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Hele–Shaw flow on hyperbolic surfaces

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Abstract

Consider a complete simply connected hyperbolic surface. The classical Hadamard theorem asserts that at each point of the surface, the exponential mapping from the tangent plane to the surface defines a global diffeomorphism. This can be interpreted as a statement relating the metric flow on the tangent plane with that of the surface. We find an analogue of Hadamard's theorem with metric flow replaced by Hele–Shaw flow, which models the injection of (two-dimensional) fluid into the surface. The Hele–Shaw flow domains are characterized implicitly by a mean value property on harmonic functions. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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1. Introduction

Let $\boldsymbol{\Omega}$ be a simply connected two-dimensional Riemannian manifold with a C^∞-smooth metric $\mathrm{d}\mathbf{s}$. We can always introduce local *isothermal coordinates* near any point of $\boldsymbol{\Omega}$, that is, such coordinates (x, y) that the metric is represented in the form $\mathrm{d}\mathbf{s}(x,y)^2 = \omega(x,y)(\mathrm{d}x^2 + \mathrm{d}y^2)$, for some positive weight function ω. And since $\boldsymbol{\Omega}$ is orientable as a two-dimensional simply-connected manifold, these local isothermal coordinates can serve as conformal charts for a complex structure on $\boldsymbol{\Omega}$ (see [1, pp. 124–126]). The Kœbe uniformization theorem then says that $\boldsymbol{\Omega}$ is conformally equivalent to one of the three sets: the Riemann sphere $\mathbb{S} = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$, or the open unit disk $\mathbb{D}$. This equivalence, together with the choice of isothermal coordinates, allows us to identify $\boldsymbol{\Omega}$ with one of the above three sets Ω supplied with the isothermal Riemannian metric

$$\mathrm{d}\mathbf{s}(z)^2 = \omega(z)|\mathrm{d}z|^2, \quad z \in \Omega \setminus \{\infty\}. \tag{1.1}$$

Here, ω is a weight function which is strictly positive and C^∞-smooth in $\Omega \setminus \{\infty\}$ and vanishes at infinity if $\Omega = \mathbb{S}$. The Gaussian curvature corresponding to the above isothermal metric (1.1) is given by the expression

$$\boldsymbol{\kappa}(z) = -\frac{2}{\omega(z)}\Delta(\log\omega)(z), \quad z \in \Omega \setminus \{\infty\},$$

where Δ stands for the normalized Laplacian:

$$\Delta = \Delta_z = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \quad z = x + \mathrm{i}y.$$

The Riemannian manifold $\boldsymbol{\Omega}$ is said to be *hyperbolic* if the Gaussian curvature is negative everywhere. For the metric (1.1), this means that $\log\omega$ should be subharmonic in $\Omega \setminus \{\infty\}$. Such functions ω are called *logarithmically subharmonic*. We note

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that this rules out the Riemann sphere as a possibility for Ω, because $\log \omega$ tends to $-\infty$ at infinity, which is not possible for a subharmonic function that is not identically $-\infty$. This observation – that the sphere cannot be supplied with a hyperbolic metric – can also be easily derived from the Gauss–Bonnet formula [4, p. 417]. In what follows, we deal exclusively with *hyperbolic* surfaces $\boldsymbol{\Omega}$.

1.1. Metric flow

Fix a point $z_0 \in \boldsymbol{\Omega}$. The classical exponential mapping Exp_{z_0} assigns to each vector η in the tangent plane $T_{z_0}(\boldsymbol{\Omega})$ a point $\operatorname{Exp}_{z_0}(\eta)$ on the geodesic starting from z_0 and having a tangent vector η at z_0 so that the length of the portion of this geodesic between z_0 and $\operatorname{Exp}_{z_0}(\eta)$ equals $|\eta|$, the length of the vector η. The classical theorem of Jacques Hadamard [15], [16, pp. 729–775] then states that the exponential mapping Exp_{z_0} is a diffeomorphism from the tangent plane $T_{z_0}(\boldsymbol{\Omega})$ onto $\boldsymbol{\Omega}$, provided that the metric is complete (see also [21, p. 74]). We recall that the metric is said to be *complete* if all Cauchy sequences are convergent. If the metric is incomplete, the exponential mapping is a diffeomorphism from some star-shaped (with respect to the origin) open region $\boldsymbol{\Omega}^\sharp$ of the tangent plane $T_{z_0}(\boldsymbol{\Omega})$ onto the region $\boldsymbol{\Omega}^*$ of points of $\boldsymbol{\Omega}$ geodesically visible from z_0.

We can interpret Hadamard's theorem as supplying "polar coordinates" centered about z_0 on all of $\boldsymbol{\Omega}$. Considering the metric disks

$$\mathbf{B}(z_0,t)=\{z\in\Omega\colon\ \mathbf{d}(z,z_0)<t\},$$

where $\mathbf{d}(z,z_0)$ expresses the distance on the surface $\boldsymbol{\Omega}$ between z and z_0, we see that the exponential mapping Exp_{z_0} is a diffeomorphism which takes a disk of radius t about the origin in the tangent plane (provided that the disk is small enought to be contained in $\boldsymbol{\Omega}^\sharp$) onto the metric disk $\mathbf{B}(z_0,t)$. This mapping can be considered as the result of a flow of a photonic gas consisting of noniteracting infinitesimal particles which are injected at a constant rate at the point z_0, and moving with constant speed along geodesics. Assuming the speed of the particles to be 1, we find that the region they cover at time $t\in\,]0,+\infty[$ is the metric disk $\mathbf{B}(z_0,t)$. The boundary curves $\partial\mathbf{B}(z_0,t)$ are perpendicular to the geodesics emanating from z_0.

If the original manifold is real-analytic, then the exponential mapping is also real-analytic [21, p. 58].

1.2. Hele–Shaw flow

We shall find the analogue of Hadamard's theorem for a different kind of flow, called Hele–Shaw flow. As before, we fix a point $z_0 \in \boldsymbol{\Omega}$. We inject a two-dimensional Newtonian fluid at a constant rate at z_0 into the surface $\boldsymbol{\Omega}$. There is no fluid present at the starting time, but immediately thereafter we have a small but growing blob around the injection point. The two-dimensional fluid is a limit case of a three-dimensional one, where we thicken the surface uniformly and then let the thickness tend to zero. Let $\mathbf{D}(z_0,t)$ be the blob at time t, for positive times t. Physics dictates that the rate of growth should be determined by the local fluid pressure. This means that the rate of growth at a boundary point $z\in\partial\mathbf{D}(z_0,t)$ is proportional to the gradient $\boldsymbol{\nabla}_z\mathbf{G}(z,z_0)$ of the Green function $\mathbf{G}(z,z_0)$ for the Laplace–Beltrami operator $\boldsymbol{\Delta}$. Following Richardson [33], we call the movement of the free boundary between the fluid and vacuum *Hele–Shaw flow*. From a mathematical point of view, the above recipe for how the boundary moves outward suffers from certain limitations. First, it only tells how the boundary moves once we have a well-behaved nonempty blob in place. It does not say how to obtain $\mathbf{D}(z_0,t)$ for t close to 0, because $t=0$ is a singular point from this point of view. But it turns out that the domains $\mathbf{D}(z_0,t)$ admit other equivalent characterizations in terms of certain quadrature identities and obstacle problems, which permits us to study the Hele–Shaw flow rigorously.

Let us now see how to obtain an equivalent description of the Hele–Shaw flow. If the metric on the domain Ω, representing the surface $\boldsymbol{\Omega}$, is given by (1.1), then the normalized two-dimensional volume form is

$$\mathrm{d}\boldsymbol{\Sigma}(z)=\omega(z)\,\mathrm{d}\Sigma(z),\qquad z\in\Omega,$$

where $\mathrm{d}\Sigma(z)$ stands for the normalized area measure in the plane:

$$\mathrm{d}\Sigma(z)=\frac{1}{\pi}\,\mathrm{d}x\,\mathrm{d}y,\qquad z=x+\mathrm{i}y.$$

The Laplace–Beltrami operator on $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Delta}_z=\frac{1}{\omega(z)}\Delta_z,\qquad z\in\Omega.$$

Suppose a function h is harmonic in a neighborhood of $\bar{\mathbf{D}}(z_0,t)$. Let t' be slightly larger than t, but such that h is harmonic on $\bar{\mathbf{D}}(z_0,t')$ as well. Assuming that the boundaries are smooth, we recall the recipe for how the boundary moves, and apply Green's theorem:

$$\int\limits_{\mathbf{D}(z_0,t')\setminus\mathbf{D}(z_0,t)} h(z)\,\mathrm{d}\boldsymbol{\Sigma}(z)=(t'-t)\int\limits_{\partial\mathbf{D}(z_0,t)} h(z)\,\frac{\partial}{\partial\mathbf{n}(z)}\mathbf{G}(z,z_0)\,\mathrm{d}\boldsymbol{\sigma}(z)+\mathrm{o}(t'-t)=(t'-t)\,h(z_0)+\mathrm{o}(t'-t).$$

Here, $\mathrm{d}\boldsymbol{\sigma}$ stands for the normalized arc length measure on $\partial\mathbf{D}(z_0,t)$. Letting t' approach t, we obtain in the limit that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}(z_0,t)} h(z)\,\mathrm{d}\boldsymbol{\Sigma}(z)=h(z_0).$$

We integrate this relationship with respect to t, recalling that $\mathbf{D}(z_0,t)$ should be empty for $t=0$, and obtain

$$th(z_0)=\int_{\mathbf{D}(z_0,t)} h(z)\,\mathrm{d}\boldsymbol{\Sigma}(z), \tag{1.2}$$

where we recall that h should be harmonic in a neighborhood of $\overline{\mathbf{D}}(z_0,t)$. The identity (1.2) is known as the *mean value property*, and we shall call the sets $\mathbf{D}(z_0,t)$ *mean value disks* on the hyperbolic surface $\boldsymbol{\Omega}$. After identification of $\boldsymbol{\Omega}$ with Ω, this property reads as

$$th(z_0)=\int_{\mathbf{D}(z_0,t)} h(z)\,\omega(z)\,\mathrm{d}\Sigma(z); \tag{1.3}$$

here, we think of $\mathbf{D}(z_0,t)$ as a subset of Ω. We show in the present paper that the domains $\mathbf{D}(z_0,t)$, satisfying (1.3), do exist (for t from certain interval $t\in\,]0,T[$). They will be obtained as noncoincidence sets for certain obstacle problems, and we shall see that these domains are all simply connected and have real analytic boundaries, provided that the weight function ω is real analytic. Moreover, we shall see that for $t\in\,]0,T[$, the domains satisfying (1.3) are unique up to addition or removal of area-null sets.

The boundaries $\partial\mathbf{D}(z_0,t)$ of the above domains $\mathbf{D}(z_0,t)$ form a one-parameter family of imbedded closed curves in the plane. We consider also the biorthogonal family of curves, the so-called *Hele–Shaw geodesics*, emanating from the origin and orthogonal to $\partial\mathbf{D}(z_0,t)$ at any point. Physically, these geodesics correspond to the trajectories of fluid particles injected at the origin at time $t=0$. Considering these two families of curves as coordinate lines for "Hele–Shaw polar coordinates", we obtain what we call the *Hele–Shaw exponential mapping* HSexp_{z_0}.

It turns out that this mapping admits a rigorous definition and that there exists an analogue of Hadamard's theorem for it. Namely, HSexp_{z_0} is a real-analytic diffeomorphism defined in some disk, centered at the origin, of the tangent plane $T_{z_0}(\boldsymbol{\Omega})$ into $\boldsymbol{\Omega}$ such that it maps disks $\{\eta\in T_{z_0}(\boldsymbol{\Omega}):\ |\eta|<r\}$ centered at the origin (which form, in fact, the Hele–Shaw flow on the tangent plane) onto the Hele–Shaw domains $\mathbf{D}(z_0,t)$, with $t=r^2$, and rays emanating from the origin (Hele–Shaw geodesics on $T_{z_0}(\boldsymbol{\Omega})$) are mapped to Hele–Shaw geodesics on $\boldsymbol{\Omega}$ passing through z_0. In case where $\boldsymbol{\Omega}$ is complete, HSexp_{z_0} is a global diffeomorphism from $T_{z_0}(\boldsymbol{\Omega})$ onto $\boldsymbol{\Omega}$.

Assume that the tangent plane $T_{z_0}(\boldsymbol{\Omega})$ is identified with the complex plane $\mathbb{C}$ supplied with the usual Euclidean norm. Then the classical exponential mapping has the following asymptotics near z_0:

$$\mathrm{Exp}_{z_0}(z)=z_0+\omega(0)^{-1/2}z+\mathrm{O}\big(|z|^2\big)\quad\text{as } |z|\to 0.$$

We shall see that the *Hele–Shaw exponential mapping* that we are about to define has a similar property. For $0<r<+\infty$, $\mathbb{D}(z_0,r)$ stands for the Euclidean disk

$$\mathbb{D}(z_0,r)=\big\{z\in\mathbb{C}:\ |z-z_0|<r\big\}.$$

Theorem 1.1 (Main Theorem, global version). *Assume that $\boldsymbol{\Omega}$ is a complete hyperbolic simply connected surface identified with the planar domain Ω which is either the unit disk $\mathbb{D}$ or the whole plane $\mathbb{C}$, and the metric on Ω is given by* (1.1) *in terms of the weight function ω. We assume that $\boldsymbol{\Omega}$ is a real-analytic surface, which means that ω is real-analytic and strictly positive. Fix a point $z_0\in\boldsymbol{\Omega}$. Then, for each $0<t<+\infty$, there exists a precompact subdomain $\mathbf{D}(z_0,t)$ of $\boldsymbol{\Omega}$ with the mean value property* (1.3) *for all bounded harmonic functions h on $\mathbf{D}(z_0,t)$, and as such, it is unique up to addition or removal of an area-null set. Moreover, there exists a unique C^1-diffeomorphism* HSexp_{z_0} *from the complex plane $\mathbb{C}$ onto $\boldsymbol{\Omega}$* (*the Hele–Shaw exponential mapping*) *such that*

- $\mathrm{HSexp}_{z_0}(0)=z_0$,
- *each ray $\{z\in\mathbb{C}\setminus\{0\}:\ \arg z=\theta\}$ is mapped by* HSexp_{z_0} *onto a curve in $\boldsymbol{\Omega}$ which points in the same direction as the ray at z_0,*
- HSexp_{z_0} *maps each pair consisting of a concentric circle about the origin and a straight line passing through the origin onto a pair of orthogonal curves, and*
- *for each $0<r<+\infty$, the domain* $\mathrm{HSexp}_{z_0}(\mathbb{D}(0,r))$ *equals the Hele–Shaw flow domain $\mathbf{D}(z_0,r^2)$, up to addition or removal of area-null sets.*

It has the following additional properties:

- HSexp_{z_0} *is a real-analytic diffeomorphism from* $\mathbb{C}$ *onto* $\boldsymbol{\Omega}$, *and*
- $\mathrm{HSexp}_{z_0}(z) = z_0 + \omega(0)^{-1/2} z + \mathrm{O}(|z|^2)$ *as* $|z| \to 0$.

It is interesting to compare the Hele–Shaw disks with the metric disks encountered earlier: $\mathbf{D}(z_0, r^2) \subset \mathbf{B}(z_0, r)$ for all $0 < r < +\infty$. For flat surfaces $\boldsymbol{\Omega}$, we have equality, $\mathbf{D}(z_0, r^2) = \mathbf{B}(z_0, r)$, but the introduction of negative curvature makes the Hele–Shaw disks smaller in comparison. This is intuitively reasonable, because the (normalized) area of the Hele–Shaw disk $\mathbf{D}(z_0, r^2)$ equals r^2, whereas that of the metric disk $\mathbf{B}(z_0, r)$ exceeds r^2 on negatively curved surfaces.

We turn to the "local" version of the above theorem, which applies to incomplete manifolds as well. As in the case of metric flow, there appears a part of the surface that is "Hele–Shaw visible" from the given point $z_0 \in \boldsymbol{\Omega}$.

Theorem 1.2 (Main Theorem, local version). *Let the setting be as in the formulation of Theorem* 1.1, *with the exception that the surface* $\boldsymbol{\Omega}$ *need no longer be a complete manifold. Then there exists a parameter* T *with* $0 < T \leqslant +\infty$, *such that the assertions of Theorem* 1.1 *hold, with the following modifications. The Hele–Shaw flow domains* $\mathbf{D}(z_0, t)$ *are well-defined for* $0 < t < T$, *and in case* $T < +\infty$, *they fail to exist as precompact Jordan domains with the mean value property* (1.3) *for* $T < t < +\infty$. *The property* $\mathrm{HSexp}_{z_0}(\mathbb{D}(0, r)) = \mathbf{D}(z_0, r^2)$ *then holds for* $0 < r < \sqrt{T}$, *and* HSexp_{z_0} *is a diffeomorphism* $\mathbb{D}(0, \sqrt{T}) \to \boldsymbol{\Omega}^\sharp$, *where* $\boldsymbol{\Omega}^\sharp = \mathrm{HSexp}_{z_0}(\mathbb{D}(0, \sqrt{T}))$ *is a subdomain of* $\boldsymbol{\Omega}$ *containing* z_0, *which is not precompact in* $\boldsymbol{\Omega}$. *It constitutes the "Hele–Shaw visible" part of* $\boldsymbol{\Omega}$. *In case* $T = +\infty$, *we have* $\boldsymbol{\Omega}^\sharp = \boldsymbol{\Omega}$.

In the context of the above local theorem, it seems natural to call $\boldsymbol{\Omega}$ *Hele–Shaw complete* if $T = +\infty$. Then every (metrically) complete surface is Hele–Shaw complete; however, there are also simple examples of incomplete surfaces that are Hele–Shaw complete. Intuitively, the reason why the two concepts diverge is that metric completeness requires an infinite (geodesic) *distance* to the "point at infinity", whereas Hele–Shaw completeness requires there to be an infinite *area* covered by the flow to reach the same point.

The metric flow has a hyperbolic flavor so that the information about an obstruction for the flow has a finite speed of propagation, whereas the Hele–Shaw flow is parabolic, because the information travels instantaneously. Nevertheless, the "exponential mapping" from each has the same kind of basic properties.

Remarks. (a) The above-mentioned results, with the obvious modifications, are probably valid in the context of C^∞-smooth surfaces as well. However, we do not have a proof of this statement.

(b) It would be interesting to have a similar study carried out for higher-dimensional manifolds.

(c) We should point out that mean value inequalities for subharmonic functions on manifolds can be found in the literature. However, if we look only for mean value inequalities for subharmonic functions that imply a mean value *equality* for harmonic functions, then our only choice are the Hele–Shaw domains. For instance, the mean value inequality of Schoen and Yau [36, p. 75] for metric balls does not have this property unless the manifold has constant curvature.

1.3. Wrapped Hele–Shaw flow

Toward the end of the paper, we develop the concept of *wrapped Hele–Shaw flow*, where the flow domains are guaranteed to be simply connected; the price we pay for this is that the actual flow takes place on a Riemann surface sheeted over the given surface. Wrapped Hele–Shaw flow then makes sense and may be continued indefinitely on compact surfaces **S** as well. Our main theorem applies to the covering surface $\boldsymbol{\Omega}$ of the hyperbolic compact surface **S**, and the wrapped flow is the projection of the ordinary Hele–Shaw flow on $\boldsymbol{\Omega}$ to **S**. The "wrapped Hele–Shaw geodesics" extend infinitely in both directions, and we can ask whether they possess ergodic properties.

1.4. Applications

The reason we initially got interested in Hele–Shaw flow is that it offers a powerful method for investigating the biharmonic operator on hyperbolic surfaces. It turns out that on a Hele–Shaw flow domain, the biharmonic Green function is positive [18,19]. In the proof of that result, the biharmonic Green function is represented via the so-called Hadamard variational formula (a special case of the well-known Duhamel principle) as an integral over the flow of slightly simpler functions, analogous to the Poisson kernel for the Laplacian. Those functions, referred to as harmonic compensators, are then in their turn represented as integrals over the flow of harmonic reproducing kernel functions. The negative Gaussian curvature leads to quite specific information regarding the harmonic reproducing kernel functions on flow domains, which is then used to derive the positivity of the harmonic compensator. In a second step, we get that the biharmonic Green function is positive as well. The positivity of the biharmonic Green function on Hele–Shaw flow domains for hyperbolic surfaces has important applications to the factorization theory of the Bergman space [19,20].

