}. \tag{4.22}$$

At this point, we would like to use (4.4) to expand the the left hand side of (4.22) in terms of diagonal contributions. Unfortunately, the function L_θ is in general not holomorphic and so we cannot use the expansion (4.4) right away. Orthogonal projection techniques offer a remedy to this problem.

Let $\mathbf{P}_{\alpha,-\theta}$ denote the orthogonal projection from $L^2_{0,\alpha,-\theta}(\mathbb{D}^2)$ onto its subspace $A^2_{0,\alpha,-\theta}(\mathbb{D}^2)$ of holomorphic functions; moreover, write

$$\mathbf{P}^\perp_{\alpha,-\theta}=I-\mathbf{P}_{\alpha,-\theta}.$$

Since $g(w)\Phi_\theta(z,w)$ is holomorphic, we get

$$\mathbf{P}^\perp_{\alpha,-\theta}[g(w)\Phi_\theta(z,w)]=0,$$

so that we realize that it follows from (4.22) that

$$\big\|g(w)\Phi_\theta(z,w)+\mathbf{P}_{\alpha,-\theta}[g(w)L_\theta(z,w)]\big\|^2_{0,\alpha,-\theta}\\ \le\frac{\alpha+1}{\theta(\alpha-2\theta+1)}\|g\|^2_{\alpha-2\theta}-\big\|\mathbf{P}^\perp_{\alpha,-\theta}[g(w)L_\theta(z,w)]\big\|^2_{0,\alpha,-\theta}. \tag{4.23}$$

Using a combination of norm expansion techniques and explicit computations, we are able to estimate the right-hand side of this inequality in terms of the function g. More precisely, we have

$$\frac{\alpha+1}{\theta(\alpha-2\theta+1)}\|g\|^2_{\alpha-2\theta} - \left\|\mathbf{P}^{\perp}_{\alpha,-\theta}[g(w)L_\theta(z,w)]\right\|^2_{0,\alpha,-\theta} = K\,\|g\|^2_{\alpha-2\theta} + \mathcal{O}(\|g\|^2_{\alpha-\theta}),$$

where $K = K(\alpha,\theta)$ is given by an explicit expression involving certain generalized hypergeometric functions.

Now, the functions $g\Phi_\theta$ and $\mathbf{P}_{\alpha,-\theta}[gL_\theta]$ are holomorphic in $\mathbb{D}^2$, and we may apply the diagonal norm expansion. The result is

$$\left\|g(w)\Phi_\theta(z,w) + \mathbf{P}_{\alpha,-\theta}[g(w)L_\theta(z,w)]\right\|^2_{\alpha,-\theta} = \sum_{N=0}^{\infty}\frac{1}{\sigma_N}\left\|b_N g^{(N+1)} + \sum_{k=0}^{N} a_{k,N}\boldsymbol{\partial}_z^{N-k}[\Phi_{k,\theta}g]\right\|^2_{\alpha-2\theta+2N+2}. \tag{4.24}$$

Here, we have introduced the functions

$$\Phi_{k,\theta}(z) = \oslash[\boldsymbol{\partial}_z^k\Phi_\theta], \quad k = 0,1,2,\ldots;$$

and we write σ_N for $\sigma(\alpha,-\theta+N)$. Finally, we state the main result of [19].

Theorem 4.5. *Fix α,θ with $-1+2\theta<\alpha<+\infty$ and $0<\theta\le 1$. Then, for any $g\in A^2_{\alpha-2\theta}(\mathbb{D})$, we have*

$$\sum_{N=0}^{\infty}\frac{1}{\sigma_N}\left\|b_N g^{(N+1)} + \sum_{k=0}^{N} a_{k,N}\boldsymbol{\partial}_z^{N-k}[\Phi_{k,\theta}g]\right\|^2_{\alpha-2\theta+2N+2} \le K\|g\|^2_{\alpha-2\theta} + \mathcal{O}(\|g\|^2_{\alpha-\theta}). \tag{4.25}$$

Here, $K = K(\alpha,\theta)$, $\sigma_N = \sigma(\alpha,-\theta+N)$, $a_{k,N}$ and b_N are given by certain explicit expressions.

One should note that $\oslash[\boldsymbol{\partial}_z^k\Phi_\theta]$ can always be computed in terms of the original function $\varphi\in\mathcal{S}$ (see section 5 of [19] for details).

