OPEN PROBLEMS IN THE FUNCTION THEORY OF THE BERGMAN SPACE

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1. The basic project

Let $L^2_a(\mathbb{D})$ be the usual Bergman space of square area integrable analytic functions on the open unit disk $\mathbb{D}$, with norm

$$||f||_{L^2} = \left(\int_{\mathbb{D}} |f(z)|^2 dS(z) \right)^{1/2}.$$

Here, dS denotes area measure in $\mathbb{C}$, normalized by a constant factor:

$$dS(z) = dxdy/\pi, \qquad z = x + iy.$$

A closed subspace J of $L^2_a(\mathbb{D})$ is said to be z-invariant, or simply invariant, provided the product zf belongs to J whenever $f \in J$. Here, we use the standard notation z for the coordinate function:

$$z(\lambda) = \lambda, \qquad \lambda \in \mathbb{D}.$$

A sequence $A = \{a_j\}_j$ of points in $\mathbb{D}$, is said to be an $L^2_a(\mathbb{D})$ zero sequence if there exists a function in $L^2_a(\mathbb{D})$ that vanishes precisely on the sequence A, counting multiplicities.

Two big projects for this space are as follows.

Problem 1.1. *Characterize the invariant subspaces of* $L^2_a(\mathbb{D})$.

Problem 1.2. *Characterize the* $L^2_a(\mathbb{D})$ *zero sequences.*

Of these two problems, the second one is more likely to have a definite answer than the first one. In fact, Problem 1.1 seems to be as difficult as the famous invariant subspace problem in Hilbert space. I was told by Stefan Richter that it is a consequence of the dilation theory of Apostol, Bercovici, Foiaş, and Pearcy [**1**] that if we know that given two z-invariant subspaces I, J in $L^2_a(\mathbb{D})$, with $I \subset J$, and $\dim(J \ominus I) = +\infty$, there exists another invariant subspace K, other than I and J, but contained in J and containing I, then every bounded linear operator on a separate Hilbert space has a nontrivial invariant subspace. Still, there is hope that

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partial results can be obtained, for instance, we might be better able to characterize an invariant subspace if we know something about its weak spectrum (see [**11, 22**] for a definition). Aharon Atzmon has obtained a description of invariant subspaces in $L^2_a(\mathbb{D})$ with one-point spectra; the proof is the standard one based on the resolvent transform (look at, for instance, [**12, 11**]). Another question which we might be able to handle asks for a description of the maximal invariant subspaces in $L^2_a(\mathbb{D})$; see Section 10 for details.

There has been some progress on Problem 1.2. Charles Horowitz [**14**] obtained some interesting results, for instance, he proved that there are $L^2_a(\mathbb{D})$ zero sequences $A = \{a_j\}_j$ of non-Blaschke type, that is, having

$$\sum_j (1 - |a_j|) = +\infty,$$

and that every subsequence of a zero sequence is a zero sequence as well. More recently, Emile LeBlanc [**19**] and Gregory Bomash [**3**] obtained probabilistic conditions on zero sets, and Kristian Seip [**23**] obtained a complete description of sampling and interpolating sequences, based on Boris Korenblum's [**16, 17**] work for A^{-n}. Seip [**24**] now has a description of the zero sets for $L^2_a(\mathbb{D})$, which is complete but for a small gap. A full characterization of zero sets may now be within reach.

Associated with the above two projects is the problem of factoring functions in the Bergman space in an effective way. For this, we need the concepts of inner functions and Blaschke products, suitably modified to this setting.

Definition 1.3. *A function $G \in L^2_a(\mathbb{D})$ is said to be an inner divisor for $L^2_a(\mathbb{D})$ if*

$$h(0) = \int_{\mathbb{D}} h(z)|G(z)|^2 dS(z)$$

holds for all bounded harmonic functions h on $\mathbb{D}$.

We note here that if normalized area measure dS on $\mathbb{D}$ is replaced by normalized arc length measure in the above definition, we have a rather unusual, though equivalent, definition of the concept of an inner function in $H^2(\mathbb{D})$. The Hardy space $H^2(\mathbb{D})$ consists by definition of all analytic functions f in the unit disk $\mathbb{D}$ satisfying

$$\|f\|_{H^2} = \left(\sup_{0<r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta/2\pi \right)^{1/2} < +\infty.$$

In analogy with finite Blaschke products, we define finite zero divisors as follows.

Definition 1.4. *An inner divisor for $L^2_a(\mathbb{D})$ is said to be a finite zero divisor for $L^2_a(\mathbb{D})$ if it extends continuously to the closed unit disk $\overline{\mathbb{D}}$. If A is its finite zero set in $\mathbb{D}$, counting multiplicities, we shall denote this function by G_A.*

It is implicit in the above definition that a finite zero divisor for $L^2_a(\mathbb{D})$ only has finitely many zeros in $\mathbb{D}$, and that it is determined uniquely, up to a unimodular constant multiple, by its finite sequence of zeros. These facts may be derived from the results obtained in [**6**]. It is, moreover, known that to every finite sequence A in $\mathbb{D}$, there exists a finite zero divisor G_A vanishing precisely on A inside $\mathbb{D}$.

Definition 1.5. *An inner divisor G is said to be a zero divisor for $L^2_a(\mathbb{D})$ if it is the limit (as $N \to +\infty$) of a sequence of finite zero divisors G_{A_N}, with $A_1 \subset A_2 \subset A_3 \subset \ldots$, in the topology of uniform convergence on compact subsets of $\mathbb{D}$.*

We note in passing that a zero divisor for $L^2_a(\mathbb{D})$ is uniquely determined, up to multiplication by a unimodular constant factor, by its sequence of zeros, counting multiplicities. We shall frequently write G_A for the zero divisor associated with the zero sequence A.

