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Sharpening Hölder's inequality

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ABSTRACT

We strengthen Hölder's inequality. The new family of sharp inequalities we obtain might be thought of as an analog of the Pythagorean theorem for the L^p-spaces. Our treatment of the subject matter is based on Bellman functions of four variables.

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1. Introduction

1.1. The Cauchy–Schwarz inequality and the Pythagorean theorem

Let $\mathcal{H}$ be a Hilbert space (over the complex or the reals) with an inner product $\langle \cdot, \cdot \rangle$. The Pythagorean theorem asserts

$$\left| \left\langle f, \frac{e}{\|e\|} \right\rangle \right|^2 + \|\mathbf{P}_{e^\perp} f\|^2 = \|f\|^2, \qquad e, f \in \mathcal{H}, \;\; e \neq 0. \tag{1.1}$$

Here, $\mathbf{P}_{e^\perp}$ denotes the orthogonal projection onto the orthogonal complement of a non-trivial vector e. At this point, we note that since $\|\mathbf{P}_{e^\perp} f\| \geqslant 0$, the identity (1.1) implies the Cauchy–Schwarz inequality

$$|\langle f, e \rangle| \leqslant \|f\| \, \|e\|, \qquad e, f \in \mathcal{H}.$$

We also note that (1.1) leads to Bessel's inequality:

$$\sum_{n=1}^{N} |\langle f, e_n \rangle|^2 \leqslant \|f\|^2, \qquad f \in \mathcal{H},$$

for an orthonormal system $e_1, \ldots, e_N$ in $\mathcal{H}$.

We may think of (1.1) as of an expression of the precise loss in the Cauchy–Schwarz inequality. Our aim in this paper is to find an analogous improvement for the well-known Hölder inequality for L^p norms. Before we turn to the analysis of L^p spaces, we need to replace the norm of the projection, $\|\mathbf{P}_{e^\perp} f\|$, by an expression which does not rely on the Hilbert space structure. It is well known that

$$\|\mathbf{P}_{e^\perp} f\| = \inf_{\alpha} \|f - \alpha e\|, \tag{1.2}$$

where α ranges over all scalars (real or complex).

1.2. Background on Hölder's inequality for L^θ

We now consider $L^\theta(X, \mu)$, where (X, μ) is a standard σ-finite measure space. We sometimes focus our attention on finite measures, but typically the transfer to the more general σ-finite case is an easy exercise. The functions are assumed complex valued. Throughout the paper we assume the summability exponents are in the interval $(1, +\infty)$, in particular, $1 < \theta < +\infty$. We reserve the symbol p for the range $[2, \infty)$ and q for $(1, 2]$ (we also usually assume that p and q are dual in the sense $\frac{1}{p} + \frac{1}{q} = 1$). Also, to simplify the presentation, we assume μ has no atoms.

Our point of departure is Hölder's inequality, which asserts that in terms of the sesquilinear form

$$\langle f, g\rangle_\mu := \int_X f\bar{g}\,\mathrm{d}\mu,$$

we have

$$|\langle f, g\rangle_\mu| \leqslant \|f\|_{L^\theta(\mu)}\|g\|_{L^{\theta'}(\mu)}, \qquad f\in L^\theta(X,\mu),\;\; g\in L^{\theta'}(X,\mu), \quad \frac{1}{\theta}+\frac{1}{\theta'}=1, \tag{1.3}$$

so that $\theta' = \theta/(\theta-1)$ is the dual exponent. Hölder's inequality was found independently by Rogers [17] and Hölder [9]. It is well-known that, for non-zero functions, equality occurs in Hölder's inequality (1.3) if and only if f has the form $f = \alpha\mathbf{N}_{\theta'}(g)$ for a scalar $\alpha\in\mathbb{C}$. Here, $\mathbf{N}_r$ denotes the nonlinear operator

$$\mathbf{N}_r(h)(x) = \begin{cases} |h(x)|^{r-2}h(x), & h(x)\neq 0;\\ 0, & h(x)=0, \end{cases} \qquad r\in(1,\infty).$$

Such operators appear naturally in the context of generalized orthogonality for L^p spaces (see, e.g., Chapter 4 of Shapiro's book [18]). We note that $\mathbf{N}_\theta$ and $\mathbf{N}_{\theta'}$ are each other's inverses, since $\mathbf{N}_\theta(\mathbf{N}_{\theta'}(h)) = h$ and $\mathbf{N}_{\theta'}(\mathbf{N}_\theta(h)) = h$ for an arbitrary function h. In addition, $\mathbf{N}_{\theta'}$ maps $L^{\theta'}(X,\mu)$ to $L^\theta(X,\mu)$ with good control of norms:

$$\|\mathbf{N}_{\theta'}(h)\|^\theta_{L^\theta(\mu)} = \int_X |h|^{\theta(\theta'-1)}\,\mathrm{d}\mu = \int_X |h|^{\theta'}\,\mathrm{d}\mu = \|h\|^{\theta'}_{L^{\theta'}(\mu)}, \qquad h\in L^{\theta'}(X,\mu).$$

1.3. Possible improvement of Hölder's inequality

We rewrite Hölder's inequality (1.3) in the form

$$\left|\left\langle f, \frac{g}{\|g\|_{L^{\theta'}(\mu)}}\right\rangle_\mu\right|^r \leqslant \|f\|^r_{L^\theta(\mu)}, \qquad f\in L^\theta(X,\mu),\;\; g\in L^{\theta'}(X,\mu),\;\; g\neq 0, \tag{1.4}$$

where r is real and positive. The natural choices for r are $r=\theta$ and $r=\theta'$. This looks a lot like the Pythagorean theorem, only that the projection term is missing. Indeed, if $\theta=\theta'=r=2$, the inequality (1.4) expresses exactly Pythagorean theorem (1.1) with the projection term suppressed (with $e=g$). How can we find a replacement of the projection term for arbitrary $\theta\neq 2$? The key to this lies in the already observed fact that we have equality in (1.4) if and only if $f=\alpha\mathbf{N}_{\theta'}(g)$ for a scalar $\alpha\in\mathbb{C}$. So, the inequality

$$\left|\left\langle f, \frac{g}{\|g\|_{L^{\theta'}(\mu)}}\right\rangle_\mu\right|^r + c\inf_\alpha \|f-\alpha\mathbf{N}_{\theta'}(g)\|_{L^\theta(\mu)} \leqslant \|f\|^r_{L^\theta(\mu)},$$
$$f\in L^\theta(X,\mu),\;\; g\in L^{\theta'}(X,\mu),\;\; g\neq 0,$$

might hold for some positive constant c. To see things more clearly, let us agree to write $e = \mathbf{N}_{\theta'}(g) \in L^p(X,\mu)$ and insert this into (1.4):

$$\left|\left\langle f, \frac{\mathbf{N}_\theta(e)}{\|e\|^{\theta-1}_{L^\theta(\mu)}}\right\rangle_\mu\right|^r \leqslant \|f\|^r_{L^\theta(\mu)}, \qquad e, f \in L^\theta(X,\mu),\ \ e \neq 0.$$

Now, looking at (1.2), knowing that equality in the previous inequality holds only when f is a scalar multiple of e, we posit the inequality

$$\left|\left\langle f, \frac{\mathbf{N}_\theta(e)}{\|e\|^{\theta-1}_{L^\theta(\mu)}}\right\rangle_\mu\right|^r + c_{\theta,r} \inf_\alpha \|f - \alpha e\|^r_{L^\theta(\mu)} \leqslant \|f\|^r_{L^\theta(\mu)}, \qquad e, f \in L^\theta(X,\mu),\ \ e \neq 0, \tag{1.5}$$

for some constant $c_{\theta,r}$, $0 \leqslant c_{\theta,r} \leqslant 1$ (this constant does not depend on f or e, it depends on θ and r only). The inequality (1.5) cannot hold for any constant $c_{\theta,r} > 1$. Indeed, to see this, it suffices to pick nontrivial e and f such that

$$\inf_\alpha \|f - \alpha e\|^r_{L^\theta(\mu)} = \|f\|^r_{L^\theta(\mu)},$$

which means that the minimum is attained at $\alpha = 0$. This is easy to do for any given e by simply replacing the function f by $f - \alpha^\star e$ (here, $\alpha = \alpha^\star$ is a point where the infimum is attained). The inequality (1.5) is appropriate for iteration, using a sequence of functions $e_1, e_2, e_3, \ldots$, as in the case of Bessel's inequality, but if $c_{\theta,r} < 1$ there is an exponential decay of the coefficients in the analogue of Bessel's inequality. We note that the inequality (1.5) holds trivially for $c_{\theta,r} = 0$ by Hölder's inequality, and gets stronger the bigger $c_{\theta,r}$ is allowed to be.

Definition 1.1. Let $c^\star_{\theta,r}$ denote the largest possible value of $c_{\theta,r}$ such that (1.5) remains valid for any complex-valued f and e.

Clearly, $0 \leqslant c^\star_{\theta,r} \leqslant 1$. As we will see later, $c^\star_{\theta,r} < 1$ unless $\theta = 2$.

We slightly transfer the nonlinearity in (1.5) from e to f, and posit the inequality

$$\left|\left\langle \mathbf{N}_\theta(f), \frac{e}{\|e\|_{L^\theta(\mu)}}\right\rangle_\mu\right|^{\frac{r}{\theta-1}} + d_{\theta,r} \inf_\alpha \|f - \alpha e\|^r_{L^\theta(\mu)} \leqslant \|f\|^r_{L^\theta(\mu)}, \qquad e, f \in L^\theta(X,\mu),\ e \neq 0, \tag{1.6}$$

where $d_{\theta,r} \geqslant 0$ is real. We emphasize that (1.6) is independent of (1.5), or that at least the dependence is not straightforward. These two inequalities provide two different ways of sharpening Hölder's inequality.

If we again argue that we can find nontrivial functions e and f such that the infimum in (1.6) is attained at $\alpha = \alpha^\star = 0$, then it is immediate that (1.6) cannot be valid generally unless $d_{\theta,r} \leqslant 1$, that is, $d_{\theta,r}$ must be confined to $0 \leqslant d_{\theta,r} \leqslant 1$.

Definition 1.2. Let $d^\star_{\theta,r}$ denote the largest possible value of $d_{\theta,r}$ such that (1.6) remains valid for any complex-valued f and e.

Remark 1.3. The constants $c^\star_{\theta,r}$ and $d^\star_{\theta,r}$ do not depend on the measure space (X,μ) as long as the measure μ is continuous. There are essentially two cases, the finite mass case and the infinite mass σ-finite case. By normalization, the finite mass case becomes the probability measure case, and all the standard probability measure spaces without point masses are isomorphic. The infinite mass σ-finite case is then treated as a the limit of the finite mass case. We explain some details later on in the proof of Proposition 1.12 at the end of Section 2. Although this is deferred until later, we will need it in what follows.

In certain ranges of r, we cannot get more than Hölder's inequality. We describe these restrictions in two lemmas below.

Lemma 1.4. *If $r<\theta$, then $c^\star_{\theta,r}=d^\star_{\theta,r}=0$.*

Proof. We consider the finite mass case, and renormalize to have mass 1. For the σ-finite case, we would instead just cut off a piece of our measure space of mass 1 and consider functions that vanish off that piece. In the mass 1 case, we apply standard probability measure space theory, and take $X=[0,1]$, with μ as Lebesgue measure. Moreover, we let e be the constant function $e=1$ while f is given by the formula

$$f(x)=\begin{cases}2, & x\in[0,\varepsilon);\\ 0, & x\in[\varepsilon,2\varepsilon);\\ 1, & x\in[2\varepsilon,1].\end{cases}$$

Here ε is a real parameter with $0<\epsilon<\frac{1}{2}$. By symmetry and convexity with respect to α,

$$\inf_\alpha \|f-e-\alpha e\|^r_{L^\theta}=\|f-e\|^r_{L^\theta}=(2\varepsilon)^{\frac{r}{\theta}}.$$

Moreover, we observe that $\langle f,e\rangle=1$ while $\|f\|^\theta_{L^\theta}=\varepsilon 2^\theta+(1-2\varepsilon)$. Thus, (1.5) leads to

$$1+c_{\theta,r}(2\varepsilon)^{\frac{r}{\theta}}\leqslant\Big(\varepsilon 2^\theta+(1-2\varepsilon)\Big)^{\frac{r}{\theta}}.$$

We subtract 1, divide by $\varepsilon^{\frac{r}{\theta}}$, compute the limit as $\varepsilon\to 0$ and obtain 0 on the right-hand side if $r<\theta$. The conclusion that the optimal constant is $c^\star_{\theta,r}=0$ is immediate from this. The same choice of f and e also gives that $d^\star_{\theta,r}=0$. $\square$

Lemma 1.5. *If $r<2$, then $c^\star_{\theta,r}=d^\star_{\theta,r}=0$.*

Proof. We prove the lemma for the case of constants $d^\star_{\theta,r}$ this time. We take as before $X = [0,1]$ and μ as Lebesgue measure. We consider the functions $e = 1$ and $f = 1 + th$, where h is a bounded real-valued function with symmetric distribution (by which we mean that the functions h and $-h$ have one and the same distribution) and t is a real parameter which will tend to zero. By the symmetry of h,

$$\inf_\alpha \|f - \alpha e\|^r_{L^\theta} = \inf_\beta \|th - \beta e\|^r_{L^\theta} = \|th\|^r_{L^\theta} = t^r\|h\|^r_{L^\theta}.$$

Again by the symmetry of h it follows that $\langle h, e\rangle = 0$, and we obtain as $t \to 0$ that

$$|\langle \mathbf{N}_\theta(f), e\rangle|^{\frac{r}{\theta-1}} = \Big(\langle (1+th)^{\theta-1}, e\rangle\Big)^{\frac{r}{\theta-1}} = \Big(1 + t(\theta-1)\langle h, e\rangle + \mathrm{O}(t^2)\Big)^{\frac{r}{\theta-1}} = 1 + \mathrm{O}(t^2)$$

and, similarly,

$$\|f\|^r_{L^\theta} = 1 + \mathrm{O}(t^2)$$

as t tends to zero. By plugging these asymptotic identities back into (1.6), we arrive at

$$d^\star_{\theta,r} t^r \|h\|^r_{L^\theta} = \mathrm{O}(t^2),$$

which either proves that $r \geqslant 2$ or that $d^\star_{\theta,r} = 0$. The case with the constants $c^\star_{\theta,r}$ is similar. □

The inequalities (1.5) and (1.6) tend to get sharper the smaller r is. Indeed, since for fixed γ with $0 \leqslant \gamma \leqslant 1$, the inequality

$$A^r + \gamma B^r \leqslant 1 \quad \text{with} \quad 0 \leqslant A, B \leqslant 1,$$

for a fixed positive $r = r_0$ implies the same inequality for all $r \geqslant r_0$, it follows that for fixed θ, the functions $r \mapsto c^\star_{\theta,r}$ and $r \mapsto d^\star_{\theta,r}$ grow with r, and, moreover, if one of these constants already equals 1, then in (1.5) or alternatively (1.6) we should use the smallest possible r so that this remains true because that represents the strongest assertion. Our Lemmas 1.4 and 1.5 suggest that the cases $r = 2$ for $\theta \leqslant 2$ and $r = \theta$ for $\theta \geqslant 2$ might be the most interesting. It appears that one may compute the constants $c^\star_{p,p}$ and $d^\star_{p,p}$ for $p > 2$. Here are our two main results.

