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ON THE UNIVERSAL INTEGRAL MEANS SPECTRUM OF CONFORMAL MAPPINGS NEAR THE ORIGIN

HÅKAN HEDENMALM AND SERGUEI SHIMORIN

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Abstract. We improve the local estimate near the origin of the integral means spectrum for conformal mappings obtained in our paper from 2005. We also study some algebraic aspects of higher order forms associated with the given conformal mapping.

1. Introduction

Notation. The standard weighted Bergman space $\mathcal{H}_\alpha(\mathbb{D})$ is a Hilbert space over the complex field $\mathbb{C}$ defined for real α with $-1 < \alpha < +\infty$; it consists of all complex-valued holomorphic functions f on the open unit disk $\mathbb{D}$ subject to the norm boundedness condition

$$\|f\|_\alpha^2 = \int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A_\alpha(z) < +\infty,$$

where $\mathrm{d}A_\alpha$ is the probability measure (on $\mathbb{D}$)

$$\mathrm{d}A_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \mathrm{d}A(z),$$

and $\mathrm{d}A$ is normalized area measure:

$$\mathrm{d}A(z) = \mathrm{d}x\mathrm{d}y/\pi, \qquad z = x + \mathrm{i}y.$$

For a positive integer n and a real number x, we introduce the standard Pochhammer symbol

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1).$$

It is agreed that $(x)_0 = 1$.

A conformal mapping $\varphi : \mathbb{D} \to \mathbb{C}$ is said to be normalized if $\varphi(0) = 0$ and $\varphi'(0) = 1$; we write $\varphi \in \mathcal{S}$. For a fixed $\tau \in \mathbb{C}$, it is easy to show that

$$\int_{-\pi}^{\pi} \left| \left[\varphi'(re^{\mathrm{i}t})\right]^\tau \right| \frac{\mathrm{d}t}{2\pi} = O\left(\frac{1}{(1-r)^\beta}\right), \qquad r \to 1^-,$$

for some positive real number β, which depends on τ. The infimum of all such β is denoted by $\beta_\varphi(\tau)$. We define the universal integral means spectrum as the function

$$\mathrm{B}_{\mathcal{S}}(\tau) = \sup_{\varphi \in \mathcal{S}} \beta_\varphi(\tau).$$

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The Prawitz-type inequality. Fix $0 < \beta < +\infty$ and $0 < \theta \le 1$, and consider a normalized conformal mapping $\varphi \in \mathcal{S}$ on the unit disk $\mathbb{D}$. The main technical result obtained [2] is that for any $g \in \mathcal{H}_\alpha(\mathbb{D})$,

$$\begin{aligned}(1.1)\qquad &\sum_{N=0}^{+\infty} \frac{\Gamma(\beta+2\theta+1)\Gamma(\beta+2N+2)}{(N+1-\theta)\Gamma(\beta+\theta+N+1)\Gamma(\beta+\theta+N+2)} \\ &\qquad\times \left\| b_N\, g^{(N+1)}(z) + \sum_{k=0}^{N} a_{k,N}\, \boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\, g(z)\big] \right\|^2_{\beta+2N+1} \\ &\qquad\le \left[\frac{(\beta+2\theta)\Gamma(2\theta+1)}{2\theta\beta\big[\Gamma(\theta+1)\big]^2} + \varkappa(\beta+2\theta-1,\theta) \right] \|g\|^2_{\beta-1} + O\big(\|g\|^2_{\beta+\theta-1}\big),\end{aligned}$$

where

$$b_N = \frac{(-1)^{N+1}(1-\theta)_{N+1}}{(N+1)!\,(\beta+N+1)_{N+1}},$$

$$a_{k,N} = \frac{(-1)^{N-k}}{k!(N-k)!}\ \frac{(-\theta+k+2)_{N-k}}{(\beta+N+k+2)_{N-k}},$$

and

$$\begin{aligned}\varkappa(\alpha,\theta) = &\frac{(1-\theta)\Gamma(\alpha+2)\Gamma(\alpha+2-2\theta)}{\Gamma(\alpha+2-\theta)\Gamma(\alpha+3-\theta)} \\ &\times \sum_{N=0}^{+\infty} \big(\alpha+3-2\theta+2N\big)\, \frac{(1-\theta)_N(2-\theta)_N\big[(\alpha+2-2\theta)_N\big]^2}{(\alpha+2-\theta)_N(\alpha+3-\theta)_N\big[(N+1)!\big]^2}.\end{aligned}$$

Some more notation is still needed to make sense of (1.1).