Given an inner divisor G for $L^2_a(\mathbb{D})$, we denote by Φ_G the function

$$\Phi_G(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\left(|G(\zeta)|^2 - 1\right) dS(\zeta), \qquad z \in \mathbb{D};$$

here, $\Gamma(z,\zeta)$ stands for the Green function for the Laplacian Δ:

$$\Gamma(z,\zeta) = \log\left|\frac{\zeta - z}{1 - \bar{\zeta}z}\right|^2, \qquad (z,\zeta) \in \mathbb{D}^2.$$

Throughout this paper, we use the slightly nonstandard Laplacian

$$\Delta_z = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \qquad z = x + iy,$$

and we regard locally integrable functions u on $\mathbb{D}$ as distributions via the dual action

$$\langle \phi, u\rangle = \int_{\mathbb{D}} u(z)\phi(z) dS(z),$$

for test functions ϕ. The function Φ_G solves the boundary value problem

$$\begin{cases} \Delta\Phi_G(z) & = |G(z)|^2 - 1, \qquad z \in \mathbb{D}, \\ \Phi_G(z) & = 0, \qquad z \in \mathbb{T}, \end{cases}$$

and it is interesting to note that in terms of the function Φ_G, the condition that G be an inner divisor may be written in a more explicit form: $\nabla\Phi_G = 0$ on $\mathbb{T}$ (in a weak sense if Φ_G is not continuously differentiable up to the boundary $\mathbb{T}$). Here, ∇ denotes the gradient operator.

The following result was proved in [**6, 7**].

Theorem 1.6. *If G is an inner divisor for $L^2_a(\mathbb{D})$, then the function Φ_G meets*

$$0 \le \Phi_G(z) \le 1 - |z|^2, \qquad z \in \mathbb{D},$$

and we have the isometry

$$\|Gf\|^2_{L^2} = \|f\|^2_{L^2} + \int_{\mathbb{D}} |f'(z)|^2 \Phi_G(z)\, dS(z),$$

valid for all $f \in H^2(\mathbb{D})$. As a consequence, we have

$$\|f\|_{L^2} \le \|Gf\|_{L^2} \le \|f\|_{H^2}, \qquad f \in H^2(\mathbb{D}).$$

Problem 1.7. *Does the isometry in Theorem 1.6 extend to all $f \in L^2_a(\mathbb{D})$? If not, then for which inner divisors G is this so? It is clear that finite zero divisors have this property.*

Through several steps, the details of which we do not wish to discuss here, Theorem 1.6 has the following consequence (see [**6, 7**]).

Corollary 1.8. *Let A be a zero sequence for the space $L^2_a(\mathbb{D})$. Then there exists a zero divisor G_A vanishing precisely on A inside $\mathbb{D}$, and it has the property that every function $f \in L^2_a(\mathbb{D})$ vanishing on A admits a factoring $f = G_A \cdot g$, with $g \in L^2_a(\mathbb{D})$, and $\|g\|_{L^2} \le \|f\|_{L^2}$.*

2. A conjecture on zero divisors

It is a consequence of Theorem 1.6 that if G is an inner divisor for $L^2_a(\mathbb{D})$, we have

$$\|f\|_{L^2} \le \|Gf\|_{L^2}, \qquad f \in H^2(\mathbb{D}),$$

which may be written as $1 \prec G$, in the notation introduced by Boris Korenblum [**18**]. The precise definition of $G \prec H$, for $G, H \in L^2_a(\mathbb{D})$, is

$$\|Gf\|_{L^2} \le \|Hf\|_{L^2}, \qquad f \in H^\infty(\mathbb{D}).$$

The zero divisor for the empty zero sequence is $G_\emptyset = 1$, so we may interpret $1 \prec G_A$ as $G_\emptyset \prec G_A$. Here, G_A is the zero divisor associated with a zero sequence A. Maybe $G_\emptyset \prec G_A$ should be thought of as a consequence of the fact that $\emptyset \subset A$ holds for all A.

Conjecture 2.1. *If A, B are two zero sequences for $L^2_a(\mathbb{D})$ having $A \subset B$, then $G_A \prec G_B$.*

This conjecture would be proved if we could demonstrate the following. We write here Φ_A instead of Φ_{G_A}, for a given zero sequence A.

Conjecture 2.2. *If A, B are two finite zero sequences for $L^2_a(\mathbb{D})$ having $A \subset B$, then $\Phi_A \le \Phi_B$ holds on $\mathbb{D}$.*

Remark 2.3. If Conjecture 2.2 holds, a limit process argument then asserts that $\Phi_A(z) \le \Phi_B(z)$, $z \in \mathbb{D}$, holds for general zero sequences A, B with $A \subset B$. □

The following result connects Conjecture 2.2 with Problem 1.7.

Proposition 2.4. *The isometry*

$$\|G_A f\|^2_{L^2} = \|f\|^2_{L^2} + \int_{\mathbb{D}} |f'(z)|^2 \Phi_A(z)\, dS(z),$$

holds for all holomorphic functions f on $\mathbb{D}$, and all finite sequences A in $\mathbb{D}$. If A is an infinite zero sequence for the space $L^2_a(\mathbb{D})$, all we know is that

$$\|f\|^2_{L^2} + \int_{\mathbb{D}} |f'(z)|^2 \Phi_A(z)\, dS(z) \le \|G_A f\|^2_{L^2},$$

holds for f analytic in $\mathbb{D}$. However, assuming the validity of Conjecture 2.2, we have equality for all f holomorphic in $\mathbb{D}$, and all zero sequences A:

$$\|G_A f\|_{L^2}^2 = \|f\|_{L^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 \Phi_A(z)\, dS(z).$$

Proof. Suppose first the sequence A is finite. If f is analytic on $\mathbb{D}$, and r has $0 < r < 1$, consider the dilation f_r of f,

$$f_r(z) = f(rz), \qquad z \in \mathbb{D},$$

which clearly belongs to the space $H^2(\mathbb{D})$. By the isometry of Theorem 1.6, we have

$$\|G_A f_r\|_{L^2}^2 = \|f_r\|_{L^2}^2 + \int_{\mathbb{D}} |f_r'(z)|^2 \Phi_A(z)\, dS(z).$$