Theorem 1.6. *Suppose $2 < p < +\infty$. Then the optimal constant $c^\star_{p,p}$ in the inequality (1.5) may be computed as*

$$c^\star_{p,p} = (p-1)\Big(\frac{s_0}{1+s_0}\Big)^{p-2},$$

where s_0 is the unique positive solution of the equation

$$(p-1)s_0^{p-2}+(p-2)s_0^{p-1}=1. \tag{1.7}$$

Using Taylor's formula, we may find the asymptotic expansion for $c^\star_{p,p}$ as $p\to 2+$:

$$c^\star_{p,p}=1-(p-2)\Big(-1+\log\frac{1+w}{w}\Big)+O\big((p-2)^2\big),$$

where $w=W(\frac{1}{e})$. Here, W denotes the Lambert W function, i.e. the solution of the equation $W(z)e^{W(z)}=z$.

Theorem 1.7. *Let $2<p<+\infty$. Then the optimal constant $d^\star_{p,p}$ is given by the formula*

$$d^\star_{p,p}=(q-1)c^\star_{p,p}=\Big(\frac{s_0}{1+s_0}\Big)^{p-2},$$

where s_0 is given by (1.7) and q is dual to p, that is, $\frac{1}{p}+\frac{1}{q}=1$.

The case of exponent $\theta<2$ appears to be somewhat more elementary. At least the theorem stated below is considerably easier to obtain than the two theorems above.

Theorem 1.8. *The optimal constant $d^\star_{q,r}$ is given by the formula*

$$d^\star_{q,r}=1$$

when $1<q\leqslant 2$ and $2\leqslant r<+\infty$. Moreover, $d^\star_{p,r}=1$ holds in the range $2<p<+\infty$ if and only if $r\geqslant 2(p-1)$.

Our method allows us to compute the constants $c^\star_{\theta,r}$ and $d^\star_{\theta,r}$ for the case of arbitrary r and p. However, the answer does not seem to have a short formulation. At least, we provide sharp constants for all endpoint cases, and also indicate the domain where $d^\star_{p,r}=1$. Fig. 1.3.1 shows two diagrams which illustrate what we know about the optimal constants $c^\star_{\theta,r}$ and $d^\star_{\theta,r}$.

We have assumed μ has no atoms. The reader may however observe that the inequalities (1.5) and (1.6) remain valid if μ is allowed to have atoms. The problem is that these inequalities may lose their sharpness. They are however sharp for the standard infinite dimensional sequence spaces ℓ_θ. In contrast, they are far from being sharp in the case where μ has two atoms of mass $\frac{1}{2}$. The reason is that the corresponding Bellman functions are no longer concave since one cannot concatenate two functions.

Remark 1.9. (a) Our results sharpen Hölder's inequality. There is a constant interest in sharper forms of the classical inequalities, for which the optimizers have already been described. Such sharpenings may be viewed as *stability results*: the new inequality says that if the equality almost holds, then the functions are close to the optimizers. See [5] for the Hausdorff–Young and Young's convolutional inequalities, [6] for the Riesz–Sobolev inequality, [2] for various martingale inequalities, and [4] for Sobolev-type embedding

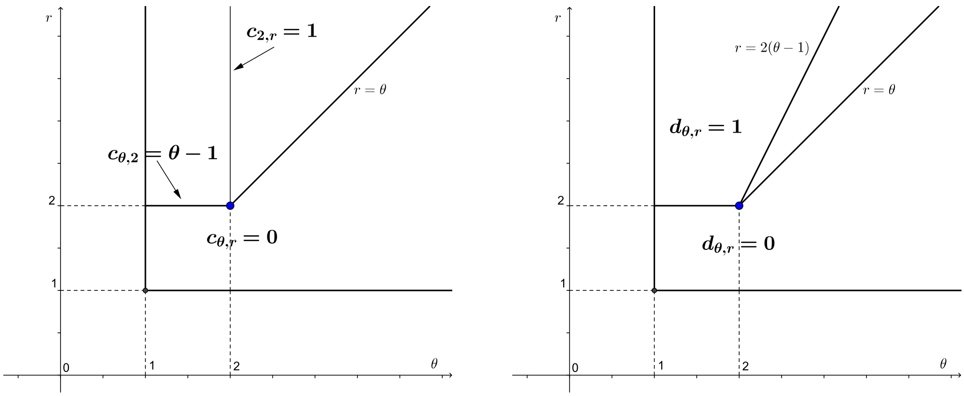

Fig. 1.3.1. Values of $c^\star_{\theta,r}$ and $d^\star_{\theta,r}$.

theorems. The latter paper also suggests a theoretical approach to the stability phenomenon. The list of references is far from being complete.

(b) The stability of Minkowski's inequality (which is closely associated with Hölder's inequality) is related to the notion of uniform convexity of Lebesgue spaces introduced by Clarkson [7]. He also found the sharp estimates for modulus of convexity of L^θ in the case $\theta \geqslant 2$. The sharp form of the uniform convexity inequalities for $\theta < 2$ was given by Hanner's inequalities which were first obtained by Beurling, see the classical paper of Hanner [8] as well as a more modern exposition [14]. The technique of the latter paper is very close to what we are using in our work. We should also mention that it is well-known that the notion of uniform convexity may be extended to Schatten classes in place of the L^p spaces, and moreover, sharp results such as Hanner's inequality might be available in this more general setting (see [1]). It is interesting to ask whether something of this sort is available also for the sharper forms of Hölder's inequalities.

(c) In some applications of Hölder's inequality, the instance of exponent $p = 2$ might not be applicable but for instance $p > 2$ close to 2 is. A case in point is the paper by Baranov and Hedenmalm [3]. It would be of interest to see what the sharpened forms derived here will be able to lead to in terms of strengthened results in that context.

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1.5. Bellman function

For a measurable subset E of an interval I and a summable f unction $f\colon I \to \mathbb{C}$, we denote the average of f over E by $\langle f \rangle_E = |E|^{-1} \int_E f(s)\, \mathrm{d}m(s)$. The symbol m denotes the Lebesgue measure and $|E| = m(E)$ by definition.

Definition 1.10. The constant $c^{\star,\mathbb{R}}_{\theta,r}$ is the largest possible constant $c_{\theta,r}$ such that the inequality (1.5) holds true for all real-valued functions f and e. The constant $d^{\star,\mathbb{R}}_{\theta,r}$ is the largest possible constant $d_{\theta,r}$ such that the inequality (1.6) holds true for all real-valued functions f and e.

We will express the constants $c^{\star,\mathbb{R}}_{\theta,r}$ and $d^{\star,\mathbb{R}}_{\theta,r}$ in terms of two Bellman functions, which are very similar. We introduce the main one.

Definition 1.11. Let $I \subset \mathbb{R}$ be a finite interval. Consider the function $\mathbb{B}_{c,+} : \mathbb{R}^4 \to \mathbb{R}$,

$$\mathbb{B}_{c,+}(x_1,x_2,x_3,x_4) = \sup\Big\{ \langle fg\rangle_I \;\Big|\; f,g \text{ real-valued}, \langle f\rangle_I = x_1, \langle g\rangle_I = x_2, \langle |f|^\theta\rangle_I = x_3, \langle |g|^{\theta'}\rangle_I = x_4 \Big\}.$$

Proposition 1.12. *For any θ and any r, we have that*

$$c^{\star,\mathbb{R}}_{\theta,r} = \left(\sup_{x_1\in(-1,1)} \frac{\mathbb{B}^r_{c,+}(x_1,0,1,1)}{1-|x_1|^r} \right)^{-1}.$$

Definition 1.13. Let $I \subset \mathbb{R}$ be a finite interval. Consider the function $\mathbb{B}_{d,+} : \mathbb{R}^4 \to \mathbb{R}$,

$$\mathbb{B}_{d,+}(x_1,x_3,x_4,x_5) = \sup\Big\{ \langle g\rangle_I \;\Big|\; f,g \text{ real-valued}, \langle f\rangle_I = x_1, \langle |f|^\theta\rangle_I = x_3, \langle |g|^{\theta'}\rangle_I = x_4, \langle fg\rangle_I = x_5 \Big\}.$$

Remark 1.14. The function $\mathbb{B}_{d,+}$ depends on four variables x_1, x_3, x_4, x_5. Though such a choice of variables might seem strange, it will appear to be very natural later. In particular, it makes the link between $\mathbb{B}_{d,+}$ and $\mathbb{B}_{c,+}$ more transparent, see Corollary 2.5 below.

Proposition 1.15. *For any θ and r,*

$$d^{\star,\mathbb{R}}_{\theta,r} = \left(\sup_{x_1\in(-1,1)} \frac{\mathbb{B}^r_{d,+}(x_1,1,1,0)}{1-|x_1|^{\frac{\theta' r}{\theta}}} \right)^{-1}.$$

We will compute the functions $\mathbb{B}_{c,+}$ and $\mathbb{B}_{d,+}$ for $\theta = p \geqslant 2$. The answer is rather complicated, so we will state it slightly later (see Theorems 3.3 and 3.5 below). These functions are solutions of specific minimization problems on subdomains of $\mathbb{R}^4$, they allow geometric interpretation.

The computation of the constants $c^{\star,\mathbb{R}}_{p,p}$ and $d^{\star,\mathbb{R}}_{p,p}$ leads to the proof of Theorems 1.6 and 1.7 via the propositions below.

Proposition 1.16. *For any $\theta \in (1,\infty)$ and any r we have $c^{\star,\mathbb{R}}_{\theta,r} = c^{\star}_{\theta,r}$.*

Proposition 1.17. *For any $\theta \in (1, \infty)$ and any r we have $d^{\star,\mathbb{R}}_{\theta,r} = d^{\star}_{\theta,r}$.*

Though the present paper is self-contained, it employs the heuristic experience acquired by the authors as a result of studying other problems. The closest one is the Bellman function in [14]. See [12] for a more geometric point of view and [10], [11], and [13] for a study of a related problem. We also refer the reader to [15], [16], [19], and [20] for history and basics of the Bellman function theory. It would appear that all the previously computed sharp Bellman functions were either two or three dimensional. Our functions $\mathbb{B}_{c,+}$ and $\mathbb{B}_{d,+}$ depend on four variables.

Organization of the paper In Section 2, we study simple properties of the functions $\mathbb{B}_{c,+}$ and $\mathbb{B}_{d,+}$ and prove our Propositions 1.12 and 1.15. We also introduce more Bellman functions here. In Section 3, we compute $\mathbb{B}_{c,+}$ and describe its foliation. Section 4 contains the computation of the constants $c^{\star,\mathbb{R}}_{p,r}$. In Section 5, we prove Proposition 1.16, in particular, we prove Theorem 1.6. Then, in Section 6, we find $\mathbb{B}_{d,+}$. In fact, it can be expressed in terms of $\mathbb{B}_{c,+}$ and its companion minimal function by a change of variables. Next, we compute $d^{\star,\mathbb{R}}_{p,r}$ in Section 7 and prove Proposition 1.17 in Section 8. We finish the paper with the elementary proof of Theorem 1.8 given in Section 9.

2. Basic properties of Bellman functions

We assume $p \geqslant 2$. By Hölder's inequality, there does not exist f or g such that $\langle |f|^p\rangle_I < |\langle f\rangle_I|^p$ or $\langle |g|^q\rangle_I < |\langle g\rangle_I|^q$. Therefore, the function $\mathbb{B}_{c,+}$ is equal to $-\infty$ outside the set

$$\Omega_c = \Big\{x \in \mathbb{R}^4 \,\Big|\, x_3 \geqslant |x_1|^p, x_4 \geqslant |x_2|^q\Big\}.$$

On the other hand, since for any $x \in \Omega_c$ there exist functions f and g such that $x_1 = \langle f\rangle_I$, $x_2 = \langle g\rangle_I$, $x_3 = \langle |f|^p\rangle_I$, and $x_4 = \langle |g|^q\rangle_I$, we have $\mathbb{B}_{c,+}(x) > -\infty$ for $x \in \Omega_c$. We call Ω_c the natural domain (or simply the domain) of $\mathbb{B}_{c,+}$. The set

$$\partial_{\mathrm{skel}}\Omega_c = \Big\{x \in \mathbb{R}^4 \,\Big|\, x_3 = |x_1|^p, x_4 = |x_2|^q\Big\}$$

is called the skeleton of Ω_c (this is the set of the extreme points of Ω_c; note that it is only a small part of the topological boundary). Similarly,

$$\Omega_d = \Big\{x \in \mathbb{R}^4 \,\Big|\, x_4 \geqslant 0,\ |x_1| \leqslant x_3^{1/p},\ |x_5| \leqslant x_3^{1/p}x_4^{1/q}\Big\}$$

is the domain of $\mathbb{B}_{d,+}$ and

$$\partial_{\mathrm{skel}}\Omega_d = \Big\{x \in \mathbb{R}^4 \,\Big|\, x_4 \geqslant 0,\ |x_1| = x_3^{1/p},\ |x_5| = x_3^{1/p}x_4^{1/q}\Big\}$$

is the skeleton of Ω_d.

Lemma 2.1. *The functions $\mathbb{B}_{c,+}$ and $\mathbb{B}_{d,+}$ satisfy the following properties*:

- *They do not depend on the interval I*;
- *They satisfy the boundary conditions*

$$\mathbb{B}_{c,+}(x_1, x_2, |x_1|^p, |x_2|^q) = x_1 x_2;$$
$$\mathbb{B}_{d,+}(x_1, |x_1|^p, |x_2|^q, x_1 x_2) = x_2;$$

- *They are positively homogeneous, for positive λ_1 and λ_2, we have*:

$$\mathbb{B}_{c,+}(\lambda_1 x_1, \lambda_2 x_2, \lambda_1^p x_3, \lambda_2^q x_4) = \lambda_1\lambda_2 \mathbb{B}_{c,+}(x);$$
$$\mathbb{B}_{d,+}(\lambda_1 x_1, \lambda_1^p x_3, \lambda_2^q x_4, \lambda_1\lambda_2 x_5) = \lambda_2 \mathbb{B}_{d,+}(x);$$

- *They are pointwise minimal among all concave functions on their domains that satisfy the same boundary conditions.*

This lemma is very standard, see Propositions 1 and 2 in [14], where a completely similar statement is proved for another Bellman function.

Minimal concave functions can be described in terms of convex hulls. It is convenient to consider companion functions for $\mathbb{B}_{c,+}$ and $\mathbb{B}_{d,+}$.

Definition 2.2. Let $I \subset \mathbb{R}$ be a finite interval, $L^p = L^p(I, m)$. Consider the functions $\mathbb{B}_{c,-}$ and $\mathbb{B}_{d,-}$ given by the formulas

$$\mathbb{B}_{c,-}(x_1, x_2, x_3, x_4) = \inf\Big\{\langle fg\rangle_I \,\Big|\, f, g \text{ real-valued}, \langle f\rangle_I = x_1, \langle g\rangle_I = x_2, \langle |f|^p\rangle_I = x_3, \langle |g|^q\rangle_I = x_4\Big\};$$
$$\mathbb{B}_{d,-}(x_1, x_3, x_4, x_5) = \inf\Big\{\langle g\rangle_I \,\Big|\, f, g \text{ real-valued}, \langle f\rangle_I = x_1, \langle |f|^p\rangle_I = x_3, \langle |g|^q\rangle_I = x_4, \langle fg\rangle_I = x_5\Big\}.$$

Remark 2.3. The functions $\mathbb{B}_{c,-}$ and $\mathbb{B}_{d,-}$ satisfy the properties similar to those listed in Lemma 2.1: the first three properties remain the same, and in the last property, one should replace “minimal concave” with “maximal convex”.