We have now accomplished our first goal: to find a parametrized inequality which holds uniformly in the class $\mathcal{S}$. As we shall see, the free parameter $0<\theta\le 1$ will play a crucial role in our investigations. We devote the next section to the applications of the theorem to the study of the universal integral means spectrum.

Estimating $\mathrm{B}_{\mathcal{S}}(\tau)$ near the origin. Let us see how our inequality can be used in the study of the universal integral means spectrum. Suppose we throw away all but the first term in the series expansion in (4.25), put $\alpha = \beta + 2\theta - 1$ and plug in the function $g = [\varphi']^{\tau/2}$. We then obtain

$$\left\| C_1 \boldsymbol{\partial}_z\{[\varphi']^{\tau/2}\} + C_2 \frac{\varphi''}{\varphi'}[\varphi']^{\tau/2} \right\|^2_{\beta+1} \le K \left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1} + \mathcal{O}\big(\left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1+\theta}\big), \tag{4.26}$$

where $C_1 = C_1(\beta, \theta)$ and $C_2 = C_2(\beta, \theta)$ are explicit expressions. Next, we note that

$$\boldsymbol{\partial}_z \{[\varphi']^{\tau/2}\} = \frac{\tau}{2} \frac{\varphi''}{\varphi'} [\varphi']^{\tau/2}$$

and that by (4.2) with $\nu = \theta$

$$\left\|\boldsymbol{\partial}_z[\varphi']^{\tau/2}\right\|^2_{\beta+1} = (\beta + 1)(\beta + 2) \left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1} + \mathcal{O}\big(\left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1+\theta}\big).$$

We implement this in the inequality (4.26) and obtain

$$A \left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1} + \mathcal{O}\big(\left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1+\theta}\big) \le K \left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1} + \mathcal{O}\big(\left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1+\theta}\big), \tag{4.27}$$

where $A = A(\beta, \theta, \tau)$ is an explicit expression and $K = K(\beta, \theta)$ is (essentially) as before (we really should write $K(\beta+2\theta-1, \theta)$ instead). This implies that if we can find, for τ fixed, β and θ such that

$$M(\beta, \theta, \tau) - K(\beta, \theta) > 0,$$

while at the same time $\|[\varphi']^{\tau/2}\|^2_{\beta-\theta+1} < +\infty$ for all $\varphi \in \mathcal{S}$, then $B_{\mathcal{S}}(\tau) \le \beta$. Initial control of the big-oh term can be achieved by using, for instance, the pointwise bound

$$|[\varphi'(z)]^\tau| \le \frac{(1 + |z|)^{2|\tau|-\operatorname{Re}\tau}}{(1 - |z|)^{2|\tau|+\operatorname{Re}\tau}}, \quad z \in \mathbb{D}.$$

In this fashion, the authors of [19] obtain the estimate

$$\limsup_{\tau\to 0} \frac{B_{\mathcal{S}}(\tau)}{|\tau|^2} \le \frac{1}{2}.$$

Let us take into account one more term in the norm expansion. We begin by taking into account the first two terms in (4.25), again plugging in $g = [\varphi']^{\tau/2}$. After some calculations, we arrive at the inequality

$$M \left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1} + \left\| A_1 \frac{\varphi'''}{\varphi'} [\varphi']^{\tau/2} + A_2 \left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{\tau/2} \right\|^2_{\beta+3} \le K \left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1} + \mathcal{O}\big(\left\|[\varphi']^{\tau/2}\right\|^2_{\beta-1+\theta}\big), \tag{4.28}$$

where, as before, we can write down explicit expressions for the constants $A_i = A_i(\beta,\theta,\tau)$, $i = 1, 2$. One checks that for θ close to 0 (and $\beta,\tau \sim 0$ as well), A_1 is almost zero, while A_2 is not. The inequality

$$\left\| \frac{\varphi'''}{\varphi'}\,[\varphi']^{\tau/2} \right\|_{\beta+3}^2 \le C\,\big\| [\varphi']^{\tau/2} \big\|_{\beta-1}^2$$