1.5. Notations

For a smooth domain Ω in $\mathbb{C}$, the Sobolev space $W^2(\Omega)$ consists of all functions in $L^2(\Omega)$ whose distributional partial derivatives up to order 2 are also in $L^2(\Omega)$. By the Sobolev–Morrey imbedding theorem, the functions in $W^2(\Omega)$ are in $C^{0,\alpha}(\overline{\Omega})$, the space of Hölder continuous functions on $\overline{\Omega}$, for each exponent α, $0<\alpha<1$. We shall say that a function u is W^2-smooth in $\overline{\Omega}$ if it is from the class $W^2(\Omega')$ for some domain Ω' containing $\overline{\Omega}$.

The space $C^{1,1}(\overline{\Omega})$ consists of all continuously differentiable functions on $\overline{\Omega}$, whose first-order partial derivatives are Lipschitz continuous. It coincides with the Sobolev space $W^{2,\infty}(\Omega)$ of functions in $L^\infty(\Omega)$ whose partial derivatives (taken in the distributional sense) of order less than or equal to 2 are also in $L^\infty(\Omega)$. The functions in the latter space may need to be redefined on a set of zero area measure to fit into the first-mentioned space.

If D and Ω are planar domains, then $D \Subset \Omega$ means that D is precompactly contained in Ω. For a point $w\in\mathbb{C}$ and a positive real parameter r, we let

$$\mathbb{D}(w,r)=\bigl\{z\in\mathbb{C}\colon\ |z-w|<r\bigr\}$$

denote the open circular disk of radius r about w. To shorten the notation, we sometimes write $\mathbb{D}(r)$ for $\mathbb{D}(0,r)$. The characteristic function of a set E is denoted by 1_E.

The symbols ∂_z and $\bar\partial_z$ denote the standard Wirtinger differential operators:

$$\partial_z=\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right)\quad\text{and}\quad \bar\partial_z=\frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right).$$

Then our normalized Laplacian can be written $\Delta_z=\partial_z\bar\partial_z$. We use $\mathrm{d}\sigma$ for normalized arc length measure: $\mathrm{d}\sigma(z)=|\mathrm{d}z|/(2\pi)$. We also write $|E|_\sigma$ for the associated normalized length of a subset E of a rectifiable curve. Similarly, we write $|E|_\Sigma$ for the normalized area of a Borel subset of $\mathbb{C}$.

2. The Hele–Shaw obstacle problem

This section is devoted to the existence of domains $D(t)$ satisfying the mean value identity (1.3). We also establish some of the basic properties of these domains.

It is known that the domains appearing in the study of the classical Hele–Shaw flows can be described in terms of certain variational inequalities, or, equivalently, in terms of obstacle problems (see, for instance, [13]). We use similar arguments, but in the context of the presence of a weight.

Let Ω be a Jordan domain in $\mathbb{C}$ with C^∞-smooth boundary $\partial\Omega$, and let ω be a C^∞-smooth weight function which is strictly positive in $\overline{\Omega}$. We assume that $z_0=0\in\Omega$.

Let $G=G_\Omega$ stand for the Green function for the Laplacian Δ in Ω. We set

$$V_t(z)=tG(z,0)-\int_\Omega G(z,\zeta)\omega(\zeta)\,\mathrm{d}\Sigma(\zeta),\qquad z\in\Omega.$$

This function solves the boundary value problem

$$\Delta V_t=t\delta_0-\omega\quad\text{in } \Omega \text{ (in the sence of distributions)};\qquad V_t|_{\partial\Omega}=0.$$

It is superharmonic in $\Omega\setminus\{0\}$ and it has a negative logarithmic singularity at the origin. Away from the origin, it is C^∞-smooth. The function V_t has superharmonic majorants, for instance, sufficiently large positive constants. Then V_t also has a smallest superharmonic majorant on Ω, because

- if we are given two superharmonic functions, the minumum of them is superharmonic as well, and
- the limit of a decreasing sequence of superharmonic functions is also superharmonic, provided it does not collapse to $-\infty$.

We denote this smallest superharmonic majorant by $\widehat{V}_t$. The set $D(t;\omega)$ is then defined as

$$D(t;\omega)=\bigl\{z\in\Omega\colon\ V_t(z)<\widehat{V}_t(z)\bigr\}. \tag{2.1}$$

We define also the function

$$U_t(z) = \widehat{V}_t(z) - V_t(z), \quad z \in \Omega. \tag{2.2}$$

The obstacle problem makes sense for all values of the parameter t, $0 < t < +\infty$; therefore, the function U_t and the domains $D(t;\omega)$ are well-defined for any positive t. In what follows we shall often drop the dependence on ω and simply write $D(t)$ instead of $D(t;\omega)$. We classify $D(t)$ as a *Hele–Shaw domain* when $D(t) \Subset \Omega$, and as a *generalized Hele–Shaw domain* when $D(t)$ is too big for this to happen. A physical interpretation is that we get generalized Hele–Shaw flow when t is so big that the boundary of $D(t)$ touches $\partial\Omega$, in which case for even bigger t the liquid is allowed to stack up on the common boundary $\partial D(t) \cap \partial\Omega$.

We want to show that the domains $D(t)$ satisfy the mean value identity (1.3).

Proposition 2.1. *Fix a t, $0 < t < +\infty$. Then the superharmonic envelope function $\widehat{V}_t$ is in $C^{1,1}(\overline{\Omega})$. It assumes the value $\widehat{V}_t = 0$ on $\partial\Omega$.*

Proof. The $C^{1,1}$-regularity of $\widehat{V}_t$ follows from general regularity theory for obstacle problems. See, for example, Chapter 1 in [10], or the paper [6] by Caffarelli and Kinderlehrer. Perhaps a word should be said about why $\widehat{V}_t$ vanishes on $\partial\Omega$. The function

$$G[-\omega](z) = -\int_{\Omega} G(z,\zeta)\,\omega(\zeta)\,\mathrm{d}\Sigma(\zeta)$$

is a superharmonic majorant to V_t, and it vanishes on $\partial\Omega$. The function $\widehat{V}_t$ is sandwiched between V_t and $G[-\omega]$, and both these functions vanish on $\partial\Omega$, which leads to the conclusion $\widehat{V}_t|_{\partial\Omega} = 0$. $\square$

As a consequence, we see that the function U_t is continuous, and, therefore, all the sets $D(t)$ are open.

Proposition 2.2. *Fix a t, $0 < t < +\infty$. Then $\widehat{V}_t$ is harmonic in $D(t)$. Moreover, we have $\Delta U_t = \omega 1_{D(t)} - t\,\delta_0$ on Ω, in the sense of distributions.*

Proof. First, we show that $\widehat{V}_t$ is harmonic in $D(t)$. To see this, we apply the standard Perron process argument. Indeed, if it were not harmonic on some small circular disk in $D(t)$, we could replace it in this disk by a harmonic function with the same boundary values on the small circle, and obtain a function which is smaller (by the maximum principle), and still superharmonic in Ω. This new function remains a majorant to V_t if the disk is small enough, in violation of the definition of $\widehat{V}_t$ as the smallest superharmonic majorant to V_t.

Therefore,

$$\Delta U_t = \Delta\big(\widehat{V}_t - V_t\big) = -\Delta V_t = \omega - t\delta_0 \quad \text{on } D(t).$$

On $\Omega \setminus D(t)$, $\widehat{V}_t$ and V_t coincide, and hence their first- and second-order derivatives coincide almost everywhere there, in view of [26, p. 53] and the preceding proposition. In particular, $\Delta U_t = 0$ almost everywhere in $\Omega \setminus D(t)$. From the known regularity of U_t, the assertion is immediate. $\square$

Theorem 2.3. *Suppose the domain $D(t) = D(t;\omega)$ is precompact in Ω. Then for any function h harmonic in $\overline{D}(t)$,*

$$t h(0) = \int_{D(t)} h(z)\omega(z)\,\mathrm{d}\Sigma(z).$$

In fact, the above mean value identity holds under the slightly weaker assumption on h that it be W^2-smooth on $\overline{D}(t)$ and harmonic in $D(t)$. Indeed, if a function u is subharmonic in $D(t)$ and W^2-smooth in $\overline{D}(t)$, then

$$t u(0) \leqslant \int_{D(t)} u(z)\omega(z)\,\mathrm{d}\Sigma(z).$$

Proof. Let the function h be harmonic in a neighborhood of $\overline{D}(t)$. Green's formula – applied to some smooth domain D' containing $\overline{D}(t)$ but contained precompactly in the region of harmonicity of h – yields, together with the previous proposition,

$$\int_{D(t)} h(z)\omega(z)\,\mathrm{d}\Sigma(z) - t h(0) = \int_{D'} h(z)\Delta U_t(z)\,\mathrm{d}\Sigma(z) = 0,$$

since $U_t = 0$ off $D(t)$. The integration on the right-hand side is to be interpreted in the generalized sense of distribution theory. The sought-after mean value property is immediate.

The same arguments, together with the fact that U_t is positive, prove the second assertion of the theorem. □

Following Gustafsson [13], we say that a planar domain D satisfies *the moment inequality* if

$$tu(0) \leqslant \int_D u(z)\omega(z)\,\mathrm{d}\Sigma(z) \tag{2.3}$$

for any u subharmonic in D and W^2-smooth in $\overline{D}$, and D satisfies the *mean value identity* if

$$tu(0) = \int_D u(z)\omega(z)\,\mathrm{d}\Sigma(z) \tag{2.4}$$

for any u harmonic in D and W^2-smooth in $\overline{D}$ (such functions are automatically bounded on D). These properties are very important for the study of Hele–Shaw flows, as is demonstrated by two following propositions.

Proposition 2.4. *Fix a t, $0 < t < +\infty$. Assume that a domain $D \Subset \Omega$ containing the origin is such that the moment inequality* (2.3) *holds for some parameter t for any function u subharmonic in D and W^2-smooth in $\overline{D}$. Then*:

(a) $D(t) \subset \operatorname{int}(\overline{D})$. *Here,* int *denotes the operation of taking the interior of a set,*
(b) *the sets D and $D(t)$ differ by a set of zero area measure.*

Proof. Consider the function

$$U(z) = -tG(z,0) + \int_D G(z,\zeta)\omega(\zeta)\,\mathrm{d}\Sigma(\zeta), \quad z \in \Omega.$$

By (2.3), it is positive throughout Ω, and by the regularizing properties of the Green potential, it is continuous on $\overline{\Omega} \setminus \{0\}$. It vanishes off $\overline{D}$, since $G(z,\cdot)$ is harmonic in $\overline{D}$ for $z \in \Omega \setminus \overline{D}$. By continuity, U also vanishes off $\operatorname{int}(\overline{D})$. The function $V_t + U$ is a superharmonic majorant to V_t; therefore, $U_t \leqslant U$, and we have (a).

To obtain (b), consider a smoothly bordered domain D' such that $D \Subset D' \Subset \Omega$. We have $\Delta U = \omega 1_D - t\delta_0$ in the sense of distributions on D', and $U = 0$ near the boundary of D'. We set

$$v(z) = \int_{D' \setminus D} G(z,\zeta)\,\mathrm{d}\Sigma(\zeta).$$

This function is harmonic in D and W^2-smooth in $\overline{D}$; therefore, by (2.3),

$$0 = \int_D v(z)\omega(z)\,\mathrm{d}\Sigma(z) - tv(0) = \int_{D'} v\Delta U\,\mathrm{d}\Sigma = \int_{D'} \Delta v U\,\mathrm{d}\Sigma = \int_{D' \setminus D} U\,\mathrm{d}\Sigma.$$

This shows that $U = 0$ almost everywhere in $\Omega \setminus D$, and, hence, $U_t = 0$ almost everywhere in $\Omega \setminus D$. Therefore, we have $|D(t) \setminus D|_\Sigma = 0$. We should check that $|D \setminus D(t)|_\Sigma = 0$ as well. To this end, we note that D and $D(t)$ have the same weighted area:

$$\int_D \omega(z)\,\mathrm{d}\Sigma(z) = \int_{D(t)} \omega(z)\,\mathrm{d}\Sigma(z) = t$$

(we substitute constant functions to the moment inequalities). The assertion (b) is now immediate. □

We see that the moment inequality characterizes domains $D(t)$ uniquely up to sets of zero area measure. But in the case where $D(t)$ is a Jordan domain, it is uniquely characterized by the mean value identity (2.4) alone, as follows from the next proposition. The argument is taken from Gustafsson's paper [14].

Proposition 2.5. *Suppose t, $0 < t < +\infty$, is such that we have $D(t) \Subset \Omega$. Suppose also that some domain $D \Subset \Omega$ satisfies the mean value identity* (2.4)*. Then there exists a domain $D^* \supset D$ such that $|D^* \setminus D|_\Sigma = 0$ and $\partial D^* \subset \overline{D}(t)$. In particular, if $D(t)$ is a Jordan domain, then any domain D satisfying* (2.4) *coincides with $D(t)$ up to addition or removal of an area-null set.*

Proof. As in the proof of the previous proposition, we consider the function

$$U(z) = -tG(z,0) + \int_D G(z,\zeta)\omega(\zeta)\,\mathrm{d}\Sigma(\zeta), \quad z \in \Omega.$$

This time, it need not be positive, but we have $U|_{\Omega\setminus\overline{D}} = 0$. Moreover, if we consider the domain $D^* = D \cup \{z \in \Omega\colon U(z) \neq 0\}$ and the function

$$v(z) = \int_{D^*\setminus D} G(z,\zeta)\operatorname{sign}\bigl[U(z)\bigr]\,\mathrm{d}\Sigma(z),$$

we then have

$$0 = \int_D v(z)\omega(z)\,\mathrm{d}\Sigma(z) - tv(0) = \int_\Omega v\Delta U\,\mathrm{d}\Sigma = \int_\Omega \Delta v U\,\mathrm{d}\Sigma = \int_{D^*\setminus D} |U|\,\mathrm{d}\Sigma.$$

Thus, we have $U = 0$ almost everywhere in $D^* \setminus D$, so that $|D^* \setminus D|_\Sigma = 0$.

The function $U - U_t$ is subharmonic in D^* (since $\Delta(U - U_t) = \omega \cdot (1_{D^*} - 1_{D(t)})$), and it has $U - U_t \leqslant 0$ on ∂D^*, since U vanishes there and U_t is positive. Therefore, we have $U \leqslant U_t$ in D^*, and we conclude that $U \leqslant 0$ in $D^* \setminus D(t)$. Now, assume that there exists a point $z_0 \in \partial D^* \setminus \overline{D}(t)$. We have then $U(z_0) = 0$ and, on the other hand, $U(z) \leqslant 0$ in some neighborhood $N(z_0)$ of the point z_0. Since U is subharmonic away from the origin, we must have $U(z) = 0$ in $N(z_0)$, and hence $\Delta U = 0$ in $N(z_0)$, which is impossible since $\Delta U = \omega 1_D - t\delta_0 = \omega 1_{D^*} - t\delta_0$. □

The following series of propositions establishes some basic properties of the domains $D(t)$ and functions $V_t, \widehat{V}_t$, and U_t.

Proposition 2.6. *For any positive t, the domain $D(t)$ is connected.*

Proof. It is clear that the origin is an interior point of $D(t)$, because $V_t(z)$ tends to $-\infty$ as z tends to 0. If $D(t)$ is disconnected, then we can find a connectivity component – call it $D_*(t)$ – which does not contain the origin. As the origin is an interior point of $D(t)$, the connected open set $D_*(t)$ is at a positive distance from it. Moreover, we have that $\partial D_*(t) \subset \overline{\Omega} \setminus D(t)$, because if a sequence of points of $D_*(t)$ has a limit point in $D(t)$, then all point sufficiently near the limit point are in $D_*(t)$ as well, making the point interior for $D_*(t)$. On $D_*(t)$, $\widehat{V}_t$ is harmonic, and on $\partial D_*(t)$, it equals the function V_t. As V_t is superharmonic on $D_*(t)$ (after all, it is superharmonic on $\Omega \setminus \{0\}$), we obtain from the maximum principle that $\widehat{V}_t \leqslant V_t$ on $D_*(t)$, in clear violation of the definition of the set $D(t)$. □

We turn to the basic monotonicity properties of the Hele–Shaw flow.

Proposition 2.7 (Monotonicity). *For t, $0 < t < +\infty$, the function $U_t = \widehat{V}_t - V_t$ increases with the parameter t. Also, if the weight ω is increased, U_t decreases, for fixed t. As a consequence, the domain $D(t;\omega)$ increases with increasing t, and decreases with increasing weight ω.*

Proof. Since $D(t;\omega)$ is defined as the set where U_t is strictly positive, it suffices to prove monotonicity properties of U_t. Let t, t' be related as follows: $0 < t < t' < +\infty$. We check that $V_t - V_{t'}$ is superharmonic, so that the function $\widehat{V}_{t'} - V_{t'} + V_t$ is superharmonic, too. The latter function also majorizes V_t, and hence $\widehat{V}_t \leqslant \widehat{V}_{t'} - V_{t'} + V_t$. It follows that U_t increases with t.

A similar argument shows that U_t decreases as the weight ω increases. The details are as follows. Let ω' be a bigger weight than ω: $\omega \leqslant \omega'$ on Ω, and let V_t' be the potential associated with ω':

$$V_t'(z) = tG(z,0) - \int_\Omega G(z,\zeta)\omega'(\zeta)\,\mathrm{d}\Sigma(\zeta).$$

The function $V_t' - V_t$ is then superharmonic, because $\Delta(V_t' - V_t) = \omega - \omega' \leqslant 0$. It follows that the function $\widehat{V}_t - V_t + V_t'$ is superharmonic, too, and it clearly majorizes V_t'. It is immediate that $\widehat{V}_t' \leqslant \widehat{V}_t - V_t + V_t'$, which leads to $U_t' \leqslant U_t$ (obvious notation), as asserted.

The assertion regarding the domain $D(t;\omega)$ is an immediate consequence of the above monotonicity properties of U_t. □

We need to know that $D(t) \Subset \Omega$, at least for small positive t. On the other hand, for large positive t, the whole domain gets filled: $D(t) = \Omega$. This is accomplished by the following proposition.

Proposition 2.8. *Let m be the minimum value of ω on $\overline{\Omega}$, and M the maximum value. Then the following assertions are valid.*

(a) *If t, $0 < t < +\infty$, is so small that the circular disk $\mathbb{D}(0, \sqrt{t/m}\,)$ is contained in Ω, then $D(t)$ is sandwiched as follows*: $\mathbb{D}(0, \sqrt{t/M}\,) \subset D(t) \subset \mathbb{D}(0, \sqrt{t/m}\,)$.
(b) *For sufficiently large positive t, we have $D(t) = \Omega$.*

Proof. The assertion (a) follows from Proposition 2.7, by comparing the weight ω with the constant weights m and M, for which the Hele–Shaw flow consists of circular disks about 0.

To prove (b), it suffices to observe that for sufficiently large positive t, $V_t(z) < 0$ for all $z \in \Omega$, and in this case $\widehat{V}_t(z) \equiv 0$. □

Formally speaking, the Hele–Shaw domains $D(t;\omega)$ may depend on the choice of the underlying domain Ω. The next proposition shows that this is not the case, provided that $D(t)$ is precompact in Ω.