Since A is finite, we know that G_A is bounded away from 0 and ∞ in a small neighborhood of the circle $\mathbb{T}$ [**6**], and by Theorem 1.6, $\Phi_A \geq 0$. Therefore, if we let r tend to 1 in the above identity, with the understanding that if one side takes the value $+\infty$, then so does the other, we obtain in the limit

$$\|G_A f\|_{L^2}^2 = \|f\|_{L^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 \Phi_A(z)\, dS(z).$$

We may now concentrate on the case when A is an infinite sequence. We then write $A = \{a_j\}_{j=1}^{\infty}$, and denote by A_N the finite subsequence $\{a_j\}_{j=1}^{N}$. By the above argument, we have the isometry

$$\|G_{A_N} f\|_{L^2}^2 = \|f\|_{L^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 \Phi_{A_N}(z)\, dS(z), \qquad f \in \mathcal{O}(\mathbb{D}), \tag{2.1}$$

for all positive integers N, if $\mathcal{O}(\mathbb{D})$ denotes the Fréchet space of all holomorphic functions on $\mathbb{D}$. To obtain the stated inequality for general zero sequences A, we write $g = G_A\, f$, and apply (2.1) to the function g/G_{A_N}, to get

$$\|g\|_{L^2}^2 = \|g/G_{A_N}\|_{L^2}^2 + \int_{\mathbb{D}} |\left(g/G_{A_N}\right)'(z)|^2 \Phi_{A_N}(z)\, dS(z).$$

Letting $N \to +\infty$, an application of Fatou's lemma yields

$$\|g/G_A\|_{L^2}^2 + \int_{\mathbb{D}} |\left(g/G_A\right)'(z)|^2 \Phi_A(z)\, dS(z) \leq \|g\|_{L^2}^2.$$

Remembering that g was the function $G_A\, f$, the claimed inequality

$$\|f\|_{L^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 \Phi_A(z)\, dS(z) \leq \|G_A f\|_{L^2}^2$$

follows.

We now proceed to obtain the isometry under the assumption of Conjecture 2.2. By Conjecture 2.2 and the monotone convergence theorem, the right hand side of the identity (2.1) converges to

$$\|f\|_{L^2}^2+\int_{\mathbb{D}}|f'(z)|^2\Phi_A(z)\,dS(z)$$

as $N\to+\infty$, and by Fatou's lemma,

$$\|G_Af\|_{L^2}\le\limsup_{N\to+\infty}\|G_{A_N}f\|_{L^2},\qquad f\in\mathcal{O}(\mathbb{D}).$$

We conclude that

$$\|G_Af\|_{L^2}^2\le\|f\|_{L^2}^2+\int_{\mathbb{D}}|f'(z)|^2\Phi_A(z)\,dS(z),\qquad f\in\mathcal{O}(\mathbb{D}).$$

The proof of Proposition 2.4 is complete. □

Remark 2.5. If we denote by $\mathcal{H}(A)$ the Hilbert space of holomorphic functions in $\mathbb{D}$ with norm

$$\|f\|_{\mathcal{H}(A)}^2=\|f\|_{L^2}^2+\int_{\mathbb{D}}|f'(z)|^2\Phi_A(z)\,dS(z),$$

we may reformulate part of the assertion of Proposition 2.4 as saying that if f is holomorphic in $\mathbb{D}$, and A is a zero sequence for $L^2_a(\mathbb{D})$, then $G_Af\in L^2_a(\mathbb{D})$ implies $f\in\mathcal{H}(A)$, and $\|f\|_{\mathcal{H}(A)}\le\|G_Af\|_{L^2}$. In view of Theorem 1.6, we have the isometry $\|f\|_{\mathcal{H}(A)}=\|G_Af\|_{L^2}$ for all $f\in H^2(\mathbb{D})$, and consequently, for all f in the closure of $H^2(\mathbb{D})$ in $\mathcal{H}(A)$. □

3. Connections with potential theory and partial differential equations

Peter Duren, Dmitry Khavinson, Harold Shapiro, and Carl Sundberg have a proof of Theorem 1.6 (see [**5, 7**]), which is based on the fact that the biharmonic Green function

$$U(z,\zeta)=|z-\zeta|^2\Gamma(z,\zeta)+\left(1-|z|^2\right)\left(1-|\zeta|^2\right),\qquad (z,\zeta)\in\mathbb{D}^2,$$

is positive on the bidisk $\mathbb{D}\times\mathbb{D}$. Here, $\Gamma(z,\zeta)$ is the usual Green function for Δ:

$$\Gamma(z,\zeta)=\log\left|\frac{\zeta-z}{1-\bar\zeta z}\right|^2,\qquad (z,\zeta)\in\mathbb{D}^2.$$

The biharmonic Green function solves the PDE boundary value problem

$$\begin{cases}\Delta_z^2U(z,\zeta) & =\delta_\zeta(z),\qquad z\in\mathbb{D},\\ U(z,\zeta),\ \nabla_zU(z,\zeta) & =0,\qquad z\in\mathbb{T},\end{cases}$$

for a given point $\zeta\in\mathbb{D}$. Conjectures 2.1 and 2.2 would be demonstrated if we could prove the assertion below.

Conjecture 3.1. *The Green function for the singular fourth order elliptic operator $\Delta|G|^{-2}\Delta$ is positive on $\mathbb{D}\times\mathbb{D}$ for every finite zero divisor G for the space $L^2_a(\mathbb{D})$. Here we mean by the Green function the solution $U_G(z,\zeta)$ to the problem*

$$\begin{cases} \Delta_z|G(z)|^{-2}\Delta_z U_G(z,\zeta) &= \delta_\zeta(z), \qquad z\in\mathbb{D},\\ U_G(z,\zeta),\,\nabla_z U_G(z,\zeta) &= 0, \qquad z\in\mathbb{T}.\end{cases}$$

