

Normal random matrices

Y. Ameur, H. Hedenmalm, N. Makarov

January 26, 2011

“Haar” measure of normal matrices

Consider $\mathcal{M}_n(\mathbb{C}) \equiv \mathbb{C}^{n^2}$ the space of all $n \times n$ matrices with complex entries. A matrix $M \in \mathcal{M}_n(\mathbb{C})$ is normal if $M^*M = MM^*$, where M^* denotes the adjoint. Let $\mathcal{N}_n(\mathbb{C})$ denote the subspace of $\mathcal{M}_n(\mathbb{C})$ of normal matrices. There is a natural measure (analogous to Haar measure, but the normal matrices are not a group!) on $\mathcal{N}_n(\mathbb{C})$ induced by the geometry of \mathbb{C}^{n^2} , and this measure induces a measure on the eigenvalue distribution of a matrix in $\mathcal{N}_n(\mathbb{C})$. Let

$$\Delta(\lambda_1, \dots, \lambda_n) = \prod_{i,j:1 \leq i < j \leq n} (\lambda_i - \lambda_j)$$

be the vandermonian; then the measure is

$$|\Delta(\lambda_1, \dots, \lambda_n)|^2 dA(\lambda_1) \cdots dA(\lambda_n),$$

where dA is area measure in the plane, divided by π .

Electron cloud interpretation

The eigenvalues $(\lambda_1, \dots, \lambda_n)$ with the Haar-type measure follow the same law as Coulomb gas, i.e., a cloud of n electrons in space with repulsive forces between them. In the absence of a localizing potential, the electrons will disperse and it will be hard to find any at all. We introduce a localizing potential Q , which we rescale by a factor m , to form mQ . The energy associated with position $(\lambda_1, \dots, \lambda_n)$ is therefore

$$\begin{aligned}\mathcal{E}(\lambda_1, \dots, \lambda_n) &:= \sum_{i,j:1 \leq i < j \leq n} \log \frac{1}{|\lambda_i - \lambda_j|^2} + m \sum_{j=1}^n Q(\lambda_j) \\ &= \log \frac{1}{|\Delta(\lambda_1, \dots, \lambda_n)|^2} + m \sum_{j=1}^n Q(\lambda_j).\end{aligned}$$

The Gibbs model

We shall use the Gibbs model

$$d\mathbb{P}(\lambda_1, \dots, \lambda_n) := \frac{1}{Z} e^{-\mathcal{E}(\lambda_1, \dots, \lambda_n)} dA^{\otimes n}(\lambda_1, \dots, \lambda_n),$$

with a fixed inverse temperature which suits us (for algebraic reasons). Here, Z is a normalizing constant, and

$$dA^{\otimes n}(\lambda_1, \dots, \lambda_n) := dA(\lambda_1) \cdots dA(\lambda_n).$$

The normalizing constant Z is called the *partition function*:

$$Z := \int_{\mathbb{C}^n} e^{-\mathcal{E}(\lambda_1, \dots, \lambda_n)} dA^{\otimes n}(\lambda_1, \dots, \lambda_n).$$

Minimization of energy

Intuitively, we might compare with a glass of water, where the confining shape of the glass, approximate equidistribution of water molecules, and the force of gravity leave a flat horizontal surface. This would suggest that

- minimization of the potential energy (i.e., Q) given that a certain quantity of electrons are to fit in should be the essential thing. However, in the present Coulomb gas model long-range electrostatic repulsion between electrons plays an essential role, and minimization of the total energy \mathcal{E} is what is important! We rescale:

$$\mathcal{E}^\# := \frac{\mathcal{E}}{n^2} = \frac{1}{n^2} \sum_{i,j:i \neq j} \log \frac{1}{|\lambda_i - \lambda_j|} + \frac{m}{n} \left\{ \frac{1}{n} \sum_j Q(\lambda_j) \right\}.$$

Continuous limit of energy minimization

In the continuous limit as $n \rightarrow +\infty$ while $m/n \rightarrow 1$, we are led to the minimization problem

$$\inf_{\sigma} I[\sigma], \quad I[\sigma] := \int_{\mathbb{C}^2} \log \frac{1}{|z-w|^2} d\sigma(z)d\sigma(w) + 2 \int_{\mathbb{C}} Q d\sigma,$$

where σ is assumed to be a Borel probability measure. If Q is continuous with

$$Q(z) - \log |z|^2 \rightarrow +\infty \quad \text{as } |z| \rightarrow +\infty,$$

a theorem of Frostman guarantees that there exists a *unique* minimizer $\hat{\sigma}$ with compact support. We write $S := \text{supp } \hat{\sigma}$, and call it a *droplet*.