Proposition 2.9. *Fix a t, $0 < t < +\infty$. Let Ω' be an open subset of Ω, containing the origin. Let $\widehat{V}_t'$ denote the least superharmonic majorant to $V_t|_{\Omega'}$ on Ω', and put*

$$D'(t) = \left\{z \in \Omega' \colon\ V_t(z) < \widehat{V}_t'(z)\right\}.$$

We then have in general $\widehat{V}_t' \leqslant \widehat{V}_t|_{\Omega'}$, and $D'(t) \subset D(t) \cap \Omega'$. In the other direction, we have the following:

(a) *if $D(t) \subset \Omega'$, then $\widehat{V}_t' = \widehat{V}_t|_{\Omega'}$ and $D'(t) = D(t)$, and*
(b) *if $D'(t) \Subset \Omega'$, then $\widehat{V}_t' = \widehat{V}_t|_{\Omega'}$ and $D'(t) = D(t)$.*

Proof. The assertions that $\widehat{V}_t' \leqslant \widehat{V}_t|_{\Omega'}$ and $D'(t) \subset D(t) \cap \Omega'$ are self-evident in view of the definitions of these objects in terms of least superharmonic majorants. We turn to the assertion (a), that we have the equalities $\widehat{V}_t' = \widehat{V}_t|_{\Omega'}$ and $D'(t) = D(t)$ provided that $D(t) \subset \Omega'$. Given that $D(t) \subset \Omega'$, we construct a function $\widetilde{V}_t$ on Ω by setting it equal to $\widehat{V}_t'$ on $D(t)$, and V_t on $\Omega \setminus D(t)$. It is clear that $\widetilde{V}_t \leqslant \widehat{V}_t$ on Ω. The function $\widetilde{V}_t$ equals $\widehat{V}_t'$ on Ω', and is therefore superharmonic there; off the closure of $D(t)$ it is also superharmonic, because V_t is superharmonic there. We wish to show that $\widetilde{V}_t$ is superharmonic throughout Ω. It is well known that a function is superharmonic on Ω if we have the appropriate mean value inequality on sufficiently small circles about each point of Ω. We just need to check this for points $z_1 \in (\Omega \setminus \Omega') \cap \overline{D}(t) \subset \partial D(t)$. Let ε, $0 < \varepsilon$, be so small that $\mathbb{D}(z_1, \varepsilon) \Subset \Omega$, and calculate, using the superharmonicity of $\widehat{V}_t$,

$$\frac{1}{\varepsilon} \int\limits_{\partial\mathbb{D}(z_1,\varepsilon)} \widetilde{V}_t(z)\, \mathrm{d}\sigma(z) \leqslant \frac{1}{\varepsilon} \int\limits_{\partial\mathbb{D}(z_1,\varepsilon)} \widehat{V}_t(z)\, \mathrm{d}\sigma(z) \leqslant \widehat{V}_t(z_1).$$

Since $z_1 \in \partial D(t)$, we have $\widehat{V}_t(z_1) = V_t(z_1)$, whence $\widetilde{V}_t(z_1) = \widehat{V}_t(z_1)$, and the mean value inequality has been established. The minimality of $\widehat{V}_t$ now forces the equality $\widetilde{V}_t = \widehat{V}_t$. The assertion $D'(t) = D(t)$ is immediate.

The assertion (b) is proved in an analogous fashion. □

2.1. Less smooth obstacles

Suppose for the moment that the weight ω is not as smooth as before, say that we only know it is in $L^p(\Omega)$ for some p, $1 < p < +\infty$, and that $0 \leqslant \omega$ holds throughout Ω. Let us see what conclusions remain from the previous considerations. Clearly, we can still form the potential function V_t, which is of Sobolev class $W^{2,p}$ away from the origin in Ω, and the superharmonic envelope function $\widehat{V}_t$ can also be formed, and it is, by the same arguments from Kinderlehrer and Stampacchia [26], in $W^{2,p}(\Omega)$. The Sobolev–Morrey imbedding theorem shows that $W^{2,p}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$, for some α, $0 < \alpha < 1$ (in fact, for $1 < p < 2$, we can take $\alpha = 2(p-1)/p$). This means that the defining function $U_t = \widehat{V}_t - V_t$ for the sets $D(t)$ is continuous on $\overline{\Omega} \setminus \{0\}$, and hence that the sets $D(t)$ are open, for all t, $0 < t < +\infty$. The same arguments as in Proposition 2.1 show that $\widehat{V}_t = 0$ on $\partial\Omega$. Theorem 2.3 and Propositions 2.4(a), 2.6, 2.7, and 2.9 hold without changes. If ω is bounded away from 0, then Proposition 2.4(b) holds as well, and the comparison argument of Proposition 2.8 shows that $D(t) \Subset \Omega$ for sufficiently small positive t.

3. Basic continuity properties

We work in the context of the previous section. Ω is a Jordan domain with C^∞-smooth boundary. The weight ω is strictly positive and C^∞-smooth in $\overline{\Omega}$. The sets $D(t) = D(t, \omega)$ are defined by (2.1).

Let T be the supremum of all t, $0 < t < +\infty$, for which $D(t)$ is precompact in Ω. The following proposition establishes a continuity property of the weighted Hele–Shaw flow.

Proposition 3.1. *Fix a t, $0 < t < T$. For any given ε, $0 < \varepsilon$, there exists a $\delta = \delta(\varepsilon)$, $0 < \delta < T - t$, such that if t' is confined to the interval $t < t' < t + \delta$, then we have the inclusion*

$$D(t') \subset D(t) + \mathbb{D}(0, \varepsilon) = \big\{ z + \zeta \colon z \in D(t),\ \zeta \in \mathbb{D}(0, \varepsilon) \big\}.$$

Proof. Let $D_\varepsilon(t) = D(t) + \mathbb{D}(0, \varepsilon)$ be the ε-fattened domain, and $D_{2\varepsilon}$, $D_{3\varepsilon}$ be defined similarly. All these domains are open and connected. We suppose that ε is so small that $D_{3\varepsilon}(t) \Subset \Omega$. Further, let $\varpi_{t,\varepsilon}$ stand for harmonic measure (supported on the boundary) for the domain $D_\varepsilon(t)$ with respect to the interior point 0. Let $u \in W^2(\Omega)$ be real-valued and subharmonic in $D_{2\varepsilon}(t)$. By the mean value inequality for subharmonic functions, we then have

$$u(0) \leqslant \int\limits_{\partial D_\varepsilon(t)} u(z)\, \mathrm{d}\varpi_{t,\varepsilon}(z). \tag{3.1}$$

Let ψ be a real-valued C^∞-smooth function in $\mathbb{C}$, subject to the following restrictions:

- ψ is radial: $\psi(z) = \psi(|z|)$,
- $0 \leqslant \psi$ throughout $\mathbb{C}$,
- $0 < \psi(z)$ holds if and only if $z \in \mathbb{D}$, and
- $\int_{\mathbb{C}} \psi(z)\, \mathrm{d}\Sigma(z) = 1$.

We now define the dilated function ψ_ε:

$$\psi_\varepsilon(z) = \varepsilon^{-2} \psi\big(\varepsilon^{-1} z\big), \quad z \in \mathbb{C}.$$

We use it to mollify the harmonic measure ϖ_ε, setting

$$\nu_{t,\varepsilon}(z) = \psi_\varepsilon * \varpi_{t,\varepsilon}(z) = \int\limits_{\partial D_\varepsilon(t)} \psi_\varepsilon(z - \zeta)\, \mathrm{d}\varpi_{t,\varepsilon}(\zeta), \quad z \in \mathbb{C},$$

which expression represents a positive C^∞-smooth function with support contained in $\overline{D_{2\varepsilon}(t)} \setminus D(t)$. By Sobolev's imbedding theorem, the function u is continuous on $\overline{\Omega}$. From the submean value property for circles and the radial symmetry of the mollifier ψ_ε, we obtain

$$u(\zeta) \leqslant \int\limits_{\mathbb{D}(0,\varepsilon)} u(\zeta + z) \psi_\varepsilon(z)\, \mathrm{d}\Sigma(z) = \int\limits_{\Omega} u(z) \psi_\varepsilon(z - \zeta)\, \mathrm{d}\Sigma(z), \quad \zeta \in \overline{D_\varepsilon(t)},$$

whence it follows that

$$\int\limits_{\partial D_\varepsilon(t)} u(\zeta)\, \mathrm{d}\varpi_{t,\varepsilon}(\zeta) \leqslant \int\limits_{\partial D_\varepsilon(t)} \int\limits_{\Omega} u(z) \psi_\varepsilon(z - \zeta)\, \mathrm{d}\Sigma(z)\, \mathrm{d}\varpi_{t,\varepsilon}(\zeta) = \int\limits_{\Omega} u(z) \nu_{t,\varepsilon}(z)\, \mathrm{d}\Sigma(z).$$

As we combine this with (3.1), we arrive at

$$u(0) \leqslant \int\limits_{\Omega} u(z) \nu_{t,\varepsilon}(z)\, \mathrm{d}\Sigma(z). \tag{3.2}$$

Note that the mollifier ψ_ε can be assumed to enjoy the estimate $\sup_{\mathbb{C}} \psi_\varepsilon \leqslant 2\varepsilon^{-2}$, which leads to the same behavior for $\nu_{t,\varepsilon}$: $\sup_{\mathbb{C}} \nu_{t,\varepsilon} \leqslant 2\varepsilon^{-2}$. The weight ω is bounded away from 0 in Ω, and so the quantity

$$\theta(\varepsilon) = \sup_{z \in \mathbb{C}} \frac{\nu_{t,\varepsilon}(z)}{\omega(z)}$$

is finite for any ε; in fact, it has the asymptotics $\theta(\varepsilon) = \mathrm{O}(\varepsilon^{-2})$ as $\varepsilon \to 0$.

Now, we introduce a new weight

$$\omega_{t,\varepsilon}(z) = \omega(z)1_{D(t)}(z) + \theta(\varepsilon)^{-1}\nu_{t,\varepsilon}(z) + \omega(z)1_{\Omega\setminus D_{2\varepsilon}(t)}(z), \quad z \in \Omega,$$

which is smaller than the original weight ω. It is less regular than ω, but Propositions 2.4(a) and 2.7 are applicable, by the remarks of the previous section. In view of (3.2) and the moment inequality property of $D(t)$ (see Theorem 2.3), we get

$$\bigl(t + \theta(\varepsilon)^{-1}\bigr)u(0) \leqslant \int_{D_{2\varepsilon}(t)} u(z)\omega_{t,\varepsilon}(z)\,\mathrm{d}\Sigma(z),$$

for all $u \in W^2(\Omega)$ that are subharmonic in $D_{2\varepsilon}(t)$. This, however, is the moment inequality for the weight $\omega_{t,\varepsilon}$, which shows, by Proposition 2.4(a), that

$$D\bigl(t + \theta(\varepsilon)^{-1}; \omega_{t,\varepsilon}\bigr) \subset \operatorname{int} \overline{D_{2\varepsilon}(t)}.$$

By the comparison principle (Proposition 2.7), we have the inclusion

$$D\bigl(t + \theta(\varepsilon)^{-1}; \omega\bigr) \subset D\bigl(t + \theta(\varepsilon)^{-1}; \omega_{t,\varepsilon}\bigr) \subset \operatorname{int} \overline{D_{2\varepsilon}(t)} \subset D_{3\varepsilon}(t).$$

The assertion is now immediate. □

It is a consequence of Proposition 3.1 that the reason why the flow stops at the parameter value $t = T$ is that then, the boundary $\partial D(t)$ hits the outer boundary $\partial\Omega$. We also need to know that the flow moves at a positive speed, at least in a situation with fairly regular boundary.

Proposition 3.2. *Fix t, $0 < t < T$, and suppose that the flow domain $D(t)$ is simply connected with C^2-smooth boundary, with the possible exception of finitely many so-called contact points. Near each contact point, we assume the boundary consists of two C^2-smooth curves tangent to each other at the point, and that $D(t)$ is what remains when we cut out the thin two-sided wedge located between the two curves.*

Then, to each given δ, $0 < \delta < T - t$, there exists an $\varepsilon = \varepsilon(\delta)$, $0 < \varepsilon$, such that – up to sets of zero area – we have the inclusion

$$D_\varepsilon(t) \subset D(t + \delta),$$

where $D_\varepsilon(t) = D(t) + \mathbb{D}(0, \varepsilon)$.

The geometric situation is illustrated in Fig. 1.

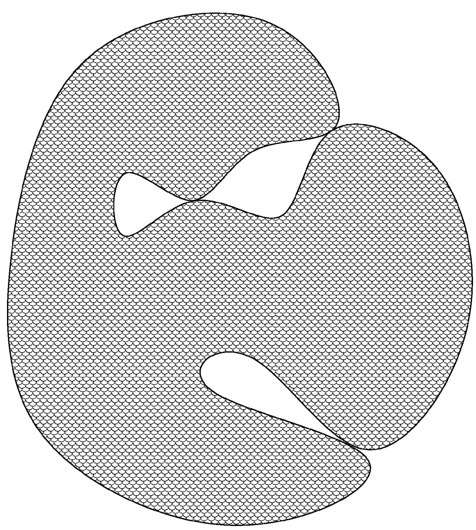

Fig. 1. A flow domain $D(t)$ with three contact points.

Proof. For ε with $0<\varepsilon$, note that we can actually write

$$D_\varepsilon(t)=\overline{D(t)}+\mathbb{D}(0,\varepsilon),$$

and suppose ε is so small that $D_\varepsilon(t)\Subset\Omega$. Let ϖ denote the harmonic measure on $\partial D(t)$ corresponding to the domain $D(t)$ and the interior point 0. The C^2-smoothness of $\partial D(t)$ (although somewhat degenerate at the contact points) entails that ϖ is comparable to normalized arc length measure σ, in symbols $\varpi\asymp\sigma|_{\partial D(t)}$, in the sense that there exist positive constants A and B such that

$$A\sigma|_{\partial D(t)}\leqslant\varpi\leqslant B\sigma|_{\partial D(t)}.$$

To see this, we can use the conformal invariance of harmonic measure, and the Kellogg–Warschawski theorem on conformal maps [31, p. 49]. Let ψ_ε be the function $\psi_\varepsilon(z)=\varepsilon^{-2}\psi(\varepsilon^{-1}z)$, where

$$\psi(z)=\frac{1}{2}\big(1-|z|^2\big)^{-1/2}1_{\mathbb{D}}(z),\quad z\in\mathbb{C}.$$

The function ψ_ε is positive and supported on $\overline{\mathbb{D}}(0,\varepsilon)$, and it has $L^1(\mathbb{C})$-norm 1; it will serve as a mollifier for our purposes. We form the convolution

$$\nu_\varepsilon(z)=\psi_\varepsilon*\varpi(z)=\int\limits_{\partial D(t)}\psi_\varepsilon(z-\zeta)\,\mathrm{d}\varpi(\zeta),\quad z\in\mathbb{C},$$

and, as we shall see later, the mollifier ψ_ε is tailored in such a way that the function ν_ε is bounded away from 0 on the open subset $\partial D(t)+\mathbb{D}(0,\varepsilon)$ of Ω, which constitutes a sort of "snake" around $\partial D(t)$; the function ν_ε vanishes off the snake. To be more precise, we shall see that there exist two positive constants A and B (not the same as above), independent of ε, such that

$$\frac{A}{\varepsilon}\leqslant\nu_\varepsilon(z)\leqslant\frac{B}{\varepsilon},\quad z\in\partial D(t)+\mathbb{D}(0,\varepsilon).\tag{3.3}$$

For the moment we shall assume that the above estimate is valid, and indicate how to proceed to obtain the desired assertion. We consider the quantity

$$\theta(\varepsilon)=\inf\left\{\frac{\nu_\varepsilon(z)}{\omega(z)}:\ z\in D_\varepsilon(t)\setminus D(t)\right\},$$

and observe that $D_\varepsilon(t)\setminus D(t)$ is contained in the snake $\partial D(t)+\mathbb{D}(0,\varepsilon)$, so that by estimate (3.3) and the fact that ω is bounded, this function is nonzero. In fact, it behaves like ε^{-1} as $\varepsilon\to0$.

As in the previous proof, we introduce a new weight

$$\omega_\varepsilon(z)=\omega(z)1_{D(t)}(z)+\theta(\varepsilon)^{-1}\nu_\varepsilon(z)+\omega(z)1_{\Omega\setminus D_\varepsilon(t)}(z),\quad z\in\Omega,$$

which is larger than ω. It is bounded away from zero in Ω, so that Proposition 2.4 is applicable. As in the proof of Proposition 3.1, we have the moment inequality for ν_ε,

$$u(0)\leqslant\int\limits_{\Omega}u(z)\nu_\varepsilon(z)\,\mathrm{d}\Sigma(z),$$

valid for all functions $u\in W^2(\Omega)$ that are subharmonic in $D_\varepsilon(t)$. In view of the fact that ν_ε vanishes off the snake $\partial D(t)+\mathbb{D}(0,\varepsilon)$, and, hence, off $D_\varepsilon(t)$, it follows from the moment inequality for $D(t)$ (see Theorem 2.3) that

$$\big(t+\theta(\varepsilon)^{-1}\big)u(0)\leqslant\int\limits_{D_\varepsilon(t)}u(z)\omega_\varepsilon(z)\,\mathrm{d}\Sigma(z)$$

holds, for all $u\in W^2(\Omega)$ that are subharmonic in $D_\varepsilon(t)$. This is the moment inequality for the weight ω_ε, and applying Proposition 2.4(b) (with $D_\varepsilon(t)$ instead of D and ω_ε instead of ω), and Proposition 2.7, we get – up to sets of zero area – the inclusion

$$D_\varepsilon(t)=D\big(t+\theta(\varepsilon)^{-1};\omega_\varepsilon\big)\subset D\big(t+\theta(\varepsilon)^{-1};\omega\big).$$

The assertion of the proposition is immediate from this.