The following argument indicates why Conjectures 2.1 and 2.2 would be easy consequences of Conjecture 3.1. Let A and B be two finite sequences of points in the disk, having $A\subset B$. The difference function $\Psi=\Phi_B-\Phi_A$ solves

$$\begin{cases} \Delta\Psi(z) &= |G_B(z)|^2-|G_A(z)|^2, \qquad z\in\mathbb{D},\\ \Psi(z) &= 0, \qquad z\in\mathbb{T},\\ \nabla\Psi(z) &= 0, \qquad z\in\mathbb{T},\end{cases}$$

and if we divide both sides of the top line by $|G_A(z)|^2$, we get

$$|G_A(z)|^{-2}\Delta\Psi(z)=|G_B(z)/G_A(z)|^2-1, \qquad z\in\mathbb{D}.$$

Since we have overdetermined boundary values, we are at liberty to apply another Laplacian, to get

$$\begin{cases} \Delta|G_A(z)|^{-2}\Delta\Psi(z) &= |(G_B/G_A)'(z)|^2, \qquad z\in\mathbb{D},\\ \Psi(z),\,\nabla\Psi(z) &= 0, \qquad z\in\mathbb{T}.\end{cases}$$

Note here that we used the fact that the quotient $G_B(z)/G_A(z)$ is holomorphic on $\mathbb{D}$. Finally, we see that in terms of the Green function $U_{G_A}(z,\zeta)$, we may express Ψ as

$$\Psi(z)=\int_D U_{G_A}(z,\zeta)|(G_B/G_A)'(\zeta)|^2 dS(\zeta), \qquad z\in\mathbb{D},$$

and the positivity of Ψ is now immediate, provided Conjecture 3.1 holds.

In [**9**], I calculated the above Green function in the special case that all the zeros of G are located at the origin. I have, moreover, also computed this Green function for G having a single zero with arbitrary location in $\mathbb{D}$. These computations confirm the validity of Conjecture 3.1. The precise formula in the case of a single zero is given in the following statement.

Proposition 3.2. *Let a be a point in $\mathbb{D}$, and let G_a be the finite zero divisor for $L^2_a(\mathbb{D})$ associated with a zero at a,*

$$G_a(z)=\frac{1}{\sqrt{2-|a|^2}}\,\frac{(a-z)(2-|a|^2-\bar a z)}{(1-\bar a z)^2}, \qquad z\in\mathbb{D}.$$

We write H_a for the function

$$H_a(z)=-\frac{z}{\sqrt{2-|a|^2}}\,\frac{z-a(2-|a|^2)}{1-\bar a z}, \qquad z\in\mathbb{D},$$

which has $H_a'(z) = G_a(z)$. Then the formula for the Green function $U_{G_a}(z,\zeta)$ is

$$U_{G_a}(z,\zeta) = \left|\frac{H_a(z) - H_a(\zeta)}{z-\zeta}\right|^2 U(z,\zeta) + \frac{(1-|a|^2)^4}{(2-|a|^2)^2} \cdot \frac{(1-|z|^2)^2(1-|\zeta|^2)^2}{|1-\bar{a}z|^2|1-\bar{a}\zeta|^2}, \qquad (z,\zeta) \in \mathbb{D}^2,$$

where $U(z,\zeta)$, as above, denotes the Green function for Δ^2.

The positivity of $U_{G_a}(z,\zeta)$ on $\mathbb{D}\times\mathbb{D}$ is an immediate consequence of the above formula. By an argument analogous to the one used in the proof of Theorem 7.2 [**9**], we see that Proposition 3.2 implies that Conjectures 2.1 and 2.2 hold for one-point zero sets A.

Conjecture 3.3. *The Green function $U^\alpha(z,\zeta)$ for the weighted biharmonic operator $\Delta(1-|z|^2)^{-\alpha}\Delta$ is positive on $\mathbb{D}\times\mathbb{D}$ for all real parameter values α having $-1 < \alpha \le 1$.*

Just as is the case with Conjecture 3.1, Conjecture 3.3 has implications for a Bergman space. The relevant space is the weighted Bergman space $L^2_a(\mathbb{D},\alpha)$, consisting of all holomorphic functions in $\mathbb{D}$, satisfying

$$\|f\|_{L^2(\alpha)} = \left((\alpha+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha dS(z)\right)^{1/2},$$

and if Conjecture 3.3 does hold, we would know that

$$\|f\|_{L^2(\alpha)} \le \|G^\alpha f\|_{L^2(\alpha)}, \qquad f \in H^2(\mathbb{D}), \tag{3.1}$$

for inner divisors G^α associated with the space $L^2_a(\mathbb{D},\alpha)$, by which we mean that

$$h(0) = (\alpha+1)\int_{\mathbb{D}} h(z)|G(z)|^2(1-|z|^2)^\alpha dS(z)$$

should hold for all bounded harmonic functions h on $\mathbb{D}$. It is known that the assertion of Conjecture 3.3 fails for parameter values α with $1 < \alpha$ [**13**], and that the factoring assertion (3.1) holds for $\alpha = 1$ [**8**].

Sergeĭ Shimorin [**26**] has obtained (3.1) for parameter values $-1 < \alpha < 0$, but I do not know if he can prove the validity of Conjecture 3.3 for these parameter values. I have checked that Conjecture 3.3 holds for $\alpha = 1$ myself. This is, in fact, a consequence of the following identity, and the inequality

$$\Gamma(z,\zeta) = \log\left|\frac{\zeta - z}{1-\bar{\zeta}z}\right|^2 > -\frac{1}{2}(1-|z|^2)(1-|\zeta|^2)\left(|z-\zeta|^{-2} + |1-\bar{\zeta}z|^{-2}\right),$$

for $(z,\zeta)\in\mathbb{D}^2$.

Proposition 3.4. *The Green function $U^1(z,\zeta)$ for the weighted biharmonic operator $\Delta(1-|z|^2)^{-1}\Delta$ has the explicit form*

$$U^1(z,\zeta) = \left(|z-\zeta|^2 - \frac{1}{4}\left|z^2-\zeta^2\right|^2\right)\Gamma(z,\zeta) + \frac{1}{8}(1-|z|^2)(1-|\zeta|^2) \times \left(7 - |z|^2 - |\zeta|^2 - |\zeta z|^2 - 4\,\mathrm{Re}\,(\bar{\zeta}z) - 2\,(1-|z|^2)(1-|\zeta|^2)\,\mathrm{Re}\,\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}\right),$$

for $(z,\zeta)\in\mathbb{D}^2$.