Obstacle problem theory for smooth potentials

Suppose Q is smooth. Let SH_1 denote the cone of real-valued subharmonic functions h in the plane with

$$h(z) \leq \log^+ |z|^2 + O(1).$$

Let \widehat{Q} denote the largest minorant to Q from the cone SH_1 . Then essentially

$$S = \{z : \widehat{Q}(z) = Q(z)\},$$

and

$$d\hat{\sigma} = \Delta \widehat{Q} \, dA = 1_S \Delta Q \, dA.$$

Comparison with Hermitian ensembles

If we consider the degenerate case when $Q = +\infty$ on $\mathbb{C} \setminus \mathbb{R}$, we get the usual Hermitian ensembles (the eigenvalues are forced to be real). This can be thought of as a limit of smooth potentials

$$\tilde{Q}(x + iy) := Q(x) + ay^2,$$

where we let $a \rightarrow +\infty$. We expect that the droplets S_a tend to a compact subset of \mathbb{R} as $a \rightarrow +\infty$, where the eigenvalues accumulate, and that the local vertical width of S_a corresponds to the local density of eigenvalues in the Hermitian ensemble. The relation

$$d\hat{\sigma} = \Delta \hat{Q} dA$$

should survive also in the Hermitian case, although the right hand side must be understood in the sense of distribution theory. E.g., the Wigner semi-circle law comes from an obstacle problem with $Q(x) = x^2$ along the real line and $Q = +\infty$ elsewhere in \mathbb{C} .

The problem at hand

To describe, at a statistical level, the location of the eigenvalues (or electrons), as we let $m, n \rightarrow +\infty$, while $m = n + o(n)$ (sometimes we make the more precise assumption $m = n + o(1)$).

The specific choice we made of the inverse temperature gives us correlation kernel structure. That is, the whole process is determined by the *correlation kernel* $L(z, w)$, which depends on n, m, Q , which has the form

$$L(z, w) := K(z, w) e^{-\frac{m}{2}(Q(z)+Q(w))},$$

where $K(z, w)$ is the *reproducing kernel* for the space of polynomials of degree $< n$ with inner product norm

$$\|f\|^2 = \int_{\mathbb{C}} |f|^2 e^{-mQ} dA < +\infty.$$

The determinant

$$\det \left([L(z_i, z_j)]_{i,j=1}^k \right)$$

describes the intensity of finding a k -tuple of electrons at the points z_1, \dots, z_k . E.g., $L(z, z)$ describes the density of electrons in position z .

Reproducing kernel expansion

Reproducing kernel expansions have a long history, rooted in the works of Hörmander, Fefferman, Boutet de Monvel, Sjöstrand, Berndtsson, etc. We use the recent version due to Berman, Berndtsson, and Sjöstrand to get the following.

THM. We have, for $n \geq m - 1$,

$$K(z, z)e^{-mQ(z)} = m\Delta Q(z) + \frac{1}{2}\Delta \log \Delta Q(z) + O(m^{-1/2}),$$

on any compact subset Σ of the interior of S with $\Delta Q > 0$ on Σ .

There exists a polarized version of this diagonal approximation:

$$K_{m,n}(z, w)e^{-mQ^*(z,w)} = m\Delta^* Q^*(z, w) + \frac{1}{2}\Delta^* \log \Delta^* Q^*(z, w) + O(m^{-1/2}e^{(m/2)[Q(z)+Q(w)-2\operatorname{Re}Q^*(z,w)]}).$$

The probability measure

$$dB^{(w)}(z) = \frac{|K(z, w)|^2}{K(w, w)} e^{-mQ(z)} dA(z)$$

we call the Berezin measure. For $w \in S$ it converges to a point mass at w as $m, n \rightarrow +\infty$ while $n = m + O(1)$, while for $w \in \mathbb{C} \setminus S_\tau$ it converges to harmonic measure for w in the domain $\mathbb{C} \setminus S$. In case w is a bulk point (i.e., it is in the interior of S with $\Delta Q(w) > 0$), one can show that the Berezin measure – suitably blown up so that the scale $m^{-1/2}$ becomes 1 – tends to a radially symmetric Gaussian in the plane.

The observation that the Berezin measure – rescaled – tends to the Gaussian at interior points with $\Delta Q > 0$, corresponds to the blown-up process converging to $\text{Gin}(\infty)$, with correlation kernel

$$L_\infty(z, w) = e^{z\bar{w}} e^{-\frac{1}{2}(|z|^2 + |w|^2)}.$$

This corresponds to the reproducing kernel for the Bargmann-Fock space. The stochastic process is translation invariant with infinitely many points equidistributed in the entire plane.

Boundary point blow-up

In case of the usual Ginibre ensemble, with reproducing kernel

$$K(z, w) = m \sum_{j=0}^{n-1} \frac{(mz\bar{w})^j}{j!},$$

we can make explicit calculations. The droplet S is the closed unit disk, so the boundary is the unit circle. If we blow up at a boundary point, the reproducing kernel tends to the reproducing kernel for a naturally defined subspace of the Bargmann-Fock space. The concrete expression involves the error function. This is most likely universal for smooth boundary points of S , for other (real-analytic) weights Q .

Boundary point blow-up II

The analysis of the Ginibre ensemble suggested that for interior points and for boundary points, the limit of the blow-ups of the correlation kernel is determined by the reproducing kernel of a Hilbert space of entire functions. Probably this is universal. In fact, for GUE we have the sine kernel at bulk points, which is the reproducing kernel for the Paley-Wiener space. And at the boundary we have the Airy process, with a different local scaling of $m^{-2/3}$. The Airy kernel is also associated with a space of entire functions. Moreover, the different typical distance $m^{-2/3}$ comes from the fact that the Wigner semi-circle law has zero density at the boundary point, with a square-root type approach.