(valid for some suitable constant C) together with (4.28) now tells us that (with $\theta,\beta,\tau \sim 0$) we may neglect the term with A_1 as a factor, and thus bound

$$\left\| \left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{\tau/2} \right\|_{\beta+3}^2$$

in terms of $\|[\varphi']^{\tau/2}\|_{\beta-1}^2$ plus a bounded term. By the Cauchy-Schwarz inequality, we can find a constant $C_3 = C_3(\beta,\theta,\tau)$ such that

$$\big\|\boldsymbol{\partial}_z[\varphi']^{\tau/2}\big\|_{\beta+1}^2 \le C_3 \left\| \left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{\tau/2} \right\|_{\beta+3} \big\|[\varphi']^{\tau/2}\big\|_{\beta-1}. \tag{4.29}$$

Finally, we obtain

$$\begin{aligned}(\beta+1)(\beta+2)\,\big\|[\varphi']^{\tau/2}\big\|_{\beta-1}^2 &= \big\|\boldsymbol{\partial}_z\{[\varphi']^{\tau/2}\}\big\|_{\beta+1}^2 + \mathcal{O}\big(\big\|[\varphi']^{\tau/2}\big\|_{\beta-1+\theta}\big)\\ &\le C_3 \left\| \left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{\tau/2} \right\|_{\beta+1} \big\|[\varphi']^{\tau/2}\big\|_{\beta-1} + \mathcal{O}\big(\big\|[\varphi']^{\tau/2}\big\|_{\beta-1+\theta}\big),\end{aligned}$$

and this gives a condition on β and θ (expressed in terms of all our constants) for the norm $\|[\varphi']^{\tau/2}\|_{\beta-1}^2$ to be bounded by some constant times $\|[\varphi']^{\tau/2}\|_{\beta-1+\theta}^2$. By choosing our parameters as above, we obtain the estimate

$$\limsup_{\tau\to 0} \frac{B_{\mathcal{S}}(\tau)}{|\tau|^2} \le 0.43649\ldots.$$

If we take one more term into account, this may be improved to (see [20])

$$\limsup_{\tau\to 0} \frac{B_{\mathcal{S}}(\tau)}{|\tau|^2} = 0.3798\ldots.$$

We should mention that it is possible to use the inequality (4.25) as a starting point for a numerical study of the universal integral means spectrum, at least for real values of τ. A numerical implementation based on the first two terms can be found in the paper [19], while three terms were used in the numerical study in [32]. The numerical implementation is based on optimization techniques, using the fact that the coefficients A_1, A_2 in (4.28) vary with the choice of the parameter θ. Deeper properties are needed to uncover the full strength of (4.28).

Underlying ideas and remaining difficulties. Now that we have seen how the methods developed in [31] and [19] work, let us try to see where the underlying ideas come from and why our techniques do not yet work as well as we would like them to.

The *Bloch space* $\mathcal{B}(\mathbb{D})$ consists of functions f which are holomorphic in the unit disk and have bounded norm in the sense that

$$\|f\|_{\mathcal{B}(\mathbb{D})} = \sup\left\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\right\} < +\infty. \tag{4.30}$$

It is a well-known fact that a Bloch space function belongs to $A^2_\alpha(\mathbb{D})$ for any given $\alpha > -1$. Sometimes it is also of interest to consider the *little Bloch space* $\mathcal{B}_0(\mathbb{D})$. This is the subspace of $\mathcal{B}(\mathbb{D})$ that consists of functions f with

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0. \tag{4.31}$$

It follows from (3.2) that if $\varphi \in \mathcal{S}$, then $\log \varphi' \in \mathcal{B}(\mathbb{D})$. Moreover, if a function f in the Bloch space has sufficiently small Bloch norm (actually, it suffices that the Bloch norm is ≤ 1), then there exists a function $\varphi \in \mathcal{S}$ such that $f = \log \varphi'$ (see [12] and [27]).