4. Generators of zero-based subspaces

As before, we write G_A for the zero divisor associated with a zero sequence A.

Problem 4.1. *Does G_A generate*

$$\mathcal{I}(A) = \big\{ f \in L^2_a(\mathbb{D}) : f = 0 \quad on \quad A \big\}$$

as an invariant subspace? In other words, do the functions G_A, $zG_A, z^2G_A, \dots$ span a dense subspace of $\mathcal{I}(A)$?

This is so for Blaschke sequences A. The general case would follow if the following holds.

Problem 4.2. *Does $G_{A\cup\{0\}}/G_A$ always belong to the Smirnov space on $\mathbb{D}$? The Smirnov space consists of all quotients f/g, where $f, g \in H^\infty(\mathbb{D})$, and g is an outer function.*

Assuming the validity of Conjecture 2.1, we have $G_A \prec G_{A\cup\{0\}}$, which entails that

$$|zG_A(z)| \le |G_{A\cup\{0\}}(z)|, \qquad z \in \mathbb{D},$$

holds. To see this, check it for finite sequences A (the argument for this is analogous to what was used in Proposition 1.3 [**6**]), and we then approximate a general zero sequence with finite subsequences. It follows that $G_{A\cup\{0\}}/G_A$ belongs to the Nevanlinna class of holomorphic quotients of bounded analytic functions. In view of this, it does not seem too far-fetched to ask whether it belongs to the Smirnov space.

We really do not understand the process of adding another zero. Still, for the function $L_A(z) = \overline{G_A(0)}G_A(z)$, there is the iterative formula

$$L_{A\cup\{\beta\}}(z) = L_A(z) - \frac{L_A(\beta)}{L_{\varphi_\beta(A)}(0)}\, \varphi'_\beta(z)\, L_{\varphi_\beta(A)}(\varphi_\beta(z)),$$

where φ_β denotes the Möbius mapping

$$\varphi_\beta(z) = \frac{\beta - z}{1 - \overline{\beta} z}, \qquad z \in \mathbb{D}.$$

The starting point for the iterative process is $L_\emptyset = 1$, and the formula connecting G_A with L_A may be written $G_A(z) = L_A(z)/\sqrt{L_A(0)}$.

Remark 4.3. We should shed some light on the connection between Problems 4.1 and 4.2. To do this, we assume for simplicity that the point 0 does not belong to the given zero sequence A, and denote by Y_n the orthogonal projection onto $\mathcal{I}(A \cup O_n)$ of the function z^n. Here, O_n stands for the the sequence that consists of n copies of the point 0. The assumption that 0 not belong to A prevents Y_n from collapsing to 0. We now claim that the functions Y_n, $n = 0, 1, 2, \dots$, span a dense subspace of $\mathcal{I}(A)$. To this end, suppose $f \in \mathcal{I}(A)$ is orthogonal to all the functions Y_n. It is convenient here to introduce $X_n = z^n - Y_n$, which for each n is orthogonal to $\mathcal{I}(A \cup O_n)$, by the way we defined the element Y_n. If we knew f belonged to $\mathcal{I}(A \cup O_n)$ for some particular n, we would then also have $\langle f, X_n\rangle_{L^2} = 0$, and since by assumption $\langle f, Y_n\rangle_{L^2} = 0$, we see that

$$f^{(n)}(0) = (n+1)!\, \langle f, z^n\rangle_{L^2} = 0.$$

We conclude that f must also belong to $\mathcal{I}(A \cup O_{n+1})$. The initial assumption $f \in \mathcal{I}(A \cup O_n)$ is fulfilled for $n = 0$, so by induction, f belongs to the intersection of all the spaces $\mathcal{I}(A \cup O_n)$, which is $\{0\}$. This shows that $f = 0$, and hence the claim is verified. It is known [**6, 5**] that

$$G_{A\cup O_n} = Y_n/\|Y_n\|_{L^2},$$

so by the above argument, the functions $G_{A\cup O_n}$, $n = 0, 1, 2, \ldots$, span a dense subspace of $\mathcal{I}(A)$. If we could only demonstrate that every $G_{A\cup O_n}$ belongs to the invariant subspace generated by G_A, we would be able to answer Problem 4.1 in the affirmative. This is where a solution to Problem 4.2 would come in handy. If the answer to Problem 4.2 is indeed yes, then we can claim that $G_{A\cup O_{n+1}}/G_{A\cup O_n}$ belongs to the Smirnov space $\mathfrak{N}_+(\mathbb{D})$ for all $n = 0, 1, 2, \ldots$, and by the multiplicative properties of $\mathfrak{N}_+(\mathbb{D})$, we arrive at $g_n = G_{A\cup O_n}/G_A \in \mathfrak{N}_+(\mathbb{D})$, for $n = 0, 1, 2, \ldots$. The Nevanlinna theory now permits us to find a sequence $\{g_{n,m}\}_{m=1}^{\infty}$ of functions in $H^\infty(\mathbb{D})$, which meets

$$|g_{n,m}(z)| \le |g_n(z)|, \qquad z \in \mathbb{D},$$

and $g_{n,m}(z) \to g_n(z)$ pointwise in $\mathbb{D}$. By the dominated convergence theorem, the functions $G_A \cdot g_{n,m}$ converge to $G_{A\cup O_n} = G_A \cdot g_n$ as $m \to +\infty$, in the norm of $L^2_a(\mathbb{D})$. It is now immediate that $G_{A\cup O_n}$ belongs to the invariant subspace generated by G_A, and hence G_A generates all of $\mathcal{I}(A)$ as an invariant subspace. $\square$

If the answer to the following problem is yes, it would imply an affirmative answer to Problem 4.1.

Problem 4.4. *Suppose ω is a continuous function on $\mathbb{D}$ satisfying*

$$(1 - |z|^2)^2 \le \omega(z) \le 2(1 - |z|^2), \qquad z \in \mathbb{D}.$$

Must then the polynomials be dense in the weighted Bergman space $L^2_a(\mathbb{D}, \omega)$ of all holomorphic functions in $\mathbb{D}$ having

$$\int_{\mathbb{D}} |f(z)|^2 \omega(z)\, dS(z) < +\infty\,?$$

What if we make the regularity assumption on ω that $\Delta^2\omega \ge 4$ on $\mathbb{D}$?