Lemma 2.4. *Let ω be a closed convex subset of $\mathbb{R}^n$, let $\partial\omega$ be the set of its extreme points. Consider a continuous function $f\colon \partial\omega \to \mathbb{R}$. Let $\mathcal{B}_f^+$ and $\mathcal{B}_f^-$ be the minimal convex and the maximal concave functions on ω that coincide with f on $\partial\omega$. The intersection of the subgraph of $\mathcal{B}_f^+$ and the epigraph of $\mathcal{B}_f^-$ coincides with the closure of the convex hull* (*in $\mathbb{R}^{n+1}$*) *of the graph of f*:

$$\Big\{(x, y) \in \omega \times \mathbb{R} \,\Big|\, \mathcal{B}_f^-(x) \leqslant y \leqslant \mathcal{B}_f^+(x)\Big\} = \overline{\operatorname{conv}\Big\{(x, f(x)) \,\Big|\, x \in \partial\omega\Big\}}.$$

We will not prove this lemma. It is a slight generalization of Proposition 3 of [14] (in [14], we worked with strictly convex compact domains). It has an immediate corollary that allows to express $\mathbb{B}_{d,+}$ and $\mathbb{B}_{d,-}$ in terms of $\mathbb{B}_{c,+}$ and $\mathbb{B}_{c,-}$.

Corollary 2.5. *Let the five dimensional body* $\mathbb{K}$ *be the convex hull of the two dimensional surface*

$$\Big\{(t_1, t_2, |t_1|^p, |t_2|^q, t_1t_2) \in \mathbb{R}^5 \;\Big|\; t_1, t_2 \in \mathbb{R}\Big\}.$$

Then, on one hand,

$$\mathbb{K} = \Big\{x \in \mathbb{R}^5 \;\Big|\; (x_1, x_2, x_3, x_4) \in \Omega_c,\ \mathbb{B}_{c,-}(x_1, x_2, x_3, x_4) \leqslant x_5 \leqslant \mathbb{B}_{c,+}(x_1, x_2, x_3, x_4)\Big\}$$

and on the other hand

$$\mathbb{K} = \Big\{x \in \mathbb{R}^5 \;\Big|\; (x_1, x_3, x_4, x_5) \in \Omega_d,\ \mathbb{B}_{d,-}(x_1, x_3, x_4, x_5) \leqslant x_2 \leqslant \mathbb{B}_{d,+}(x_1, x_3, x_4, x_5)\Big\}.$$

Corollary 2.5 says that the graphs of the four Bellman functions we consider are parts of the boundary of a certain convex hull. We invoke the Carathéodory theorem (see Corollary 1 in [14]) to see for each point $x \in \Omega$ there exists not more than five points x_j in the set $\partial_{\mathrm{skel}}\Omega$ such that $x \in \operatorname{conv}\{x_j\}_j$ and $\mathbb{B}$ (which is any of our Bellman functions) is linear on $\operatorname{conv}\{x_j\}$ (see Corollary 1 in [14]).[3] Thus, Ω splits into subsets ω, which might be one, two, three, or four dimensional, such that $\mathbb{B}$ is linear on each ω. Such a splitting is called the foliation of $\mathbb{B}$. The function $\mathbb{B}$ can be easily computed once we establish its foliation.

We also state without proof that for any ω there exists a unique affine function $\mathrm{D}\mathbb{B}(\omega)$ whose graph is the supporting plane of the subgraph of $\mathbb{B}$ at each point $x \in \omega$. In other words, the gradient of $\mathbb{B}$ is one and the same for all the points inside each subdomain of the foliation. We state this principle without proof since, first, it is not needed for the formal proof (however, it helps us to find the answer), second, it requires additional smoothness assumptions (which our particular problem does satisfy).

Our strategy of the proof will be to try to find the affine functions $\mathrm{D}\mathbb{B}$ and then reconstruct the function $\mathbb{B}$ from them. In a sense, we find the convex conjugate of $\mathbb{B}$ rather than the function itself.

Proof of Proposition 1.12. We rewrite inequality (1.5) as

$$\inf_{\alpha\in\mathbb{R}} \|f - \alpha e\|^r_{L^\theta(\mu)} \leqslant c^{-1}_{\theta,r}\Big(\|f\|^r_{L^\theta(\mu)} - \big|\langle f, e|e|^{\theta-2}\rangle_\mu\big|^r\Big), \quad \|e\|_{L^\theta(\mu)} = 1.$$

By the Hahn–Banach Theorem, the latter inequality is equivalent to

[3] This is not quite correct: our domain is not compact and we are not allowed to use the Carathéodory theorem; we do not rely on this reasoning formally.

$$\big|\langle f, g\rangle_\mu\big|^r \leqslant c_{\theta,r}^{-1}\Big(\|f\|_{L^\theta(\mu)}^r - \big|\langle f, e|e|^{\theta-2}\rangle_\mu\big|^r\Big),$$
$$\|e\|_{L^\theta(\mu)} = 1,\ \|g\|_{L^{\theta'}(\mu)} = 1 \text{ and } \langle e, g\rangle_\mu = 0. \tag{2.1}$$

Without loss of generality, we may assume that $e \neq 0$ almost everywhere. Consider the measure $d\tilde{\mu} = |e|^\theta d\mu$. This is a continuous probability measure. We also consider modified functions $\tilde{f} = \frac{f}{e} \in L^\theta(\tilde{\mu})$ and $\tilde{g} = \frac{ge}{|e|^\theta} \in L^{\theta'}(\tilde{\mu})$. Clearly,

$$\langle \tilde{f}, \tilde{g}\rangle_{\tilde{\mu}} = \langle f, g\rangle_\mu, \quad \|\tilde{f}\|_{L^\theta(\tilde{\mu})} = \|f\|_{L^\theta(\mu)}, \quad \|\tilde{g}\|_{L^{\theta'}(\tilde{\mu})} = \|g\|_{L^{\theta'}(\mu)},$$
$$\langle 1, \tilde{g}\rangle_{\tilde{\mu}} = \langle e, g\rangle_\mu, \quad \text{and} \quad \langle \tilde{f}, 1\rangle_{\tilde{\mu}} = \langle f, e|e|^{\theta-2}\rangle_\mu$$

and (2.1) turns into

$$\big|\langle \tilde{f}, \tilde{g}\rangle_{\tilde{\mu}}\big|^r \leqslant c_{\theta,r}^{-1}\big(\|\tilde{f}\|_{L^\theta(\tilde{\mu})}^r - \big|\langle \tilde{f}\rangle_{\tilde{\mu}}\big|^r\big).$$

It remains to identify the standard probability space $(X, \tilde{\mu})$ with $([0,1], m)$ and use Definition 1.11. □

The proof of Proposition 1.15 is completely similar.

3. The computation of $\mathbb{B}_{c,+}$

3.1. Statement of results

We will use three auxiliary functions φ, λ, and ρ. The function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is simple:

$$\varphi(R) = R|R|^{p-2}.$$

The function $\lambda\colon \mathbb{R} \to \mathbb{R}$ is given by the formula

$$\lambda(R) = \begin{cases} \frac{1}{1+R} - \frac{p-1}{1+\varphi(R)}, & R \neq -1; \\ -\frac{p-2}{2}, & R = -1. \end{cases} \tag{3.1}$$

Note that λ is a continuous function. The function ρ will be defined after the following lemma.

Lemma 3.1. *There exists $R_0 \in (0,1)$ such that*

$$R_0^{\frac{p-1}{2}} + R_0^{-\frac{p-1}{2}} = (p-1)\big(R_0^{\frac{1}{2}} + R_0^{-\frac{1}{2}}\big). \tag{3.2}$$

The function λ is decreasing on $(-1, R_0)$ and increasing on $(R_0, 1)$.

We will prove this technical lemma at the end of the subsection.

Definition 3.2. Define the function $\rho\colon [-1,1] \to [R_0, 1]$ by the formula $\lambda(\rho(R)) = \lambda(R)$ when $R \in [-1, R_0]$ and $\rho(R) = R$ when $R \in [R_0, 1]$.

Note that $\lambda(-1) = \lambda(1)$. The function ρ first decreases from 1 downto R_0 on $[-1, R_0]$ and then increases back to 1 on $[R_0, 1]$.

Theorem 3.3. *For any $R \in [-1,1]$, $a_1, a_2 \in \mathbb{R}$ such that $a_1a_2 > 0$, the function $\mathbb{B}_{c,+}$ is linear on the segment $\ell_c(a_1, a_2, R)$ with the endpoints*

$$a = (a_1, a_2, |a_1|^p, |a_2|^q), \qquad b = (-\rho(R)a_1, -\varphi(R)a_2, |\rho(R)a_1|^p, |R|^p|a_2|^q). \tag{3.3}$$

In other words,

$$\mathbb{B}_{c,+}(\tau a + (1-\tau)b) = a_1a_2(\tau + (1-\tau)\rho(R)\varphi(R)), \quad \tau \in [0,1]. \tag{3.4}$$

If $R \in [R_0, 1]$, then $\mathbb{B}_{c,+}(x) = x_3^{\frac{1}{p}} x_4^{\frac{1}{q}}$ on $\ell_c(a_1, a_2, R)$. The segments $\ell_c(a_1, a_2, R)$ with $a_1 > 0$, $a_2 > 0$, and $R \in [-1,1]$ cover the domain

$$\Big\{x \in \Omega \,\Big|\, -1 < \frac{x_2}{x_4^{\frac{1}{q}}} \leqslant \frac{x_1}{x_3^{\frac{1}{p}}} < 1\Big\}$$

entirely. The segments $\ell_c(a_1, a_2, R)$ with $a_1 < 0$, $a_2 < 0$, and $R \in [-1,1]$ cover the domain

$$\Big\{x \in \Omega \,\Big|\, -1 < \frac{x_1}{x_3^{\frac{1}{p}}} \leqslant \frac{x_2}{x_4^{\frac{1}{q}}} < 1\Big\}$$

entirely.

We see that Theorem 3.3 describes the foliation of $\mathbb{B}_{c,+}$: the segments $\ell_c(a_1, a_2, R)$ are the one dimensional subsets of Ω_c such that $\mathbb{B}_{c,+}$ is linear on each of them. Later we will see that this is not all the truth: some of these segments form linearity domains of dimension three (see Section 3.4 below).

The formula

$$\mathbb{B}_{c,+}(x_1, x_2, x_3, x_4) = -\mathbb{B}_{c,-}(-x_1, x_2, x_3, x_4) \tag{3.5}$$

leads to the corollary.

Corollary 3.4. *The segments $\ell_c(a_1, a_2, R)$ with $a_1a_2 < 0$ cover the interior of Ω_c entirely. The function $\mathbb{B}_{c,-}$ is linear on each of these segments. Moreover,*

$$\partial\mathbb{K} = \Big(\cup_{R\in[-1,1],a_1,a_2\in\mathbb{R}} \mathfrak{L}(a_1, a_2, R)\Big) \cup \\ \Big\{(x_1, x_2, |x_1|^p, x_4, x_1x_2) \,\Big|\, x_4 \geqslant |x_2|^q\Big\} \cup \Big\{(x_1, x_2, x_3, |x_2|^q, x_1x_2) \,\Big|\, x_3 \geqslant |x_1|^p\Big\},$$

where the segments $\mathfrak{L}(a_1,a_2,R)$ are given by the formula

$$\mathfrak{L}(a_1,a_2,R) = \Big((\tau-(1-\tau)\rho(R))a_1, (\tau-(1-\tau)\varphi(R))a_2, (\tau+(1-\tau)\rho(R)^p)|a_1|^p, \\ (\tau+(1-\tau)|R|^p)|a_2|^q, (\tau+(1-\tau)\rho(R)\varphi(R))a_1a_2\Big), \quad \tau\in[0,1].$$

Theorem 3.5. *For any $R\in[-1,1]$, $a_1,a_2\in\mathbb{R}$ such that $a_2>0$, the function $\mathbb{B}_{d,+}$ is linear on the segment $\ell_d(a_1,a_2,R)$ with the endpoints*

$$a = (a_1, |a_1|^p, |a_2|^q, a_1a_2), \qquad b = (-\rho(R)a_1, |\rho(R)a_1|^p, |R|^p|a_2|^q, \rho(R)\varphi(R)a_1a_2).$$

The function $\mathbb{B}_{d,-}$ is linear on the segments $\ell_d(a_1,a_2,R)$ with $a_2<0$.

Theorem 3.3 has two assertions. The proof of each of them is presented in its own subsection (Subsections 3.2 and 3.3). Theorem 3.5 is proved in Section 8.

Proof of Lemma 3.1. We rewrite (3.2) as $\kappa(R_0)=0$, where

$$\kappa(R) = (p-1)|R|^{\frac{p}{2}-1}(1+R) - 1 - R|R|^{p-2}, \quad R\in(-1,1).$$

We differentiate κ to find

$$\kappa'(R) = (p-1)|R|^{\frac{p}{2}-2}\Big(\Big(\frac{p}{2}-1\Big)\operatorname{sign}R + \frac{p}{2}|R| - |R|^{\frac{p}{2}}\Big).$$

This is greater than 0 when $R\in(0,1)$. We also have $\kappa(0)=-1<0$ and $\kappa(1)=2(p-2)>0$. Therefore, κ has a unique root R_0 in $(0,1)$. The function κ changes sign from negative to positive at R_0.

We also compute

$$\lambda'(R) = \frac{(p-1)^2|R|^{p-2}}{\big(1+R|R|^{p-2}\big)^2} - \frac{1}{(1+R)^2} = \frac{\kappa(R)\Big((p-1)|R|^{\frac{p}{2}-1}(1+R) + \big(1+R|R|^{p-2}\big)\Big)}{\big(1+R|R|^{p-2}\big)^2(1+R)^2}. \tag{3.6}$$

Thus, $\operatorname{sign}(\lambda'(R)) = \operatorname{sign}(\kappa(R))$.

Note that $\kappa'(R)<0$ when $R\in(-1,0)$. Moreover, $\kappa(-1)=0$, so $\kappa(R)<0$ when $R\in(-1,0)$. We conclude that $\lambda'(R)<0$ for $R\in(-1,0)$.

Thus, λ decreases on $(-1,R_0)$ and increases on $(R_0,1)$. □

3.2. First assertion of Theorem 3.3

To prove that the function $\mathbb{B}_{c,+}$ is linear on a certain segment $\ell(a,b)$ that connects two points a and b on $\partial_{\mathrm{skel}}\Omega_c$, we will construct an affine function Ψ (depending on a and b) such that

$$\Psi(a) = \mathbb{B}_{c,+}(a),\ \Psi(b) = \mathbb{B}_{c,+}(b),\ \text{and } \Psi(x) \geqslant \mathbb{B}_{c,+}(x) \quad \text{for any } x \in \Omega_c.$$

By the third and fourth statements of Lemma 2.1, it suffices to verify the inequality $\Psi(x) \geqslant \mathbb{B}_{c,+}(x)$ for $x \in \partial_{\mathrm{skel}}\Omega_c$ only. Let

$$\Psi(x) = t_0 + t_1x_1 + t_2x_2 + t_3x_3 + t_4x_4. \tag{3.7}$$

In view of the above-mentioned properties, Ψ majorizes $\mathbb{B}_{c,+}$ if and only if the function $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$, given by the formula

$$\Phi(x_1,x_2) = \Psi(x_1,x_2,|x_1|^p,|x_2|^q) - x_1x_2 = t_0 + t_1x_1 + t_2x_2 + t_3|x_1|^p + t_4|x_2|^q - x_1x_2, \tag{3.8}$$

is non-negative. Moreover, if $\Phi(a_1,a_2) = \Phi(b_1,b_2) = 0$, then $\mathbb{B}_{c,+}$ is linear on the segment $\ell(a,b)$.

Lemma 3.6. *The function Ψ defined by* (3.7) *majorizes $\mathbb{B}_{c,+}$ on Ω_c if and only if the following two conditions hold*:

1) $t_3, t_4 > 0$;
2) $H(x_1) \geqslant 0$ *for any $x_1 \in \mathbb{R}$, where*

$$H(x_1) = t_0 - \frac{1}{p}\frac{|x_1 - t_2|^p}{(qt_4)^{p-1}} + t_3|x_1|^p + t_1x_1. \tag{3.9}$$

Proof. Fix $x_2 \neq t_1$ and consider the asymptotic behavior of Φ as x_1 tends to infinity. The senior term of Φ should be non-negative, which leads to $t_3 \geqslant 0$. The equality $t_3 = 0$ is impossible since in this case Φ is a linear with respect to x_1 function with non-zero senior coefficient. Such a function cannot be non-negative on the entire axis. Thus, $t_3 > 0$. Similarly, $t_4 > 0$.

