THE KLEIN-GORDON EQUATION AND QUADRATURE

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Abstract. We obtain a quadrature identity which characterizes the solutions to the Klein-Gordon equation in $1+1$ dimensions.

1. Introduction

1.1. **The Laplace equation.** In this work, we will consider the (hyperbolic) Klein-Gordon equation in the $(1+1)D$ case. Let us first briefly review the instance of harmonic functions, which solve the (elliptic) 2D Laplace equation $\Delta u = 0$, where

$$\Delta := \partial_x^2 + \partial_y^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

is the Laplacian. It is well-known that harmonic functions may be characterized in terms of the mean value property

$$\int_{\mathbb{D}(x_0,y_0,r)} u(x,y)\,\mathrm{d}A(x,y) = \pi r^2\, u(x_0,y_0), \tag{1.1.1}$$

where $\mathbb{D}(x_0,y_0,r)$ denotes the open disk of radius $r>0$ centered at (x_0,y_0) and $\mathrm{d}A(x,y)=\mathrm{d}x\mathrm{d}y$ is area measure. To be precise, if u is C^2-smooth in a domain $\Omega\subset\mathbb{R}^2$, then u is harmonic there if and only if (1.1.1) holds for any triple (x_0,y_0,r) with the property that the closure of the disk $\mathbb{D}(x_0,y_0,r)$ is contained in Ω.

1.2. **The Klein-Gordon equation.** We now turn to the $(1+1)D$ Klein-Gordon equation

$$\Box u + \lambda u = 0, \tag{1.2.1}$$

where $\lambda \neq 0$ is real, and

$$\Box := \partial_x^2 - \partial_y^2$$

is the wave operator. Here, we reserve the right to interpret either x or y as the time parameter, whenever convenient. For the Klein-Gordon equation, this means that x is time when $\lambda>0$, whereas y is time when $\lambda<0$. We remark that the excluded instance $\lambda=0$ of (1.2.1) is the well-known wave equation.

The wave equation is associated with characteristic directions, parallel to the lines $y=x$ and $y=-x$. Together, these form light cones. We will use such light cones to form *diamonds*. We begin with a diamond which emanates from the origin, and is characterized by two positive real parameters a,b. In words, we just follow the two characteristic directions from the origin and stop at two points with positive y values, with coordinates $(x,y)=(a,a)$ and $(x,y)=(-b,b)$. At those two points, we switch and instead follow the characteristic direction not followed previously and stop when the two points intersect at $(x,y)=(a-b,a+b)$. The diamond $Q(a,b)$ is then the region enclosed by the two paths. The *vertices* of the diamond $Q(a,b)$ consist of the four points $(0,0)$, (a,a), $(-b,b)$, $(a-b,a+b)$, and we call $(0,0)$ and $(a-b,a+b)$ startpoint and endpoint, respectively, whereas (a,a) and $(-b,b)$ are middle points. We write $\mathcal{V}(Q)$ for the set of four vertices of the diamond $Q=Q(a,b)$, and declare that $\sigma:\mathcal{V}(Q)\to\{-1,1\}$, the sign function, assumes the value 1 at the startpoint and the endpoint, and the value -1 at both middle points. These notions are easily extended in a translation invariant manner, so that we may speak of diamonds emanating from a general point (x_0,y_0). The solutions to the Klein-Gordon (1.2.1)

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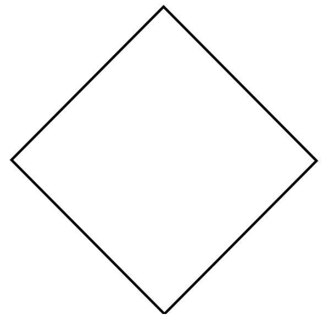

FIGURE 1.2.1. Illustration of a diamond.