It is the philosophy of this paper that the study of integral means spectra amounts to understanding the function $f = \log \varphi'$ as an element of the quotient space $\mathcal{B}(\mathbb{D})/\mathcal{B}_0(\mathbb{D})$ for $\varphi \in \mathcal{S}$. Since for $\lambda \in \mathbb{C}$ we have $[\varphi']^\lambda = \exp(\lambda f)$, we are asking for which α $\exp(\lambda f)$ belongs to $A^2_\alpha(\mathbb{D})$. One way to approach this is to use (4.2) to achieve inequalities of the type

$$\begin{aligned} \left\|\exp(\lambda f)\right\|^2_\alpha &= c(\alpha)\left\|\boldsymbol{\partial}_z \exp(\lambda f)\right\|^2_{\alpha+2} + \mathcal{O}(1) \\ &= c(\alpha)\|\lambda f' \exp(\lambda f)\|^2_{\alpha+2} + \mathcal{O}(1) \\ &\leq c(\alpha)|\lambda|\|\mathbf{M}_{f'}\|\|\exp(\lambda f)\|^2_\alpha + \mathcal{O}(1), \end{aligned} \tag{4.32}$$

with

$$1 - c(\alpha)|\lambda|\|\mathbf{M}_{f'}\| > 0$$

and the $\mathcal{O}(1)$-term given by the norm of $\exp(\lambda f)$ in some weighted Bergman space to which the function belongs a priori. Here, $\mathbf{M}_F$ denotes the operator of multiplication by the function F, and $\|\mathbf{M}_{f'}\|$ is the operator norm $A^2_\alpha(\mathbb{D}) \to A^2_{\alpha+2}(\mathbb{D})$. In order to be able to choose α as small as possible, we need to get a good bound for the multiplier norm. The inequality (4.15) fits into this scheme; in that case we choose $\lambda = -1/2$ and consider second order derivatives instead. In that situation, the corresponding multiplier

$$\mathbf{M}_{f''-\frac{1}{2}(f')^2}$$

is precisely the Schwarzian derivative $S[\varphi]$.

More generally, we are provided with an entire collection of inequalities, which can be exploited to obtain similar estimates, by (4.25). In order to make good use of the information given to us by this result, we would like to use as many of the terms in the series expansion as possible. The success of the method now relies on our ability to somehow compare the norms appearing on the left-hand side of the fundamental inequality with the norm of a suitably chosen function $g \in A^2_{\beta-1}(\mathbb{D})$. That is, we now want to compare

$$\left\| b_N g^{(N+1)} + \sum_{k=0}^{N} a_{k,N}\, \boldsymbol{\partial}_z^{N-k}[g\Phi_{k,\theta}] \right\|^2_{\beta+2N-1}$$

with $\|g\|^2_{\beta-1}$. This, however, is in general quite difficult. As we noticed before, the case with the first term only ($N = 0$) is easy to handle. Let us therefore return to the case where we take two terms in our series expansion into account. Setting

$$f = \log \varphi' \quad \text{and} \quad g = \exp\left[\frac{\tau}{2} f\right],$$

we may rewrite (4.28) in the form

$$A\,\|g\|^2_{\beta-1} + B\,\left\|\left\{f'' + \eta[f']^2\right\} g\right\|^2_{\beta+3} \le \|g\|^2_{\beta-1} + \mathcal{O}(\|g\|^2_{\beta-1+\theta}). \tag{4.33}$$

As usual, the constants $A = A(\beta, \theta, \tau)$, $B = B(\beta, \theta, \tau)$, and $\eta = \eta(\beta, \theta, \tau)$ are explicit expressions. We are now faced with the problem of comparing the term

$$\left\|\left\{f'' + \eta\,[f']^2\right\} g\right\|^2_{\beta+3}$$

with the norm of the function $g \in A^2_{\beta-1}(\mathbb{D})$, and we want to do this for different values of the constant η. Of course, if η is such that the expression $\{f'' + \eta[f']^2\}g$ can be thought of as a multiple of a higher order derivative of g (as was the case in our discussion involving the Schwarzian), then we can immediately apply (4.2) and get a good estimate in terms of $\|g\|^2_{\beta-1}$. There was also the rather ad-hoc method involving the Cauchy-Schwarz inequality (corresponding to the degenerate case when η tends to infinity). But it would appear that in order to really make effective use of (4.33), we should need to have inequalities of the type

$$\begin{aligned} c_1(\eta,\eta')\,\left\|\left\{f'' + \eta'\,[f']^2\right\}\, g\right\|^2_{\beta+3} + \mathcal{O}(1) &\\ \le \left\|\left\{f'' + \eta[f']^2\right\} g\right\|^2_{\beta+3} &\\ \le c_2(\eta,\eta')\,\left\|\left\{f'' + \eta'[f']^2\right\} g\right\|^2_{\beta+3} + \mathcal{O}(1) & \end{aligned} \tag{4.34}$$

with good constants $c_1(\eta,\eta')$ and $c_2(\eta,\eta')$. If the above were true, we could exchange one η for another in order to reach a situation where the result (4.2) applies. The estimates we would obtain of the universal integral means spectrum would of course depend on the quality of the constants in (4.34).