We turn to the technical work of verifying the estimate (3.3). Since $\varpi\asymp\sigma|_{\partial D(t)}$, it suffices to obtain it with μ_ε in place of ν_ε, where

$$\mu_\varepsilon(z)=\psi_\varepsilon*\sigma|_{\partial D(t)}(z)=\int\limits_{\partial D(t)}\psi_\varepsilon(z-\zeta)\,\mathrm{d}\sigma(\zeta),\quad z\in\mathbb{C}.$$

Let $\gamma(z,t,\varepsilon)$ be the curve $\{\zeta\in\mathbb{C}\colon\ z+\varepsilon\zeta\in\partial D(t)\}$, which is a magnified and translated version of the boundary $\partial D(t)$. A change of variables yields the identity

$$\mu_\varepsilon(z)=\frac{1}{2\varepsilon}\int\limits_{\mathbb{D}\cap\gamma(z,t,\varepsilon)}\frac{\mathrm{d}\sigma(\zeta)}{(1-|\zeta|^2)^{1/2}},\quad z\in\mathbb{C},$$

with the usual agreement that the integral over the empty set is 0. The requirement that $z\in D(t)+\mathbb{D}(0,\varepsilon)$ is equivalent to having $\mathbb{D}\cap\gamma(z,t,\varepsilon)\neq\emptyset$, so that we need to show that

$$A\leqslant\int\limits_{\mathbb{D}\cap\gamma(z,t,\varepsilon)}\frac{\mathrm{d}\sigma(\zeta)}{(1-|\zeta|^2)^{1/2}}\leqslant B \tag{3.4}$$

holds for some positive constants A, B, whenever the integration is over a nonempty set. The set $\mathbb{D}\cap\gamma(z,t,\varepsilon)$ then consists of finitely many curve segments, each of which enters at some point of $\mathbb{T}$ and exits at another. We need only be concerned with very small ε, in which case the curve segments of $\mathbb{D}\cap\gamma(z,t,\varepsilon)$ are pretty much straight lines, being blow-ups of C^2-smooth curves. As a matter of fact, unless we are blowing up near a contact point, there is only one curve, and near a contact point, we have two, so "finitely many" can be replaced by "one or two". If there are two curves, it is enough to obtain an estimate (3.4) for each of them, so it is enough to treat the case of a single curve segment. The curvature of the curve segment $\gamma^\sharp=\mathbb{D}\cap\gamma(z,t,\varepsilon)$ is uniformly of the size $\mathrm{O}(\varepsilon)$ as $\varepsilon\to0$. We recall that if we parametrize $\mathbb{D}\cap\gamma(z,t,\varepsilon)$ by $\zeta=\gamma(t)$, where t runs over a bounded open interval I of $\mathbb{R}$, in such a way that the parametrization is at constant speed 1, $|\dot\gamma(t)|\equiv1$, then the curvature is expressed by $|\ddot\gamma(t)|$ (we use dots to indicate differentiation with respect to t). We fix the parameter interval $I=]t_0,t_1[\subset\mathbb{R}$ by requiring that $t_0<0<t_1$ and that $\gamma(0)=\min_{t\in I}|\gamma(t)|$. We calculate

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigl(|\gamma(t)|^2\bigr)=2\bigl|\dot\gamma(t)\bigr|^2+2\operatorname{Re}\ddot\gamma(t)\bar\gamma(t)=2+2\operatorname{Re}\ddot\gamma(t)\bar\gamma(t),\quad t\in]t_0,t_1[,$$

where the right-hand side is $2+\mathrm{O}(\varepsilon)$ as $\varepsilon\to0$. Hence, for small ε,

$$\frac{3}{2}\leqslant\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigl(|\gamma(t)|^2\bigr)\leqslant\frac{5}{2},\quad t\in]t_0,t_1[.$$

The first derivative of $|\gamma(t)|^2$ vanishes for $t=0$, so that an integration of the previous estimate yields

$$\frac{3}{2}t\leqslant\frac{\mathrm{d}}{\mathrm{d}t}\bigl(|\gamma(t)|^2\bigr)\leqslant\frac{5}{2}t,\quad t\in[0,t_1[,$$

with an analogous estimate in the remaining interval $]t_0,0]$. At the right endpoint t_1, the curve intersects the unit circle, and we have $|\gamma(t_1)|=1$. Another integration starting from t_1 gives as result that

$$\frac{3}{4}\bigl(t_1^2-t^2\bigr)\leqslant1-\bigl|\gamma(t)\bigr|^2\leqslant\frac{5}{4}\bigl(t_1^2-t^2\bigr),\quad t\in[0,t_1].$$

We have an analogous estimate on the remaining interval $[t_0,0]$. Since

$$\int\limits_0^{t_1}\frac{\mathrm{d}t}{(t_1^2-t^2)^{1/2}}=\int\limits_{t_0}^{0}\frac{\mathrm{d}t}{(t_0^2-t^2)^{1/2}}=\int\limits_0^1\frac{\mathrm{d}t}{(1-t^2)^{1/2}}=\frac{1}{2}\pi,$$

it follows that the integral expression

$$\int\limits_{\gamma^\sharp}\frac{\mathrm{d}\sigma(\zeta)}{(1-|\zeta|^2)^{1/2}}=\int\limits_{t_0}^{t_1}\frac{\mathrm{d}t}{(1-|\gamma(t)|^2)^{1/2}}$$

is kept between $2\pi/\sqrt{5}$ and $2\pi/\sqrt{3}$, which accomplishes the proof. □

4. Regularity of the boundary

In this section we prove that in the case of real-analytic weight function ω the domains $D(t)$ have regular boundaries which are represented as unions of a finite number of real analytic simple curves, having at most a finite number of double and cusp points. The proof is based on the work of Sakai [35], classifying all possible singularities of the boundary of a domain possessing a Schwarz function.

If $D \subset \mathbb{C}$ is a domain with the boundary ∂D, and $z_1 \in \partial D$, then a complex valued function S defined and continuous in $N(z_1) \cap \overline{D}$, where $N(z_1)$ is some open neighborhood of z_1, is called a *local Schwarz function* for D near z_1, if S is holomorphic in $N(z_1) \cap D$ and $S(z) = \bar{z}$ on $N(z_1) \cap \partial D$. In his Acta paper [35], Sakai proved that for a domain D possessing a local Schwarz function near any boundary point, all boundary points are classified as

- *isolated points*;
- *regular boundary points*, where nearby ∂D is a real-analytic curve, and D is situated on one side of the curve;
- *interior regular boundary points*, where nearby ∂D is an infinite (closed) subset of a real-analytic curve, and D is on both sides of the curve;
- *regular contact points*, where nearby ∂D consists of two real-analytic curves, tangent to each other at the point, and D is the complement of the thin wedgelike set between the curves;
- *cusp points*, where nearby ∂D is the image of a real-analytic curve under a second degree polynomial mapping, which is such that it produces a cusp at the point in question; the set D is located to the one side of the cusp, with the cusp pointing inward toward D.

In order to apply this classification to the boundaries $\partial D(t)$, we should prove the existence of a local Schwarz function; here, we really need to use the assumption that ω is real-analytic. The next proposition establishes this not only for $D(t)$, but also for arbitrary domains satisfying the mean value identity (2.4).

Proposition 4.1. *Assume that the weight function ω is real-analytic in Ω. Then*

(a) *If the domain $D(t)$ defined by* (2.1) *is precompact in Ω, then it possesses a local Schwarz function near any boundary point.*
(b) *If a domain $D \Subset \Omega$, containing the origin, is such that the mean value identity* (2.4) *holds for any function h harmonic in D and W^2-smooth in $\overline{D}$, then there exists a domain $D^* \supset D$ which differs from D by a set of zero area and which possesses a local Schwarz function near any boundary point.*

Proof. We first prove case (b) of the proposition. The passage from D to D^* means some process of regularization: we first have to remove boundary points which are degenerate in some sense (for example, different slits). To do this, we consider the function

$$U(z) = \int_D G(z,\zeta)\omega(\zeta)\,\mathrm{d}\Sigma(\zeta) - tG(z,0), \quad z \in \Omega.$$

Obviously, U vanishes off $\overline{D}$. We claim that it vanishes also almost everywhere off D. To see this, we take an auxiliary function

$$\phi(z) = \int_{\overline{D}\setminus D} G(z,\zeta)\,\mathrm{sign}\big[U(\zeta)\big]\,\mathrm{d}\Sigma(\zeta).$$

It is then harmonic in D and W^2-smooth in $\overline{D}$, and we have

$$0 = \int_D \phi\omega\,\mathrm{d}\Sigma - t\phi(0) = \int_\Omega \Delta U\phi\,\mathrm{d}\Sigma = \int_\Omega U\Delta\phi\,\mathrm{d}\Sigma = \int_{\overline{D}\setminus D} |U|\,\mathrm{d}\Sigma.$$

Since U is $C^{1,\alpha}$-regular off the origin for each $\alpha \in (0,1)$, we have also that $\nabla U = 0$ almost everywhere in $\Omega \setminus D$. If we now set

$$D^* = D \cup \big\{z\colon U(z) \neq 0 \text{ or } \nabla U \neq 0\big\},$$

then this new domain differs from D by an area-null set.

Next, we show the existence of a local Schwarz function for the domain D^*. Fix a point $z_1 \in \partial D^*$. The weight ω, being real-analytic, has a convergent power series expansion in some neighbourhood $N(z_1)$ of z_1:

$$\omega(z) = \sum_{m,n=0}^{\infty} \widehat{\omega}(m,n)(z-z_1)^m(\bar{z}-\bar{z}_1)^n.$$

Let

$$W(z)=\sum_{m,n=0}^{\infty}\frac{\widehat{\omega}(m,n)}{(m+1)(n+1)}(z-z_1)^{m+1}(\bar z-\bar z_1)^{n+1},$$

for z near z_1, and observe that $\Delta W(z)=\omega(z)$ there. The function $\partial_z U(z)$ is of regularity class $C^{0,\alpha}$, for $\alpha\in(0,1)$, away from 0 on $\overline{\Omega}$, in particular in $N(z_1)$, provided that the given neighborhood is small. Let R be the function

$$R(z)=\bar z_1+\frac{1}{\omega(z_1)}\partial_z\bigl(W(z)-U(z)\bigr),$$

which is well defined near z_1 and of regularity class $C^{0,\alpha}$ there. The $\bar\partial$ derivative of R is

$$\bar\partial_z R(z)=\frac{1}{\omega(z_1)}\bar\partial_z\partial_z\bigl(W(z)-U(z)\bigr)=\frac{1}{\omega(z_1)}\Delta_z\bigl(W(z)-U(z)\bigr)=\frac{\omega(z)}{\omega(z_1)}1_{\Omega\setminus D^*}(z),$$

for z near z_1. In particular, if $N(z_1)$ is a small neighborhood of z_1, the function R is holomorphic in $D^*\cap N(z_1)$. By the construction of the domain D^*, the gradient ∇U vanishes at any point of ∂D^*, so that for $z\in\partial D^*\cap N(z_1)$,

$$\begin{aligned}R(z)&=\bar z_1+\frac{1}{\omega(z_1)}\partial_z W(z)=\bar z_1+\frac{1}{\omega(z_1)}\sum_{m,n=0}^{\infty}\frac{\widehat{\omega}(m,n)}{n+1}(z-z_1)^m(\bar z-\bar z_1)^{n+1}=\bar z_1+\bar z-\bar z_1+\mathrm{O}\bigl(|z-z_1|^2\bigr)\\&=\bar z+\mathrm{O}\bigl(|z-z_1|^2\bigr),\end{aligned}\tag{4.1}$$

where $\mathrm{O}(|z-z_1|^2)$ stands for a real-analytic function of the given magnitude. Let us write $T(z,\bar z)$ for the real-analytic function near z_1 expressed by the right-hand side of (4.1), with notation that emphasizes the separate dependence of z and $\bar z$; we think of T as a holomorphic function of two complex variables near $(z_1,\bar z_1)\in\mathbb{C}^2$. We recapture what we know about the function R: for some small neighborhood $N(z_1)$ of $z_1\in\Omega\cap\partial D^*$, we have that R is Hölder continuous there; moreover, on $N(z_1)\cap D^*$, R is holomorphic, and on $N(z_1)\setminus D^*$,

$$R(z)=T(z,\bar z)=\bar z+\mathrm{O}\bigl(|z-z_1|^2\bigr).$$

By the implicit function theorem, if $N(z_1)$ is small, there exists a Hölder continuous function S on $N(z_1)$ such that $R(z)=T(z,S(z))$, which is then holomorphic on $N(z_1)\cap D^*$. This is the sought-after Schwarz function. The criterion that allows us to invoke the implicit function theorem is that $\bar\partial_z T(z_1,\bar z_1)=1\neq 0$.

In the case (a), the proof follows the same arguments, but we use the function U_t defined by (2.2) instead of U. Clearly, U_t vanishes on $\partial D(t)$, and ∇U_t vanishes there since at any point of $\partial D(t)$, the function U_t has a local minimum. The analyticity of R is a consequence of Proposition 2.2. □

In [35], Makoto Sakai mentions that the above construction of a Schwarz function is possible. We have merely filled in the details.

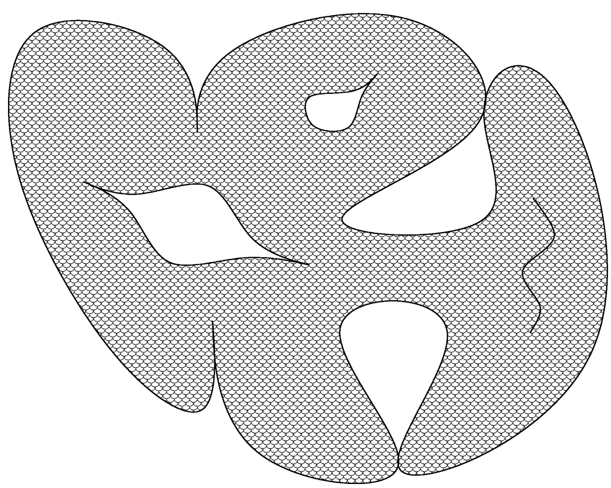

Fig. 2. A generic flow domain $D(t)$ according to Sakai.

We see that in the case of real-analytic weight ω, the boundary $\partial D(t)$ is very regular. It is certainly possible for $D(t)$ to be multiply connected, but the holes have to be pretty well-behaved. In fact, there may be at most finitely many holes with nonempty interior, and the rest of the holes are of the types finitely many (subsets of) interior real-analytic arcs, and finitely many isolated points (the ones not accounted for already). See Fig. 2 for an illustration of what $D(t)$ may look like.

5. Logarithmically subharmonic weights

In this section, we add the requirement that the weight function ω is logarithmically subharmonic, which corresponds to negative Gaussian curvature for the Riemannian metric (1.1). We also assume that the underlying domain Ω is simply connected and recall that T is the supremum of those $t > 0$ for which $D(t) \Subset \Omega$. We shall show that for $t \in]0, T[$, all the domains $D(t)$ are real-analytic Jordan domains, and that they are uniquely characterized by the mean value identity (2.4). We also prove that for $T < t < +\infty$, there exists no Jordan domain D satisfying this identity.

Throughout this section, we assume that Ω is a C^∞-smooth Jordan domain and ω is strictly positive, real-analytic and logarithmically subharmonic in Ω.

5.1. *The biharmonic Green function*

We shall use the techniques of the biharmonic boundary value problems in the unit disk. Let $\Gamma(z,\zeta)$, $z,\zeta \in \mathbb{D}$, stand for the biharmonic Green function in $\mathbb{D}$, i.e., the solution of the boundary value problem

$$\begin{cases} \Delta_z^2 \Gamma(z,\zeta) = \delta_\zeta(z) & \text{for } z \in \mathbb{D} \text{ (in the sense of distributions);} \\ \Gamma(z,\zeta) = \left|\nabla_z \Gamma(z,\zeta)\right| = 0 & \text{for } z \in \mathbb{T} = \partial\mathbb{D}. \end{cases}$$

The well-known explicit formula for $\Gamma(z,\zeta)$ is

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2 + \left(1-|z|^2\right)\left(1-|\zeta|^2\right), \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D}$$

(see [12, Chapter 7]). It is also known that $0 < \Gamma(z,\zeta)$ everywhere in $\mathbb{D}\times\mathbb{D}$.

An application of the Green formula shows that an arbitrary function $Y(z)$ which is C^4-smooth in $\overline{\mathbb{D}}$ and vanishes on the boundary $\mathbb{T} = \partial\mathbb{D}$ can be represented as

$$Y(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 Y(\zeta)\,\mathrm{d}\Sigma(\zeta) + \frac{1}{2}\int_{\mathbb{T}} H(\zeta,z)\partial_n Y(\zeta)\,\mathrm{d}\sigma(\zeta), \quad z \in \mathbb{D}, \tag{5.1}$$

where $\partial_n Y(\zeta)$ denotes the normal derivative of Y in the interior direction, and the function $H(\zeta,z)$ is given as

$$H(\zeta,z) = \Delta_\zeta \Gamma(z,\zeta) = \left(1-|z|^2\right)\frac{1-|z\zeta|^2}{|1-\bar{z}\zeta|^2}, \quad (\zeta,z) \in \mathbb{T}\times\mathbb{D}.$$

We see that the function $H(\zeta,z)$ is also positive everywhere in $\mathbb{T}\times\mathbb{D}$.

We shall use also the Green function $G_{\mathbb{D}}(z,\zeta)$ for the Laplacian in $\mathbb{D}$:

$$G_{\mathbb{D}}(z,\zeta) = \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2.$$

The assertion of the next lemma can be found essentially in [8,17].

Lemma 5.1. *Assume that a positive function $\nu \in C^\infty(\overline{\mathbb{D}})$ is subharmonic in $\mathbb{D}$ and such that the identity*

$$\int_{\mathbb{D}} h(z)\nu(z)\,\mathrm{d}\Sigma(z) = h(0) \tag{5.2}$$

holds for any harmonic polynomial h. Then

(a) *for any function $u \in W^2(\mathbb{D})$ subharmonic in $\mathbb{D}$,*

$$\int_{\mathbb{D}} u(z)\,\mathrm{d}\Sigma(z) \leqslant \int_{\mathbb{D}} u(z)\nu(z)\,\mathrm{d}\Sigma(z); \tag{5.3}$$

(b) *for any $\zeta \in \mathbb{T}$, we have $\nu(\zeta) \geqslant 1$.*

Proof. To prove (a), we consider the function

$$\Phi(z) = \int_{\mathbb{D}} G_{\mathbb{D}}(z,\zeta)\big(\nu(\zeta)-1\big)\,\mathrm{d}\Sigma(\zeta).$$

We have $\Phi|_{\mathbb{T}} = 0$, and since $\Delta\Phi = \nu - 1$ annihilates harmonic polynomials, we have also $\nabla\Phi|_{\mathbb{T}} = 0$. Therefore, Φ can be represented as

$$\Phi(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2\Phi(\zeta)\,\mathrm{d}\Sigma(\zeta) = \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta\nu(\zeta)\,\mathrm{d}\Sigma(\zeta),$$

and, therefore, it is positive in $\mathbb{D}$. We have then

$$\int_{\mathbb{D}} u(z)\big(\nu(z)-1\big)\,\mathrm{d}\Sigma(z) = \int_{\mathbb{D}} u(z)\Delta\Phi(z)\,\mathrm{d}\Sigma(z) = \int_{\mathbb{D}} \Delta u(z)\Phi(z)\,\mathrm{d}\Sigma(z) \geqslant 0,$$

which proves (a).

To prove (b), it suffices to substitute functions $u_n(z) = |r(z)|^{2n}/\|r^n\|_{L^2(\mathbb{D})}$, where $r(z) = (1+\bar\zeta z)/2$, to (5.3) and then let n tend to $+\infty$ (see [17, Proposition 1.3]). □

We shall also need the next lemma.

Lemma 5.2. *Let Y be a C^∞-smooth real-valued function on $\overline{\mathbb{D}}\setminus\{0\}$, with a logarithmic singularity at the origin, such that $\Delta^2 Y = \Delta\delta_0 - \mu$ in $\mathbb{D}$, where $\mu \in C^\infty(\overline{\mathbb{D}})$ has $0 \leqslant \mu$ in $\mathbb{D}$. Suppose that $Y|_{\mathbb{T}} = 0$, and that $\partial_n Y \leqslant 0$ on $\mathbb{T}$. Then*

$$Y(z) \leqslant \log|z|^2 + 1 - |z|^2 < 0, \quad z \in \mathbb{D}.$$

Proof. We represent the function Y by formula (5.1):

$$Y(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 Y(\zeta)\,\mathrm{d}\Sigma(\zeta) + \frac{1}{2}\int_{\mathbb{T}} H(\zeta,z)\partial_n Y(\zeta)\,\mathrm{d}\sigma(\zeta),$$

where we think of the integration in the sense of distributions. As the function $H(\zeta,z)$ is positive, the expression on the right-hand side only gets larger if we drop the second term:

$$Y(z) \leqslant \int_{\mathbb{D}} \Gamma(z,\zeta)\big(\Delta\delta_0(\zeta) - \mu(\zeta)\big)\,\mathrm{d}\Sigma(\zeta) = \Delta_\zeta\Gamma(z,0) - \int_{\mathbb{D}} \Gamma(z,\zeta)\mu(\zeta)\,\mathrm{d}\Sigma(\zeta) \leqslant \Delta_\zeta\Gamma(z,0) = \log|z|^2 + 1 - |z|^2,$$

where we have also used the fact that Γ is positive. The proof is complete. □

5.2. Ruling out holes

We assume that $D(t) \Subset \Omega$. Let $D_\bullet(t)$ stand for the simply connected domain obtained from $D(t)$ by adding all the interior holes, both the ones with nontrivial interior and the ones that are parts of arcs as well as the isolated points. Since Ω is simply connected, we have $D_\bullet(t) \Subset \Omega$. The boundary $\partial D_\bullet(t)$ is a closed real-analytic curve, with the possible exception of finitely many contact and cusp points. Let $\phi : \mathbb{D} \to D_\bullet(t)$ be the Riemann mapping which sends 0 to 0 (it is unique up to rotations of $\mathbb{D}$). By the regularity of the boundary, ϕ extends analytically to a neighborhood of $\overline{\mathbb{D}}$, with the cusp points corresponding to simple zeros of ϕ'. We then consider the domain $B = \phi^{-1}(D(t))$, and note that $\mathbb{D}\setminus B$ is a compact subset of $\mathbb{D}$. In general, B may look like what is illustrated by Fig. 3. *We shall prove that* $B = \mathbb{D}$. Introducing the weight $\nu = t^{-1}\omega\circ\phi|\phi'|^2$, which is logarithmically subharmonic in $\mathbb{D}$, we obtain from Theorem 2.3 that

$$h(0) = \int_B h(z)\nu(z)\,\mathrm{d}\Sigma(z), \tag{5.4}$$

for all functions h of the form $h = g\circ\phi$, with $g \in W^2(\Omega)$ harmonic in $D(t)$. We also have the corresponding inequality for subharmonic functions. The new weight ν is real-analytic in $\overline{\mathbb{D}}$, with zeros at the finitely many points of $\mathbb{T}$ corresponding to the cusps; elsewhere, it is strictly positive.