Notation for φ-forms. To simplify the notation, we agree to write

$$X_j[\varphi] = \frac{\varphi^{(j+1)}}{\varphi'},$$

where we frequently suppress φ. The homogeneous φ-forms of degree n are then just the homogeneous polynomials of degree n in the variables X_1, X_2, $X_3, \ldots$, with the convention that the variable X_j has degree j. The differentiation operator $\boldsymbol{\partial}_z$ then sends homogeneous polynomials of degree n to homogeneous polynomials of degree $n+1$, and algebraically, it is determined by the property of being a differentiation (the Leibniz product rule), and by the rule

$$\boldsymbol{\partial}_z X_j = X_{j+1} - X_j X_1.$$

The functions $\Phi_{k,\theta}$. For $k = 0, 1, 2, \ldots$, we have

$$(1.2)\qquad \Phi_{k,\theta}(z) = \oslash\big[\boldsymbol{\partial}_z^k \Phi_0\big](z) = (k+1-\theta)\,k! \sum_{n=0}^{k} \frac{(-1)^n(1+\theta)_n}{(n+1)!} \Psi_{k+1,n+1}(z).$$

The functions $\Psi_{k,n}$ are defined for integers k, n, with $1 \le n \le k$, by the relation

$$\Psi_{k,n}(z) = \sum_{(j_1,\ldots,j_n)\in I(k,n)} \frac{X_{j_1}[\varphi](z)\cdots X_{j_n}[\varphi](z)}{(j_1+1)!\cdots(j_n+1)!},$$

where $I(k,n)$ is the set of all n-tuples $(j_1,\ldots,j_n)$ of positive integers with $j_1+\ldots+j_n=k$. In the terminology of [2], $\Psi_{k,n}(z)$ is a monomial φ-form of degree k and bidegree n. We calculate that, for instance,

$$\Psi_{k,1}(z)=\frac{X_k[\varphi](z)}{(k+1)!},\qquad \Psi_{k,2}(z)=\sum_{l=1}^{k-1}\frac{X_l[\varphi](z)\,X_{k-l}[\varphi](z)}{(l+1)!(k-l+1)!}.$$

Applications to Makarov's law of the iterated logarithm. In this paper, we improve the estimates obtained in the recent paper [2], which allows us to strengthen the estimate of the universal integral means spectrum around the origin. In a forthcoming paper [1], these improved estimates are applied to yield a substantial reduction of the constant in Makarov's law of the iterated logarithm, from 30 to 37/10 (see, e.g., Pommerenke's book [3]). We recall that Makarov's law compares harmonic measure to Hausdorff measure for simply connected domains.

2. Main results

A special case of the Prawitz-type inequality. We consider $\beta\sim 0$ and $\theta\sim 0$ (here β should tend to zero first, and only later, θ), so that

$$b_N=\frac{(-1)^{N+1}(1-\theta)_{N+1}}{(N+1)!\,(\beta+N+1)_{N+1}}\sim\frac{(-1)^{N+1}}{(N+1)_{N+1}}$$

and

$$a_{k,N}=\frac{(-1)^{N-k}}{k!(N-k)!}\;\frac{(-\theta+k+2)_{N-k}}{(\beta+N+k+2)_{N-k}}\sim\frac{(-1)^{N-k}(k+2)_{N-k}}{k!(N-k)!(N+k+2)_{N-k}}.$$

The basic estimate (1.1) then simplifies to

$$\sum_{N=0}^{+\infty}\frac{(2N+1)!}{[(N+1)!]^2}\times\left\|b_N\,g^{(N+1)}(z)+\sum_{k=0}^{N}a_{k,N}\,\boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\,g(z)\big]\right\|^2_{\beta+2N+1}$$
$$\le\big(1+\epsilon(\beta,\theta)\big)\left[\frac{1}{\beta}+\frac{1}{\theta}\right]\|g\|^2_{\beta-1}+O\big(\|g\|^2_{\beta+\theta-1}\big),\tag{2.1}$$

where $\epsilon(\beta,\theta)\to 0$ as $\beta\to 0$. The derivation of (2.1) from (1.1) actually requires working out a generalized version of Lemma 6.5 of [2], with the third order derivative of the conformal mapping replaced by a derivative of arbitrary order.

We shall apply (2.1) to the choice

$$g(z)=g_\tau(z)=\big[\varphi'(z)\big]^{\tau/2},\qquad z\in\mathbb{D},$$

with τ complex, $\tau\sim 0$. Then

$$\boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\,g_\tau(z)\big]=g_\tau(z)\,\boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\big]+O(|\tau|),$$

as $\tau\to 0$, where the big "Oh" term is controlled as an element of $\mathcal{H}_{\beta+2N+1}(\mathbb{D})$. This means that in the series on the left hand side of (2.1), we may replace

$$b_N\,g_\tau^{(N+1)}(z)+\sum_{k=0}^{N}a_{k,N}\,\boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\,g_\tau(z)\big]$$

by

$$g_\tau(z)\sum_{k=0}^{N}a_{k,N}\,\boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\big]$$

without seriously affecting the assertion of (2.1).

