Polyanalytic Bergman Kernels

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Let $Q : \mathbb{C} \to \mathbb{R}$ be a $C^2$-smooth function satisfying

$$Q(z) \geq (1 + \epsilon) \log |z|^2, \quad |z| \geq C$$

for two positive numbers $\epsilon$ and $C$. This function will be referred to as the weight. We set

$$\text{Pol}_{q,n} := \text{span}_\mathbb{C}\{\bar{z}^r z^j \mid 0 \leq r \leq q - 1, 0 \leq j \leq n - 1\},$$

and

$$A^2_{q, mQ, n} := \text{Pol}_{q,n} \cap L^2(\mathbb{C}, e^{-mQ(z)}dA(z)).$$

Here $dA$ is the standard area measure on $\mathbb{C}$ divided by $\pi$. 
We equip $A^2_{q,mQ,n}$ with the inner product from $L^2(\mathbb{C}, e^{-mQ(z)}d(z))$. It is a finite dimensional Hilbert space and possesses the reproducing kernel

$$K_{q,mQ,n}(z, w) = \sum_j e_j(z)e_j(w),$$

where the set $\{e_j\}$ is any orthonormal basis of $A^2_{q,mQ,n}$.

We will study asymptotics of $K_{q,mQ,n}$ for fixed $q$ as $m, n \to +\infty$. Ameur, Hedenmalm and Makarov analyzed the case $q = 1$. 

Reproducing kernel of $A^2_{q,mQ,n}$
Suppose that $A_{q,mQ,n}^{2}$ is $nq$-dimensional. We use $K_{q,mQ,n}$ to define the following probability density measure on $\mathbb{C}^{nq}$:

$$dP_{q,mQ,n}(z_1, \ldots, z_{nq}) = \frac{1}{(nq)!} \det \left[K_{q,mQ,n}(z_i, z_j)\right]_{nq \times nq} e^{-m \sum_{j=1}^{nq} Q(z_j)} dA(z_1) \ldots dA(z_{nq}).$$

(This is a probability measure by a standard argument in random matrix theory.)

This is a particular instance of a determinantal point process. We identify all the copies of $\mathbb{C}$ and think this as a random configuration of $nq$ points in the plane $\mathbb{C}$.
In general, determinantal processes model non-interacting fermions in quantum mechanics. Notice the repulsion between the points!

In the special case \( Q(z) = |z|^2 \) the distribution defined by \( K_{q,m|z|^2,n} \) describes a system of non-interacting electrons in a uniform magnetic field perpendicular to the plane, so that \( q \) first energy (Landau) levels each have \( n \) particles. The strength of the magnetic field is proportional to the parameter \( m \).
We studied the case $Q(z) = |z|^2$. The kernel can be then be explicitly written down with Laguerre polynomials. Important statistical quantities related to the point process can be expressed with the kernel. For example, integrating the one-point intensity

$$K_{q,m|z|^2,n}(z,z)e^{-m|z|^2}$$

over a set $A$ gives the expected number of particles in $A$. Therefore, we want to have good asymptotics for the kernel. As $n, m \to +\infty$ and $|n - m| = O(1)$, studying the one-point intensity shows that the points tend to accumulate uniformly on the unit disk.
Figure: The determinantal process defined by the kernel $K_{m,m|z|^2,n}$ with $m = n = 61$ and $q = 3$. The simulation is based on the algorithm by Hough, Krishnapur, Peres, and Virág.
We turn to more general weights $Q$ now but concentrate mostly on the case $q = 2$. Where are the particles now? Analysis of the one-point intensity

$$\Lambda_{2,mQ,n}^1(z) = K_{2,mQ,n}(z, z)e^{-mQ(z)}.$$ 

shows that the points accumulate on a certain compact set $S$ with density $\Delta Q(z)$. The same happens in the case $q = 1$. The set $S$ is the support of the equilibrium measure from weighted log potential theory with the external weight $mQ$. 
We would like to have more detailed understanding of the interactions in the "bulk", i.e. in the interior $S$. This leads to the study the so-called connected two-point function

$$|K_{q,mQ,n}(z, w)|^2 e^{-mQ(z)-mQ(w)},$$

which measures the repulsion between a point in $z$ and $w$. 
Theorem (Haimi)

Set $q = 2$. Fix $z_0 \in \text{int}S$ satisfying $\Delta Q(z_0) > 0$ and $M > 0$. Assume that $Q$ is real-analytic in a neighborhood of $z_0$. Then, there exists a number $m_0$ such that for all $m \geq m_0$, we have

\[
\left| \frac{1}{m\Delta Q(z_0)} K_{q,mQ,n} \left( z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}}, z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}} \right) \right| \\
\times e^{-\frac{1}{2} mQ \left( z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}} \right)} - \frac{1}{2} mQ \left( z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}} \right) = |L_{q-1}^1(|\xi - \lambda|^2)| e^{-\frac{1}{2} |\xi - \lambda|^2} + O(m^{-1/2}),
\]

as $m \to +\infty$ and $n \geq m - M$. The convergence is uniform on compact sets of $\mathbb{C}^2$. Here, $L_{q-1}^1$ stands for the associated Laguerre polynomial of degree $q - 1$ and parameter 1.
Figure: A Fresnel zone pattern, exhibiting rings of zero repulsion around a given point in the bulk.
Main ingredient: polyanalytic extension of Tian-Yau-Catlin-Zelditch expansion