Remark 4.5. We shall now try to indicate the relationship between Problems 4.1 and 4.4. Note first that in view of Remark 2.5, G_A generates $\mathcal{I}(A)$ if and only if the closure of polynomials is dense in the space $\mathcal{H}(A)$. By the elementary estimates

$$\frac{2}{3}\,\|f\|^2_{L^2} \le \int_{\mathbb{D}} |f'(z)|^2(1 - |z|^2)^2 dS(z) \le 2\,\|f\|^2_{L^2},$$

valid for $f \in L^2_a(\mathbb{D})$ with $f(0) = 0$, we have that the norm in $\mathcal{H}(A)$ is comparable to

$$\|f\|^2_* = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \omega_A(z) dS(z),$$

where we denote by ω_A the function

$$\omega_A(z) = (1 - |z|^2)^2 + \Phi_A(z), \qquad z \in \mathbb{D}.$$

We see that the polynomials are dense in $\mathcal{H}(A)$ if and only if they are dense in $L^2_a(\mathbb{D},\omega_A)$. It is now clear that G_A generates the invariant subspace $\mathcal{I}(A)$ if and only if polynomial approximation is possible in $L^2_a(\mathbb{D},\omega_A)$. The constructed function ω_A has

$$(1-|z|^2)^2 \le \omega_A(z) \le (1-|z|^2)^2 + 1 - |z|^2 \le 2\,(1-|z|^2), \qquad z\in\mathbb{D},$$

by Theorem 1.6. Moreover, since $\Delta^2\Phi_A \ge 0$, we also have

$$\Delta\omega_A(z) = \Delta^2\Phi_A(z) + \Delta^2(1-|z|^2)^2 \ge \Delta^2(1-|z|^2)^2 = 4, \qquad z\in\mathbb{D}. \quad \square$$

Polynomial approximation problems are, generally speaking, rather difficult. Proposition 4.6 below represents my level of understanding on the topic. Given a positive continuous weight function ω on the unit disk $\mathbb{D}$, having

$$\int_{\mathbb{D}} \omega(z)\,dS(z) < +\infty, \tag{4.1}$$

we denote by $L^2_h(\mathbb{D},\omega)$ the Hilbert space of harmonic functions f on $\mathbb{D}$ having

$$\|f\|_{L^2(\omega)} = \left(\int_{\mathbb{D}} |f(z)|^2\omega(z)\,dS(z)\right)^{1/2} < +\infty.$$

It has been known for a long time [**20**, p. 131], [**15**, p. 343] that the analytic polynomials are dense in $L^2_a(\mathbb{D},\omega)$ for radial weights ω. The corresponding statement is also true for the space $L^2_h(\mathbb{D},\omega)$, and moreover, we can get the result for weights that do not deviate too much from radial weights. To obtain such a result, it is useful to consider for a parameter $0<\lambda<1$ and a function $f\in L^2_h(\mathbb{D},\omega)$ the dilation f_λ of f: $f_\lambda(z) = f(\lambda z)$, $z\in\mathbb{D}$, and observe that every dilation f_λ of f is definitely approximable by harmonic polynomials (or analytic polynomials, if $f\in L^2_a(\mathbb{D},\omega)$), so that if we could show that $f_\lambda\to f$ in the norm of $L^2(\mathbb{D},\omega)$, the desired conclusion would follow. Another condition which is known to assure that we have polynomial approximation is due to Dzhrbashian [**20**, p.133], and requires that the weight should (almost) fall on every radius emanating from the origin. If we merge these two ideas, we obtain the following result. First, however, we need to recall some terminology: an integrable function $\nu\ge 0$ on the unit circle $\mathbb{T}$ meets the Muckenhoupt (A_2) condition provided that

$$\mathfrak{A}_2(\nu) = \sup_I \left\{ |I|^{-2}\int_I \nu\,ds \int_I \frac{1}{\nu}\,ds \right\} < \infty,$$

the supremum being taken over all arcs I on $\mathbb{T}$, where ds denotes arc length measure on $\mathbb{T}$, normalized so that the total length of $\mathbb{T}$ is 1.

Proposition 4.6. *Suppose ω is a positive continuous function on the unit disk $\mathbb{D}$, which satisfies the integrability condition (4.1). Suppose, moreover, that $\{\lambda_j\}_1^\infty$ is a sequence of numbers in the interval $]0,1[$, converging to 1. For r, $0<r<1$, let $\omega_r(z)=\omega(rz)$, and for $0<r,s<1$, introduce the quantity*

$$\mathfrak{Q}[\omega](r,s) = \sup\{\omega(rz)/\omega(sz) : z\in\mathbb{T}\}.$$

If the weight ω satisfies

$$\limsup_{r\to 1-}\sup_j\left(\min\left\{\mathfrak{Q}[\omega](r,\lambda_j r),\mathfrak{A}_2(\omega_r)\right\}\right)<\infty,$$

then the dilations f_{λ_j} of f converge to f as $j\to\infty$ in the norm of $L^2_h(\mathbb{D},\omega)$, for every $f\in L^2_h(\mathbb{D},\omega)$. As a consequence, under this condition on ω, we see that the harmonic polynomials are dense in $L^2_h(\mathbb{D},\omega)$, and the analytic polynomials are dense in $L^2_a(\mathbb{D},\omega)$.