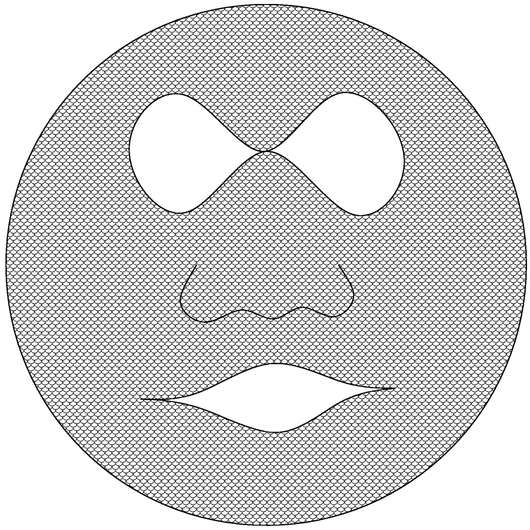

Fig. 3. The mapped flow domain B.

We can interpret the domain B as appearing from an obstacle problem. After all, the function $V_t \circ \phi$ has the least superharmonic majorant in $\mathbb{D}$, and one checks that $\widehat{V}_t \circ \phi$ is that majorant (by Proposition 2.9(a), with $\Omega' = D_\bullet(t)$, we have that $\widehat{V}_t$ is the least superharmonic majorant to V_t in $D_\bullet(t)$; the conformal invariance of the operation of taking the least superharmonic majorant then proves the claim). We calculate that

$$\Delta(V_t \circ \phi)(z) = \Delta V_t\big(\phi(z)\big)\big|\phi'(z)\big|^2 = t\delta_0(z) - \omega \circ \phi(z)\big|\phi'(z)\big|^2, \quad z \in \mathbb{D},$$

so that if we define

$$W(z) = \log|z|^2 - \int_{\mathbb{D}} G_{\mathbb{D}}(z,\zeta)\nu(\zeta)\,\mathrm{d}\Sigma(\zeta), \quad z \in \mathbb{D},$$

we obtain $W = t^{-1}V_t \circ \phi - t^{-1}P[V_t \circ \phi|_{\mathbb{T}}]$, where in general, $P[f]$ expresses the harmonic extension to $\mathbb{D}$ via the Poisson integral formula of a function f given on the boundary $\mathbb{T}$. Let $\widehat{W}$ stand for the least superharmonic majorant to W on $\mathbb{D}$; by the above, $\widehat{W} = t^{-1}\widehat{V}_t \circ \phi - t^{-1}P[V_t \circ \phi|_{\mathbb{T}}]$.

For $1 \leqslant r < +\infty$, let

$$W_r(z) = r\log|z|^2 - \int_{\mathbb{D}} G_{\mathbb{D}}(z,\zeta)\nu(\zeta)\,\mathrm{d}\Sigma(\zeta), \quad z \in \mathbb{D},$$

and let $\widehat{W}_r$ be its least superharmonic majorant on $\mathbb{D}$. We are interested in the open sets

$$B(r) = \big\{z \in \mathbb{D}:\ W_r(z) < \widehat{W}_r(z)\big\}.$$

By Proposition 2.6, the set $B(r)$ is connected for each r, and by Proposition 2.7, it increases with r. The left end point $r = 1$ corresponds to dropping the parameter: $W_1 = W$ and $B(1) = B$. The function W_r vanishes on the unit circle $\mathbb{T}$. If $W_r(z) \leqslant 0$ throughout $\mathbb{D}$, then the least superharmonic majorant is trivially $\widehat{W}_r(z) \equiv 0$. In this case, we also have $\partial_n W_r(z) \leqslant 0$ on $\mathbb{T}$ (interior normal derivative). Lemma 5.2, applied to the function $Y = r^{-1}W_r$, provides a converse to this statement: since $\Delta^2 W_r = r\Delta\delta_0 - \Delta\nu$, and ν is subharmonic, we obtain that the condition $\partial_n W_r|_{\mathbb{T}} \leqslant 0$ implies

$$W_r(z) \leqslant r\big(\log|z|^2 + 1 - |z|^2\big) < 0, \quad z \in \mathbb{D}; \tag{5.5}$$

and in this case we have $B(r) = \mathbb{D}$. In other words,

- $\sup_{\mathbb{D}} W_r \leqslant 0$ if and only if $\sup_{\mathbb{T}} \partial_n W_r \leqslant 0$, and
- if $\sup_{\mathbb{T}} \partial_n W_r \leqslant 0$, then $W_r < 0$ on $\mathbb{D}$, and $B(r) = \mathbb{D}$.

If $\sup_{\mathbb{T}} \partial_n W_r \leqslant 0$ holds for $r = 1$, then we are done, for then $B = B(1) = \mathbb{D}$. So, let us suppose instead that $0 < \sup_{\mathbb{T}} \partial_n W_1$. The formula defining W_r yields

$$\partial_n W_r(z) = -2(r-1) + \partial_n W_1(z), \quad z \in \mathbb{T},$$

and since $\partial_n W_1$ is in $C^\infty(\mathbb{T})$ and real-valued, there exists a critical value $r = r_1$, $1 < r_1 < +\infty$, such that $0 < \sup_{\mathbb{T}} \partial_n W_r$ for $1 \leqslant r < r_1$ and $\sup_{\mathbb{T}} \partial_n W_r \leqslant 0$ for $r_1 \leqslant r < +\infty$; in fact, $r_1 = 1 + \frac{1}{2}\sup_{\mathbb{T}} \partial_n W_1$.

The set $\mathbb{D} \setminus B(r)$ is the coincidence set for the obstacle problem, and each point of $\mathbb{D}$ where the smallest concave majorant of W_r touches the graph of the function definitely belongs to this set. In particular, any point of $\mathbb{D}$ where W_r attains a positive maximum value is in $\mathbb{D} \setminus B(r)$. Since we know that $\mathbb{D} \setminus B \Subset \mathbb{D}$ from Sakai's classification, and $\mathbb{D} \setminus B(r)$ gets smaller as r increases, it follows that any such maximum point is in the compact $\mathbb{D} \setminus B$. For each r with $1 \leqslant r < r_1$, the function W_r attains a positive maximum on $\mathbb{D}$. Let us say that the maximum is attained at the interior point $z(r)$, which then must belong to $\mathbb{D} \setminus B$. We choose a sequence ρ_j, $j = 1, 2, 3, \ldots$, with $1 < \rho_j < r_1$, and limit $\rho_j \to r_1$ as $j \to +\infty$. The points $z_j = z(\rho_j)$ are in $\mathbb{D} \setminus B$, and a subsequence of them tends to a point $z_\infty \in \mathbb{D} \setminus B$ (by compactness). We have that $0 < W_{\rho_j}(z_j)$, so that in the limit $0 \leqslant W_{r_1}(z_\infty)$. But for $r = r_1$, (5.5) holds, which does not permit such a point z_∞ to exist. So, something must be wrong, and that something is the assumption that B was not all of $\mathbb{D}$.

5.3. Ruling out cusps and contact points

We work in the context of the previous subsection. We have proved that $B = \mathbb{D}$, and hence that $D(t)$ is simply connected. We shall now demonstrate that the boundary of $D(t)$ fails to have cusp points. We know that (5.4) (with $B = \mathbb{D}$) holds for all h of the form $h = g \circ \phi$, with $g \in W^2(\Omega)$ harmonic on $D(t)$. If h is a C^∞-smooth function on $\overline{\mathbb{D}}$, *holomorphic* in $\mathbb{D}$, with the property that it is totally flat (meaning that all derivatives of finite order vanish) at the finitely many points of $\mathbb{T}$ which are mapped onto cusp or contact points under ϕ, then h is of the above form $g \circ \phi$, with a holomorphic g, and (5.4) holds for h. The identity (5.4) is preserved under closure with respect to the norm of $L^1(\mathbb{D})$, and an arbitrary analytic polynomial can be approximated in $L^1(\mathbb{D})$ by functions h from the above-mentioned class. Taking complex conjugates, we obtain that the weight ν possesses the reproducing property (5.2) for arbitrary harmonic polynomial h. By Lemma 5.1(b), we have $\nu(\zeta) \geqslant 1$ for any $\zeta \in \mathbb{T}$, and, in particular, $\phi'(\zeta) \neq 0$ for any $\zeta \in \mathbb{T}$, which shows that the boundary $\partial D(t)$ cannot have any cusps.

It remains to show that we cannot have contact points. But if this happened for some $t = t_0$, then by continuity properties of domains $D(t)$, given by Propositions 3.1 and 3.2, we obtain that for $t' = t_0 + \delta$, with sufficiently small δ, the domain $D(t')$ would have holes. We refer to Fig. 1 for an illustration of the situation. According to Proposition 3.1, the flow domain $D(t')$ is just a little bigger than $D(t)$ for t' close to t, and the only place where topological changes are possible is in small neighborhoods of the set of contact points. On the other hand, by Proposition 3.2, $D(t')$ does indeed contain a whole little neighborhood of each contact point, at least up to sets of zero area measure. It remains to show that if the set of contact points for $D(t)$ is nonempty, the domain $D(t')$ now has to possess holes. Let us study an *outer contact point*, where to the one side, we have the unbounded component of $\mathbb{C} \setminus D(t)$, and, to the other, a bounded one. As the contact point fuses, at least part of the bounded component remains, and it now is a hole, because there is no longer a path to the unbounded component. The existence of holes in $D(t')$ is immediate. But holes are impossible by results of the previous subsection, and this shows that the contact points are also impossible.

5.4. Uniqueness of ω-mean-value disks

We have proved that for $t \in]0, T[$ all $D(t)$ are Jordan domains. By Proposition 2.5, we conclude that they are uniquely – up to addition or removal area-null sets – characterized by the mean value identity (2.4).

Assume now that a Jordan domain $D \Subset \Omega$ satisfies this identity for some $t > T$. By Proposition 4.1(b), there exists a refined domain $D^* \supset D$ which differs from D by an area-null set and possesses a local Schwartz function near any boundary point. But since D was a Jordan domain, D^* must coincide with D, and, therefore, we get the existence of a local Schwartz function for D. By Sakai's classification, the boundary ∂D is a finite union of analytic arcs with possible finite number of singularities in the form of inner cusps. By standard approximation arguments, we obtain that the mean value identity (2.4) holds for an arbitrary function h which is holomorphic in D and continuous in $\overline{D}$. Now, let ϕ be the conformal mapping from $\mathbb{D}$ onto D, such that $\phi(0) = 0$. By regularity of the boundary ∂D, ϕ is analytic in $\overline{\mathbb{D}}$. Introducing a new weight ν in $\overline{\mathbb{D}}$, given as $\nu = t^{-1}\omega \circ \phi \cdot |\phi'|^2$, we see that it satisfies the mean value identity (5.2) for any h holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, and, therefore, for any harmonic polynomial. Applying Lemma 5.1(a), we get, for any function $u \in W^2(\mathbb{D})$ subharmonic in $\mathbb{D}$,

$$u(0) \leqslant \int_{\mathbb{D}} u(z)\,\mathrm{d}\Sigma(z) \leqslant \int_{\mathbb{D}} u(z)\nu(z)\,\mathrm{d}\Sigma(z).$$

Returning to the domain D, we obtain the moment inequality (2.3) for any u which is W^2-smooth in $\overline{D}$ and subharmonic in D. By Proposition 2.4, D differs from $D(t)$ by a set of zero area, which shows that $D(t) \Subset \Omega$ and $t \in]0, T[$. This contradiction shows that for $T < t < +\infty$, there exists no Jordan domain satisfying the mean value identity (2.4).

To summarize all preceding results, we formulate the following theorem.

Theorem 5.3. *Let Ω be a bounded and simply connected domain in $\mathbb{C}$ with C^∞-smooth boundary, and let ω be a weight function which is strictly positive, real-analytic and logarithmically subharmonic in $\overline{\Omega}$. If the domains $D(t)$ are defined by* (2.1)*, then there exists a positive number T such that $D(t) \Subset \Omega$ for $t \in]0, T[$, and that this fails for $t \in [T, +\infty[$. For $t \in]0, T[$,*

- *the domains $D(t)$ increase continuously with t, and a portion of $\partial D(t)$ approaches $\partial\Omega$ as $t \to T$;*
- *all $D(t)$ are Jordan domains with real-analytic boundaries;*
- *the domains $D(t)$ satisfy the moment inequality, and, a fortiori, the mean value identity (see Theorem 2.3); moreover, if a domain $D \Subset \Omega$ satisfies the mean value identity (2.4), then $D = D(t)$ up to an area-null set.*

For $T < t < +\infty$, there does not exist any Jordan domain D satisfying the mean value identity (2.4).

6. The evolution equation

A classical approach to the study of Hele–Shaw flows consists in considering the conformal mappings ϕ_t from the unit disk $\mathbb{D}$ onto the flow domains $D(t)$ and examining the evolution equation they satisfy. We want to derive a similar equation in the context of weighted Hele–Shaw flows, in the case where the weight function depends on an additional parameter.

As before, assume that $\Omega \subset \mathbb{C}$ is a bounded and simply connected domain with C^∞-smooth boundary. This time, we are given a one-parameter family of weight functions ϖ_s on Ω, where the parameter s is confined to some open interval $]a, b[$ with $a < b$. We assume $\varpi_s(z)$ is C^∞-smooth in both variables $(z, s) \in \Omega \times]a, b[$. Let $t(s)$, be a known C^∞-smooth strictly positive function on $]a, b[$, and assume that we have a one-parameter continuous family of Jordan domains $\mathcal{D}(s)$ with real-analytic boundaries so that $0 \in \mathcal{D}(s) \Subset \Omega$ and the mean value identity

$$\int_{\mathcal{D}(s)} h(z)\varpi_s(z)\,\mathrm{d}\Sigma(z) = t(s)h(0) \tag{6.1}$$

holds for each $s \in]a, b[$, where h is an arbitrary function harmonic in $\overline{\mathcal{D}}(s)$. We consider the conformal mappings χ_s from $\mathbb{D}$ onto $\mathcal{D}(s)$ normalized by $\chi_s(0) = 0$, $\chi_s'(0) > 0$.

To derive the evolution equation for χ_s, we consider first the special case where $\mathcal{D}(s_0) = \mathbb{D}$ for some $s_0 \in]a, b[$. For s close to s_0 and a function h harmonic in $\overline{\mathbb{D}}$, we write

$$\bigl(t(s) - t(s_0)\bigr)h(0) = \left[\int_{\mathcal{D}(s)} - \int_{\mathbb{D}}\right] h(z)\varpi_{s_0}(z)\,\mathrm{d}\Sigma(z) + \int_{\mathbb{D}} h(z)\bigl(\varpi_s(z) - \varpi_{s_0}(z)\bigr)\,\mathrm{d}\Sigma(z) + \mathrm{o}(s - s_0). \tag{6.2}$$

For s close to s_0, we represent the boundary $\partial\mathcal{D}(s)$ in polar coordinates (r, θ) as

$$r = 1 + (s - s_0)\rho\bigl(\mathrm{e}^{\mathrm{i}\theta}\bigr) + \mathrm{o}(s - s_0);$$

then the first term of the right-hand side of (6.2) has the asymptotics

$$2(s - s_0)\int_{\mathbb{T}} h(\zeta)\rho(\zeta)\varpi_{s_0}(\zeta)\,\mathrm{d}\sigma(\zeta) + \mathrm{o}(s - s_0)$$

as $s \to s_0$. Letting now s tend to s_0, we obtain in the limit from (6.2)

$$t'(s_0)h(0) = 2\int_{\mathbb{T}} h(\zeta)\rho(\zeta)\varpi_{s_0}(\zeta)\,\mathrm{d}\sigma(\zeta) + \int_{\mathbb{D}} h(z)\partial_s\varpi_s(z)\big|_{s=s_0}\,\mathrm{d}\Sigma(z), \tag{6.3}$$

where $\partial_s\varpi_s$ denotes the partial derivative of ϖ_s with respect to s. The right-most integral in this formula can be rewritten as an integral over $\mathbb{T}$. To do this, we introduce the sweeping-out operator P^*, defined on functions g from $C(\overline{\mathbb{D}})$ by

$$P^*[g](\zeta) = \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \bar{\zeta}z|^2} g(z)\,\mathrm{d}\Sigma(z), \quad \zeta \in \mathbb{T}. \tag{6.4}$$

It can be considered as an operator adjoint to the operator P of harmonic extension of functions from $\mathbb{T}$ to $\mathbb{D}$: for any functions $g,h\in C(\overline{\mathbb{D}})$ with h harmonic in $\mathbb{D}$, we have

$$\int_{\mathbb{D}} h(z)g(z)\,\mathrm{d}\Sigma(z)=\int_{\mathbb{T}} h(\zeta)P^*[g](\zeta)\,\mathrm{d}\sigma(\zeta). \tag{6.5}$$

We write equation (6.3) in terms of this sweeping-out operator:

$$t'(s_0)h(0)=\int_{\mathbb{T}} h(\zeta)\bigl[2\rho(\zeta)\varpi_{s_0}(\zeta)+P^*[\partial_s\varpi_s](\zeta)|_{s=s_0}\bigr]\,\mathrm{d}\sigma(\zeta),$$

for an arbitrary function h harmonic in $\overline{\mathbb{D}}$. The only finite Borel measure on $\mathbb{T}$ that represents for the origin is normalized arc length measure $\mathrm{d}\sigma$, which allows us to deduce from the above the expression for the rate of growth of the domains $\mathcal{D}(s)$ for s near s_0:

$$\rho(\zeta)=\frac{t'(s_0)-P^*[\partial_s\varpi_s](\zeta)|_{s=s_0}}{2\varpi_{s_0}(\zeta)}.$$

Now, using the formula for conformal mappings of near-circular domains found in Nehari's book [28, pp. 263–265], we can write the expression for the derivative $\partial_s\chi_s$ at the point $s=s_0$:

$$\frac{\partial\chi_s}{\partial s}(z)\Big|_{s=s_0}=z\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\cdot\frac{t'(s_0)-P^*[\partial_s\varpi_s](\zeta)|_{s=s_0}}{2\varpi_{s_0}(\zeta)}\,\mathrm{d}\sigma(\zeta). \tag{6.6}$$

To obtain the evolution equation for arbitrary $\mathcal{D}(s_0)$, we switch coordinates to the unit disk, and set

$$\tilde{\mathcal{D}}(s)=\chi_{s_0}^{-1}\bigl(\mathcal{D}(s)\bigr),\quad \tilde{\chi}_s=\chi_{s_0}^{-1}\circ\chi_s,\quad\text{and}\quad \tilde{\varpi}_s(z)=\varpi_s\circ\chi_{s_0}(z)\bigl|\chi'_{s_0}(z)\bigr|^2.$$

Applying (6.6), we obtain the equation

$$\frac{\partial\chi_s}{\partial s}(z)=z\chi'_s(z)\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\cdot\frac{t'(s)-P^*[(\partial_s\varpi_s)\circ\chi_s|\chi'_s|^2](\zeta)}{2\varpi_s\circ\chi_s(\zeta)|\chi'_s(\zeta)|^2}\,\mathrm{d}\sigma(\zeta). \tag{6.7}$$

The next proposition formulates the above discussion in a rigorous fashion.