- The proof of this result is based on an algorithm to compute asymptotic expansion of polyanalytic Bergman kernels. This generalizes the work of Tian-Yau-Catlin-Zelditch on analytic Bergman kernels.
- The result is stated in the following, slightly different setting. Let $\Omega$ be a domain in $\mathbb{C}$ and $Q : \Omega \to \mathbb{R}$ $C^4$-smooth strictly subharmonic function which also satisfies
  \[
  \sup_{\Omega} \frac{1}{\Delta Q} \Delta \log \frac{1}{\Delta Q} < +\infty.
  \]

  We define the Hilbert space of $q$-analytic functions on $\Omega$:
  \[
  A^2_{q,mQ} := \{ u : \bar{\partial}^q u = 0, \int_{\Omega} |u|^2 e^{-mQ} dA < +\infty \}.
  \]

  Let $K_{q,mQ}$ denote the reproducing kernel of $A^2_{q,mQ}$. 

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We now fix an arbitrary point $z_0$ in $\Omega$ and assume that $Q$ is real-analytic in the neighborhood. Then, for some $r > 0$, there exists a polarization $Q : \mathbb{D}(z_0, r) \times \mathbb{D}(z_0, r) \to \mathbb{C}$ of $Q$. This function is analytic in $z$, conjugate-analytic in $w$ and satisfies $Q(z, z) = Q(z)$. 
Consider a series of the following form, defined on a bidisk $\mathbb{D}(z_0, r) \times \mathbb{D}(z_0, r)$:

$$K^{(k)}_{2,mQ}(z, w) = \left[ m^2 \sigma_{2,2}(z, w) + m \sigma_{2,1}(z, w) + \sigma_{2,0}(z, w) + m^{-1} \sigma_{2,-1}(z, w) 
+ \cdots + m^{-k} \sigma_{2,-k}(z, w) \right] e^{mQ(z,w)},$$

(2)

where the functions $\sigma_{2,j}$ are bianalytic in the first and conjugate bianalytic in the second variable. We say $K^{(k)}_{2,mQ}$ is a local bianalytic Bergman kernel of order $k$ we have

$$|K_{2,mQ}(z, w) - K^{(k)}_{2,mQ}(z, w)| e^{-\frac{1}{2} m(Q(z)+Q(w))} = O(m^{-k-1})$$

on $\mathbb{D}(z_0, r) \times \mathbb{D}(z_0, r)$ as $m \to +\infty$. 
Theorem (Haimi, Hedenmalm)

Fix $z_0 \in \Omega$. Then, for $r > 0$ small enough, there exist local bianalytic Bergman kernels on $\mathbb{D}(z_0, r) \times \mathbb{D}(z_0, r)$ of any order.

- The proof is constructive: we provide a mechanical algorithm to compute the coefficients. The method is based on the work of Berman-Berndtsson-Sjöstrand in the analytic case. The key idea is to use a certain technique from the theory of Fourier integral operators.

- Later, Haimi found that BBS-method can be simplified in the analytic case (at least in a one complex variable setting), and then extended this to polyanalytic functions. Microlocal techniques are now replaced by Taylor expansion and integration by parts.
Let us come back to polynomials.

**Theorem (Haimi)**

*Fix a compact set* $K$ *in the interior of* $S \cap N_+$ *and constant* $M > 0$. *Set*

$$r := \frac{1}{4} \text{dist}(K, \mathbb{C} \setminus (S \cap N_+)).$$

*Then, there exist positive constants* $C, \epsilon$ *and* $m_0$ *such that for any* $z_0 \in K$ *and* $z_1 \in S$, *we have*

$$|K_{2,mQ,n}(z_0, z_1)|^2 e^{-mQ(z_0) - mQ(z_1)} \leq Cm^2 e^{-\epsilon \sqrt{m} \min\{r, |z_0 - z_1|\}}$$

*where we assume* $m \geq \max\{m_0, M - 1\}$ *and* $n \geq m - M + 1$. *The constants* $C, \epsilon$ *and* $m_0$ *only depend on* $Q, K$ *and* $M$. 
In the model case $Q(z) = |z|^2$, we also found a scaling limit for the kernel on the boundary:

$$
\frac{1}{m} K_{q,m|z|^2,n}(1 + \frac{\xi}{\sqrt{m}}, 1 + \frac{\xi}{\sqrt{m}}) \rightarrow \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{q-1} \frac{1}{r!} \int_0^{\infty} H_r(t)^2 e^{-\frac{1}{2}t^2/2} dt \quad (3)
$$

Here, $H_r$ is an Hermite polynomial. This should be thought of as a "generalized error function". Notice the connection to GUE. It is likely that the limit obtained here is universal.
If we let first $n \to +\infty$ and then $q \to +\infty$, blow-up and rescale around a bulk point, we obtain the limit $J^1(2|\xi|)/|\xi|$. Two dimensional analogue of the sine kernel.

Open question: what happens if we require for example $n + q \leq N$ and then let $N \to +\infty$.

On the boundary in the Gaussian case: integrated semicircle law.