Proof. We follow the general line of argument of [**15**, pp. 343–344]. Given an ε, $0<\varepsilon$, take ρ, $0<\rho<1$, so close to 1 that

$$\int_{\rho<|z|<1}|f(z)|^2\omega(z)\,dS(z)<\varepsilon, \tag{4.2}$$

and

$$\int_{\lambda_j\rho<|z|<\lambda_j}|f(z)|^2\omega(z)\,dS(z)<\lambda_j^2\varepsilon. \tag{4.3}$$

By choosing ρ possibly even closer to 1, we may assume that

$$\min\left\{\mathfrak{Q}[\omega](r,\lambda_j r),\mathfrak{A}_2(\omega_r)\right\}\le C,\qquad \rho<r<1,\;\; j=1,2,3,\dots,$$

for some constant C, $0<C<+\infty$. We now plan to estimate the size of

$$\int_{\rho<|z|<1}|f(\lambda_j z)|^2\omega(z)\,dS(z).$$

We fix an r, $\rho<r<1$, and note that if it is $\mathfrak{Q}[\omega](r,\lambda_j r)$ that is $\le C$, we have

$$\omega(re^{i\theta})\le C\,\omega(\lambda_j re^{i\theta}),$$

and hence

$$\int_{-\pi}^{\pi}|f(\lambda_j re^{i\theta})|^2\omega(re^{i\theta})\,d\theta\le C\int_{-\pi}^{\pi}|f(\lambda_j re^{i\theta})|^2\omega(\lambda_j re^{i\theta})\,d\theta.$$

If, on the other hand, it is $\mathfrak{A}_2(\omega_r)$ that is $\le C$, then by Benjamin Muckenhoupt's theorem [**21, p. 223**], (and the control of the constants involved [**21**, pp. 215, 224]) we have

$$\int_{-\pi}^{\pi}|f(\lambda_j re^{i\theta})|^2\omega(re^{i\theta})\,d\theta\le K(C)\int_{-\pi}^{\pi}|f(re^{i\theta})|^2\omega(re^{i\theta})\,d\theta,$$

for some constant $K(C)$ that only depends on C. No matter which is the case, we get

$$\int_\rho^1\int_{-\pi}^{\pi}|f(\lambda_j re^{i\theta})|^2\omega(re^{i\theta})\,d\theta\,rdr\le(C+K(C))\,\pi\varepsilon,$$

in view of (4.2) and (4.3). Since ρ, $0<\rho<1$, was fixed, we have that $f_{\lambda_j}\to f$ as $j\to+\infty$ uniformly on the disk $|z|<\rho$, and in particular, we can arrange so that

$$\int_{|z|<\rho}|f(\lambda_j z)-f(z)|^2\omega(z)\,dS(z)<\varepsilon,$$

for all large j, say $j\ge N(\varepsilon)$. If we combine this with the estimate of the integral on the annulus $\rho<|z|<1$, we see that

$$\int_{\mathbb{D}}|f(\lambda_j z)-f(z)|^2\omega(z)\,dS(z)<8\,(1+C+K(C))\,\varepsilon,$$

for $j\ge N(\varepsilon)$. The assertion of the proposition is now immediate. $\square$

5. A Carathéodory theorem for the Bergman space?

Recall the statement of the famous Carathéodory theorem.

Theorem 5.1. (Carathéodory) *Every $f \in H^\infty(\mathbb{D})$ with norm ≤ 1 is the normal limit of finite Blaschke products.*

The appropriate analog in a Bergman space setting is as follows.

Conjecture 5.2. *Every analytic function f on $\mathbb{D}$ with*

$$\|f\varphi\|_{L^2} \le \|\varphi\|_{H^2}, \qquad \varphi \in H^2(\mathbb{D}),$$

is the normal limit of finite zero divisors.

6. A Frostman theorem for the Bergman space?

Recall Frostman's theorem on approximation of inner functions by Blaschke products.

Theorem 6.1. (Frostman) *Every inner function is approximable in the norm of $H^\infty(\mathbb{D})$ by Blaschke products.*

Let $\mathfrak{M}(H^2, L^2_a)$ be the space of multipliers $H^2(\mathbb{D}) \to L^2_a(\mathbb{D})$, normed appropriately:

$$\|G\|_{\mathfrak{M}(H^2,L^2_a)} = \sup\left\{\|Gf\|_{L^2} : f \in H^2(\mathbb{D}),\ \|f\|_{H^2} \le 1\right\}.$$

Conjecture 6.2. *Every inner divisor for $L^2_a(\mathbb{D})$ is approximable by zero divisors in the norm of $\mathfrak{M}(H^2, L^2_a)$.*

7. Korenblum's maximum principle

Conjecture 7.1. (Korenblum) *There exists an absolute constant ε, $0 < \varepsilon < 2^{-1/2}$, such that if $f, g \in L^2_a(\mathbb{D})$ have $|f(z)| \le |g(z)|$ in the annulus $\varepsilon < |z| < 1$, then $\|f\|_{L^2} \le \|g\|_{L^2}$.*

One can rather trivially obtain an estimate like $\|f\|_{L^2} \le C(\varepsilon)\|g\|_{L^2}$, with $C(\varepsilon)$ being a constant larger than 1 tending to 1 as $\varepsilon \to 0$. Korenblum thinks $C(\varepsilon) = 1$ could be attained for some nonzero value of the parameter ε. One should view Conjecture 7.1 as a suspected property peculiar to square moduli of analytic functions. If one should try to replace this class by, for instance, the collection of exponentials of subharmonic functions, the analogous assertion that $\varphi(z) \le \psi(z)$ on the annulus $\varepsilon < |z| < 1$ should imply

$$\int_{\mathbb{D}} \exp(\varphi(z))\, dS(z) \le \int_{\mathbb{D}} \exp(\psi(z))\, dS(z)$$

for subharmonic functions φ, ψ fails, no matter how small the positive number ε is. This is so because one can take as $\varphi(z)$ the function $\log|z|$, and as $\psi(z)$ the function that is the maximum of $\log|z|$ and the constant function $\log \varepsilon$.

The condition of Conjecture 7.1 is invariant under multiplication by a bounded holomorphic function, so the assertion of Conjecture 7.1 may be rephrased as $f \prec g$.

The properties of the domination relation $\prec$ deserve to be studied in some depth.

Problem 7.2. *Suppose $f, g \in L^2_a(\mathbb{D})$ have $f \prec g$ and $g \prec f$. Must then $f = \gamma g$ for a unimodular constant γ? What if f and g are bounded functions?*

Kehe Zhu has informed me that the answer to Problem 7.2 is affirmative, and in fact a simple consequence of a property of the Berezin transform.

