Berezin quantization and normal random matrices

Y. Ameur, H. Hedenmalm, N. Makarov

August 5, 2008

"Haar" measure of normal matrices

Consider $\mathcal{M}_n(\mathbb{C}) \equiv \mathbb{C}^{n^2}$ the space of all $n \times n$ matrices with complex entries. A matrix $M \in \mathcal{M}_n(\mathbb{C})$ is normal if $M^*M = MM^*$, where M^* denotes the adjoint. Let $\mathcal{N}_n(\mathbb{C})$ denote the subspace of $\mathcal{M}_n(\mathbb{C})$ of normal matrices. There is a natural measure (analogous to Haar measure, but the normal matrices are not a group!) on $\mathcal{N}_n(\mathbb{C})$ induced by the geometry of $\mathbb{C}^{n^2}$, and this measure induces a measure on the eigenvalue distribution of a matrix in $\mathcal{N}_n(\mathbb{C})$. Let

$$\triangle(\lambda_1, \ldots, \lambda_n) = \prod_{i,j:1\le i<j\le n} (\lambda_i - \lambda_j)$$

be the vandermonian; then the measure is

$$\left|\triangle(\lambda_1, \ldots, \lambda_n)\right|^2 \mathrm{d}A(\lambda_1) \cdots \mathrm{d}A(\lambda_n),$$

where $\mathrm{d}A$ is area measure in the plane, divided by π.

Electron cloud and localization

The measure describing the density of eigenvalues may also be used to describe the joint "probability" density of a cloud of electrons. But we may not speak of any true probabilities, as the total mass of the measure is infinite. To localize the eigenvalues or electrons we need a constraining potential Q. We introduce the measure

$$d\mu_Q(\lambda_1, \ldots, \lambda_n) = \left|\triangle(\lambda_1, \ldots, \lambda_n)\right|^2 \mathrm{e}^{-m(Q(\lambda_1)+\cdots+Q(\lambda_n))} \mathrm{d}A(\lambda_1)\cdots \mathrm{d}A(\lambda_n),$$

which we easily turn into a probability measure:

$$d\mathbb{P}_Q(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_{m,n}} \left|\triangle(\lambda_1, \ldots, \lambda_n)\right|^2 \mathrm{e}^{-m(Q(\lambda_1)+\cdots+Q(\lambda_n))} \mathrm{d}A(\lambda_1)\cdots \mathrm{d}A(\lambda_n),$$

where $Z_{m,n}$ is the relevant normalization parameter (the partition function).

Electron cloud and localization II

We assume that Q is sufficiently smooth (real analytic) and grows fast enough at infinity:

$$Q(\lambda) \geq \rho \log^{+} |z|^2 + O(1),$$

where ρ is positive. In the electron cloud model, Q is associated with a constraining magnetic field ΔQ (Δ is the Laplacian).

The problem

Describe, as $m, n \to +\infty$, with $n/m \to \tau \in]0, \rho[$, the asymptotic distribution (statistically) of the eigenvalues. Sometimes we need to require more, e. g., $n = m\tau + o(1)$.

Potential theory

Let SH_τ denote the cone of real-valued subharmonic functions h in the plane with

$$h(z) \le \tau \log^+ |z|^2 + O(1).$$

Let $\widehat{Q}_\tau$ denote the largest minorant to Q from the cone SH_τ. Then the coincidence set

$$S_\tau = \{\widehat{Q}_\tau = Q\}$$

is important for the characterization of the eigenvalues.

Linear statistics

Let g be a smooth bounded function in the plane, and consider the trace

$$\mathrm{trace}_n \, g = g(\lambda_1) + \cdots + g(\lambda_n).$$

We have the following result:

THM. In probability

$$\frac{1}{n} \mathrm{trace}_n \, g \to \int_{S_\tau} g(z) \, \Delta Q(z) \, dA(z),$$

as $m, n \to +\infty$ and $n/m \to \tau$.

This means that the eigenvalues accumulate on the set S_τ with density ΔQ.

Fluctuations

We define the fluctuation:

$$\mathrm{fluct}_n g = \mathrm{trace}_n\, g - n \int_{S_\tau} g\, \Delta Q\, dA.$$

It is of interest to understand the fluctuations statistically.