Proposition 6.1. *Let Ω be a simply connected domain and $\varpi_s(z)$ a continuous family of strictly positive weights, such that the mapping $(z,s)\mapsto\varpi_s(z)$ is C^∞-smooth in both variables on $\Omega\times]a,b[$. Let $t(s)$, with $s\in]a,b[$, be a strictly positive C^∞-smooth function. Assume that there exists a one-parameter family of univalent functions $\chi_s:\mathbb{D}\to\Omega$, for $s\in]a,b[$, which extend as univalent functions to $\mathbb{D}(0,1+\varepsilon)\to\Omega$ for some positive ε, and such that the dependence on the parameter s is C^∞-smooth. Suppose these mappings χ_s satisfy the evolution equation* (6.7). *Define the domains $\mathcal{D}(s)\Subset\Omega$ by $\mathcal{D}(s)=\chi_s(\mathbb{D})$. Assume also that the mean value identity* (6.1) *holds for some fixed $s=s_1\in]a,b[$. Then the same identity holds for all $s\in]a,b[$.*

Proof. By an approximation argument, it suffices to check (6.1) for harmonic polynomials h. To do this, it suffices to prove that

$$\frac{\partial}{\partial s}\int_{\mathcal{D}(s)} h(z)\varpi_s(z)\,\mathrm{d}\Sigma(z)=t'(s)h(0) \tag{6.8}$$

holds for any $s\in]a,b[$ and any h which is harmonic in $\overline{\mathcal{D}}(s)$. After switching coordinates to the unit disk, this is equivalent to the following statement: if $\mathcal{D}(s_0)=\mathbb{D}$, $\chi_{s_0}(z)=z$, and the χ_s satisfy (6.6), then for any function h harmonic in $\overline{\mathbb{D}}$, (6.8) holds for $s=s_0$. Using Green's formula, we obtain, for real-valued h,

$$\begin{aligned}\frac{\partial}{\partial s}\int_{\mathcal{D}(s)} h(z)\varpi_s(z)\,\mathrm{d}\Sigma(z)|_{s=s_0}&=\frac{\partial}{\partial s}\int_{\mathbb{D}} h\bigl(\chi_s(w)\bigr)\varpi_s\bigl(\chi_s(w)\bigr)\bigl|\chi'_s(w)\bigr|^2\,\mathrm{d}\Sigma(w)|_{s=s_0}\\ &=\int_{\mathbb{D}}\bigl[2\,\mathrm{Re}\bigl(\partial(h\varpi_{s_0})\partial_s\chi_s|_{s=s_0}\bigr)+h\partial_s\varpi_s|_{s=s_0}+2h\varpi_{s_0}\,\mathrm{Re}\bigl(\partial_s\chi'_s\,\big|_{s=s_0}\bigr)\bigr]\,\mathrm{d}\Sigma\\ &=\int_{\mathbb{T}} hP^*[\partial_s\varpi_s]|_{s=s_0}\,\mathrm{d}\sigma+\int_{\mathbb{T}}2\,\mathrm{Re}\bigl(\bar{\zeta}\partial_s\chi_s(\zeta)|_{s=s_0}\bigr)h(\zeta)\varpi_{s_0}(\zeta)\,\mathrm{d}\sigma(\zeta),\end{aligned} \tag{6.9}$$

where ∂ stands for differentiation with respect to the independent complex variable, and ∂_s is differentiation with respect to the real parameter s. In view of (6.6), we have

$$2\,\mathrm{Re}\big(\bar\zeta\,\partial_s\chi_s(\zeta)|_{s=s_0}\big)=\frac{t'(s_0)-P^*[\partial_s\varpi_s](\zeta)|_{s=s_0}}{\varpi_{s_0}(\zeta)},\quad \zeta\in\mathbb{T},$$

and substituting this to the last integral in (6.9), we get the desired conclusion. □

We return to the context of the preceding sections; there, we have a single weight function $\varpi_s(z)=\omega(z)$, and consider $t(s)=s$. We denote by ϕ_t the conformal mapping from $\mathbb{D}$ onto $D(t)$ normalized by the conditions $\phi_t(0)=0$ and $\phi_t'(0)>0$. The evolution equation (6.7) for the mappings ϕ_t takes the form

$$\frac{\partial\phi_t}{\partial t}(z)=z\phi_t'(z)\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\cdot\frac{\mathrm{d}\sigma(\zeta)}{2\omega(\phi_t(\zeta))|\phi_t'(\zeta)|^2},\quad z\in\mathbb{D}, \tag{6.10}$$

or, equivalently,

$$2\,\mathrm{Re}\left(\bar\zeta\,\frac{\partial\phi_t(\zeta)}{\partial t}\overline{\phi_t'}(\zeta)\omega\big(\phi_t(\zeta)\big)\right)=1,\quad \zeta\in\mathbb{T}. \tag{6.11}$$

In the classical case $\omega=1$, these equations appeared originally in the paper by Vinogradov and Kufarev [38], where they proved the local existence and uniqueness of the solutions. Equation (6.11) for $\omega=1$ was also derived by Richardson [33]. Physically, the evolution equation (6.10) describes the behavior of the conformal mappings from $\mathbb{D}$ onto the Hele–Shaw flow domains. The focus of those investigations was the case of a nonempty possibly irregular blob of fluid at the initial time $t=0$, corresponding to a given conformal mapping ϕ_0.

The following theorem establishes local existence and uniqueness for the evolution equation (6.7).

Theorem 6.2. *Let Ω be a bounded simply-connected domain, and ϖ_s, with $s\in\,]a,b[$, a one-parameter family of weight functions strictly positive in Ω. Assume that $\varpi_s(z)$ is real-analytic in the variable z and C^∞-smooth in s. Also, let $t(s)$ be some C^∞-smooth strictly positive function on the interval $s\in\,]a,b[$. For some $s_0\in\,]a,b[$ and $\varepsilon>0$, let $\chi_{s_0}:\mathbb{D}(0,1+\varepsilon)\to\Omega$ be a univalent function. Then there exist positive numbers δ and ε', and a unique one-parameter family of analytic functions $\chi_s:\mathbb{D}(0,1+\varepsilon')\to\Omega$, with $s\in(s_0-\delta,s_0+\delta)$, univalent on $\mathbb{D}(0,1+\varepsilon')$, such that the χ_s depend C^∞-smoothly on s and satisfy the evolution equation* (6.7). *If, in addition, $\varpi_s(z)$ and $t(s)$ depend real-analytically on the parameter s, then χ_s also is real-analytic in s.*

The proof of this theorem follows arguments from the paper by Reissig and von Wolfersdorfer [32], where the same theorem in the classical case $\varpi_s(z)=1$, $t(s)=s$ was derived from a nonlinear version, due to Nishida and Nirenberg, of the Cauchy–Kovalevskaya theorem. For the sake of completeness, we formulate here this theorem of Nishida and Nirenberg (see [29,30]).

We need the concept of a Banach scale: $\mathbf{B}=\{B_r\}_{r\in\,]r_0,r_1]}$ is called a *Banach scale* if

- for each $r\in\,]r_0,r_1]$, B_r is a Banach space,
- for each pair $r,r'\in\,]r_0,r_1]$, with $r'<r$, B_r is a vector subspace of $B_{r'}$, and the imbedding $B_r\to B_{r'}$ is a norm contraction.

Note that the spaces B_r get smaller as r increases. In connection with a given Banach scale, we want to be able to speak of functions taking values in it. Assume that we are given a parametric collection $\mathbf{E}=\{E_r\}_{r\in\,]r_0,r_1]}$ of subsets of a set E, so that intersections and unions of the sets E_r are well-defined; it is no loss of generality to assume that E_r decreases with increasing r. We take $E=\bigcup\{E_r:\ r\in\,]r_0,r_1]\}$. Similarly, we write $B=\bigcup\{B_r:\ r\in\,]r_0,r_1]\}$, which can be thought of as an inductive limit of Banach spaces.

We speak of a function $f:E\to B$ as an $(\mathbf{E},\mathbf{B})$*-scale function* provided that $f(E_r)\subset B_r$ for each $r\in\,]r_0,r_1]$. Suppose we have a topological, differentiable, or even analytic structure on the set E. Then we say that the $(\mathbf{E},\mathbf{B})$-scale function f is *continuous* provided that each restriction $f|_{E_r}$ is a continuous map $E_r\to B_r$, for $r\in\,]r_0,r_1]$. It is C^n*-smooth* $(n=1,2,3,\ldots)$ if each restriction $f|_{E_r}$ is a C^n-smooth map $E_r\to B_r$, for $r\in\,]r_0,r_1]$. Similarly, the same definition applies to C^∞-smooth maps, as well as to holomorphic and real-analytic maps.

Theorem 6.3. *Assume we have a Banach scale $\mathbf{B}=\{B_r\}_{r\in\,]r_0,r_1]}$. Fix $s_0>0$ and another positive number R. Let $B_r(R)$ denote the open ball of radius R in B_r, and form the union $B(R)=\bigcup\{B_r(R):\ r\in\,]r_0,r_1]\}$. Assume that we have a mapping $(s,f)\mapsto\mathfrak{L}[s,f]$, defined for $s\in\,]-s_0,s_0[$ and $f\in B(R)$, which takes values in $B=\bigcup_r B_r$, and is such that for some positive constants K and C, the following conditions are fulfilled.*

(i) *For all pairs $r, r' \in]r_0, r_1]$ with $r' < r$, $\mathfrak{L}[s, f]$ is continuous in s and f as a mapping from $]{-s_0}, s_0[\times B_r(R)$ into $B_{r'}$;*
(ii) *For each $r \in]r_0, r_1[$, the continuous function $\mathfrak{L}[s, 0]$ satisfies*

$$\|\mathfrak{L}[s,0]\|_{B_r} \leqslant \frac{K}{r_1 - r};$$

(iii) *For all pairs $r, r' \in]r_0, r_1]$ with $r' < r$, all $s \in]{-s_0}, s_0[$, and all $f_1, f_2 \in B_r(R)$, we have*

$$\|\mathfrak{L}[s, f_1] - \mathfrak{L}[s, f_2]\|_{B_{r'}} \leqslant \frac{C}{r - r'}\|f_1 - f_2\|_{B_r}.$$

For positive a, let $I_r =]{-a(r_1 - r)}, a(r_1 - r)[$, where $r \in]r_0, r_1[$. We form the parametric collection $\mathbf{I} = \{I_r\}_{r \in]r_0, r_1[}$. Then for sufficiently small a the abstract Cauchy–Kovalevskaya problem

$$\frac{d\psi}{ds}(s) = \mathfrak{L}\big[s, \psi(s)\big], \qquad \psi(0) = 0, \tag{6.12}$$

has a unique solution $\psi :]{-a(r_1 - r_0)}, a(r_1 - r_0)[\to B$ that is a C^1-smooth $(\mathbf{I}, \mathbf{B})$-scale function with $\psi(s) \in B_r(R)$ whenever $s \in I_r$.

The proof of Theorem 6.3, as presented in [30], is based on the well-known Picard iterative process from the theory of ordinary differential equations. The analysis of this scheme also shows that this theorem admits an appropriate modification for the case of the complex-analytic Cauchy–Kovalevskaya problem. Namely, assume that $\mathfrak{L}[s, \psi]$ is defined for complex $s \in \mathbb{D}(0, s_0)$, and that all assumptions of the theorem are fulfilled for these complex s. Form the parametric collection $\mathbf{D} = \{\mathbb{D}(0, a(r_1 - r))\}_{r \in]r_0, r_1[}$, for sufficiently small positive a (so that in particular, $a(r_1 - r_0) < s_0$). Suppose that $\mathfrak{L}[s, f]$ is analytic in s and f in the following sense: for any complex-analytic $(\mathbf{D}, \mathbf{B})$-scale function $\phi : \mathbb{D}(0, a(r_1 - r_0)) \to B(R)$, the function $s \mapsto \mathfrak{L}[s, \phi(s)]$ is also analytic in s. Then there exists a unique analytic solution $\psi(s)$ of the above Cauchy–Kovalevskaya problem. The same is true in the real-analytic case: if $\mathfrak{L}[s, f]$ is real analytic in s and f, then the solution of (6.12) is also real-analytic in s.

We turn to the proof of Theorem 6.2.

Proof. As before, we switch coordinates to the unit disk and assume without loss of generality that $s_0 = 0$ and $\chi_{s_0}(z) = \chi_0(z) = z$. First, we rewrite the evolution equation (6.7) in terms of a new "unknown" function

$$\psi_s(z) = \frac{1}{\chi_s'(z)} - 1.$$

To this end, we introduce the following chain of nonlinear operators. Suppose the function f is holomorphic in $\mathbb{D}(0, r)$, for some radius $1 < r < +\infty$, and uniformly bounded there:

$$\sup\big\{|f(z)|\colon z \in \mathbb{D}(0, r)\big\} \leqslant R.$$

The parameter r is supposed to be close to 1, and R is assumed small. We define

$$F(z) = F[f](z) = \int_0^z \frac{dw}{1 + f(w)}, \quad z \in \mathbb{D}(0, r). \tag{6.13}$$

Since f is close to 0, we see that $F(z)$ is close to z in the uniform metric on $D(0, r)$. We need the *Herglotz transform* $\mathfrak{H}_+$:

$$\mathfrak{H}_+[g](z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} g(\zeta)\, d\sigma(\zeta), \quad z \in \mathbb{D},$$

for functions $g \in L^1(\mathbb{T})$. Next, we define a nonlinear operator $\mathfrak{T}[s, f]$ by

$$\mathfrak{T}[s, f] = \mathfrak{H}_+\left[\frac{|1 + f|^2}{2\varpi_s \circ F}\big(t'(s) - P^*\big[|1 + f|^{-2}\partial_s \varpi_s \circ F\big]\big)\right]. \tag{6.14}$$

Finally, we set

$$\mathfrak{L}[s, f](z) = z f'(z)\mathfrak{T}[s, f](z) - \big(1 + f(z)\big)\frac{d}{dz}\big(z\mathfrak{T}[s, f](z)\big), \quad z \in \mathbb{D}.$$

The nonlinear operator $\mathfrak{L}[s, f]$ is well-defined provided that $|s|$ and R are sufficiently small. Since

$$F[\psi_s](z) = \int_0^z \frac{\mathrm{d}w}{1+\psi_s(w)} = \int_0^z \chi_s'(w)\,\mathrm{d}w = \chi_s(z),$$

we realize that by the way we have tailored the operator $\mathfrak{L}[s, f]$, the evolution equation (6.7) becomes equivalent to the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi_s = \mathfrak{L}[s, \psi_s], \quad \psi_0 = 0. \tag{6.15}$$

Of course, the functions χ_s we obtain from ψ_s are univalent in $\mathbb{D}(0, r)$, as their derivatives are close to 1.

We make the same choice of a Banach scale as in [32]: **B** is formed by the spaces B_r, for $r \in]r_0, r_1]$, where r_0, r_1 are some fixed (radial) parameters such that $1 < r_0 < r_1 < +\infty$ and both r_0 and r_1 are close to 1. The space B_r consists of all functions holomorphic and continuous up to the boundary in the circular disk $\mathbb{D}(0, r)$. We use the uniform norm on $\mathbb{D}(0, r)$ as the norm in B_r.

In order to apply Theorem 6.3, we need to find an analytic extension of the function $\mathfrak{T}[s, f](z)$ to the disk $\mathbb{D}(0, r)$. We start by finding an analytic extension of the expression

$$P^*\left[\frac{\partial_s\varpi_s \circ F}{|1+f|^2}\right](\zeta)$$

from the unit circle to an annulus $1 < |\zeta| < r$. Since $\varpi_s(z)$ is real-analytic in z in the closed disk $\overline{\mathbb{D}}$, it can be represented in the form

$$\varpi_s(z) = \varpi^\sharp(z, \bar z, s),$$

where $\varpi^\sharp$ is some function holomorphic in the first two variables in some neighborhood of the set $\Delta = \{(z, z)\colon z \in \overline{\mathbb{D}}\} \subset \mathbb{C}^2$ (the diagonal of the closed bidisk $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$). We also have $\partial_s\varpi_s(z) = \partial_s\varpi^\sharp(z, \bar z, s)$. Using the notation

$$\zeta^* = 1/\bar\zeta$$

for the point reflected with respect to the unit circle $\mathbb{T}$, we define for $f \in B_r(R)$ (the open ball in B_r of radius R),

$$\mathfrak{W}_s(\zeta) = \mathfrak{W}[s, f](\zeta) = \frac{1}{\zeta}\int_\zeta^{\zeta^*} \frac{\partial_s\varpi^\sharp(F(w), \overline{F}(\zeta^*), s)}{(1+f(w))(1+\overline{f}(\zeta^*))}\,\mathrm{d}w, \quad r^{-1} < |\zeta| < r,$$

where $F(w) = F[f](w)$ was defined by (6.13), and the integration is along the straight line segment connecting ζ with ζ^*. If the bound R for the uniform norm of f on $\mathbb{D}(0, r)$ is small enough, and r is sufficiently close to 1, then the function $\mathfrak{W}_s(\zeta)$ is well-defined and continuous in the annulus $A_r = \{\zeta \in \mathbb{C}\colon r^{-1} \leqslant |\zeta| \leqslant r\}$. Its restriction to the circle $\mathbb{T}$ is

$$\mathfrak{W}_s(\zeta) = 0 \quad \text{for } \zeta \in \mathbb{T},$$

and, moreover, it solves the following $\bar\partial$-problem:

$$\bar\partial_\zeta\mathfrak{W}_s(\zeta) = -\frac{1}{\zeta\bar\zeta^2}\cdot\frac{\partial_s\varpi^\sharp(F(\zeta^*), \overline{F}(\zeta^*), s)}{(1+f(\zeta^*))(1+\overline{f}(\zeta^*))} = -\frac{1}{\zeta\bar\zeta^2}\cdot\frac{\partial_s\varpi_s \circ F(\zeta^*)}{|1+f(\zeta^*)|^2}, \quad \zeta \in A_r.$$

We introduce the transformed function

$$\mathfrak{X}_s(\zeta) = \mathfrak{X}[s, f](\zeta) = \int_{\mathbb{D}} \frac{z}{\zeta - z}\frac{\partial_s\varpi_s \circ F(z)}{|1+f(z)|^2}\,\mathrm{d}\Sigma(z) - \int_{\mathbb{C}\setminus\overline{\mathbb{D}}} \frac{\partial_s\varpi_s \circ F(w^*)}{|1+f(w^*)|^2}\,\frac{\mathrm{d}\Sigma(w)}{w\bar w^2(\zeta - w)} - \mathfrak{W}_s(\zeta), \quad \zeta \in A_r.$$

This function is continuous in the annulus A_r and, by the above properties of the function $\mathfrak{W}_s(\zeta)$, we have that the function $\mathfrak{X}_s(\zeta)$ meets the following conditions:

- $\mathfrak{X}_s(\zeta) = P^*[(\partial_s\varpi_s \circ F)/|1+f|^2](\zeta)$ for $|\zeta| = 1$;
- $\mathfrak{X}_s(\zeta)$ is holomorphic in ζ in the annulus $1 < |\zeta| < r$.

Finally, we define

$$\mathfrak{Y}_s(\zeta) = \mathfrak{Y}[s, f](\zeta) = \frac{(1+f(\zeta))(1+\overline{f}(\zeta^*))(t'(s) - \mathfrak{X}_s(\zeta))}{2\varpi^\sharp(F(\zeta), \overline{F}(\zeta^*), s)}.$$

This function is continuous in A_r, holomorphic in ζ in the annulus $1 < |\zeta| < r$, and for $\zeta \in \mathbb{T}$, it coincides with the expression in brackets in formula (6.14) defining $\mathfrak{T}[s, f]$. Now, we see that the formula

$$\mathfrak{T}[s, f](z) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\partial\mathbb{D}(0,r)} \frac{\zeta + z}{\zeta - z} \mathfrak{Y}[s, f](\zeta) \frac{\mathrm{d}\zeta}{\zeta}, \quad z \in \mathbb{D}(0, r),$$

defines an analytic extension of $\mathfrak{T}[s, f](z)$ to the disk $\mathbb{D}(0, r)$, since Cauchy's formula permits us to change the path of integration. Consequently, $\mathfrak{L}[s, f](z)$ admits an analytic extension to the same disk $\mathbb{D}(0, r)$.