8. Cyclic vectors and Shapiro's problem

The space $A^{-\infty}$ consists of all analytic functions f on the unit disk $\mathbb{D}$ satisfying the growth condition

$$|f(z)| \le C(f,\alpha)\,(1-|z|^2)^{-\alpha}, \qquad z \in \mathbb{D},$$

for some positive constants α and $C(f,\alpha)$. The function theory aspects of this space were illuminated extensively by Boris Korenblum in his Acta paper [**16**]; one rather trivial but interesting observation is that $A^{-\infty}$ is a topological algebra with respect to pointwise multiplication and the natural injective limit topology. In his second Acta paper [**17**], Korenblum described completely the closed ideals in $A^{-\infty}$. The Bergman space $L^2_a(\mathbb{D})$ is clearly a subspace of $A^{-\infty}$, but it is not an algebra. The invariant subspaces are the $L^2_a(\mathbb{D})$ analogs of the closed ideals in $A^{-\infty}$. In order to gain some understanding of invariant subspaces, the concept of a *cyclic vector* is basic.

Definition 8.1. *A function $f \in L^2_a(\mathbb{D})$ is cyclic in $L^2_a(\mathbb{D})$ if the functions $f, zf, z^2f, \ldots$ span a dense subspace of $L^2_a(\mathbb{D})$.*

Problem 8.2. *Describe the cyclic elements of $L^2_a(\mathbb{D})$.*

A natural question when one tries to attack Problem 8.1 is the following.

Problem 8.3. (Korenblum) *It is known that every cyclic element of $L^2_a(\mathbb{D})$ generates a dense ideal in $A^{-\infty}$, or in other words, it is cyclic in $A^{-\infty}$. Does the converse hold, that is, if $f \in L^2_a(\mathbb{D})$ is cyclic in $A^{-\infty}$, must then f be cyclic in $L^2_a(\mathbb{D})$?*

It is known (see [**25**]) that the answer to Problem 8.3 is yes if we add the assumption that the function f belong to the Nevanlinna class of holomorphic quotients of bounded analytic functions. This in its turn follows rather easily from the case when f is assumed bounded. Leon Brown and Boris Korenblum [**4**] have shown that if the function f belongs to a slightly smaller Bergman space $L^p_a(\mathbb{D})$, $2 < p < +\infty$, then the cyclicity of $f \in L^2_a(\mathbb{D})$ in $A^{-\infty}$ implies cyclicity in $L^2_a(\mathbb{D})$. If a function $f \in L^2_a(\mathbb{D})$ satisfies

$$|f(z)| \ge \varepsilon\left(1-|z|^2\right)^N, \qquad z \in \mathbb{D}, \tag{8.1}$$

for some positive numbers ε, N, then f is invertible in $A^{-\infty}$, and hence cyclic in $A^{-\infty}$.

Problem 8.4. (Shapiro) *Suppose $f \in L^2_a(\mathbb{D})$ satisfies* (8.1)*. Must then f be cyclic in $L^2_a(\mathbb{D})$?*

9. Invariant subspaces generated by zero sets

In [**10**], two zero based invariant subspaces $\mathcal{I}(A)$ and $\mathcal{I}(B)$ were constructed, the zero sequences A and B being disjoint, which are at a positive angle from each other. That entails that their sum $I = \mathcal{I}(A)+\mathcal{I}(B)$ is a closed invariant subspace of $L^2_a(\mathbb{D})$, which has the funny property that zI has codimension 2 in I. In particular, I is not the whole space, as would be natural, considering that the functions in I have no zeros in common. In [**10**], it was shown that given two disjoint zero sequences A and B, we either have that $\mathcal{I}(A)+\mathcal{I}(B)$ is dense in $L^2_a(\mathbb{D})$, or its closure I is an invariant subspace with the property that zI has codimension 2 in I.

Problem 9.1. *Determine for which pairs of disjoint zero sequences A and B the subspace $\mathcal{I}(A)+\mathcal{I}(B)$ is dense in $L^2_a(\mathbb{D})$.*

It was shown in [**10**] that the subspace $\mathcal{I}(A)+\mathcal{I}(B)$ *is* dense in $L^2_a(\mathbb{D})$ provided that the sequences A and B do not both accumulate at every point on the boundary $\mathbb{T}$. Moreover, it seems that one can show that $\mathcal{I}(A)+\mathcal{I}(B)$ is *not* dense in $L^2_a(\mathbb{D})$ if and only if $A\cup B$ is not a zero sequence, and the completion of $L^2_a(\mathbb{D})$ with respect to the norm

$$\|f\|_{A,B} = \Big(\|f+\mathcal{I}(A)\|_{L^2_a(\mathbb{D})/\mathcal{I}(A)} + \|f+\mathcal{I}(A)\|_{L^2_a(\mathbb{D})/\mathcal{I}(A)}\Big)^{1/2}, \qquad f\in L^2_a(\mathbb{D}),$$

is a Hilbert space of holomorphic functions in $\mathbb{D}$. Thus what remains is to rephrase the latter condition in terms of geometric properties of the sequences A and B.

10. Maximal invariant subspaces

Let us agree to say that an invariant subspace I in $L^2_a(\mathbb{D})$ is maximal provided every invariant subspace containing it is either I or the whole space $L^2_a(\mathbb{D})$. If I is maximal, then $L^2_a(\mathbb{D})/I$ is a Hilbert space lacking nontrivial invariant subspaces with respect to the induced operator $z[I] : L^2_a(\mathbb{D})/I \to L^2_a(\mathbb{D})/I$, so that if I has codimension larger than 1 (it must then have codimension $+\infty$), we would have an operator on infinite dimensional Hilbert space with only trivial invariant subspaces. If I is maximal and has codimension 1, it has the form $I = \{f\in L^2_a(\mathbb{D}) : f(\lambda)=0\}$ for some $\lambda\in\mathbb{D}$.

Problem 10.1. *Must every maximal invariant subspace of $L^2_a(\mathbb{D})$ have codimension 1?*

At this point in time (July, 1993), I believe that I can give an affirmative answer to the above problem.

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