We fix x_1 and see that the function $x_2 \mapsto \Phi(x_1,x_2)$ attains its minimum at the point

$$x_2 = \operatorname{sign}(x_1 - t_2)\left|\frac{x_1 - t_2}{qt_4}\right|^{p-1}.$$

We plug this expression back into Φ and find

$$\min_{x_2} \Phi(x_1,x_2) = \Phi\left(x_1, \operatorname{sign}(x_1 - t_2)\frac{|x_1 - t_2|^{p-1}}{(qt_4)^{p-1}}\right) = H(x_1). \quad \square$$

We study the equations

$$\Phi(a_1,a_2) = \Phi(b_1,b_2) = 0 \tag{3.10}$$

under the condition $\Phi \geqslant 0$. In particular, (a_1,a_2) and (b_1,b_2) are minima of Φ. Thus,

$$\frac{\partial\Phi}{\partial x_1}(a_1,a_2)=\frac{\partial\Phi}{\partial x_1}(b_1,b_2)=0; \tag{3.11}$$

$$\frac{\partial\Phi}{\partial x_2}(a_1,a_2)=\frac{\partial\Phi}{\partial x_2}(b_1,b_2)=0. \tag{3.12}$$

The derivative of Φ satisfies $\frac{\partial\Phi}{\partial x_1}(x_1,x_2)=t_1+p|x_1|^{p-2}x_1t_3-x_2$. Note that (3.11) does not have solutions for which only one of the identities $a_1=b_1$ or $a_2=b_2$ hold (i.e. either $a_1=b_1$ and $a_2=b_2$ or $a_1\neq b_1$ and $a_2\neq b_2$). Consequently, $a_1\neq b_1$ and $a_2\neq b_2$. We solve (3.11) for t_1 and t_3:

$$t_1=\frac{a_2b_1|b_1|^{p-2}-a_1b_2|a_1|^{p-2}}{b_1|b_1|^{p-2}-a_1|a_1|^{p-2}}; \tag{3.13}$$

$$t_3=\frac{b_2-a_2}{p(b_1|b_1|^{p-2}-a_1|a_1|^{p-2})}. \tag{3.14}$$

Similarly, we solve (3.12) for t_2 and t_4:

$$t_2=\frac{a_1b_2|b_2|^{q-2}-a_2b_1|a_2|^{q-2}}{b_2|b_2|^{q-2}-a_2|a_2|^{q-2}}; \tag{3.15}$$

$$t_4=\frac{b_1-a_1}{q(b_2|b_2|^{q-2}-a_2|a_2|^{q-2})}. \tag{3.16}$$

We also have the system (3.10) that defines t_0. We do not need the expression for t_0, however, the compatibility condition is of crucial importance:

$$(b_1-a_1)t_1+(b_2-a_2)t_2+(|b_1|^p-|a_1|^p)t_3+(|b_2|^q-|a_2|^q)t_4+(a_1a_2-b_1b_2)=0. \tag{3.17}$$

Using formulas (3.13) and (3.14), we get

$$(|b_1|^p-|a_1|^p)pt_3+(b_1-a_1)t_1=b_1b_2-a_1a_2.$$

Similarly, formulas (3.15) and (3.16) lead to

$$(|b_2|^p-|a_2|^p)qt_4+(b_2-a_2)t_2=b_1b_2-a_1a_2.$$

With these identities in hand, we rewrite (3.17) as

$$\frac{1}{q}(b_1-a_1)t_1+\frac{1}{p}(b_2-a_2)t_2=0. \tag{3.18}$$

We treat the t_j as functions of a and b in what follows. We summarize our computations in the following lemma.

Lemma 3.7. *The function $\mathbb{B}_{c,+}$ is linear on $\ell(a,b)$ if and only if the parameters t_1, t_2, t_3, t_4 given by* (3.13)*,* (3.15)*,* (3.14)*, and* (3.16) *satisfy the conditions below.*

1) $t_3 > 0$, $t_4 > 0$.
2) $H(x_1) \geqslant 0$ *for any* $x_1 \in \mathbb{R}$.
3) *Equality* (3.18) *holds.*

To prove the first assertion of Theorem 3.3, we need to restate the conditions of Lemma 3.7. As for the first condition $t_3 > 0$ and $t_4 > 0$, it is equivalent to $(b_1 - a_1)(b_2 - a_2) > 0$, see formulas (3.14) and (3.16). We introduce new variables

$$R_1 = -\frac{b_1}{a_1}, \qquad R_2 = -\frac{b_2|b_2|^{q-2}}{a_2|a_2|^{q-2}}. \tag{3.19}$$

We express t_1, t_2, t_3, t_4 in the new variables (recall $\varphi(R) = R|R|^{p-2}$):

$$t_1 = a_2\frac{\varphi(R_1) - \varphi(R_2)}{1 + \varphi(R_1)}\,; \qquad t_2 = a_1\frac{R_2 - R_1}{1 + R_2}\,; \tag{3.20}$$

$$pt_3 = \frac{a_2}{a_1|a_1|^{p-2}} \cdot \frac{1 + \varphi(R_2)}{1 + \varphi(R_1)}\,; \qquad qt_4 = \frac{a_1}{a_2|a_2|^{q-2}} \cdot \frac{1 + R_1}{1 + R_2}\,. \tag{3.21}$$

Note that division by a_1 and a_2 is eligible since $a_1 a_2 > 0$ in Theorem 3.3.

Lemma 3.8. *The third condition in Lemma 3.7 is equivalent to*

$$\lambda(R_1) = \lambda(R_2). \tag{3.22}$$

Proof. We divide (3.18) by $a_1 a_2$, express everything in terms of R_1 and R_2, and obtain

$$\frac{1}{q}(1 + R_1)\frac{\varphi(R_1) - \varphi(R_2)}{1 + \varphi(R_1)} + \frac{1}{p}(1 + \varphi(R_2))\frac{R_2 - R_1}{1 + R_2} = 0,$$

which, after division by $(1 + R_1)(1 + \varphi(R_2))/p$ appears to be $\lambda(R_1) = \lambda(R_2)$. $\square$

Lemma 3.9. *If a and b are such that $t_3, t_4 > 0$, then H does not change sign on the real line.*

Proof. We have chosen the t_i in such a way that $H(a_1) = H(b_1)$. Moreover, $H'(a_1) = H'(b_1) = 0$ since $\nabla\Phi(a_1, a_2) = \nabla\Phi(b_1, b_2) = 0$. Therefore, H'' has at least two distinct roots on (a_1, b_1). Note that the function

$$H''(x_1) = (p - 1)\left(pt_3|x_1|^{p-2} - \frac{|x_1 - t_2|^{p-2}}{(qt_4)^{p-1}}\right)$$

is either equal to zero, or has no more than two roots. If H'' is identically zero, then H vanishes on $\mathbb{R}$ as well. In the other case, H has unique local extremum on (a_1, b_1). In this case, H'' has exactly two roots. Therefore, $H''(a_1)$ and $H''(b_1)$ have the same sign and H does not change sign on $\mathbb{R}$. $\square$

Lemma 3.10. *Let a and b be such that t_3, $t_4 > 0$. If $R_1 = R_2$, then H is identically zero. If $R_1 \neq R_2$, then $H \geqslant 0$ is equivalent to $\psi(R_2) > \psi(R_1)$, where $\psi(R) = \frac{|1+\varphi(R)|}{|1+R|^{p-1}}$.*

Proof. Identity (3.21) leads to

$$pt_3(qt_4)^{p-1} = \frac{\psi(R_2)}{\psi(R_1)}.$$

If $R_1 = R_2$, then $t_1 = t_2 = 0$ and $pt_3(qt_4)^{p-1} = 1$. Thus, H is identically equal to zero in this case.

If $R_1 \neq R_2$, then $t_2 \neq 0$. In the case $pt_3(qt_4)^{p-1} = 1$, the signs of H at $-\infty$ and $+\infty$ differ, which contradicts Lemma 3.9. If $pt_3(qt_4)^{p-1} > 1$, then H is positive at the infinities, if $pt_3(qt_4)^{p-1} < 1$ it is negative. □

The first assertion of Theorem 3.3 is almost proved. Indeed, consider any segment $\ell_c(a_1, a_2, R)$ with $a_1 a_2 > 0$. Define R_1 and R_2 by formulas (3.19), i.e. $R_1 = \rho(R)$ and $R_2 = R$, and note that $R_1 \geqslant R_2$. Then, $\lambda(R_1) = \lambda(R_2)$ (by Definition 3.2) and $\psi(R_2) > \psi(R_1)$ since ψ decreases on $(-1, 1)$. Thus, by Lemmas 3.8 and 3.10, the function Ψ defined by (3.7), (3.20), and (3.21) satisfies the requirements of Lemma 3.7. This means $\mathbb{B}_{c,+}$ is linear on $\ell_c(a_1, a_2, R)$. It remains to combine formulas (3.3) and (3.4) to see that $\mathbb{B}_{c,+}(x) = x_3^{\frac{1}{p}} x_4^{\frac{1}{q}}$ if $x \in \ell_c(a_1, a_2, R)$ and $R \geqslant R_0$.

3.3. Second assertion of Theorem 3.3

Definition 3.11. Consider the mapping

$$T\colon \Omega_c \setminus \Big(\{x \mid x_1 = x_3 = 0\} \cup \{x \mid x_2 = x_4 = 0\}\Big) \to [-1, 1]^2$$

given by the rule

$$T\colon (x_1, x_2, x_3, x_4) \mapsto (x_1 x_3^{-\frac{1}{p}}, x_2 x_4^{-\frac{1}{q}}). \tag{3.23}$$

The skeleton $\partial_{\mathrm{skel}}\Omega_c$ is mapped onto the vertices $(\pm 1, \pm 1)$. Due to the homogeneity of $\mathbb{B}_{c,+}$ (Lemma 2.1), it suffices to compute the values of $\mathbb{B}_{c,+}$ for x such that $x_3 = x_4 = 1$; the values at all other points may be restored from them:

$$\mathbb{B}_{c,+}(x) = x_3^{\frac{1}{p}} x_4^{\frac{1}{q}} \mathbb{B}_{c,+}(x_1 x_3^{-\frac{1}{p}}, x_2 x_4^{-\frac{1}{q}}, 1, 1)\,.$$

Therefore, to prove the second assertion of Theorem 3.3, it suffices to show that the union of $T(\ell_c(a_1, a_2, R))$ over all a_1, a_2, and R such that $a_1 > 0$, $a_2 > 0$, and $R \in (-1, 1)$, covers the triangle

$$\{(y_1, y_2) \mid -1 < y_2 < y_1 < 1\}. \tag{3.24}$$

Note that $T(\ell_c(a_1, a_2, 1)) = \{(y_1, y_1) \mid y_1 \in [-1, 1]\}$.

Lemma 3.12. *The function η defined by the formula*

$$\eta\colon (\tau, R) \mapsto \Big(\eta_1(\tau, R), \eta_2(\tau, R)\Big) \overset{\text{def}}{=} \left(\frac{-\tau + (1-\tau)\rho(R)}{(\tau + (1-\tau)\rho(R)^p)^{\frac{1}{p}}}, \frac{-\tau + (1-\tau)\varphi(R)}{(\tau + (1-\tau)|R|^p)^{\frac{1}{q}}} \right)$$

maps $(0, 1) \times (-1, 1)$ onto the triangle (3.24) bijectively.

Remark 3.13. For R fixed, the image of the mapping $\tau \mapsto \eta(\tau, R)$ coincides with $T(\ell_c(a_1, a_2, R))$, here $a_1 > 0$ and $a_2 > 0$.

Proof. First, we show that for any $(y_1, y_2) \in (-1, 1)^2$ such that $y_2 < y_1$, there exist $R \in (-1, 1)$ and $\tau \in (0, 1)$ such that $y_i = \eta_i(\tau, R)$, $i = 1, 2$.

Note that $\frac{\partial \eta_1}{\partial \tau}(\tau, R) < 0$:

$$\begin{aligned} \frac{\partial \eta_1}{\partial \tau} &= \Big[\frac{1}{p}\big(\tau - (1-\tau)\rho\big)(1-\rho^p) - (1+\rho)\big(\tau + (1-\tau)\rho^p\big)\Big]\big(\tau + (1-\tau)\rho^p\big)^{-1-\frac{1}{p}} = \\ &= -\Big[\Big(\frac{1}{q} + \rho + \frac{1}{p}\rho^p\Big)\tau + \Big(\frac{1}{p}(1-\rho^p)\rho + (1+\rho)\rho^p\Big)(1-\tau)\Big]\big(\tau + (1-\tau)\rho^p\big)^{-1-\frac{1}{p}} \\ &< 0. \end{aligned} \tag{3.25}$$

Moreover, $\eta_1(0, R) = 1$, $\eta_1(1, R) = -1$. Therefore, for any $y_1 \in (-1, 1)$ and $R \in [-1, 1]$, there exists a unique value $\tau \in (0, 1)$ such that $\eta_1(\tau, R) = y_1$. Let us denote this value by the symbol $\tau_1(R, y_1)$. The function $\tau_1(\cdot, y_1)$ is continuous for any fixed $y_1 \in (-1, 1)$. What is more, $\eta(\tau, 1) = (1 - 2\tau, 1 - 2\tau)$ and $\eta(\tau, -1) = (1 - 2\tau, -1)$. Thus, $\tau_1(-1, y_1) = \tau_1(1, y_1) = \frac{1-y_1}{2}$.

Fix y_1 for a while. Note that the function

$$Y_2\colon R \mapsto \eta_2(\tau_1(R, y_1), R)$$

is continuous and

$$Y_2(-1) = \eta_2\Big(\frac{1-y_1}{2}, -1\Big) = -1 < y_2 < y_1 = \eta_2\Big(\frac{1-y_1}{2}, 1\Big) = Y_2(1).$$

Thus, for some $R \in (-1, 1)$ we have $Y_2(R) = y_2$. This means that the identity $\eta(\tau, R) = (y_1, y_2)$ holds for $\tau = \tau_1(R, y_1)$ and this specific choice of R. Thus, we have proved that the union of $T(\ell_c(a_1, a_2, R))$ covers the triangle (3.24).

Let us now show that the function Y_2 increases provided y_1 is fixed. If this is true, then for any $y_1 \in (-1, 1)$ and $y_2 \in (-1, y_1)$, there exists unique value R such that $Y_2(R) = y_2$. Moreover, the same monotonicity leads to the inequality $\eta_2 \leqslant \eta_1$. Thus, η is bijective.

So it remains to prove the mentioned monotonicity. We compute the derivative:

$$Y_2'(R) = \Big(\frac{\partial\eta_1}{\partial\tau}\Big)^{-1}\cdot\Delta(R), \qquad \Delta(R) \stackrel{\text{def}}{=} \frac{\partial\eta_2}{\partial R}\frac{\partial\eta_1}{\partial\tau} - \frac{\partial\eta_2}{\partial\tau}\frac{\partial\eta_1}{\partial R}.$$

We have already proved $\frac{\partial\eta_1}{\partial\tau} < 0$ (see (3.25)).