Routine estimates which are left to the reader show that the operator $\mathfrak{L}[s, f]$, as an operator acting in the scale B introduced above satisfies all requirements of Theorem 6.3, provided that the constant R is sufficiently small, and r_0, r_1 are sufficiently close to 1. Applying Theorem 6.3, we obtain the existence and uniqueness of solutions ψ_s of (6.15). Finally, if $\varpi_s(z)$ and $t(s)$ are real-analytic in s, all operators introduced above depend real-analytically on s and f, which shows that the solutions ψ_s, and consequently, χ_s, are also real-analytic in s.

The proof of Theorem 6.2 is complete. □

Theorem 6.2 immediately applies to the evolution equation (6.10) for the conformal mappings $\phi_t : \mathbb{D} \to D(t)$, where $D(t)$ is the Hele–Shaw flow domain for parameter value t. Taking into account Proposition 6.1 and uniqueness of the domains $D(t)$, we arrive at the following theorem.

Theorem 6.4. *In the context of Theorem* 5.3, *for* $t \in]0, T[$, *let* ϕ_t *be the conformal mapping from* $\mathbb{D}$ *onto* $D(t)$, *normalized by the conditions* $\phi_t(0) = 0$ *and* $\phi_t'(0) > 0$. *Then* $(z, t) \mapsto \phi_t(z)$ *is real-analytic in* t *and holomorphic in* z, *for* $(z, t) \in \overline{\mathbb{D}} \times]0, T[$. *Moreover, the mappings* ϕ_t *satisfy the evolution equation* (6.10).

7. The exponential mapping near the origin

In this section, we define the Hele–Shaw exponential mapping locally near the origin so that it satisfies those assertions of Theorem 1.2 which are local to the origin.

We work in the context of Theorems 5.3 and 6.4, and make the additional normalizing assumption that $\omega(0) = 1$. For positive real s, we consider the following domains:

$$\mathcal{D}(s) = \frac{1}{s} D(s^2) = \{z \in \mathbb{C}\colon\ sz \in D(s^2)\}.$$

For the parameter value $s = 0$, we put $\mathcal{D}(0) = \mathbb{D}$. The domains $\mathcal{D}(s)$ are characterized by the mean value identity (6.1) with respect to the weight function $\varpi_s(z) = \omega(sz)$ and with $t(s) = 1$:

$$\int\limits_{\mathcal{D}(s)} h(z)\varpi_s(z)\, \mathrm{d}\Sigma(z) = h(0), \tag{7.1}$$

where h ranges over all bounded harmonic functions on $\mathcal{D}(s)$; this also applies to $s = 0$. Clearly, $\varpi_s(z)$ and $t(s)$ are real-analytic in the variable s, so that we may apply Theorem 6.2 with $s_0 = 0$ and $\chi_0(z) = z$. This gives us a one-parametric family of functions χ_s, where $s \in]{-\delta}, \delta[$ for some positive δ, defined and univalent in a common disk $\mathbb{D}(0, 1 + \varepsilon)$ with $\varepsilon > 0$, and depending real-analytically on s, which satisfy the evolution equation (6.7). By Proposition 6.1 and the uniqueness of the the domains $D(t)$ satisfying the mean value identity (Theorem 5.3), the functions χ_s map $\mathbb{D}$ conformally onto the domains $\mathcal{D}(s)$. In particular, we have

$$\chi_s(z) = s^{-1}\phi_{s^2}(z), \quad 0 < s < \sqrt{T}; \tag{7.2}$$

here, as before, ϕ_t is the conformal mapping from $\mathbb{D}$ onto $D(t)$, and we notice that the formula naturally extends χ_s up to the parameter value $\sqrt{T}$, as indicated. We can define $\mathcal{D}(s)$ for negative s as well, setting it equal to the image of the unit disk $\mathbb{D}$ under χ_s; initially, this makes sense at least for $-\delta < s < 0$. By Proposition 6.1, the mean value identity (7.1) holds for negative s as well, and since the function $\varpi_s(z)$ does not change under a change of signs in both z and s, we obtain, from the uniqueness of mean value disks, that

$$\mathcal{D}(-s) = -\mathcal{D}(s) = \{z \in \mathbb{C}\colon\ -z \in \mathcal{D}(s)\}. \tag{7.3}$$

In view of the requirement that $\chi_s'(0) > 0$ should hold for all s, negative as well as positive, we also get the symmetry relation

$$\chi_{-s}(-z) = -\chi_s(z); \tag{7.4}$$

in particular, χ_s and $\mathcal{D}(s)$ are well-defined for $-\sqrt{T} < s < \sqrt{T}$.

For s close to 0, the boundary of the domain $\mathcal{D}(s)$ can be represented in polar coordinates (r,θ) as follows:

$$r = r(\theta,s) = 1 + \sum_{l=1}^{+\infty} R_l(\theta)s^l, \tag{7.5}$$

where each $R_l(\theta)$ is a real-analytic function of the real parameter θ. The next proposition establishes some general properties of the functions $R_l(\theta)$.

Proposition 7.1. *For any $l = 1,2,3,\ldots,$ the function $R_l(\theta)$ is periodic with period 2π, and in fact, it is a trigonometric polynomial of degree at most l. Furthermore, it satisfies $R_l(\theta+\pi) = (-1)^l R_l(\theta)$ for all real θ.*

Proof. By the symmetry property (7.3) of the domains $\mathcal{D}(s)$, the function $r(\theta,s)$ is unperturbed by the transformation $(\theta,s) \mapsto (\theta+\pi,-s)$, from which we easily read off that $R_l(\theta+\pi) = (-1)^l R_l(\theta)$.

Next, we assume that $s > 0$ is small. By the mean value identity (7.1) applied to the function $h(z) = z^m$, for $m = 0,1,2,3,\ldots,$ we get

$$\int_{\mathcal{D}(s)} z^m \omega(sz)\,\mathrm{d}\Sigma(z) = \delta_{m,0}, \quad m = 0,1,2,3,\ldots,$$

where the delta is the standard Kronecker symbol. A similar relation holds for $h(z) = \bar{z}^m$. Writing the integrals in polar coordinates, we obtain

$$\int_0^{2\pi}\int_0^{r(\theta,s)} 2\rho^{|m|+1}\mathrm{e}^{-\mathrm{i}m\theta}\omega\big(s\rho\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\rho\frac{\mathrm{d}\theta}{2\pi} = \delta_{m,0}, \quad m\in\mathbb{Z}. \tag{7.6}$$

We now consider the Taylor series expansion of ω near the origin:

$$\omega(z) = \sum_{j,k=0}^{+\infty} \omega_{j,k} z^j \bar{z}^k.$$

Substituting it into (7.6), we get

$$\int_0^{2\pi}\int_0^{r(\theta,s)} 2\sum_{j,k=0}^{+\infty} \omega_{j,k} s^{j+k}\rho^{j+k+|m|+1}\mathrm{e}^{\mathrm{i}(j-k-m)\theta}\,\mathrm{d}\rho\frac{\mathrm{d}\theta}{2\pi} = \delta_{m,0}, \quad m\in\mathbb{Z},$$

which is equivalent to

$$\sum_{j,k=0}^{+\infty} \frac{2\omega_{j,k}\,s^{j+k}}{j+k+|m|+2}\big(r(\cdot,s)^{j+k+|m|+2}\big)^\wedge(k-j+m) = \delta_{m,0}, \quad m\in\mathbb{Z}. \tag{7.7}$$

Here, the symbol $(\cdot)^\wedge(j)$ means the jth Fourier coefficient of the expression with respect to the suppressed variable. Finally, we want to substitute the expansion (7.5) into this formula. We shall use multi-index notation: a letter α will denote a multi-index

$$\alpha = (\alpha_0,\alpha_1,\alpha_2,\ldots),$$

where $\alpha_j \in \mathbb{Z}_+ = \{0,1,2,3,\ldots\}$ and only a finite number of the α_j are different from zero. We write

$$|\alpha| = \sum_j \alpha_j; \qquad \|\alpha\| = \sum_j j\alpha_j; \qquad \alpha! = \prod_j \alpha_j!; \qquad R^\alpha(\theta) = \prod_j R_j(\theta)^{\alpha_j}.$$

In the last formula, we take $R_0(\theta) \equiv 1$. By the multinomial theorem,

$$r(\theta,s)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!}R^\alpha(\theta)s^{\|\alpha\|},$$

so that as we substitute this expansion into (7.7), we get

$$\sum_{j,k,\alpha}\left\{\frac{(j+k+|m|+1)!}{\alpha!}\omega_{j,k}\widehat{R^\alpha}(k-j+m)s^{j+k+\|\alpha\|}\colon\ 0\leqslant j,k<+\infty, |\alpha| = j+k+|m|+2\right\} = \frac{1}{2}\delta_{m,0}.$$

Replacing the summation variable k by $p = j + k$, this becomes

$$\sum_{j,p,\alpha}\left\{\frac{(p+|m|+1)!}{\alpha!}\omega_{j,p-j}\widehat{R^\alpha}(p-2j+m)s^{p+\|\alpha\|}\colon 0\leqslant j\leqslant p<+\infty, |\alpha|=p+|m|+2\right\}=\frac{1}{2}\delta_{m,0}.$$

Identification of coefficients of powers of s leads to

$$\sum_{j,p,\alpha}\left\{\frac{(p+|m|+1)!}{\alpha!}\omega_{j,p-j}\widehat{R^\alpha}(p-2j+m)\colon 0\leqslant j\leqslant p<+\infty, |\alpha|=p+|m|+2,\ p+\|\alpha\|=l\right\}$$
$$=\frac{1}{2}\delta_{m,0}\delta_{l,0}, \tag{7.8}$$

for $l = 0, 1, 2, 3, \ldots$. The summation over j on the left-hand side can be interpreted as a convolution multiplication. To make this precise, we introduce the functions W_p,

$$W_p(\theta)=\sum_{j=0}^{p}\omega_{j,p-j}\mathrm{e}^{\mathrm{i}(2j-p)\theta},\quad p=0,1,2,3,\ldots,\ \theta\in\mathbb{R};$$

each W_p is a trigonometric polynomial of degree at most p. We have

$$\sum_{j=0}^{p}\omega_{j,p-j}\widehat{R^\alpha}(p-2j+m)=\big(W_p R^\alpha\big)^\wedge(m).$$

Thus, for $l = 1, 2, 3, \ldots$, (7.8) takes the form

$$\sum_{p,\alpha}\left\{\frac{(p+|m|+1)!}{\alpha!}\big(W_p R^\alpha\big)^\wedge(m)\colon 0\leqslant p<+\infty, |\alpha|=p+|m|+2,\ p+\|\alpha\|=l\right\}=0. \tag{7.9}$$

In the sum on the left-hand side, only $R_0, R_1, \ldots, R_l$ actually occur in R^α, due to the restrictions given on α. Moreover, only one term contains R_l as a "factor": this happens for the index parameters $p = 0$ and $\alpha = \alpha(l) = (p+|m|+1, 0, \ldots, 0, 1, 0, \ldots)$, where the 1 occurs at the lth place; the term in question thus equals $\widehat{R_l}(m)$. As a consequence, we obtain

$$\widehat{R_l}(m)=-\sum_{p,\alpha}\left\{\frac{(p+|m|+1)!}{\alpha!}\big(W_p R^\alpha\big)^\wedge(m)\colon 0\leqslant p<+\infty, |\alpha|=p+|m|+2,\ p+\|\alpha\|=l,\ \alpha\neq\alpha(l)\right\}. \tag{7.10}$$

This equation holds for any integer m, and it gives us a recursive formula for the coefficients $\widehat{R_l}(m)$. A simple induction argument now shows that $R_l(\theta)$ is a trigonometric polynomial of degree at most l, because $W_p R^\alpha$ is a trigonometric polynomial of degree less than or equal to the degree of W_p (which is $\leqslant p$) plus the degree of R^α, which is at most $\|\alpha\|$, by the induction assumption. □

For s close to 0, say $s \in]-\delta, \delta[$, the mapping $\chi_s(z)$ is conformal in a disk $\mathbb{D}(0, 1+\varepsilon)$, for some fixed $\varepsilon > 0$ independent of s, and depends real-analytically on the parameter s. Moreover, they are close to the identity mapping (uniformly on $\mathbb{D}(0, 1+\varepsilon)$, say). Hence, the inverse mapping $\psi_s = \chi_s^{-1}$ possesses analogous properties: it, too, is conformal on some open disk containing $\overline{\mathbb{D}}$. As a conformal mapping, ψ_s is characterized by the properties $\psi_s(0) = 0$, $\psi_s'(0) > 0$, and $\psi_s(\mathcal{D}(s)) = \mathbb{D}$. Consider the Taylor expansion of $\psi_s(z)/z$ with respect to s:

$$\frac{\psi_s(z)}{z}=1+\sum_{l=1}^{+\infty}p_l(z)s^l. \tag{7.11}$$

The next proposition establishes that the functions p_l possess rather special properties.

Proposition 7.2. *For any $l = 1, 2, 3, \ldots$, the function p_l is a polynomial of degree at most l, and it satisfies $p_l(-z) = (-1)^l p_l(z)$, that is, it is even for even l and odd for odd l.*

Proof. By the symmetry property (7.4) of χ_s, we have $\psi_{-s}(-z) = -\psi_s(z)$, which implies $p_l(-z) = (-1)^l p_l(z)$.

The fact that ψ_s maps $\partial\mathcal{D}(s)$ onto $\mathbb{T}$ is conveniently expressed by

$$\big|\psi_s\big(r(\theta,s)\mathrm{e}^{\mathrm{i}\theta}\big)\big|^2=1,$$

where $r(\theta,s)$ is the function parametrizing $\partial\mathcal{D}(s)$ given by (7.5), so that we have

$$\left|\sum_{l=0}^{+\infty} p_l\big(r(\theta,s)\mathrm{e}^{\mathrm{i}\theta}\big)s^l\right|^2 = r(\theta,s)^{-2}, \tag{7.12}$$

provided that we set $p_0(z)\equiv 1$. Writing

$$r(\theta,s)^{-2} = \sum_{l=0}^{+\infty} q_l(\theta)s^l,$$

we obtain from (7.5) and Proposition 7.1 that $q_0(\theta)\equiv 1$ and that for any $l=1,2,3,\ldots,$ q_l is a trigonometric polynomial of degree at most l satisfying $q_l(\theta+\pi)=(-1)^l q_l(\theta)$. Each function p_l is analytic in a neighborhood of the closed disk $\overline{\mathbb{D}}$, so that it has a convergent power series expansion

$$p_l(z) = \sum_{m=0}^{+\infty} \widehat{p_l}(m)z^m, \quad z\in\overline{\mathbb{D}},$$

and substituting this into (7.12), we arrive at

$$\sum_{j,k=0}^{+\infty}\sum_{m,n=0}^{+\infty} s^{j+k}\widehat{p_j}(m)\overline{\widehat{p_k}}(n)\,r(\theta,s)^{m+n}\mathrm{e}^{\mathrm{i}(m-n)\theta} = \sum_{l=0}^{+\infty} q_l(\theta)s^l.$$

We recall from the proof of the previous proposition the multi-index expansion

$$r(\theta,s)^{m+n} = \sum_{|\alpha|=m+n}\frac{(m+n)!}{\alpha!}R^\alpha(\theta)s^{\|\alpha\|}.$$

Inserting this information into the above identity, we obtain

$$\sum_{j,k,m,n=0}^{+\infty}\sum_{|\alpha|=m+n}\frac{(m+n)!}{\alpha!}\widehat{p_j}(m)\overline{\widehat{p_k}}(n)R^\alpha(\theta)s^{j+k+\|\alpha\|}\mathrm{e}^{\mathrm{i}(m-n)\theta} = \sum_{l=0}^{+\infty} q_l(\theta)s^l.$$

Comparing Taylor coefficients, we have, for $l=1,2,3,\ldots,$

$$\sum_{j,k,m,n,\alpha}\left\{\frac{(m+n)!}{\alpha!}\widehat{p_j}(m)\overline{\widehat{p_k}}(n)R^\alpha(\theta)\mathrm{e}^{\mathrm{i}(m-n)\theta}\colon\ 0\leqslant j,k,m,n<+\infty,\ |\alpha|=m+n,\ \|\alpha\|=l-j-k\right\} = q_l(\theta).$$

The sum on the left-hand side actually only involves the functions $p_0, p_1,\ldots,p_l$, and not higher index p_j's. Moreover, we can separate out the terms involving p_l, because they occur precisely when $\alpha=(m+n,0,0,\ldots)$ and $(j,k)=(l,0)$ or $(j,k)=(0,l)$:

$$\begin{aligned}
2\,\mathrm{Re}\big[p_l\big(\mathrm{e}^{\mathrm{i}\theta}\big)\big] &= \sum_{m=0}^{+\infty}\widehat{p_l}(m)\mathrm{e}^{\mathrm{i}m\theta} + \sum_{n=0}^{+\infty}\overline{\widehat{p_l}}(n)\mathrm{e}^{-\mathrm{i}n\theta}\\
&= q_l(\theta) - \sum_{j,k,m,n,\alpha}\bigg\{\frac{(m+n)!}{\alpha!}\widehat{p_j}(m)\overline{\widehat{p_k}}(n)R^\alpha(\theta)\mathrm{e}^{\mathrm{i}(m-n)\theta}\colon\\
&\qquad 0\leqslant j,k,m,n<+\infty,\ |\alpha|=m+n,\ \|\alpha\|=l-j-k,\\
&\qquad (j,k)\neq(l,0),\ (j,k)\neq(0,l)\bigg\}.
\end{aligned}$$

A simple induction argument over l now shows the desired assertion that p_l has is a polynomial of degree at most l. After all, R^α is a trigonometric polynomial of degree at most $\|\alpha\|$, so that the degree of each term in the sum on the right-hand side has degree at most $m+n+|m-n|=2\max\{m,n\}$. And under the induction assumption, each such term vanishes unless $m<l$ and $n<l$. As q_l is a trigonometric polynomial of degree at most l, it follows that the real part of $p_l(\mathrm{e}^{\mathrm{i}\theta})$ is a trigonometric polynomial of degree at most l as well. But then $p_l(z)$ must be an ordinary polynomial of degree $\leqslant l$. The proof is complete. □

The next item is a lemma from the theory of power series.

Lemma 7.3. *Let $f(z,w)$ be a function holomorphic in a neighborhood of $(0,0)$ in $\mathbb{C}^2$. Suppose that its power series expansion is of the special form*

$$f(z,w)=\sum_{m=0}^{+\infty}\sum_{n=0}^{m}a_{m,n}z^m w^n, \tag{7.13}$$

which converges near $(0,0)$. Let g be defined by the relation $g(z,w)=f(z,w/z)$ for $z\neq 0$. Then g admits an extension to a function holomorphic in both variables in a neighborhood of the origin $(0,0)$.