are characterized by the quadrature property

$$\sum_{\nu\in\mathcal{V}(Q)}\sigma(\nu)u(\nu)=\frac{\lambda}{2}\int_Q u\mathrm{d}A. \tag{1.2.2}$$

for all diamonds Q precompactly contained in the domain Ω where u solves (1.2.1). A precise statement is the following: If u is C^2-smooth in a domain $\Omega\subset\mathbb{R}^2$, then u solves (1.2.1) if and only if the quadrature identity (1.2.2) holds for all diamonds Q whose closure is contained in Ω. The quadrature property (1.2.2) shows some similarity with the mean value property for harmonic functions. This time, only diamonds are allowed, and a weighted sum at the vertices replaces the value at the center. This may offer some clues as to why the solutions to the Klein-Gordon equation apparently exhibit some unexpected uniqueness properties along the characteristics, see [2], [1], [3], [4]. The proof of the equivalence of the Klein-Gordon equation (1.2.1) with the quadrature identity (1.2.2) is elementary, but seems not to have been observed previously. It does not obviously generalize to higher dimensions. There are several possible variants, but most look less attractive than (1.2.2).

2. The equivalence of the differential equation and the quadrature equation

2.1. Change-of-variables. We make the change of variables

$$\begin{cases}\xi=\frac{1}{2}(x+y),\\ \eta=\frac{1}{2}(y-x),\end{cases}$$

so that

$$\partial_\xi\partial_\eta=\partial_y^2-\partial_x^2,$$

and the differential equation (1.2.1) reads in the new coordinates

$$\partial_\xi\partial_\eta u=\lambda u,$$

Theorem 2.1.1. *Let u be C^2-smooth in a domain $\Omega\subset\mathbb{R}^2$. Then the following are equivalent:*
(i) $\Box u+\lambda u=0$ holds on Ω,
(ii) The quadrature identity (1.2.2) holds for every diamond Q with $\operatorname{clos}Q\subset\Omega$.

Proof. If we let Q be the diamond with vertices $(0,0)$, (a,a), $(-b,b)$, and $(a-b,a+b)$, we get from the change-of-variables formula that

$$\begin{aligned}-\int_Q\Box u(x,y)\,\mathrm{d}A(x,y)&=\int_0^b\int_0^a 2\partial_\xi\partial_\eta u\,\mathrm{d}\xi\mathrm{d}\eta\\ &=2u|_{\xi=a,\eta=b}-2u|_{\xi=a,\eta=0}-2u|_{\xi=0,\eta=b}+2u|_{\xi=0,\eta=0}\\ &=2u(a-b,a+b)-2u(a,a)-2u(-b,b)+2u(0,0)=2\sum_{\nu\in\mathcal{V}(Q)}\sigma(\nu)u(\nu).\end{aligned} \tag{2.1.1}$$

Now, if we assume (i) holds, then $-\Box u = \lambda u$ on Q, so that the left-hand side equals

$$\lambda \int_Q u \mathrm{d}A,$$

and the quadrature identity (1.2.2) follows for the given diamond Q. By translation invariance, Q may be assumed to be a general diamond contained inside Ω, and (ii) follows.

As for the reverse direction, we assume (ii), and observe that in view of (2.1.1), the quadrature formula maintains that

$$\int_Q (\Box u + \lambda u) \mathrm{d}A = 0$$

for every diamond. If we let $|Q|_A$ denote the area of the diamond in question, we see that

$$\frac{1}{|Q|_A} \int_Q (\Box u + \lambda u) \mathrm{d}A = 0$$

holds, and the left-hand side expresses the average of the continuous function $\Box u + \lambda u$ on Q. By letting the diamond Q shrink to a point, say $(x_0, y_0) \in \Omega$, the left-hand side converges to the value at the point, which is $\Box u(x_0, y_0) + \lambda u(x_0, y_0)$, whereas the right-hand side stays 0. It now follows that the differential equation holds at the given point: $\Box u(x_0, y_0) + \lambda u(x_0, y_0) = 0$. The point is arbitrary, so the differetial equation holds throughout Ω, and (i) follows. $\Box$

References

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