Let $\mathcal{A}^{-\gamma}(\mathbb{D})$ denote the Banach space of functions f holomorphic in $\mathbb{D}$, subject to the norm boundedness condition

$$\|f\|_{\mathcal{A}^{-\gamma}(\mathbb{D})} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\gamma\,|f(z)| < +\infty.$$

Here, we assume $0 < \gamma < +\infty$ (the space is trivial for negative γ). Moreover, let $\mathcal{A}_0^{-\gamma}(\mathbb{D})$ denote the closed subspace of functions f with

$$f(z) = o\biggl(\frac{1}{(1-|z|^2)^\gamma}\biggr), \quad \text{as} \quad |z|\to 1.$$

It is well known that if $f \in \mathcal{B}(\mathbb{D})$, then $f' \in \mathcal{A}^{-1}(\mathbb{D})$, $f'' \in \mathcal{A}^{-2}(\mathbb{D})$, and so on. This implication is actually reversible. Moreover, $f \in \mathcal{B}_0(\mathbb{D})$ if and only if $f' \in \mathcal{A}_0^{-1}(\mathbb{D})$, which happens if and only if $f'' \in \mathcal{A}_0^{-2}(\mathbb{D})$. This means that the mapping $f \mapsto f''$ induces an isomorphism $\mathcal{B}(\mathbb{D})/\mathcal{B}_0(\mathbb{D}) \cong \mathcal{A}^{-2}(\mathbb{D})/\mathcal{A}_0^{-2}(\mathbb{D})$. What then about the nonlinear map

$$f \mapsto f'' + \eta\,[f']^2,$$

for some complex η, where it is assumed that $f = \log\varphi'$ with $\varphi \in \mathcal{S}$? Does it induce a correspondence between the spaces $\mathcal{B}(\mathbb{D})/\mathcal{B}_0(\mathbb{D})$ and $\mathcal{A}^{-2}(\mathbb{D})/\mathcal{A}_0^{-2}(\mathbb{D})$? Answering this question is a first step toward understanding when an estimate like (4.34) is possible. Pleasantly, Eero Saksman obtained the following.

Theorem 4.6. (Saksman) *For each fixed complex number η, $f \in \mathcal{B}(\mathbb{D})$ holds if and only if*

$$f''(z) + \eta\,[f'(z)]^2 \in \mathcal{A}^{-2}(\mathbb{D}).$$

However, it is not always true that

$$f''(z) + \eta\,[f'(z)]^2 \in \mathcal{A}_0^{-2}(\mathbb{D})$$

implies that $f \in \mathcal{B}_0(\mathbb{D})$. The main source of counterexample is the following:

$$\varphi_\lambda(z) = \frac{1}{\lambda}\bigl\{(1+z)^\lambda - 1\bigr\}, \qquad z \in \mathbb{D},$$

where λ is a complex parameter (for $\lambda = 0$, we pick $\varphi_0(z) = \log(1+z)$). One verifies that $\varphi_\lambda \in \mathcal{S}$ if and only if

$$|\lambda - 1| \le 1 \quad \text{or} \quad |\lambda + 1| \le 1.$$

We calculate that

$$f_\lambda(z) = \log \varphi'_\lambda(z) = (\lambda - 1)\log(1+z),$$

which induces a nontrivial element of $\mathcal{B}(\mathbb{D})/\mathcal{B}_0(\mathbb{D})$ for $\lambda \neq 1$. A short calculation yields

$$f''_\lambda(z) + \eta\,[f'_\lambda(z)]^2 = (1-\lambda)[1-\eta(1-\lambda)]\,\frac{1}{(1+z)^2}, \qquad z \in \mathbb{D},$$

which *vanishes* provided that

$$\frac{1}{\eta} = 1 - \lambda.$$

So for $1 \le |\eta| < +\infty$, we see that $f'' + \eta[f']^2$ may be in $\mathcal{A}_0^{-2}(\mathbb{D})$ while $f \notin \mathcal{B}_0(\mathbb{D})$. However, for $|\eta| < 1$, there is still hope that $f'' + \eta[f']^2 \in \mathcal{A}_0^{-2}(\mathbb{D})$ might imply that $f \in \mathcal{B}_0(\mathbb{D})$. For the Schwarzian derivative (the case $\eta = -\frac{1}{2}$), it seems to be so (see [2]).