We investigate Δ. We compute the remaining partial derivatives:

$$\frac{\partial\eta_1}{\partial R} = \tau(1-\tau)\rho'\big(1+\rho^{p-1}\big)\big(\tau+(1-\tau)\rho^p\big)^{-1-\frac{1}{p}}, \tag{3.26}$$

$$\frac{\partial\eta_2}{\partial\tau} = \Big[\frac{1}{q}\big(\tau-(1-\tau)\varphi\big)(1-|R|^p)-(1+\varphi)\big(\tau+(1-\tau)|R|^p\big)\Big]\big(\tau+(1-\tau)|R|^p\big)^{-1-\frac{1}{q}}, \tag{3.27}$$

$$\frac{\partial\eta_2}{\partial R} = (p-1)\tau(1-\tau)(1+R)|R|^{p-2}\big(\tau+(1-\tau)|R|^p\big)^{-1-\frac{1}{q}}. \tag{3.28}$$

We plug these expressions into the formula for Δ:

$$\begin{aligned}\Delta\cdot&\big(\tau+(1-\tau)|R|^p\big)^{1+\frac{1}{q}}\big(\tau+(1-\tau)\rho^p\big)^{1+\frac{1}{p}} =\\ &= \tau(1-\tau)\Big\{(p-1)(1+R)|R|^{p-2}\Big[\frac{1}{p}\big(\tau-(1-\tau)\rho\big)(1-\rho^p)-(1+\rho)\big(\tau+(1-\tau)\rho^p\big)\Big]-\\ &\qquad -\rho'(1+\rho^{p-1})\Big[\frac{1}{q}\big(\tau-(1-\tau)\varphi\big)(1-|R|^p)-(1+\varphi)\big(\tau+(1-\tau)|R|^p\big)\Big]\Big\}.\end{aligned} \tag{3.29}$$

When $R\in[R_0,1]$, we have $\rho = R$. Thus, the expression in the formula (3.29) is

$$\begin{aligned}\tau(1-\tau)&\Big\{(p-1)(1+R)R^{p-2}\Big[\frac{1}{p}(\tau-(1-\tau)R)\big(1-R^p\big)-(1+R)\big(\tau+(1-\tau)R^p\big)\Big]-\\ &-(1+R^{p-1})\Big[\frac{1}{q}\big(\tau-(1-\tau)R^{p-1}\big)\big(1-R^p\big)-\big(1+R^{p-1}\big)\big(\tau+(1-\tau)R^p\big)\Big]\Big\} =\\ &= \frac{1}{p}\tau(1-\tau)\big(\tau+(1-\tau)R^p\big)\Big[\big(1+R^{p-1}\big)^2-(p-1)^2R^{p-2}(1+R)^2\Big] < 0\end{aligned} \tag{3.30}$$

by Lemma 3.1, since the latter expressions in the brackets has the sign opposite to $\lambda'(R)$, see (3.6). Thus, we have proved that Y_2 increases on $(R_0,1)$ for any fixed y_1.

It remains to consider the case $R\in(-1,R_0)$. The expression in the braces in (3.29) is a linear function of τ. Thus, it suffices to investigate its signs at the endpoints $\tau=0$ and $\tau=1$.

When $\tau=1$, the expression in the first brackets in (3.29) is negative:

$$\frac{1-\rho^p}{p}-(1+\rho) = -\frac{1}{q}-\frac{\rho^p}{p}-\rho < 0.$$

The expression in the second brackets in (3.29) is equal to

$$\frac{1-|R|^p}{q} - (1+\varphi) = -\Big(\frac{1}{p} + \frac{|R|^p}{q} + R|R|^{p-2}\Big) \leqslant 0.$$

Finally, $\rho' < 0$, therefore, the expression in the braces in (3.29) is negative.

It remains to study the case $\tau = 0$. In this case, our expression is equal to

$$\begin{aligned}\Upsilon = -(p-1)(1+R)|R|^{p-2}\Big[\frac{\rho}{p}\big(1-\rho^p\big) + (1+\rho)\rho^p\Big] \\ + \rho'(1+\rho^{p-1})\Big[\frac{\varphi}{q}\big(1-|R|^p\big) + (1+\varphi)|R|^p\Big].\end{aligned} \tag{3.31}$$

Now we will compute the expressions in the brackets in (3.31) and the derivative ρ' separately. Let us first rewrite the identity (3.1) in a more convenient form:

$$\frac{p-1}{1+\varphi} = \frac{1-(1+R)\lambda}{1+R}, \qquad \varphi = \frac{(1+R)(p-1+\lambda)-1}{1-(1+R)\lambda}. \tag{3.32}$$

Similarly, we rewrite the identity $\lambda(\rho) = \lambda$ as

$$\frac{p-1}{1+\rho^{p-1}} = \frac{1-(1+\rho)\lambda}{1+\rho}, \qquad \rho^{p-1} = \frac{(1+\rho)(p-1+\lambda)-1}{1-(1+\rho)\lambda}. \tag{3.33}$$

We also re-express $\lambda'(R)$:

$$\begin{aligned}\lambda'(R) = \frac{(p-1)^2}{(1+\varphi)^2}|R|^{p-2} - \frac{1}{(1+R)^2} = \Big(\frac{1-(1+R)\lambda}{1+R}\Big)^2\frac{\varphi}{R} - \frac{1}{(1+R)^2} = \\ = \frac{\big(1-(1+R)\lambda\big)\big((1+R)(p-1+\lambda)-1\big)}{R(1+R)^2} - \frac{1}{(1+R)^2} = \\ = \frac{1}{R(1+R)^2}\Big((1+R)(p-1+\lambda)\big(1-(1+R)\lambda\big) - 1 + (1+R)\lambda - R\Big) = \\ = \frac{(p-1+\lambda)\big(1-(1+R)\lambda\big)-1+\lambda}{R(1+R)}.\end{aligned} \tag{3.34}$$

The identity $\lambda(R) = \lambda(\rho(R))$ leads to

$$\rho' = \frac{\lambda'(R)}{\lambda'(\rho)} = \frac{\rho(1+\rho)}{R(1+R)} \cdot \frac{(p-1+\lambda)\big(1-(1+R)\lambda\big)-1+\lambda}{(p-1+\lambda)\big(1-(1+\rho)\lambda\big)-1+\lambda}. \tag{3.35}$$

We rewrite the expression in the first brackets of (3.31):

$$\begin{aligned}\frac{\rho}{p}\big(1-\rho^p\big)+(1+\rho)\rho^p = \frac{\rho+\rho^p}{p} + \frac{\rho^p+\rho^{p+1}}{q} = \frac{\rho}{q}\Big(\frac{1+(1+\rho)(p-1)}{p-1}(1+\rho^{p-1}) - (1+\rho)\Big) = \\ = \frac{\rho}{q}\Big(\frac{1+(1+\rho)(p-1)}{1-(1+\rho)\lambda}(1+\rho) - (1+\rho)\Big) = \frac{\rho(1+\rho)}{q(1-(1+\rho)\lambda)}\big(1+(1+\rho)(p-1)-1+(1+\rho)\lambda\big) = \\ = \frac{\rho(1+\rho)^2(p-1+\lambda)}{q\big(1-(1+\rho)\lambda\big)}.\end{aligned} \tag{3.36}$$

And we also rewrite the expression in the second brackets of (3.31):

$$\frac{\varphi}{q}(1-|R|^p)+(1+\varphi)|R|^p=\frac{\varphi+|R|^p}{q}+\frac{|R|^p(1+\varphi)}{p}=\frac{\varphi(1+R)}{q}+\frac{\varphi R(1+\varphi)}{p}=$$
$$=\frac{\varphi(1+R)}{q}+\frac{\varphi R(p-1)(1+R)}{p\big(1-(1+R)\lambda\big)}=\frac{\varphi(1+R)}{q}\Big(1+\frac{R}{1-(1+R)\lambda}\Big)=\frac{\varphi(1+R)^2(1-\lambda)}{q\big(1-(1+R)\lambda\big)}. \tag{3.37}$$

Combining (3.35), (3.36), and (3.37), we re-express (3.31) as

$$\Upsilon=-\frac{(p-1)(1+R)|R|^{p-2}\rho(1+\rho)^2(p-1+\lambda)}{q\big(1-(1+\rho)\lambda\big)}+$$
$$+\frac{\rho(1+\rho)}{R(1+R)}\cdot\frac{(p-1+\lambda)\big(1-(1+R)\lambda\big)-1+\lambda}{(p-1+\lambda)\big(1-(1+\rho)\lambda\big)-1+\lambda}\cdot\frac{(p-1)(1+\rho)}{\big(1-(1+\rho)\lambda\big)}\cdot\frac{\varphi(1+R)^2(1-\lambda)}{q\big(1-(1+R)\lambda\big)}=$$
$$=\frac{(p-1)(1+R)|R|^{p-2}\rho(1+\rho)^2}{q\big(1-(1+\rho)\lambda\big)}\times$$
$$\left\{-(p-1+\lambda)+\frac{(p-1+\lambda)\big(1-(1+R)\lambda\big)-1+\lambda}{(p-1+\lambda)\big(1-(1+\rho)\lambda\big)-1+\lambda}\cdot\frac{1-\lambda}{1-(1+R)\lambda}\right\}. \tag{3.38}$$

Note that λ attains its maximal value at the endpoints of $[-1,1]$. It equals $-\frac{p-2}{2}<0$ there, thus, $\lambda<0$. Consequently, the quantities $1-(1+R)\lambda$ and $1-(1+\rho)\lambda$ are non-negative. Thus, the first multiple in (3.38) is non-negative.

Consider the expression inside the braces in (3.38) now. Note that both denominators are positive. We have just discussed the second, as for the first, its sign coincides with the sign of $\lambda'(\rho)$ (see (3.34)), which is positive by Lemma 3.1. Multiplying the expression in the braces by the denominators, we get

$$\operatorname{sign}(\Upsilon)=\operatorname{sign}\Big\{\Big((p-1+\lambda)\big(1-(1+R)\lambda\big)-1+\lambda\Big)(1-\lambda)-$$
$$-(p-1+\lambda)\Big((p-1+\lambda)\big(1-(1+\rho)\lambda\big)-1+\lambda\Big)\big(1-(1+R)\lambda\big)\Big\}=$$
$$-\operatorname{sign}\Big\{(1-\lambda)^2-2(1-\lambda)(p-1+\lambda)\big(1-(1+R)\lambda\big)+$$
$$(p-1+\lambda)^2\big(1-(1+R)\lambda\big)\big(1-(1+\rho)\lambda\big)\Big\}=$$
$$-\operatorname{sign}\Big\{\Big((1-\lambda)-(p-1+\lambda)\big(1-(1+R)\lambda\big)\Big)^2+$$
$$(p-1+\lambda)^2\big(1-(1+R)\lambda\big)\Big[1-(1+\rho)\lambda-1+(1+R)\lambda\Big]\Big\}=$$

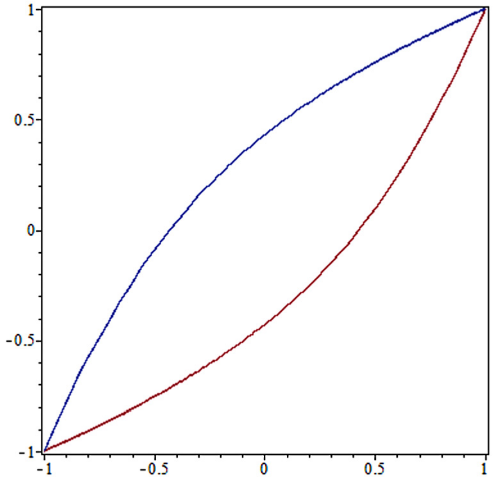

Fig. 3.4.1. The curves η_- and η_+ bound the T-image of the domain where $\mathbb{B}_{c,+}(x) = x_3^{\frac{1}{p}} x_4^{\frac{1}{q}}$.

$$- \operatorname{sign}\Big\{ \Big((1-\lambda) - (p-1+\lambda)\big(1-(1+R)\lambda\big)\Big)^2 + \\ (p-1+\lambda)^2\big(1-(1+R)\lambda\big)(R-\rho)\lambda \Big\} = -1,$$

since $\lambda < 0$ and $R < \rho$.

We proved that $\Upsilon < 0$ for $R \in (-1, R_0)$. Therefore, $\Delta < 0$, and $Y_2' > 0$. $\square$

3.4. The structure of the foliation

Proposition 3.14. *The interior of Ω_c is foliated by one and three dimensional extremals of $\mathbb{B}_{c,+}$. The three dimensional domains are parametrized by $t_3 > 0$, each such domain is the convex hull of the curve γ_{t_3}:*

$$\gamma_{t_3} : x_1 \mapsto (x_1, pt_3\varphi(x_1), |x_1|^p, |x_1|^p(pt_3)^q), \quad x_1 \in \mathbb{R}.$$

The T-image of each such linearity domain is the subdomain of $[-1,1]^2$ bounded by the curve $\eta_- = \eta(\cdot, R_0)$ and its symmetric with respect to $(0,0)$ image η_+ (see Figure 3.4.1).

The remaining part of the domain is covered by one dimensional extremals. The T-image of each such segment is either the curve $\eta(\cdot, R)$, $R \in (-1, R_0)$, or its symmetric with respect to $(0,0)$ image.

Lemma 3.15. *Let $t_3 > 0$. The function $\mathbb{B}_{c,+}$ is linear on the convex hull of γ_{t_3}, moreover, $\mathbb{B}_{c,+} = x_3^{\frac{1}{p}} x_4^{\frac{1}{q}}$ there.*

Let $a, b, c \in \partial_{\mathrm{skel}}\Omega_c$ be three distinct points such that their convex hull does not lie inside the topological boundary of Ω_c. If $\mathbb{B}_{c,+}$ coincides with a linear function $t_0 + t_1x_1 +$

$t_2x_2 + t_3x_3 + t_4x_4$ on the said convex hull, then $t_0 = t_1 = t_2 = 0$, $pt_3(qt_4)^{p-1} = 1$, and $a, b, c \in \gamma_{t_3}$.

Proof. Note that

$$\mathbb{B}_{c,+}(x) = x_1x_2 = x_3^{\frac{1}{p}} x_4^{\frac{1}{q}} = t_3x_3 + t_4x_4, \quad \text{where } t_4 \text{ is such that} \quad pt_3(qt_4)^{p-1} = 1,$$

for any $x \in \gamma_{t_3}$. What is more, for any $x \in \partial_{\mathrm{skel}}\Omega_c$, there is an inequality

$$\mathbb{B}_{c,+}(x) = x_1x_2 \leqslant \big|pt_3|x_1|^p\big|^{\frac{1}{p}} \big|qt_4|x_2|^q\big|^{\frac{1}{q}} \leqslant t_3|x_1|^p + t_4|x_2|^q = t_3x_3 + t_4x_4.$$

Thus, $\mathbb{B}_{c,+}(x) \leqslant t_3x_3 + t_4x_4$ for $x \in \Omega_c$. On the other hand, it follows from concavity of $\mathbb{B}_{c,+}$ that $\mathbb{B}_{c,+}(x) \geqslant t_3x_3+t_4x_4$ on $\mathrm{conv}(\gamma_{t_3})$. Therefore, $\mathbb{B}_{c,+}(x) = t_3x_3+t_4x_4 = x_3^{\frac{1}{p}} x_4^{\frac{1}{q}}$ on $\mathrm{conv}(\gamma_{t_3})$ since $x_4 = (pt_3)^q x_3$ there. The first assertion of the lemma is proved.