Computation of the involved φ-forms. For $N = 0, 1, 2, \ldots$, let Λ_{N+1} denote the φ-form

$$\Lambda_{N+1}(z) = \lim \sum_{k=0}^{N} a_{k,N}\, \boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\big],$$

the limit being taken as $\beta \to 0$ and $\theta \to 0$; it is a homogeneous φ-form of degree $N + 1$.

We rewrite

$$a_{k,N} \sim \frac{(-1)^{N+k}(k+2)_{N-k}}{k!(N-k)!(N+k+2)_{N-k}} = \frac{(-1)^N(N+1)!}{(2N+1)!}\,\frac{(-1)^k(N+k+1)!}{k!(k+1)!(N-k)!},$$

and hence it follows from (1.2) that

$$\text{(2.2)}\quad \Lambda_{N+1}(z) = \frac{(-1)^N(N+1)!}{(2N+1)!}\sum_{k=0}^{N}\sum_{n=0}^{k}\frac{(-1)^{n+k}(N+k+1)!}{(n+1)k!(N-k)!}\,\boldsymbol{\partial}_z^{N-k}\Psi_{k+1,n+1}(z).$$

We now begin to read off equation (2.2), for $N = 0, 1, 2, 3$. For $N = 0$, we find that

$$\Lambda_1(z) = \Psi_{1,1} = \frac{\varphi''}{2\varphi'} = \frac{1}{2}\,X_1,$$

while for $N = 1$, we get

(2.3)

$$\Lambda_2(z) = -\frac{2!}{3!}\left\{\frac{2!}{1\cdot 0!1!}\boldsymbol{\partial}_z\Psi_{1,1} - \frac{3!}{1\cdot 1!0!}\Psi_{2,1} + \frac{3!}{2\cdot 1!0!}\Psi_{2,2}\right\} = \frac{[\varphi'']^2}{12[\varphi']^2} = \frac{1}{12}\,X_1^2.$$

Next, for $N = 2$, we have

$$\text{(2.4)}\quad \begin{aligned}\Lambda_3(z) = \frac{3!}{5!}\bigg\{&\frac{3!}{1\cdot 0!2!}\boldsymbol{\partial}_z^2\Psi_{1,1} - \frac{4!}{1\cdot 1!1!}\boldsymbol{\partial}_z\Psi_{2,1} + \frac{4!}{2\cdot 1!1!}\boldsymbol{\partial}_z\Psi_{2,2}\\ &+ \frac{5!}{1\cdot 2!0!}\Psi_{3,1} - \frac{5!}{2\cdot 2!0!}\Psi_{3,2} + \frac{5!}{3\cdot 2!0!}\Psi_{3,3}\bigg\}\\ &= \frac{1}{40}\left\{\frac{\varphi'''\varphi''}{[\varphi']^2} - \frac{[\varphi'']^3}{[\varphi']^3}\right\} = \frac{1}{40}\big(X_2X_1 - X_1^3\big),\end{aligned}$$

while for $N = 3$, we get

$$\text{(2.5)}\qquad \Lambda_4(z) = \frac{1}{5040}\Big(12\,X_3X_1 - 16\,X_2^2 + 24\,X_2X_1^2 - 27\,X_1^4\Big),$$

and for $N = 4$,

$$\text{(2.6)}\qquad \Lambda_5(z) = \frac{1}{18144}\Big(3\,X_4X_1 - 5X_3X_2 + 20\,X_2^2X_1 - 45\,X_2X_1^3 + 27\,X_1^5\Big).$$

Careful but tedious computations show that for $N = 5$,

(2.7)

$$\begin{aligned}\Lambda_6(z) = \frac{1}{665280}\Big\{&6X_5X_1 - 48X_4X_2 + 54X_4X_1^2 + 45X_3^2 - 120X_3X_2X_1 + 40X_2^3\\ &- 90X_3X_1^3 + 270X_2^2X_1^2 - 360X_2X_1^4 + 225X_1^6\Big\},\end{aligned}$$

while for $N = 6$,

(2.8)
$$\Lambda_7(z) = \frac{1}{2471040}\Big\{X_6X_1 - 7X_5X_2 + 7X_5X_1^2 + 7X_4X_3 + 14X_4X_2X_1 - 42X_4X_1^3 - 35X_3^2X_1 + 105X_3X_2X_1^2 + 70X_2^3X_1 - 420X_2^2X_1^3 + 525X_2X_1^5 - 225X_1^7\Big\}.$$

It is not hard to verify that

$$\boldsymbol{\partial}_z\Lambda_{N+1} = d_N\,\Lambda_{N+2} \tag{2.9}$$

holds for $N = 1,\ 3,\ 5$, where the constants d_N are as follows:

$$d_N = \frac{2(N+1)(2N+3)}{N+2}. \tag{2.10}$$

Indeed, further calculations for $N = 7,\ 9$ *support the conjecture that (*2.9*) holds in general for odd positive* N.