For higher order terms, the picture is much more complicated. Nevertheless, we believe that clearing up of the above issues for the second term should go a long way toward understand all the other terms as well.

5. Recent progress on bounded functions

The universal integral means spectrum near 2. Here, we mention some recent results of Anton Baranov and the first-named author. Their work is inspired by the deep paper [22] of Jones and Makarov. We recall that the main result of [22] concerning the universal means spectrum reads as follows.

Theorem 5.1. *We have*

$$\mathrm{B}_{\mathcal{S}_b}(2-t) = 1 - t + \mathcal{O}(t^2), \quad \text{as} \quad t \to 0.$$

In the paper [1], a similar theorem is obtained. While slightly weaker for real arguments, the result holds for complex values of τ as well.

Theorem 5.2. *We have*

$$\mathrm{B}_{\mathcal{S}_b}(2-\tau) \le 1 - \operatorname{Re}\tau + \left(\frac{9\mathrm{e}^2}{2} + o(1)\right)|\tau|^2 \log|\tau|, \quad \text{as} \quad |\tau| \to 0.$$

The basic identity. As before, let $\varphi \in \mathcal{S}_b$ be a conformal mapping. The proof is based on the following elementary identity:

$$\begin{aligned}\log\frac{z(\varphi(z)-\varphi(\zeta))}{(z-\zeta)\varphi(z)} - \zeta(1-|\zeta|^2)\left[\frac{\varphi'(\zeta)}{\varphi(\zeta)-\varphi(z)} - \frac{1}{\zeta - z}\right] \\ + \log\big(1-\bar z\zeta\big) + \bar z\zeta\,\frac{1-|\zeta|^2}{1-\bar z\zeta} \\ = \zeta^2\int_{\mathbb{D}}\frac{\varphi'(w)}{\varphi(w)-\varphi(z)}\,\frac{\bar\zeta-\bar w}{(1-\bar w\zeta)^2}\,\mathrm{d}A(w).\end{aligned}$$

This identity is related to the Grunsky inequality in a manner explained in [1]. As a matter of fact, the diagonal restriction of the above identity is essential:

$$\begin{aligned}\log\frac{z\varphi'(z)}{\varphi(z)} - z(1-|z|^2)\,\frac{\varphi''(z)}{\varphi'(z)} + \log\big(1-|z|^2\big) + |z|^2 \\ = z^2\int_{\mathbb{D}}\frac{\varphi'(w)}{\varphi(w)-\varphi(z)}\,\frac{\bar z-\bar w}{(1-\bar w z)^2}\,\mathrm{d}A(w).\end{aligned}$$

It is well-known that

$$-z(1-|z|^2)\,\frac{\varphi''(z)}{\varphi'(z)} + |z|^2$$

is bounded by a universal constant, so that we get

$$\begin{aligned}\log\frac{z\varphi'(z)}{\varphi(z)} + \log\big(1-|z|^2\big) + \mathcal{O}(1) \\ = z^2\int_{\mathbb{D}}\frac{\varphi'(w)}{\varphi(w)-\varphi(z)}\,\frac{\bar z-\bar w}{(1-\bar w z)^2}\,\mathrm{d}A(w).\end{aligned} \tag{5.1}$$

