The polyanalytic Ginibre ensembles

Haakan Hedenmalm (joint with Antti Haimi)

Royal Institute of Technology, Stockholm

haakanh@kth.se

June 30, 2011

Background

Let

$$\triangle(\lambda_1, \ldots, \lambda_n) = \prod_{i,j:1\leq i<j\leq n} (\lambda_j - \lambda_i)$$

be the vandermondian. Let $Q : \mathbb{C} \to \mathbb{R}$ be a (smooth) confining potential, with a certain minimal growth at infinity. We put

$$\mathrm{d}\mathbb{P}(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z} |\triangle(\lambda_1, \ldots, \lambda_n)|^2 \mathrm{e}^{-m[Q(\lambda_1)+\ldots+Q(\lambda_n)]}$$

where Z is a normalizing constant to get a probability measure. This models a fermionic cloud under a confining potential. The model is also known as *Coulomb gas*. We should think of $m = n$ and that we look for asymptotics as n goes to infinity.

Results for analytic ensembles

By adjusting the methods of K. Johansson (DMJ 1997) for ensembles on the real line to the complex case, we obtained the following.

Theorem

(Hedenmalm-Makarov, 2004) With probability 1, the sum of point masses $\sum_{i=1}^{n} \mathrm{d}\delta_{\lambda_i}$ *tends to* $1_S \Delta Q \mathrm{d}A$ *as* $n \to +\infty$*. Here,* S *is the support of the equilibrium measure, which may be obtained from an obstacle problem.*

Let f be a smooth compactly supported real-valued test function on the interior of S. Let $\mathrm{fluct}_n f := f(\lambda_1) + \ldots + f(\lambda_n) - n \int_S f \Delta Q \mathrm{d}A$.

Theorem

(Ameur-Hedenmalm-Makarov, 2009) As n *tends to infinity, the variable* $\mathrm{fluct}_n f$ *tends to a Gaussian normal* $N(e_f, v_f)$ *with mean* $e_f = (2\pi)^{-1} \int_S f \Delta \log \Delta Q \mathrm{d}A$ *and variance* $v_f = (4\pi)^{-1} \int_S |\nabla f|^2 \mathrm{d}A$*.*

The reproducing kernel connection

Let $K_n(z,w)$ denote the reproducing kernel of the space of polynomials in z of degree $\leq n-1$ with respect to the inner product of $L^2(\mathbb{C}, e^{-nQ}\mathrm{d}A)$. Then the k-intensity of the Coulomb gas process is given by ($k \leq n$ here)

$$\det[K_n(z_i,z_j)e^{-n[Q(z_i)+Q(z_j)]/2}]_{i,j=1}^k;$$

the n-intensity is up to proportionality constant the original density of states. The k-intensity describes the likelihood density of finding a k-tuple of points in position $(z_1,\ldots,z_k)$. Here, we just need the 1-point intensity $K_n(z_1,z_1)e^{-nQ(z_1)}$ and the 2-point density $[K_n(z_1,z_1)K_n(z_2,z_2)-|K_n(z_1,z_2)|^2]e^{-n[Q(z_1)+Q(z_2)]}$.

The Berezin density

The reproducing kernel K_n is associated with the orthogonal projection onto a the space of polynomials of degree $\leq n-1$. In a sense, the polynomial space is the quantized model and the weighted L^2-space is the classical analogue. In an effort to produce a more robust model of quantization, F. A. Berezin suggested to replace the kernel $K_n(z,w)$ by

$$B_n^{\langle z\rangle}(w) = \frac{|K_n(z,w)|^2}{K_n(z,z)} e^{-nQ(w)}$$

which defines a probability density, and acts boundedly on $L^\infty(\mathbb{C})$.

Theorem

(Ameur, Hedenmalm, Makarov) For bulk point z_0, the dilated probability density $\xi \mapsto n^{-1}B_n^{\langle z\rangle}(z_0 + m^{-1/2}\xi)$ converges as n tends to infinity to the Gaussian $\Delta Q(z_0)e^{-|\xi|^2\Delta Q(z_0)}$.