THM. Suppose g is real-valued and vanishes near the boundary of S_τ, and near points where $\Delta Q = 0$. Then, as $m, n \to +\infty$ with $n = m\tau + o(1)$, $\mathrm{fluct}_n g$ tends to a normal distribution with mean

$$\frac{1}{2} \int_{S_\tau} g\, \Delta \log \Delta Q\, \mathrm{d}A$$

and variance

$$\frac{1}{2} \int_{S_\tau} |\nabla g|^2 \mathrm{d}A.$$

Method: reproducing kernels

Let $\mathfrak{P}_n$ denote the complex-linear space of all polynomials of degree $\le n-1$. We embed this space in the Hilbert space $\mathfrak{L}_{mQ}$ with norm

$$\|f\|_{mQ}^2 = \int_{\mathbb{C}} |f|^2 \mathrm{e}^{-mQ} \mathrm{d}A < +\infty.$$

Write

$$\mathfrak{P}_{m,n} = \left\langle \mathfrak{P}_n, \|\cdot\|_{mQ} \right\rangle.$$

Point evaluations are bounded in $\mathfrak{P}_{m,n}$, so there exists a function $k_w \in \mathfrak{P}_{m,n}$ such that

$$f(w) = \langle f, k_w \rangle_{mQ}, \qquad w \in \mathbb{C}.$$

The function

$$K_{m,n}(z,w) = k_w(z)$$

is the reproducing kernel.

Marginal probability, determinants, and reproducing kernels

The k-point marginal probability measure $\mathbb{P}^k_{m,n}$ is characterized by

$$\int_{\mathbb{C}^k} f(\lambda_1,\ldots,\lambda_k)\,\mathrm{d}\mathbb{P}^k_{m,n}(\lambda_1,\ldots,\lambda_k) = \int_{\mathbb{C}^n} f(\lambda_1,\ldots,\lambda_k)\,\mathrm{d}\mathbb{P}_{m,n}(\lambda_1,\ldots,\lambda_n)$$

(we just integrate out the variables $\lambda_{k+1},\ldots,\lambda_n$). Then

$$\mathrm{d}\mathbb{P}^k_{m,n}(\lambda_1,\ldots,\lambda_k) = \frac{(n-k)!}{n!}\det\big(K_{m,n}(\lambda_i,\lambda_j)\big)^k_{i,j=1} \times \mathrm{e}^{-m(Q(\lambda_1)+\ldots+Q(\lambda_k))}\mathrm{d}A(\lambda_1)\cdots\mathrm{d}A(\lambda_k).$$

Note that $\mathbb{P}^n_{m,n} = \mathbb{P}_{m,n}$!

Reproducing kernel expansion

Reproducing kernel expansions have a long history, rooted in the works of Hörmander, Fefferman, Boutet de Monvel, Sjöstrand, Berndtsson, etc. We use the recent version due to Berman, Berndtsson, and Sjöstrand to get the following.

THM. We have, for $n \geq m\tau - 1$,

$$K_{m,n}(z,z)\mathrm{e}^{-mQ(z)} = m\Delta Q(z) + \frac{1}{2}\Delta \log \Delta Q(z) + O(m^{-1/2}),$$

on any compact subset K of the interior of S_τ with $\Delta Q > 0$ on K.

There exists a polarized version of this diagonal approximation:

$$K_{m,n}(z,w)\mathrm{e}^{-mQ^*(z,w)} = m\Delta^* Q^*(z,w) + \frac{1}{2}\Delta^* \log \Delta^* Q^*(z,w) \\ + O\Big(m^{-1/2}\mathrm{e}^{(m/2)[Q(z)+Q(w)-2\mathrm{Re}Q^*(z,w)]}\Big).$$

Berndtsson-Hörmander $\bar{\partial}$-estimates

We recall Hörmander's classical estimate: there exists a solution u_* to $\bar{\partial} u = f$ with

$$\int_{\mathbb{C}} |u_*|^2 e^{-\phi} dA \leq \int_{\mathbb{C}} |f|^2 e^{-\phi} \frac{dA}{\Delta\phi},$$

provided $\Delta\phi > 0$. There is no loss of generality to assume that u_* is the solution with smallest left hand side norm. What if we would like to estimate instead

$$\int_{\mathbb{C}} |u_*|^2 e^{\varrho-\phi} dA$$

where u_* remains the norm minimal solution with respect to ϕ, and ϱ is some kind of disturbance?