By Lemma 3.7, the parameters t_1, t_2, t_3, t_4 should fall under (3.13), (3.14), (3.15), and (3.16) for each of the pairs (a, b), (a, c), and (b, c). Consequently, the numbers a_1, b_1, and c_1 are distinct since a, b, c are distinct (if, say, $a_1 = b_1$, then the equations (3.13), (3.14), (3.15), and (3.16) for the pairs (a, c) and (b, c) lead to the conclusion $a_2 = b_2$). What is more, $t_3 > 0$ and $t_4 > 0$. The function H given in (3.9), satisfies $H(a_1) = H(b_1) = H(c_1) = H'(a_1) = H'(b_1) = H'(c_1) = 0$, and thus equals to zero (see the proof of Lemma 3.9). Therefore, $t_0 = t_1 = t_2 = 0$ and $pt_3(qt_4)^{p-1} = 1$. Further, $t_3|a_1|^p + t_4|a_2|^q = B(a) = a_1a_2$. This is the case of equality in Young's inequality, which leads to $a_2 = pt_3\varphi(a_1)$. This means $a \in \gamma_{t_3}$. Similarly, b and c also lie on the same curve. □

Remark 3.16. The convex hulls of the curves γ_{t_3}, $t_3 > 0$, are pairwise disjoint.

Lemma 3.17. *The image of the convex hull of γ_{t_3} under T is the region bounded by η_- and η_+.*

Proof. Let the T-image of $x \in \mathrm{conv}\,\gamma_{t_3}$ lie below the main diagonal of $[-1, 1]^2$. We know that for some $R \in [-1, 1]$, the curve $\eta(\cdot, R)$ passes through $T(x)$. Therefore, there exists a chord with the endpoints a and b, defined by (3.3), which contains x. Lemma 3.15 and Remark 3.16 lead to the inclusion $a, b \in \gamma_{t_3}$. Thus,

$$\frac{a_2}{\varphi(a_1)} = pt_3 = \frac{b_2}{\varphi(b_1)} = \frac{\varphi(R)a_2}{\varphi(\rho(R))\varphi(a_1)},$$

which leads to $\rho(R) = R$, which is $R \in [R_0, 1]$.

If $R \in [R_0, 1]$, then with the choice $a_2 = \varphi(a_1)pt_3$, the points a and b given by (3.3), lie on γ_{t_3}. Consequently, the chord that connects them lies inside $\mathrm{conv}(\gamma_{t_3})$. Therefore, the part of the set $T(\mathrm{conv}(\gamma_{t_3}))$ that lies below the diagonal of $[-1, 1]^2$, coincides with the set $\{\eta(\tau, R)\colon \tau \in [0, 1], R \in [R_0, 1]\}$. It remains to notice that the latter set is exactly the region bounded by η_- and the main diagonal (this follows from the bijectivity of T and the monotonicity of Y_2, see the proof of Lemma 3.12). $\square$

Proof of Proposition 3.14. Proposition 3.14 follows from Lemmas 3.15 and 3.17. $\square$

4. The computation of $c^{\star,\mathbb{R}}_{p,r}$

By Proposition 1.12 and Lemma 2.1,

$$(c^{\star,\mathbb{R}}_{p,r})^{-1} = \sup_{|x_1|^p < x_3} \frac{|\mathbb{B}_{c,+}(x_1, 0, x_3, x_4)|^r}{(x_3^{r/p} - |x_1|^r)x_4^{r/q}}.$$

Consider the segment $\ell_c(a_1, a_2, R)$ with the endpoints $a = (-1, -1, 1, 1)$ and $b = (\rho(R), \varphi(R), \rho(R)^p, |R|^p)$, defined by the parameter $R \in (-1, 1)$. We may restrict our attention to such segments only, due to homogeneity considerations. Note that if $R < 0$, then $\ell_c(a_1, a_2, R)$ does not contain points x such that $x_2 = 0$. If $R \geqslant 0$ such a point $\tau a + (1-\tau)b$ corresponds to the value $\tau = \frac{\varphi(R)}{1+\varphi(R)}$ and its coordinates are

$$x_1 = \frac{\rho(R) - \varphi(R)}{1 + \varphi(R)}, \quad x_2 = 0, \quad x_3 = \frac{\rho(R)^p + \varphi(R)}{1 + \varphi(R)}, \quad x_4 = \frac{R^p + \varphi(R)}{1 + \varphi(R)}. \tag{4.1}$$

The value of $\mathbb{B}_{c,+}$ at this point is

$$\mathbb{B}_{c,+}(\tau a + (1-\tau)b) = \frac{(\rho(R) + 1)\varphi(R)}{1 + \varphi(R)} \tag{4.2}$$

according to Theorem 3.3.

Thus, we need to maximize

$$M_r(R) = \frac{|\mathbb{B}_{c,+}(x_1, 0, x_3, x_4)|^r}{x_3^{r/p} - |x_1|^r} \cdot x_4^{-r/q} = \\ \left(\frac{(\rho(R)+1)\varphi(R)}{1+\varphi(R)}\right)^r \cdot \frac{(1+\varphi(R))^{r/q}}{(R^p + \varphi(R))^{r/q}\Big(\big(\frac{\rho(R)^p+\varphi(R)}{1+\varphi(R)}\big)^{r/p} - \big|\frac{\varphi(R)-\rho(R)}{1+\varphi(R)}\big|^r\Big)},$$

when $R \in [0, 1]$. We set $r = p$ and concentrate on the proof of Theorem 1.6 in the real-valued case. Let also $M = M_p$. Note that $\rho(R) \geqslant R \geqslant R^{p-1} = \varphi(R) \geqslant 0$. We slightly modify the expression $M(R)$. Consider the function

$$Q\colon (t, s) \mapsto \frac{(s^p + t)(1+t)^{p-1} - (s-t)^p}{t(1+s)^p}, \quad s \geqslant t > 0.$$

We extend it to the case $t = 0$ by continuity.

By an elementary computation,

$$\frac{1}{M(R)} = Q\big(R^{p-1}, \rho(R)\big) \cdot \left(\frac{1+R}{1+R^{p-1}}\right)^{p-1} \geqslant Q\big(R^{p-1}, \rho(R)\big).$$

Let us show that the latter expression attains its minimum at $R = 0$. Clearly,

$$\frac{1}{M(0)} = Q\big(0, \rho(0)\big).$$

We investigate Q. Fix $s > 0$ and consider the function $h_1(t) \stackrel{\text{def}}{=} (1+s)^p Q(t,s)$. By continuity, $h_1(0) = (p-1)s^p + 1 + ps^{p-1}$. We want to show $h_1(t) \geqslant h_1(0)$ when $t \geqslant 0$. This is equivalent to $h_2(t) \stackrel{\text{def}}{=} th_1(t) - th_1(0) \geqslant 0$. We differentiate h_2 and get $h_2'(0) = 0$ and

$$h_2''(t) = \frac{\partial^2}{\partial t^2}\Big(t(1+s)^p Q(t,s)\Big) = \frac{\partial^2}{\partial t^2}\Big((s^p + t)(1+t)^{p-1} - (s-t)^p\Big) = \\ 2(p-1)(1+t)^{p-2} + (p-1)(p-2)(s^p+t)(1+t)^{p-3} - p(p-1)(s-t)^{p-2}. \quad (4.3)$$

By Young's inequality,

$$\frac{2}{p}(1+t)^{p-2} + \frac{p-2}{p} s^p (1+t)^{p-3} \geqslant s^{\frac{p-2}{p}p}(1+t)^{\frac{2}{p}(p-2) + \frac{p-2}{p}(p-3)} \\ = s^{p-2}(1+t)^{\frac{(p-1)(p-2)}{p}} \geqslant s^{p-2} \geqslant (s-t)^{p-2}.$$

Taking into account (4.3), this leads to $h_2''(t) \geqslant 0$. Thus, the function h_2 is convex when $t \geqslant 0$ and satisfies the identities $h_2(0) = h_2'(0) = 0$. Consequently, $h_2(t) \geqslant 0$, $h_1(t) \geqslant h_1(0)$, and

$$Q(t,s) \geqslant Q(0,s) = \frac{(p-1)s^p + 1 + ps^{p-1}}{(1+s)^p}.$$

Let us find the minimum (with respect to s) of $Q(0,s)$. We differentiate:

$$\frac{\partial}{\partial s} Q(0,s) = \frac{p(p-1)s^{p-2}}{(1+s)^{p-1}} - p\frac{(p-1)s^p + 1 + ps^{p-1}}{(1+s)^{p+1}} = p\frac{(p-1)s^{p-2} + (p-2)s^{p-1} - 1}{(1+s)^{p+1}}.$$

There exists a unique positive point where the latter expression changes sign from negative to positive, call it s_0. Then s_0 satisfies (1.7), which might be further rewritten in terms of λ

$$(p-1)s_0^{p-2}(1+s_0) = 1 + s_0^{p-1};$$

$$(p-1) - \frac{p-1}{1+s_0^{p-1}} = \frac{(p-1)s_0^{p-1}}{1+s_0^{p-1}} = \frac{s_0}{1+s_0} = 1 - \frac{1}{1+s_0};$$

$$\lambda(s_0) = \frac{1}{1+s_0} - \frac{p-1}{1+s_0^{p-1}} = 2-p = \lambda(0).$$

In other words, $s_0 = \rho(0)$. Thus, for any $s \geqslant t \geqslant 0$ we have proved the chain of inequalities

$$Q(t,s) \geqslant Q(0,s) \geqslant Q(0,\rho(0)),$$

which leads to

$$\frac{1}{M(R)} \geqslant Q\big(R^{p-1}, \rho(R)\big) \geqslant Q(0,\rho(0)) = \frac{1}{M(0)}.$$

Therefore,

$$(c_{p,p}^{\star,\mathbb{R}})^{-1} = M(0) = \frac{1}{Q(0,s_0)} = \frac{(1+s_0)^p}{(p-1)s_0^p + ps_0^{p-1}+1} = \frac{(1+s_0)^p}{s_0 + s_0^p + s_0^{p-1}+1} = \frac{(1+s_0)^{p-1}}{1+s_0^{p-1}} = \frac{1}{(p-1)}\cdot\Big(\frac{1+s_0}{s_0}\Big)^{p-2},$$

which proves Theorem 1.6 for the case of real-valued functions.

5. Proof of Proposition 1.16

Let $\theta \in [2,\infty)$, let $p=\theta$, $q = \frac{p}{p-1}$; the case $\theta \in (1,2)$ is completely similar.

Definition 5.1. Consider the function $\mathbb{B}_{c,+}^{\mathbb{C}} \colon \mathbb{C}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ given by the formula

$$\mathbb{B}_{c,+}^{\mathbb{C}}(z_1,z_2,x_3,x_4) = \sup\Big\{ \operatorname{Re}(\langle f\bar g\rangle_I) \;\Big|\; \langle f\rangle_I = z_1, \langle g\rangle_I = z_2, \langle |f|^p\rangle_I = x_3, \langle |g|^q\rangle_I = x_4 \Big\}.$$

The natural domain for $\mathbb{B}_{c,+}^{\mathbb{C}}$ is

$$\Omega_{c,\mathbb{C}} = \Big\{ x = (z_1,z_2,x_3,x_4) \in \mathbb{C}^2 \times \mathbb{R}^2 \;\Big|\; x_3 \geqslant |z_1|^p, x_4 \geqslant |z_2|^q \Big\}.$$

Similarly to Lemma 2.1, the function $\mathbb{B}_{c,+}^{\mathbb{C}}$ is the minimal among concave functions $G\colon \Omega_{c,\mathbb{C}} \to \mathbb{R}$ that satisfy the boundary conditions

$$G(z_1,z_2,|z_1|^p,|z_2|^q) = \operatorname{Re}(z_1\bar z_2).$$

We also have complex homogeneity,

$$\mathbb{B}_{c,+}^{\mathbb{C}}(\zeta z_1,\zeta z_2,x_3,x_4) = \mathbb{B}_{c,+}^{\mathbb{C}}(z_1,z_2,x_3,x_4), \quad |\zeta| = 1.$$

Similarly to Proposition 1.12,

$$c^{\star}_{p,r} = \Bigg(\sup_{|z_1|\in(0,1)} \frac{\big(\mathbb{B}^{\mathbb{C}}_{c,+}(z_1,0,1,1)\big)^r}{1-|z_1|^r} \Bigg)^{-1} = \Bigg(\sup_{z_1\in(0,1)} \frac{\big(\mathbb{B}^{\mathbb{C}}_{c,+}(z_1,0,1,1)\big)^r}{1-z_1^r} \Bigg)^{-1},$$

the latter identity follows from the complex homogeneity above. We will prove the identity

$$\mathbb{B}^{\mathbb{C}}_{c,+}(x) = \mathbb{B}_{c,+}(x), \quad x \in \Omega_c.$$

The inequality $\mathbb{B}_{c,+}(x) \leqslant \mathbb{B}^{\mathbb{C}}_{c,+}(x)$ is evident. For the reverse inequality, it suffices to show that the function

$$(z_1, z_2, x_3, x_4) \mapsto \mathbb{B}_{c,+}(\mathrm{Re}(z_1), \mathrm{Re}(z_2), x_3, x_4)$$

majorizes $\mathbb{B}^{\mathbb{C}}_{c,+}$ on $\Omega_{c,\mathbb{C}}$. The said function is concave, so, it suffices to verify the inequality on $\partial_{\mathrm{skel}}\Omega_{c,\mathbb{C}}$:

$$\mathbb{B}_{c,+}(\mathrm{Re}(z_1), \mathrm{Re}(z_2), |z_1|^p, |z_2|^q) \geqslant \mathrm{Re}(z_1\bar{z}_2), \quad z_1, z_2 \in \mathbb{C}.$$

The maximal value of $\mathrm{Re}(z_1\bar{z}_2)$ provided $x_1 = \mathrm{Re}(z_1)$, $x_2 = \mathrm{Re}(z_2)$, $x_3 = |z_1|^p$, and $x_4 = |z_2|^q$ are fixed, is $x_1x_2 + \sqrt{\big(x_3^{\frac{2}{p}} - x_1^2\big)\big(x_4^{\frac{2}{q}} - x_2^2\big)}$. Thus, it suffices to prove

$$\mathbb{B}_{c,+}(x_1, x_2, x_3, x_4) \geqslant x_1x_2 + \sqrt{\big(x_3^{\frac{2}{p}} - x_1^2\big)\big(x_4^{\frac{2}{q}} - x_2^2\big)}. \tag{5.1}$$

We invoke Theorem 3.3 and verify this inequality on each chord $\ell_c(a_1, a_2, R)$ individually.

Pick some $R \in [-1, 1]$ and consider the chord $\ell_c(a_1, a_2, R)$ with the endpoints $a = (-1, -1, 1, 1)$ and $b = (\rho(R), \varphi(R), \rho(R)^p, |R|^p)$ (by homogeneity, we may consider such chords only). We also pick a point $x = \tau a + (1-\tau)b$ on $\ell_c(a_1, a_2, R)$, here $\tau \in (0, 1)$. If $R \in [R_0, 1]$, then $\rho = R$. Therefore, by Theorem 3.3, $\mathbb{B}_{c,+}(x) = x_3^{\frac{1}{p}} x_4^{\frac{1}{q}}$, and the inequality (5.1) is simple (this is nothing more than $|z_1z_2| \geqslant \mathrm{Re}(z_1\bar{z}_2)$). So, we assume $R \in [-1, R_0]$ in what follows.