Next, we fix $Q(z) = |z|^2$ so that we are in the Ginibre setting. Then the spectral droplet S is the closed unit disk, and the bulk is the open unit disk $\mathbb{D}$. We let $K_{n,q}$ be the reproducing kernel for the subspace of polynomials in z and $\bar{z}$, where the degree in z is $\leq n-1$ and the degree in $\bar{z}$ is $\leq q-1$. We consider the point process with k-point intensity given by

$$\det[K_{n,q}(z_i, z_j)]_{i,j=1}^{k}$$

and call it the *q-polyanalytic Ginibre ensemble*. The nq-point density the joint probability distribution for the process (after rescaling). A typical sample from this process with $q = 3$ and $n = m = 61$ is supplied in the figure below.

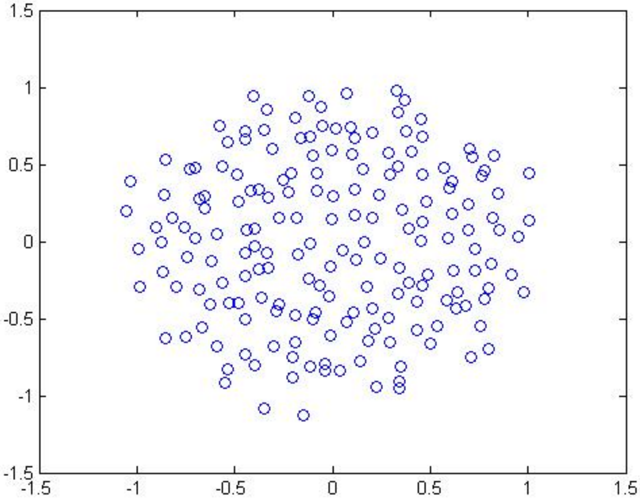

Figure: Polyanalytic Ginibre process with $q = 3$ and $m = n = 61$

Lemma

For $q \leq n$, the kernel $K_{n,q}$ is given by $K_{n,q}(z,w) = K^{I}_{n,q}(z,w) + K^{II}_{n,q}(z,w)$, *where*

$$K^{I}_{n,q}(z,w) = n \sum_{r=0}^{q-1} \sum_{i=0}^{n-r-1} \frac{r!}{(r+i)!} (nz\bar{w})^{i} L^{i}_{r}(n|z|^2) L^{i}_{r}(n|w|^2)$$

and

$$K^{II}_{n,q}(z,w) = n \sum_{j=0}^{q-2} \sum_{k=1}^{q-j-1} \frac{j!}{(k+j)!} (\bar{z}w)^{k} L^{k}_{j}(n|z|^2) L^{k}_{j}(n|w|^2).$$

Definition

If $(\lambda_1, \ldots, \lambda_{nq})$ have joint probability density from the q-polyanalytic Ginibre ensemble, and $z_0 \in \mathbb{C}$, the process $(\xi_1, \ldots, \xi_{nq})$ given by $\lambda_j = z_0 + n^{-1/2}\xi_j$ is called the local blow-up process at z_0 tol scale $n^{-1/2}$.

Theorem

(Haimi-Hedenmalm) For bulk points $z_0 \in \mathbb{D}$, the local blow-up process at z_0 to scale$n^{-1/2}$ is for large n approximately given by the intensities with correlation kernel $L^1_{q-1}(|\xi - \eta|^2)e^{\xi\bar{\eta}}e^{-(|\xi|^2+|\eta|^2)/2}$.

Corollary

(Haimi-Hedenmalm) At bulk points $z_0 \in \mathbb{D}$, the local blow-up process at z_0 to scale $(qn)^{-1/2}$ for large q and much bigger n is approximately given by the intensities with correlation kernel $|\xi|^{-1}J_1(2|\xi|)$.

Remark: The above correlation kernel is the analogue of the sine kernel in the 1D setting.

Theorem

At boundary points $z_0 \in \mathbb{T} = \partial\mathbb{D}$, WLOG $z_0 = 1$, the local blow-up process to scale $(q/n)^{1/2}$ has, for big q and much larger n, the 1-point function approximately given by ($-1 \leq \mathrm{Re}\,\xi \leq 1$ here)

$$\frac{2}{\pi}\int_{-1}^{-\mathrm{Re}\,\xi}\sqrt{1-t^2}\mathrm{d}t.$$

Remark: So the density of particles is nontrivial in the annulus

$$1-(q/m)^{1/2} \leq |z| \leq 1+(q/m)^{1/2};$$

inside the annulus the density is approximately a positive constant, and outside it approximately vanishes.

References

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