Berndtsson-Hörmander $\bar{\partial}$-estimates II

A sequence of results beginning with Agmon and continuing with Berndtsson shows that this is possible, provided the disturbance ϱ is not too big. Here, we need to add a growth restriction at infinity. We suppose Σ is a compact subset of $\mathbb{C}$, and that ϕ, $\widehat{\phi}$, ϱ have:

- $\widehat{\phi} \le \phi$ on $\mathbb{C}$, and $\phi \le \widehat{\phi}$ on Σ,

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- $\widehat{\phi} + \varrho$ is strictly subharmonic on $\mathbb{C}$,
- ϱ is locally constant on $\mathbb{C} \setminus \Sigma$,
- there exists $\kappa \in]0, 1[$ such that

$$\frac{|\bar{\partial}\varrho|^2}{\Delta\widehat{\phi} + \Delta\varrho} \leq \kappa^2 \quad \text{on} \quad \Sigma.$$

The perturbed Hörmander estimate

THM. Suppose f is a smooth function with $\operatorname{supp} f \subset \Sigma$. Then u_*, the norm minimal solution to $\bar\partial u = f$ with respect to the weight $\mathrm{e}^{-\phi}$ subject to the growth bound $O(|z|^{n-1})$ at infinity, satisfies

$$\int_{\mathbb{C}} |u_*|^2 \mathrm{e}^{\varrho-\phi} \mathrm{d}A \le \frac{1}{(1-\kappa)^2} \int_{\Sigma} |f|^2 \mathrm{e}^{\varrho-\phi} \frac{\mathrm{d}A}{\Delta\widehat{\phi} + \Delta\varrho}.$$

Off-diagonal damping of reproducing kernels

From the Berndtsson-Hörmander type estimate, we may derive the following.

THM. Let S_τ^α be the subset of S_τ where $\Delta Q > \alpha$, and let $d(\cdot,\cdot)$ denote Euclidean distance in the plane. Then, for positive α, we get, for $w \in S_\tau^\alpha$,

$$|K_{m,n}(z,w)|^2 \mathrm{e}^{-m(\widehat{Q}_\tau(z)+Q(w))} \leq Cm^2 \, \mathrm{e}^{-\epsilon\sqrt{m}\, d(w,\{z\}\cup\mathbb{C}\setminus S_\tau^\alpha)},$$

for some positive number ϵ. Here, it is assumed that $m\tau - O(1) \leq n \leq m\tau + 1$.

Applications to random matrices

The Berndtsson-Hörmander type estimate has as its main application the effect that the measure

$$|K_{m,n}(z,w)|^2 e^{-m(Q(z)+Q(w))} dA(z)dA(w),$$

which has total mass n, get "localized" to a small neighborhood of the diagonal. This may then, together with the local expansion theorem for reproducing kernels, be used to prove that the fluctuations tend to GFF. For details, please listen to the talk by Y. Ameur.

Berezin quantization

The probablity measure

$$dB_{m,n}^{\langle w\rangle}(z)=\frac{|K_{m,n}(z,w)|^2}{K_{m,n}(w,w)}\,e^{-mQ(z)}dA(z)$$

we call the Berezin measure. For $w \in S_\tau$ it converges to a point mass at w as $m, n \to +\infty$ while $n = m\tau + O(1)$, while for $w \in \mathbb{C} \setminus S_\tau$ it cannot converge to a point mass. We conjecture that for $w \in \mathbb{C} \setminus S_\tau$, $dB_{m,n}^{\langle w\rangle}$ tends instead to harmonic measure for w in the domain $\mathbb{C} \setminus S_\tau$. In case w is a point in the interior of S_τ with $\Delta Q(w) > 0$, one can show that the Berezin measure – suitably blown up so that the scale $m^{-1/2}$ becomes 1 – tends to a radially symmetric Gaussian in the plane.

Berezin quantization – boundary points

Let us consider $Q(z) = |z|^2$ and $\tau = 1$. Then $S_\tau = S_1 = \bar{\mathbb{D}}$, the unit disk, and take $w = 1$. One shows that the blow-up of the Berezin measure converges to

$$\frac{1}{\pi} \mathrm{e}^{-|z|^2} \left| \int_z^{+\infty} \mathrm{e}^{-s^2/2} \mathrm{d}s \right|^2 \mathrm{d}A(z),$$

which of course is not radially symmetric.