By Theorem 3.3, $\mathbb{B}_{c,+}(x) = \tau + (1-\tau)\rho(R)\varphi(R)$. The coordinates of x are

$$\begin{aligned} x_1 &= -\tau + (1-\tau)\rho, \\ x_2 &= -\tau + (1-\tau)\varphi, \\ x_3 &= \tau + (1-\tau)\rho^p, \\ x_4 &= \tau + (1-\tau)|R|^p. \end{aligned}$$

Thus, we may represent (5.1) as

$$\begin{aligned} \tau + (1-\tau)\rho\varphi \geqslant & \big((1-\tau)\rho - \tau\big)\big((1-\tau)\varphi - \tau\big) + \\ & \sqrt{(\tau + (1-\tau)\rho^p)^{\frac{2}{p}} - ((1-\tau)\rho - \tau)^2} \times \\ & \sqrt{(\tau + (1-\tau)|R|^p)^{\frac{2}{q}} - ((1-\tau)\varphi - \tau)^2}, \end{aligned}$$

which might be further rewritten as

$$\tau^2(1-\tau)^2(1+\rho)^2(1+\varphi)^2 \geqslant \Big[(\tau+(1-\tau)\rho^p)^{\frac{2}{p}} - ((1-\tau)\rho-\tau)^2\Big] \times \Big[(\tau+(1-\tau)|R|^p)^{\frac{2}{q}} - ((1-\tau)\varphi-\tau)^2\Big].$$

This is equivalent to

$$S_1(R,\tau)S_2(R,\tau) \leqslant \tau^2(1-\tau)^2, \tag{5.2}$$

where

$$S_1(R,\tau) = \frac{(\tau+(1-\tau)\rho(R)^p)^{\frac{2}{p}} - ((1-\tau)\rho(R)-\tau)^2}{(1+\rho(R))^2},$$

$$S_2(R,\tau) = \frac{(\tau+(1-\tau)|R|^p)^{\frac{2}{q}} - ((1-\tau)\varphi(R)-\tau)^2}{(1+\varphi(R))^2}.$$

Since $\rho(R) \in [R_0, 1]$, we have

$$S_1(\rho(R),\tau)S_2(\rho(R),\tau) \leqslant \tau^2(1-\tau)^2$$

because this inequality is equivalent to (5.1) with $\mathbb{B}_{c,+}(x_1,x_2,x_3,x_4)$ replaced with $x_3^{\frac{1}{p}} x_4^{\frac{1}{q}}$. Moreover, $S_1(R,\tau) = S_1(\rho(R),\tau)$ since $\rho(\rho(R)) = \rho(R)$. Therefore, it suffices to show that $S_2(R,\tau) \leqslant S_2(\rho(R),\tau)$ when $R \in (-1, R_0]$ (note that both S_1 and S_2 are non-negative).

Consider the function $S(R,\tau) = S_2(R,\tau) - S_2(\rho(R),\tau)$. Note that $S_2(R,0) = S_2(\rho(R),0) = S_2(R,1) = S_2(\rho(R),1) = 0$, which leads to $S(R,0) = S(R,1) = 0$. Let us show that the function $S(R,\cdot)$ is convex on $[0,1]$. We compute its second derivative:

$$\frac{q^2}{2(2-q)}\frac{\partial^2}{\partial\tau^2}S(R,\tau) = \Big(\frac{1-|R|^p}{1+\varphi(R)}\Big)^2(\tau+(1-\tau)|R|^p)^{\frac{2}{q}-2} - \Big(\frac{1-\rho(R)^p}{1+\rho(R)^{p-1}}\Big)^2(\tau+(1-\tau)\rho(R)^p)^{\frac{2}{q}-2}.$$

Thus, convexity of $S(R,\cdot)$ on $[0,1]$ is equivalent to

$$\Big(\frac{1+\varphi(R)}{1-|R|^p}\Big)^p(\tau+(1-\tau)|R|^p) \leqslant \Big(\frac{1+\rho(R)^{p-1}}{1-\rho(R)^p}\Big)^p(\tau+(1-\tau)\rho(R)^p).$$

This inequality is linear with respect to τ. We prove it at the endpoints $\tau = 0$ and $\tau = 1$. At these points, it turns into:

$$\Big(\frac{1+\varphi(R)}{1-|R|^p}\Big)|R| \leqslant \Big(\frac{1+\rho(R)^{p-1}}{1-\rho(R)^p}\Big)\rho(R), \tag{5.3}$$

$$\frac{1+\varphi(R)}{1-|R|^p} \leqslant \frac{1+\rho(R)^{p-1}}{1-\rho(R)^p}. \tag{5.4}$$

First, we will show (5.4). Second, we will justify $\rho(R) \geqslant |R|$. Then, inequality (5.3) will follow from (5.4).

The function $u\colon t \mapsto \frac{1+t|t|^{p-2}}{1-|t|^p}$ is increasing on $(-1,1)$:

$$u'(t) = |t|^{p-2}\frac{(p-1)+pt+|t|^p}{(1-|t|^p)^2} \geqslant 0.$$

The inequality $u(R) \leqslant u(\rho(R))$ is exactly (5.4). It remains to verify that $\rho(R) \geqslant |R|$.

If $R \geqslant 0$, then $R \leqslant R_0 \leqslant \rho(R)$. So, we consider the case $R \in [-1,0]$ only.

Lemma 5.2. *For any $t \geqslant 0$, we have $\lambda(-t) \geqslant \lambda(t)$.*

Proof. Using the definition of λ, we rewrite this as

$$\frac{1}{1-t} - \frac{1}{1+t} = \frac{2t}{1-t^2} \geqslant \frac{2(p-1)t^{p-1}}{1-t^{2(p-1)}} = \frac{(p-1)}{1-t^{p-1}} - \frac{(p-1)}{1+t^{p-1}},$$

which, in its turn, is equivalent to

$$1 - t^{2(p-1)} - (p-1)t^{p-2}(1-t^2) \geqslant 0.$$

The left hand-side vanishes at $t=1$ and decreases on $t \in (0,1)$; here is the derivative of the left hand-side:

$$\begin{aligned} &-2(p-1)t^{2p-3} - (p-1)(p-2)t^{p-3} + (p-1)pt^{p-1} \\ &= -(p-1)t^{p-3}(2t^p - pt^2 + (p-2)) \leqslant 0. \quad \square \end{aligned}$$

Thus, we have established $\lambda(-\rho(R)) \geqslant \lambda(\rho(R)) = \lambda(R)$ when $R \in (-1,0)$. By Lemma 3.1, the function λ is decreasing on $[-1,R_0]$, consequently, $-\rho(R) \leqslant R$. Therefore, $|R| \leqslant \rho(R)$. The inequality (5.4) is proved, and (5.3) follows from it.

6. The computation of $\mathbb{B}_{d,+}$

By Corollary 2.5, the union of the graphs of $\mathbb{B}_{d,+}$ and $\mathbb{B}_{d,-}$ coincides with the boundary of $\mathbb{K}$. By Corollary 3.4, the said boundary consists of the segments $\mathfrak{L}(a_1,a_2,R)$ and two additional sets. Clearly, a point $(x_1, \mathbb{B}_{d,\pm}(x), x_3, x_4, x_5)$, $x \in \operatorname{int}\Omega_d$, does not belong to any of these exceptional sets. Thus, each of the graphs of $\mathbb{B}_{d,+}$ and $\mathbb{B}_{d,-}$ on the interior of Ω_d consists of the segments $\ell_d(a_1,a_2,R)$. To prove Theorem 3.5 for real-valued functions, it suffices to show that a segment $\ell_d(a_1, a_2, R)$ cannot lie on the graph of $\mathbb{B}_{d,+}$ if $a_2 < 0$: then, similarly, the segments $\ell_d(a_1, a_2, r)$ with $a_2 > 0$ do not lie on the graph of $\mathbb{B}_{d,-}$, thus, they foliate the graph of $\mathbb{B}_{d,+}$.

Assume the contrary: let $a_2 < 0$ and let the segment $\ell_d(a_1, a_2, R)$ lie on the graph of $\mathbb{B}_{d,+}$. Then, there exist t_0, t_1, t_3, t_4, t_5 such that the subgraph of the affine function $x \mapsto t_0 + t_1x_1 + t_3x_3 + t_4x_4 + t_5x_5$ contains $\mathbb{K}$ and its graph contains the endpoints of $\ell_d(a_1, a_2, R)$. In other words, the inequality

$$\Phi_2(x_1, z) := t_0 + t_1x_1 + t_3|x_1|^p + t_4|z|^q + t_5x_1z - z \geqslant 0 \tag{6.1}$$

holds for any $x_1, z \in \mathbb{R}$; moreover, (6.1) turns into equality at the points $(x_1, z) = (a_1, a_2)$ and $(x_1, z) = (-\rho(R)a_1, -\varphi(R)a_2)$. The derivative of Φ_2 with respect to z vanishes at the points $(x_1, z) = (a_1, a_2)$ and $(x_1, z) = (-\rho(R)a_1, -\varphi(R)a_2)$. Therefore, we have a system of equations

$$t_5a_1 + qt_4a_2|a_2|^{q-2} - 1 = 0, \qquad -t_5\rho(R)a_1 - qt_4a_2|a_2|^{q-2}\varphi(R)|\varphi(R)|^{q-2} - 1 = 0.$$

Note that this system does not have solutions when $R \in [R_0, 1]$. Using the identity $\varphi(R)|\varphi(R)|^{q-2} = R$, we solve this equations for t_4 and t_5:

$$t_5 = -\frac{1 + R}{(\rho(R) - R)a_1}, \qquad qt_4 = \frac{(1 + \rho(R))}{(\rho(R) - R)a_2|a_2|^{q-2}}.$$

The function ρ is positive, moreover, for $R \in [-1, R_0)$, we have $\rho(R) > R$. Therefore, the sign of qt_4 coincides with the sign of a_2. Thus, if $a_2 < 0$, then $t_4 < 0$. This contradicts (6.1) for sufficiently large $|z|$.

7. The computation of $d^{\star,\mathbb{R}}_{p,p}$

By Proposition 1.15,

$$d^{\star,\mathbb{R}}_{p,p} = \left(\sup_{x_1 \in (-1,1)} \frac{\mathbb{B}^p_{d,+}(x_1, 1, 1, 0)}{1 - |x_1|^q} \right)^{-1}. \tag{7.1}$$

Consider the segment $\ell_d(a_1, a_2, R)$ with $a_2 > 0$, $a_1 \neq 0$ and $R \in [-1, R_0)$ (the point $(x_1, 1, 1, 0)$ lies in the interior of Ω_d). We find the point x on this segments such that $x_5 = 0$:

$$(\tau + (1 - \tau)\rho(R)\varphi(R))a_1a_2 = 0 \quad \Longleftrightarrow \quad \tau = \frac{-\rho(R)\varphi(R)}{1 - \rho(R)\varphi(R)}, \quad 1 - \tau = \frac{1}{1 - \rho(R)\varphi(R)}.$$

The real τ belongs to $[0, 1]$, which is equivalent to $R \in [-1, 0]$. Here are all the other coordinates of x as well as the value of $\mathbb{B}_{d,+}$ at x:

$$x_1 = (\tau - (1-\tau)\rho)a_1 = -\frac{\rho(1+\varphi)}{1-\rho\varphi}a_1 \tag{7.2}$$

$$x_3 = (\tau + (1-\tau)\rho^p)|a_1|^p = \frac{\rho^p - \rho\varphi}{1-\rho\varphi}|a_1|^p, \tag{7.3}$$

$$x_4 = (\tau + (1-\tau)|R|^p)a_2^q = \frac{|R|^p - \rho\varphi}{1-\rho\varphi}a_2^q, \tag{7.4}$$

$$\mathbb{B}_{d,+}(x) = (\tau - (1-\tau)\varphi)a_2 = -\frac{\varphi(1+\rho)}{1-\rho\varphi}a_2. \tag{7.5}$$

Choosing appropriate a_1 and a_2 to get $x_3 = x_4 = 1$, and plugging this back into (7.1), we see that $(d_{p,p}^{\star,\mathbb{R}})^{-1}$ coincides with the maximal value of the function $\tilde{S}$ given by the rule

$$\tilde{S}(R) = \frac{|R|^{p-1}(1+\rho)^p(\rho^{p-1}+|R|^{p-1})^{q-1}}{(|R|+\rho)^{p-1}\Big((1+\rho|R|^{p-1})(\rho^{p-1}+|R|^{p-1})^{q-1} - \rho(1-|R|^{p-1})^q\Big)}. \tag{7.6}$$

Lemma 7.1. *The function* $\tilde{S}\colon [-1,0] \to \mathbb{R}$ *attains its maximal value at zero.*

Proof. Consider the function S:

$$S(u,t) = \frac{(1+ut)(u+t^{p-1}) - t(1-u)^q(u+t^{p-1})^{2-q}}{u(1+t)^p}, \quad (u,t) \in [0,1]^2.$$

We will show $S(u,t) \geqslant S(0,\rho(0))$ for $u \in [0,1]$ and $t \in [\rho(0),1]$. The monotonicity wanted follows from this inequality:

$$\tilde{S}(R)^{-1} = S(|R|^{p-1},\rho)\cdot\frac{(\rho+|R|)^{p-1}}{\rho^{p-1}+|R|^{p-1}} \geqslant S(|R|^{p-1},\rho) \geqslant S(0,\rho(0)) = \tilde{S}(0)^{-1}.$$

Let us first show that $\frac{\partial S}{\partial u} \geqslant 0$. We compute this derivative:

$$(1+t)^p u^2 \frac{\partial S}{\partial u} = u^2\frac{\partial}{\partial u}\Big(ut + t^p + 1 + \frac{t^{p-1}}{u} - t\frac{(1-u)^q}{u}(u+t^{p-1})^{2-q}\Big) =$$

$$t(u^2 - t^{p-2}) + t(1-u)^{q-1}(u+t^{p-1})^{1-q}\Big(qu(u+t^{p-1}) - (2-q)u(1-u) + (1-u)(u+t^{p-1})\Big) =$$

$$t(u^2 - t^{p-2}) + t\Big(\frac{1-u}{u+t^{p-1}}\Big)^{q-1}(u^2 + u(q-1)(1+t^{p-1}) + t^{p-1}).$$

We make the change of variable $v = t^{p-1}$. The positivity of the derivative of S with respect to u is equivalent to the positivity of

$$F(u,v) = (1-u)^{q-1}(u^2 + u(q-1)(1+v) + v) - (u+v)^{q-1}(v^{2-q} - u^2).$$

We compute $\frac{\partial^2 F}{\partial v^2}$:

$$\begin{aligned}\frac{\partial^2 F}{\partial v^2} &= (2-q)(q-1)(u+v)^{q-3}(v^{2-q}-u^2) - 2(q-1)(2-q)(u+v)^{q-2}v^{1-q} \\ &\quad + (2-q)(q-1)(u+v)^{q-1}v^{-q} \\ &= (2-q)(q-1)(u+v)^{q-3}[v^{2-q}-u^2-2(u+v)v^{1-q}+(u+v)^2v^{-q}] \\ &= (2-q)(q-1)(u+v)^{q-3}u^2(v^{-q}-1) \geqslant 0.\end{aligned}$$

Therefore, F is convex with respect to v. We continue the computations:

$$\frac{\partial F}{\partial v}(u,1) = (1-u)^{q-1}(1+(q-1)u) - (q-1)(1+u)^{q-2}(1-u^2) - (2-q)(1+u)^{q-1} = \\ u(q-1)((1+u)^{q-1}+(1-u)^{q-1}) - ((1+u)^{q-1}-(1-u)^{q-1}).$$

Further,

$$\begin{aligned}\frac{1}{q-1}\frac{\partial^2 F}{\partial v \partial u}(u,1) &= ((1+u)^{q-1}+(1-u)^{q-1}) + u(q-1)((1+u)^{q-2}-(1-u)^{q-2}) \\ &\quad - ((1+u)^{q-2}+(1-u)^{q-2}) \\ &= uq((1+u)^{q-2}-(1-u)^{q-2}) \leqslant 0,\end{aligned}$$

which leads to

$$\frac{\partial F}{\partial v}(u,1) \leqslant \frac{\partial F}{\partial v}(0,1) = 0.$$

Since F is convex with respect to v, we have $\frac{\partial F}{\partial v} \leqslant 0$. Consequently,

$$F(u,v) \geqslant F(u,1) = (1-u)^{q-1}(1+2(q-1)u+u^2) - (1+u)^{q-1}(1-u^2) = \\ (1-u)\Big[2qu(1-u)^{q-2}+(1-u)^q-(1+u)^q\Big].$$