The improved integral means spectrum estimate. The reason why we implemented $\theta \sim 0$ is that in [2] this proved to be the most informative value of θ as regarded the estimation of the integral means spectrum $\mathrm{B}_{\mathcal{S}}(\tau)$ for τ near 0. We now want to implement at least some of the above identities for Λ_{N+1} (with $N = 0, 1, 2$) in the Prawitz-type estimate (2.1). In view of (2.4), the second and third terms (with $N = 1,\ 2$) are, essentially, for $\tau \sim 0$,

$$\frac{3}{2}\left\|\frac{1}{12}\,X_1(\varphi)^2\,[\varphi']^{\tau/2}\right\|^2_{\beta+3} \tag{2.11}$$

and

$$\frac{10}{3}\left\|\frac{1}{80}\,\boldsymbol{\partial}_z\Big[X_1(\varphi)^2[\varphi']^{\tau/2}\Big]\right\|^2_{\beta+5}. \tag{2.12}$$

Since in general,

$$\|f'\|^2_{\beta+5} \sim (\beta+5)(\beta+6)\|f\|^2_{\beta+3} \sim 30\,\|f\|^2_{\beta+3},$$

the total sum of the second and third terms is essentially

$$\left(\frac{3}{2\times 12^2} + \frac{10\times 30}{3\times 80^2}\right)\left\|X_1(\varphi)^2[\varphi']^{\tau/2}\right\|^2_{3+\beta} = \frac{5}{192}\left\|X_1(\varphi)^2[\varphi']^{\tau/2}\right\|^2_{3+\beta}.$$

So we should implement $5/192$ wherever $1/96$ occurs in the argument leading up to Theorem 6.7 in [2]. The result is the following strengthening of Theorem 6.7.

Theorem 2.1. *We have*

$$\limsup_{\mathbb{C}\ni\tau\to 0}\frac{\mathrm{B}_{\mathcal{S}}(\tau)}{|\tau|^2} \le \frac{\sqrt{24}-3}{5} = 0.3798\ldots.$$

Remarks on the structure of the involved φ-forms. If (2.9) were true for even N as well as for odd N, we would be able to immediately apply our method and involve all the terms at once, because we understand what the norm in $\mathcal{H}_{\alpha+2}(\mathbb{D})$ of the derivative of a function is (essentially) in terms of the norm of the function itself in $\mathcal{H}_\alpha(\mathbb{D})$. Of course it is not so, but, nevertheless, it is of interest to consider the difference

$$\Delta_{N+2} = \boldsymbol{\partial}_z \Lambda_{N+1} - d_N \, \Lambda_{N+2}$$

for even N. Here, we use (2.10) to define d_N for even N. We compute that

$$\Delta_2 = \frac{1}{2}\left[X_2 - \frac{3}{2}\, X_1^2\right],$$

a multiple of the Schwarzian derivative. Next, we compute that

$$\Delta_4 = \frac{7}{120}\left[X_2 - \frac{3}{2}\, X_1^2\right]^2$$

and

$$\Delta_6 = \frac{11}{36288}\left[90X_2X_1^4 - 60X_2^2X_1^2 - 6X_1^2X_4 - 5X_3^2 + 4X_4X_2 + 20X_3X_2X_1 - 45X_1^6\right].$$

By homogeneity, no information is lost if we put $X_1 = 1$ throughout. Then Δ_4 gets to be 0 if we put $X_2 = \frac{3}{2}$. What happens to Δ_6 if we put $X_2 = \frac{3}{2}$ and $X_1 = 1$? We readily see that then

$$\Delta_6 = -\frac{55}{36288}\left[X_3 - 3\right]^2,$$

again a square! This square is zero if $X_3 = 3$. If we in the computation of Δ_8 put $X_1 = 1$, $X_2 = \frac{3}{2}$, and $X_3 = 3$, we then find that Δ_8 is a constant multiple of

$$\left[X_4 - \frac{15}{2}\right]^2.$$

It appears that generally, in the computation of Δ_{2n}, we will find a constant multiple of the the square

$$\left[X_n - \frac{(n+1)!}{2^n}\right]^2,$$

if we put

$$X_j = \frac{(j+1)!}{2^j}, \qquad j = 0, 1, 2, \dots, n-1.$$

This has been verified to be correct for $n = 1, 2, 3, 4, 6$. There must be deep structural properties of Λ_{N+1}, of which this only scratches a little on the surface.

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Department of Mathematics, Royal Institute of Technology, S–100 44 Stockholm, Sweden

E-mail address: haakanh@math.kth.se

Department of Mathematics, Royal Institute of Technology, S–100 44 Stockholm, Sweden

E-mail address: shimorin@math.kth.se