Uniform Sobolev imbedding. An application of Hölder's inequality and some well-known properties of Marcinkiewicz-Zygmund integrals show that the Cauchy-type operator

$$\widetilde{\mathfrak{C}}_\varphi[f](z) = \int_{\mathbb{D}}\frac{\varphi'(w)}{\varphi(w)-\varphi(z)}\,\frac{\bar z-\bar w}{1-\bar w z}f(w)\,\mathrm{d}A(w)$$

has a uniform Sobolev imbedding property ($0<\kappa<+\infty$):

$$\int_{\mathbb{D}}\exp\left\{|\lambda|\sup_{f\in\mathrm{ball}(X_\kappa(\mathbb{D}))}\big|\,\widetilde{\mathfrak{C}}_\varphi[f](z)\,\big|^{2+\kappa}\right\}|\varphi'(z)|^2\,\mathrm{d}A(z) < +\infty, \tag{5.2}$$

provided that $\lambda\in\mathbb{C}$ has

$$|\lambda| < \frac{\kappa 4^{-\kappa}}{9\mathrm{e}(2+\kappa)}.$$

Here, $X_\kappa(\mathbb{D})$ is the Lebesgue (Banach) space

$$X_\kappa(\mathbb{D}) = L^p(\mathbb{D},\mu),$$

where

$$p = \frac{2+\kappa}{1+\kappa}, \qquad \mathrm{d}\mu(z) = (1-|z|^2)^{-\kappa/(1+\kappa)}\,\mathrm{d}A(z);$$

the norm in $X_\kappa(\mathbb{D})$ is given by

$$\|f\|_{X_\kappa(\mathbb{D})} = \left\{\int_{\mathbb{D}} |f(z)|^{(2+\kappa)/(1+\kappa)}(1-|z|^2)^{-\kappa/(1+\kappa)}\,\mathrm{d}A(z)\right\}^{(1+\kappa)/(2+\kappa)}.$$

We apply the uniform Sobolev imbedding to the function f_z given by

$$f_z(w) = \frac{g_z(w)}{\|g_z\|_{X_\kappa(\mathbb{D})}}, \qquad g_z(w) = \frac{z^2}{1-\bar{w}z};$$

that is, we pick a different f for each point $z \in \mathbb{D}$ in (5.2). It is important to note that by (5.1),

$$\widetilde{\mathfrak{C}}_\varphi[g_z](z) = \log\left[z\frac{\varphi'(z)}{\varphi(z)}\left(1-|z|^2\right)\right] + O(1), \qquad z \in \mathbb{D}. \tag{5.3}$$

Theorem 5.2 now follows from the uniform Sobolev imbedding estimate (5.2) by a convexity argument involving linear approximation. For details, we refer to [1].

It is interesting to note that the method used by Baranov and Hedenmalm uses the diagonal restriction of an *integrated version* of a Grunsky identity, while the standard method which yields so far best results at other parts of the spectrum uses a kind of diagonal restriction of the usual (weighted) Grunsky identity (for the notion of Grunsky identity, see [1]).

6. Numerical implementation

Estimating $\mathrm{B}_{\mathcal{S}_b}(t)$ near 0, 1, and 2. In this section, we present some new numerical estimates that we have obtained using the results of the paper [1]. There, it is shown that for $\varphi \in \mathcal{S}_b$, we have, for every $0 < \kappa < 1$,

$$\mathrm{B}_{\mathcal{S}_b}(2-\tau) \le 1 - \operatorname{Re}\tau + \left(\frac{9e4^\kappa}{\kappa}\right)^{1/(1+\kappa)} \frac{(1+\kappa)\Gamma(\frac{1-\kappa}{1+\kappa})}{(2+\kappa)\Gamma(\frac{1}{1+\kappa})^2}|\tau|^{(2+\kappa)/(1+\kappa)}.$$

We now choose small real τ and vary κ to obtain numerical estimates of $\mathrm{B}_{\mathcal{S}_b}$ near the point 2. In addition, we apply the methods of [19] and [32], to obtain of estimates of $\mathrm{B}_{\mathcal{S}}$ close to the origin. The fact that the universal means spectra are convex now allows us to estimate

$\mathrm{B}_{\mathcal{S}_b}(1)$ using linear interpolation between upper estimates for $\mathrm{B}_{\mathcal{S}}(t)$ and $\mathrm{B}_{\mathcal{S}_b}(2-\tau)$ for suitable (small real) values of t and τ. Using a larger collection of estimates than the one presented in [32], we have managed to obtain the estimate

$$\mathrm{B}_{\mathcal{S}_b}(1) \leq 0.4598\ldots,$$

which represents a very modest improvement of the bound $B_{\mathcal{S}_b}(1) \leq 0.4600\ldots$ given in [19].

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