The expression in the brackets is non-negative since it is equal to zero at $u=0$ and does not decrease with respect to u:

$$\begin{aligned}&\frac{1}{q}\frac{\partial}{\partial u}\Big[2qu(1-u)^{q-2}+(1-u)^q-(1+u)^q\Big] \\ &\quad = 2(1-u)^{q-2} + 2u(2-q)(1-u)^{q-3} - (1-u)^{q-1} - (1+u)^{q-1} \\ &\quad = (1+u)\Big((1-u)^{q-2}-(1+u)^{q-2}\Big) + 2u(2-q)(1-u)^{q-3} \geqslant 0.\end{aligned}$$

Thus, we have proved $F(u,v) \geqslant 0$. This leads to the inequality $\frac{\partial S}{\partial u} \geqslant 0$. Consequently,

$$S(u,t) \geqslant S(0,t) = (1+t)^{-p}(q-1+t^p+qt^{p-1}).$$

We compute the derivative of S with respect to t:

$$(1+t)^{p+1}\frac{\partial S}{\partial t}(0,t) = -p(q-1+t^p+qt^{p-1}) + p(1+t)(t^{p-1}+t^{p-2})$$
$$= p(t^{p-2}+(2-q)t^{p-1}-(q-1)).$$

This expression increases and equals to zero at $t=\rho(0)$ (see (1.7)). Therefore, $S(0,t) \geqslant S(0,\rho(0))$ when $t \geqslant \rho(0)$. This means we have proved that for any $u \in [0,1]$ and any $t \in [\rho(0),1]$ we have $S(u,t) \geqslant S(0,\rho(0))$. $\square$

Lemma 7.1 and the considerations before it lead to the proof of Theorem 1.7 in the real-valued case (recall that $s_0 = \rho(0)$):

$$d^{\star,\mathbb{R}}_{p,p} = \tilde{S}(0)^{-1} = S(0,s_0) = \Big(\frac{s_0}{1+s_0}\Big)^{p-2}.$$

8. Proof of Proposition 1.17

We consider the case $\theta > 2$, the case $\theta < 2$ is similar. We use the notation $p=\theta$ and $q=\frac{p}{p-1}$. Consider yet another Bellman function

$$\mathbb{B}^{\mathbb{C}}_{d,+}(z_1,x_3,x_4,z_5) = \sup\Big\{\operatorname{Re}(\langle g\rangle_{_I}) \,\Big|\, \langle f\rangle_{_I} = z_1, \langle |f|^p\rangle_{_I} = x_3, \langle |g|^q\rangle_{_I} = x_4, \langle f\bar{g}\rangle_{_I} = z_5\Big\}.$$

The natural domain of $\mathbb{B}^{\mathbb{C}}_{d,+}$ is

$$\Omega_{d,\mathbb{C}} = \Big\{(z_1,x_3,x_4,z_5) \in \mathbb{C}\times\mathbb{R}^2\times\mathbb{C} \,\Big|\, 0 \leqslant x_4,\ |z_1|^p \leqslant x_3,\ |x_5| \leqslant x_3^{1/p}x_4^{1/q}\Big\}.$$

As usual, $\mathbb{B}^{\mathbb{C}}_{d,+}$ is minimal among concave functions on $\Omega_{d,\mathbb{C}}$ that satisfy the boundary conditions:

$$\mathbb{B}^{\mathbb{C}}_{d,+}(z_1,|z_1|^p,|z_2|^q,z_1z_2) = \operatorname{Re} z_2, \quad z_1 \neq 0,$$
$$\mathbb{B}^{\mathbb{C}}_{d,+}(0,0,x_4,0) = x_4^{\frac{1}{q}}, \quad x_4 \geqslant 0.$$

Similar to Proposition 1.15,

$$d^{\star}_{p,r} = \bigg(\sup_{|z_1|\in(0,1)} \frac{(\mathbb{B}^{\mathbb{C}}_{d,+}(z_1,1,1,0))^r}{1-|z_1|^{\frac{qr}{p}}}\bigg)^{-1} = \bigg(\sup_{x_1\in(0,1)} \frac{(\mathbb{B}^{\mathbb{C}}_{d,+}(x_1,1,1,0))^r}{1-x_1^{\frac{qr}{p}}}\bigg)^{-1},$$

the latter identity follows from homogeneity. Similar to Section 5, we will prove that

$$\mathbb{B}^{\mathbb{C}}_{d,+}(x) = \mathbb{B}_{d,+}(x), \quad x \in \Omega_d.$$

The inequality $\mathbb{B}^{\mathbb{C}}_{d,+}(x) \geqslant \mathbb{B}_{d,+}(x)$ is evident. Thus, it suffices to show that the function

$$(z_1, x_3, x_4, z_5) \mapsto \mathbb{B}_{d,+}(\mathrm{Re}(z_1), x_3, x_4, \mathrm{Re}(z_5))$$

majorizes $\mathbb{B}^{\mathbb{C}}_{d,+}$ on $\Omega_{d,\mathbb{C}}$. The said function is concave, which allows to verify the majorization property on the skeleton of $\Omega_{d,\mathbb{C}}$ only. This can be rewritten as

$$\mathrm{Re}(z_5/z_1) \leqslant \mathbb{B}_{d,+}(\mathrm{Re}(z_1), |z_1|^p, |z_5|^q|z_1|^{-q}, \mathrm{Re}(z_5)), \qquad z_1 \neq 0.$$

For $\mathrm{Re}(z_1) = x_1, \mathrm{Re}(z_5) = x_5$, $|z_1|$, and $|z_5|$ fixed, the expression $\mathrm{Re}(z_5/z_1)$ attains the maximal value

$$\frac{x_1x_5 + \sqrt{(|z_1|^2 - x_1^2)(|z_5|^2 - x_5^2)}}{|z_1|^2}.$$

Thus, it remains to prove

$$\mathbb{B}_{d,+}(x_1, x_3, x_4, x_5) \geqslant \frac{x_1x_5 + \sqrt{(x_3^{2/p} - x_1^2)(x_4^{2/q}x_3^{2/p} - x_5^2)}}{x_3^{2/p}}, \quad x_3 \neq 0. \tag{8.1}$$

Lemma 8.1. *For any $x \in \Omega_d$*

$$(\mathbb{B}_{d,\pm}(x)x_3^{2/p} - x_1x_5)^2 \geqslant (x_3^{2/p} - x_1^2)(x_4^{2/q}x_3^{2/p} - x_5^2). \tag{8.2}$$

Proof. Let us temporarily use the notation $b = \mathbb{B}_{d,\pm}$. With this notation, (8.2) turns into

$$b^2x_3^{2/p} - 2x_1x_5b \geqslant x_4^{2/q}x_3^{2/p} - x_4^{2/q}x_1^2 - x_5^2,$$

which is equivalent to

$$(x_5 - x_1b)^2 \geqslant (x_3^{2/p} - x_1^2)(x_4^{2/q} - b^2). \tag{8.3}$$

The inequality (5.1), together with (3.5), leads to

$$\mathbb{B}_{c,-}(x_1, x_2, x_3, x_4) \leqslant x_1x_2 - \sqrt{(x_3^{2/p} - x_1^2)(x_4^{2/q} - x_2^2)}.$$

Thus, the inequality

$$(x_5 - x_1x_2)^2 \geqslant (x_3^{2/p} - x_1^2)(x_4^{2/q} - x_2^2)$$

holds for any point $x = (x_1, x_2, x_3, x_4, x_5)$ that lies on the union of the graphs of $\mathbb{B}_{c,+}$ and $\mathbb{B}_{c,-}$, which is the same as (8.3). □

Let us consider two functions

$$G^-(x) = \frac{x_1x_5 - \sqrt{(x_3^{2/p} - x_1^2)(x_4^{2/q}x_3^{2/p} - x_5^2)}}{x_3^{2/p}},$$

$$G^+(x) = \frac{x_1x_5 + \sqrt{(x_3^{2/p} - x_1^2)(x_4^{2/q}x_3^{2/p} - x_5^2)}}{x_3^{2/p}}$$

defined on the interior of Ω_d. The inequality (8.1) follows from $\mathbb{B}_{d,-} \leqslant G^- < G^+ \leqslant \mathbb{B}_{d,+}$. Lemma 8.1 says that $\mathbb{B}_{d,\pm}(x) \notin (G^-(x), G^+(x))$ for any $x \in \Omega_d$ such that $x_3 \neq 0$. The interior of Ω_d is connected, the functions $\mathbb{B}_{d,\pm}$ and $G^\pm$ are continuous on it, and moreover, $G^+ > G^-$. Therefore, either $\mathbb{B}_{d,+} \geqslant G^+$, or $\mathbb{B}_{d,+} \leqslant G^-$ everywhere. It remains to notice

$$\mathbb{B}_{d,+}(0,0,1,1) \geqslant \mathbb{B}_{d,-}(0,0,1,1) = -\mathbb{B}_{d,+}(0,0,1,1),$$

which shows, $\mathbb{B}_{d,+}(0,0,1,1) \geqslant 0 > G^-(0,0,1,1)$. Thus, $\mathbb{B}^+_{d,+} \geqslant G^+$ and (8.1) is proved.

9. The computation of $c^\star_{q,r}$ and $d^\star_{q,r}$ for $q < 2$

9.1. The computation of $c^\star_{q,r}$

Similar to Proposition 1.12,

$$c^{\star,\mathbb{R}}_{q,r} = \left(\sup_{x_2\in(-1,1)} \frac{\mathbb{B}^r_{c,+}(0,x_2,1,1)}{1-|x_2|^r}\right)^{-1} = \left(\sup_{|x_2|^q<x_4} \frac{\mathbb{B}^r_{c,+}(0,x_2,x_3,x_4)}{(x_4^{r/q}-|x_2|^r)x_3^{r/p}}\right)^{-1}. \tag{9.1}$$

On each segment $\ell_c(a_1,a_2,R)$, we find a point $x = \tau a + (1-\tau)b$ with $x_1 = 0$. In other words, $(\tau - (1-\tau)\rho) = 0$, i.e. $\tau = \frac{\rho}{1+\rho}$. Here are the other coordinates of x and the value of $\mathbb{B}_{c,+}$ there:

$$x_2 = a_2\frac{\rho-\varphi}{1+\rho}, \qquad x_3 = |a_1|^p\frac{\rho+\rho^p}{1+\rho}$$

$$x_4 = |a_2|^q\frac{\rho+|R|^p}{1+\rho}, \qquad \mathbb{B}_{c,+}(x) = a_1a_2\frac{\rho+\rho\varphi}{1+\rho}.$$

We plug these values back into (9.1):

$$\frac{1}{c^{\star,\mathbb{R}}_{q,r}} = \sup_{R\in(-1,1)} \frac{\rho^r(1+\varphi)^r}{(1+\rho)^{r/q}(\rho+\rho^p)^{r/p}\left[\left(\frac{\rho+|R|^p}{1+\rho}\right)^{r/q} - \left|\frac{\rho-\varphi}{1+\rho}\right|^r\right]}.$$

By Lemma 1.5, the latter supremum equals $+\infty$ provided $r < 2$.

We claim without proof that the point $R = -1$ is the global maximum provided $r = 2$. The limiting value at this point is $(p-1)$. Thus, $c^{\star,\mathbb{R}}_{q,2} = (q-1)$.

We also claim that 0 is the absolute maximum when $r = p$. In this case,

$$c^{\star,\mathbb{R}}_{q,p} = \frac{1+\rho(0)^{p-1}}{1+\rho(0)}.$$

9.2. Proof of Theorem 1.8

Proposition 9.1. *For any $p > 2$ we have $d^{\star}_{p,2(p-1)} = 1$.*

Proof. Let us first prove that $d^{\star,\mathbb{R}}_{p,2(p-1)} = 1$. In other words, we are going to prove the inequality

$$\big|\langle \mathbf{N}_p(f), e\rangle\big|^2 + \inf_{\alpha} \|f+\alpha e\|^{2p-2}_{L^p} \leqslant \|f\|^{2p-2}_{L^p}$$

for real-valued functions f and e, with the assumption $\|e\|_{L^p} = 1$. Let the infimum be attained at $\alpha = \alpha^\star$.

We consider the function $F\colon \mathbb{R} \to \mathbb{R}$ given by the rule

$$F(t) = \|f+te\|^{2p-2}_{L^p} - \big|\langle \mathbf{N}_p(f+te), e\rangle\big|^2 \tag{9.2}$$

and compute its derivative:

$$F'(t) = 2(p-1)\langle \mathbf{N}_p(f+te), e\rangle\Big(\|f+te\|^{p-2}_{L^p} - \int |f+te|^{p-2}e^2\Big).$$

The expression in the parentheses is non-negative by Hölder's inequality. Moreover, the function

$$t \mapsto \langle \mathbf{N}_p(f+te), e\rangle$$

has positive derivative $(p-1)\int |f+te|^{p-2}e^2$, in particular, this function has no more than one root. Thus, the value $t = \alpha^\star$ is its unique root. The function F attains its minimal value at $t = \alpha^\star$. What is more,

$$F(\alpha^\star) = \|f+\alpha^\star e\|^{2p-2}_{L^p},$$

which makes the inequality (9.2) equivalent to $F(0) \geqslant F(\alpha^\star)$.

We return to the complex-valued case. It suffices to prove a slightly weaker inequality

$$\big|\operatorname{Re}\langle \mathbf{N}_p(f), e\rangle\big|^2 + \inf_{\alpha} \|f+\alpha e\|^{2p-2}_{L^p} \leqslant \|f\|^{2p-2}_{L^p}, \quad \|e\|_{L^p} = 1.$$

The inequality (1.6) (with $r = 2(p-1)$ and $d_{p,r} = 1$) follows if one multiplies e by suitable scalar to make the scalar product $\langle \mathbf{N}_p(f), e\rangle$ real. We slightly strengthen our inequality:

$$\big|\operatorname{Re}\langle \mathbf{N}_p(f), e\rangle\big|^2 + \inf_{t\in\mathbb{R}} \|f+te\|_{L^p}^{2p-2} \leqslant \|f\|_{L^p}^{2p-2}, \quad \|e\|_{L^p} = 1.$$

This inequality can be proved with the help of a modified functions F,

$$F(t) = \|f+te\|_{L^p}^{2p-2} - \big|\operatorname{Re}\langle \mathbf{N}_p(f+te), e\rangle\big|^2,$$

exactly the same way as (9.2). □

Lemma 9.2. *For dual exponents p and q, the identities $d^\star_{p,r(p-1)} = 1$ and $d^\star_{q,r} = 1$ are equivalent.*

Proof. By the Hahn–Banach theorem, we have $d^\star_{p,r(p-1)} = 1$ if and only if the inequality

$$\big|\langle \mathbf{N}_p(f), e\rangle\big|^r + \big|\langle f, g\rangle\big|^{r(p-1)} \leqslant \|f\|_{L^p}^{r(p-1)}, \qquad \|e\|_{L^p} = \|g\|_{L^q} = 1, \quad \langle e, g\rangle = 0$$

holds true. We introduce a new function $F = \mathbf{N}_p(f)$ and restate this as

$$\big|\langle F, e\rangle\big|^r + \big|\langle \mathbf{N}_q(F), g\rangle\big|^{r(p-1)} \leqslant \|F\|_{L^q}^{r}, \qquad \|e\|_{L^p} = \|g\|_{L^q} = 1, \quad \langle e, g\rangle = 0,$$

which is equivalent to $d^\star_{q,r} = 1$. □

Corollary 9.3. *For any $q < 2$, we have $d^\star_{q,2} = 1$.*

By Lemma 1.5, $d^\star_{q,r} = 0$ holds when $r < 2$. Hence, by Lemma 9.2, we have $d^\star_{p,r} < 1$ if $r < 2(p-1)$ and $p > 2$. This proves Theorem 1.8.

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