Proof. Since $f(z,w)$ is holomorphic near $(0,0)$, its power series is absolutely convergent in a possibly smaller neighborhood of the origin:

$$\sum_{m=0}^{+\infty}\sum_{n=0}^{m}|a_{m,n}|\delta^m\varepsilon^n<+\infty,$$

provided that $\delta,\varepsilon>0$ are sufficiently small. The function $g(z,w)$ is represented by the Laurent series

$$g(z,w)=f(z,w/z)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}a_{m,n}z^{m-n}w^n, \tag{7.14}$$

which converges absolutely for, say, $|z|=\delta$ and $|w|=\delta\varepsilon$. We immediately realize that the Laurent series for $g(z,w)$ is in fact a power series, so that the result follows from the following classical statement concerning power series of two variables: if a power series converges absolutely at some point (z,w) with $z\neq 0$ and $w\neq 0$, then it converges absolutely in a small bidisk about the origin. □

We now introduce a function $\mu(z,t)$ defined first for appropriate z and $t>0$ by

$$\mu(z,t)=\frac{\sqrt{t}}{z}\psi_{\sqrt{t}}\left(\frac{z}{\sqrt{t}}\right)=1+\sum_{l=1}^{+\infty}p_l\left(\frac{z}{\sqrt{t}}\right)t^{l/2}. \tag{7.15}$$

By Proposition 7.2 and Lemma 7.3, this function admits an extension to a function analytic in (z,t) near the origin $(0,0)$. We also define

$$H(z,t)=|z|^2\bigl|\mu(z,t)\bigr|^2.$$

In terms of this function H, we get the following characterization of the boundaries $\partial D(s^2)=s\partial\mathcal{D}(s)$:

$$z\in\partial D\bigl(s^2\bigr)\quad\text{if and only if}\quad H\bigl(z,s^2\bigr)=s^2. \tag{7.16}$$

We are now ready to prove the local existence of the desired Hele–Shaw exponential mapping. We want to find a mapping $\boldsymbol{\Phi}$ (the Hele–Shaw exponential mapping HSexp_0 from Theorems 1.1 and 1.2) defined and real-analytic near the origin such that it maps each small disk $\mathbb{D}(0,r)$ diffeomorphically onto the corresponding ω-mean-value disk $D(r^2)$, and the rays $\{z\in\mathbb{C}\colon \arg z=\theta\}$ and the circles $\{z\in\mathbb{C}\colon |z|=r\}$ are mapped to orthogonal curves. Let us think of the complex variable w as represented by polar coordinates $w=r\mathrm{e}^{\mathrm{i}\theta}$, and let $\partial_r=\partial/\partial r$ denote the partial differentiation in the radial direction. Then the desired mapping $\boldsymbol{\Phi}$ must satisfy the following two conditions:

- $H(\boldsymbol{\Phi}(w),|w|^2)=|w|^2$, and
- the vector $\partial_r\boldsymbol{\Phi}(w)$ is orthogonal to the boundary $\partial D(|w|^2)$.

Differentiating the first requirement, we find that

$$2\,\mathrm{Re}\bigl[\partial_z H\bigl(\boldsymbol{\Phi}(w),|w|^2\bigr)r\partial_r\boldsymbol{\Phi}(w)\bigr]=2|w|^2\bigl[1-\partial_t H\bigl(\boldsymbol{\Phi}(w),|w|^2\bigr)\bigr],$$

whereas the second requirement is equivalent to the condition

$$\mathrm{Im}\bigl[\partial_z H\bigl(\boldsymbol{\Phi}(w),|w|^2\bigr)r\partial_r\boldsymbol{\Phi}(w)\bigr]=0.$$

Combining the above two displayed equations, we obtain a differential equation for $\boldsymbol{\Phi}$:

$$r\frac{\partial\boldsymbol{\Phi}}{\partial r}(w)=\frac{|w|^2[1-\partial_t H(\boldsymbol{\Phi}(w),|w|^2)]}{\overline{\boldsymbol{\Phi}}(w)\,\overline{\mu}(\boldsymbol{\Phi}(w),|w|^2)\nu(\boldsymbol{\Phi}(w),|w|^2)}. \tag{7.17}$$

Here, $\nu(z,s)=\mu(z,s)+z\partial_z\mu(z,s)$. We search for $\boldsymbol{\Phi}$ of the form

$$\boldsymbol{\Phi}(w)=w\big(1+\Psi(w)\big),\qquad \Psi(0)=0,$$

and notice that this implies automatically that the differential of $\boldsymbol{\Phi}$ at the origin is the identity mapping. The differential equation (7.17) for $\boldsymbol{\Phi}$ becomes the following equation for Ψ:

$$r\frac{\partial\Psi}{\partial r}(w)=\frac{1-\partial_t H(\boldsymbol{\Phi}(w),|w|^2)}{(1+\overline{\Psi}(w))\,\overline{\mu}(\boldsymbol{\Phi}(w),|w|^2)\nu(\boldsymbol{\Phi}(w),|w|^2)}-1-\Psi(w),\qquad \Psi(0)=0. \tag{7.18}$$

If u and v are Cartesian coordinates in w-plane, so that $w=u+\mathrm{i}v$, then

$$r\frac{\partial}{\partial r}=u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}.$$

It follows that Eq. (7.18) has the form

$$u\frac{\partial\Psi}{\partial u}+v\frac{\partial\Psi}{\partial v}=f\big(u,v,\Psi(u,v)\big),\qquad \Psi(0,0)=0, \tag{7.19}$$

where on the right-hand side, f is real-analytic in the variables u, v, and Ψ near the origin $(0,0,0)$, and has value $f(0,0,0)=0$. If we think of Ψ and f as $\mathbb{R}^2$-valued functions, then (7.19) is a system of two first-order partial differential equations for which the origin is a singular point. Such a kind of singularity is said to be of *Briot–Bouquet type*. In the scalar one-dimensional case, the theorem of Briot and Bouquet asserts that an equation

$$u\frac{\mathrm{d}\Psi}{\mathrm{d}u}=f\big(u,\Psi(u)\big),\qquad \Psi(0)=0,$$

with $f(0,0)=0$ and f real-analytic near the origin, has a unique solution Ψ that is real-analytic near the origin, provided that the partial derivative

$$\frac{\partial f}{\partial\Psi}(0,0)$$

is not a positive integer [23, pp. 295–296]. In our vector-valued two-dimensional case, the existence of solutions that are real-analytic near the origin for equations of the type (7.19) depends on the eigenvalues of the 2×2 Jacobi matrix

$$J=\frac{\partial f}{\partial\Psi}(0,0,0).$$

It follows from [24, Lemma 1] or [3, Theorem 1] that a sufficient condition for the existence of real-analytic solutions of (7.19) is that none of eigenvalues of J is a positive integer. In our case, if f is the right-hand side of (7.18), we have

$$f(0,0,\Psi)=-2\operatorname{Re}\Psi+\mathrm{O}\big(|\Psi|^2\big)\quad \text{as } |\Psi|\to 0,$$

which shows that the eigenvalues of J are 0 and -2. Thus, (7.18) possesses a solution Ψ that is real-analytic near the origin.

We have obtained the local existence near the origin of a real-analytic mapping $\boldsymbol{\Phi}$ satisfying (7.17). This function is a diffeomorphism on a neighborhood of the origin, because $\boldsymbol{\Phi}(w)=w+\mathrm{O}(|w|^2)$ as $w\to 0$. Tracing our arguments backwards, we see that $\boldsymbol{\Phi}$ maps $\mathbb{D}(0,r)$ onto $D(r^2)$, and maps rays and circles emanating from the origin to orthogonal curves.

We can now state the following theorem.

Theorem 7.4. *In the context of Theorem* 5.3, *there exists a unique real-analytic mapping* $\boldsymbol{\Phi}=\mathrm{HSexp}_0$ (*it is even unique in the class of* C^1*-smooth mappings*) *defined on a neighborhood of the origin such that*

- *the differential of* $\boldsymbol{\Phi}$ *at the origin is the identity mapping times* $\omega(0)^{-1/2}$,
- *for small positive* r, $\boldsymbol{\Phi}$ *maps circles* $\mathbb{T}(0,r)=\partial\mathbb{D}(0,r)$ *diffeomorphically onto* $\partial D(r^2)$, *and*
- *near the origin, each ray* $\{w\in\mathbb{C}\colon \arg w=\theta\}$ *is mapped by* $\boldsymbol{\Phi}$ *to a curve orthogonal to* $\partial D(t)$ *for each small positive* t.

Proof. Using an appropriate dilation of the plane, we can reduce the theorem to the case $\omega(0)=1$. In this case, the above discussion shows the existence of a real-analytic mapping $\boldsymbol{\Phi}$ satisfying (7.17) and having differential at the origin equal to the identity mapping. In particular, $\boldsymbol{\Phi}$ is a diffeomorphism in some neighborhood of the origin. On the other hand (7.17) implies that $\boldsymbol{\Phi}(w)\in\partial D(|w|^2)$ for all sufficiently small $|w|$, that is, $\boldsymbol{\Phi}$ is a real-analytic diffeomorphism from $|w|=r$ *into* $\partial D(r^2)$. For topological reasons, such a diffeomorphism must in fact be *onto*. The last assertion of the theorem (about orthogonal curves) follows from Eq. (7.17).

To see that we have uniqueness, assume that we have two mappings, $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$, satisfying the assertions of the theorem. Then the mapping $\boldsymbol{\Phi}_1^{-1} \circ \boldsymbol{\Phi}_2$ is a real-analytic diffeomorphism near the origin which sends all small circles $\mathbb{T}(0,r)$ onto themselves, maps each ray $\{\arg w = \theta\}$ to some ray of the same type, and it has the identity differential at the origin. This is possible only for the identity mapping, which shows that $\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_2$. □

8. Global properties of the exponential mapping

We return to the context of the introduction and summarize the results of preceding sections. Given a simply connected two-dimensional Riemannian manifold $\boldsymbol{\Omega}$ with hyperbolic real-analytic Riemannian metric, we identify it with the domain $\Omega \subset \mathbb{C}$ which is either the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$, so that the metric has the form (1.1) with a real analytic weight function ω. That the metric is hyperbolic means that ω is logarithmically subharmonic.

For any Jordan subdomain $\Omega' \Subset \Omega$ with C^∞-smooth boundary, we define the Hele–Shaw domains $D(t)$ as the noncoincidence sets (2.1) for the corresponding obstacle problem. By Proposition 2.9, $D(t)$ does not depend on the choice of the underlying domain Ω', as long as we have $D(t) \Subset \Omega'$. For any such $\Omega' \Subset \Omega$, we can define

$$T(\Omega') = \sup\bigl\{t \in]0, +\infty[:\ D(t) \Subset \Omega'\bigr\},$$

and then set

$$T = \sup\bigl\{T(\Omega'):\ \Omega' \text{ is a precompact Jordan subdomain of } \Omega\bigr\}.$$

For any $t \in]0, T[$, we have $D(t) \Subset \Omega$, and by Theorem 2.3, $D(t)$ is an ω-mean-value disk, that is, it satisfies (1.3). Moreover, by the uniqueness part of Theorem 5.3, there does not exist any Jordan domain that is ω-mean-value disks for $t \in]T, +\infty[$ in case $T < +\infty$. For $t \in]0, T[$, the ω-mean-value disks $D(t)$ are Jordan domains with real-analytic boundaries and they increase continuously with t. Moreover, they depend real-analytically on the parameter t in the following sense: if ϕ_t is the conformal mapping from $\mathbb{D}$ onto $D(t)$, normalized by the conditions $\phi_t(0) = 0$ and $\phi_t'(0) > 0$, then ϕ_t depends real-analytically on t.

By Theorem 7.4, we have a Hele–Shaw exponential mapping HSexp_0 defined and real analytic in some disk $\mathbb{D}(0,\delta)$ and such that it maps small circles $\mathbb{T}(0,r)$ diffeomorphically onto Hele–Shaw flow boundaries $\partial D(r^2)$, and near the origin, each ray $\{\arg w = \theta\}$ is mapped to a curve orthogonal to these flow boundaries. We want to extend (uniquely) HSexp_0 to the disk $\mathbb{D}(0, \sqrt{T})$ in such a way that the extended mapping is a diffeomorphism possessing the same properties. The next theorem establishes that such an extension can be found.

Theorem 8.1. *There exists a unique mapping $\boldsymbol{\Phi} = \mathrm{HSexp}_0$, defined and real-analytic in $\mathbb{D}(0, \sqrt{T})$ such that*

(a) *it maps each circle $\mathbb{T}(0,r)$ with $r \in]0, \sqrt{T}[$ diffeomorphically onto the flow boundary $\partial D(r^2)$, and*
(b) *it maps each ray $\{w \in \mathbb{D}(0, \sqrt{T}):\ \arg w = \theta\}$ to a curve that is orthogonal to all the flow boundaries $\partial D(t)$ with $t \in]0, T[$, and such that*
(c) *$\boldsymbol{\Phi}$ has differential at the origin equal to the identity mapping times the scaling factor $\omega(0)^{-1/2}$.*

This mapping is a diffeomorphism from the disk $\mathbb{D}(0, \sqrt{T})$ onto $\boldsymbol{\Phi}(\mathbb{D}(0, \sqrt{T})) = \bigcup\{D(t):\ t \in]0, T[\}$.

Proof. It is enough to find a unique mapping with desired properties defined in a disk $\mathbb{D}(0, r_0)$, for any $r_0 \in]0, \sqrt{T}[$. For $t \in]0, r_0^2]$, let $\eta_t = \phi_t^{-1}$ denote the conformal mapping inverse to ϕ_t. From the above properties of $\partial D(t)$ and hence of ϕ_t, we see that the function $\eta_t(z)$ is analytic in z and real analytic in t in a neighborhood of the set

$$E_\varepsilon = \bigl\{(z,t):\ z \in \overline{D}(t),\ t \in \bigl[\varepsilon, r_0^2\bigr]\bigr\},$$

for any $\varepsilon \in]0, r_0^2[$. On the other hand, the relation (7.15) and the definition of above functions ψ_s show that we have

$$\mu(z,t) = \sqrt{t}\,\frac{\eta_t(z)}{z}$$

for (z,t) close to the origin $(0,0)$. We use this formula to extend the domain of definition for $\mu(z,t)$ to (z,t) in a neighborhood of the sets E_ε. As before, we also define the function

$$H(z,t) = |z|^2 \bigl|\mu(z,t)\bigr|^2,$$

and we see that the functions μ and H turn out to be well-defined and real-analytic in a neighborhood of the set

$$E_0 = \bigl\{(z,t):\ z \in \overline{D}(t),\ t \in \bigl[0, r_0^2\bigr]\bigr\},$$

where we agree that $D(0)=\{0\}$. We also have the representation (7.16) for boundaries $\partial D(t)$ in terms of the function H valid for all $t\in[0,r_0^2]$.

As in the preceeding section, the desired properties of the mapping $\boldsymbol{\Phi}$ are equivalent to Eq. (7.17). By Theorem 7.4, we have already a real analytic solution $\boldsymbol{\Phi}^0(w)$ of (7.17) defined in a neighborhood of some small closed disk $\bar{\mathbb{D}}(0,\delta)$. We want to extend $\boldsymbol{\Phi}^0$ to a function $\boldsymbol{\Phi}$ which is defined in $\mathbb{D}(0,r_0)$ and still satisfies (7.17). Setting $\boldsymbol{\Phi}_\theta(r)=\boldsymbol{\Phi}(r\mathrm{e}^{\mathrm{i}\theta})$ for $r\in[\delta,r_0]$, we obtain from (7.17) the following Cauchy problem for the functions $\boldsymbol{\Phi}_\theta$:

$$\frac{\mathrm{d}\boldsymbol{\Phi}_\theta(r)}{\mathrm{d}r}=\frac{r(1-\partial_t H(\boldsymbol{\Phi}_\theta(r),r^2))}{\bar{\boldsymbol{\Phi}}_\theta(r)\bar{\mu}(\boldsymbol{\Phi}_\theta(r),r^2)\nu\big(\boldsymbol{\Phi}_\theta(r),r^2\big)},\qquad \boldsymbol{\Phi}_\theta(\delta)=\boldsymbol{\Phi}^0\big(\delta\mathrm{e}^{\mathrm{i}\theta}\big).\tag{8.1}$$

Here, as before, $\nu=\mu+z\partial_z\mu$. Taking into account the above definitions of the functions μ and H, and also the evolution equation for the conformal mappings ϕ_t in the form (6.11), we can rewrite the last equation in more elegant form:

$$\frac{\mathrm{d}\boldsymbol{\Phi}_\theta(r)}{\mathrm{d}r}=\frac{r}{\omega(\boldsymbol{\Phi}_\theta(r))}\,\frac{\overline{\eta'_{r^2}}(\boldsymbol{\Phi}_\theta(r))}{\overline{\eta_{r^2}}(\boldsymbol{\Phi}_\theta(r))},\qquad \boldsymbol{\Phi}_\theta(\delta)=\boldsymbol{\Phi}^0\big(\delta\mathrm{e}^{\mathrm{i}\theta}\big).\tag{8.2}$$

By standard theorems in the theory of Ordinary Differential Equations, for any angle θ, there exists a unique solution $\boldsymbol{\Phi}_\theta(r)$ of Eq. (8.2) defined on a maximally extended interval $[\delta,r_1(\theta)[$, and this solution $\boldsymbol{\Phi}_\theta(r)$ is real-analytic in the parameters θ and r. By Eq. (8.1), we have

$$H\big(\boldsymbol{\Phi}_\theta(r),r^2\big)=r^2.$$

It follows that for any $r\in[\delta,r_0]\cap[\delta,r_1(\theta)[$, we have $\boldsymbol{\Phi}_\theta(r)\in\partial D(r^2)$, and the pair $(\boldsymbol{\Phi}_\theta(r),r^2)$ must belong to the set E_{δ^2}, which is a compact subset of the domain of definition of the right-hand side of (8.1). This shows that we must be able to continue the solution further if $r_1(\theta)\leqslant r_0$, contradicting the maximality of the interval $[\delta,r_1(\theta)[$. Hence we must have $r_1(\theta)>r_0$, so that the solution $\boldsymbol{\Phi}_\theta(r)$ is defined in the whole interval $[\delta,r_0]$. As a result, we obtain a real-analytic extension $\boldsymbol{\Phi}$ of $\boldsymbol{\Phi}^0$ to the whole disk $\mathbb{D}(0,r_0)$.

We already noticed that for any $r\in]0,r_0]$ and any angle θ, we have $\boldsymbol{\Phi}(r\mathrm{e}^{\mathrm{i}\theta})\in\partial D(r^2)$, and, by the uniqueness theorem from Ordinary Differential Equations (using as input that $\boldsymbol{\Phi}^0$ is one-to-one), we have $\boldsymbol{\Phi}(r\mathrm{e}^{\mathrm{i}\theta_1})\neq\boldsymbol{\Phi}(r\mathrm{e}^{\mathrm{i}\theta_2})$ for any two angles with $\mathrm{e}^{\mathrm{i}\theta_1}\neq\mathrm{e}^{\mathrm{i}\theta_2}$. It follows that $\boldsymbol{\Phi}$ is a one-to-one mapping from any circle $\mathbb{T}(0,r)$ into the flow boundary $\partial D(r^2)$, and, for topological reasons, this mapping is onto. To prove that $\boldsymbol{\Phi}$ is a real-analytic diffeomorphism of $\mathbb{D}(0,r_0)$ onto $D(r_0^2)$, it remains to check that it has a nondegenerate differential at any point. Since $\boldsymbol{\Phi}$ maps circles $\mathbb{T}(0,r)$ and rays $\{w\in\mathbb{D}(0,r_0)\colon \arg w=\theta\}$ to orthogonal curves, it suffices to check that $\partial_r\boldsymbol{\Phi}(w)\neq0$ and $\partial_\theta\boldsymbol{\Phi}(w)\neq0$, where $w=r\mathrm{e}^{\mathrm{i}\theta}$. The first property follows immediately from Eq. (8.2). The second is also a consequence of this equation, since the variable θ is involved in (8.2) as an initial data, and in this case we have either $\partial_\theta\boldsymbol{\Phi}_\theta(r)\neq0$ or $\partial_\theta\boldsymbol{\Phi}_\theta(r)=0$ identically for $r\in[\delta,r_0[$, and the latter alternative is clearly impossible.

The uniqueness of $\boldsymbol{\Phi}$ can be proved by the same arguments as we used in the proof of Theorem 7.4. □

To finish off the proof of our main Theorems 1.1 and 1.2, we need to check that if the original metric on $\boldsymbol{\Omega}$ is complete, then $T=+\infty$, and that when the latter happens, HSexp_0 maps $\mathbb{C}$ onto Ω. We first compare the $\sqrt{\omega(z)}|\mathrm{d}z|$-metric discs $B(0,r)$ with the ω-mean-value disks $D(t)$. Namely, we show that $D(r^2)$ is well-defined and contained in $B(0,r)$ if the latter disk is precompactly contained in Ω. By Hadamard's theorem, in case of a complete metric, the metric disks $B(0,r)$ are all precompact in Ω, which shows that we indeed have $T=+\infty$. In case $T=+\infty$, we need to check that the union of all $D(t)$ covers the whole region Ω. It is clearly sufficient to show that an arbitrary subdomain $\Omega'\Subset\Omega$ is contained in $D(t)$ for sufficiently large t. This is easily achieved by obstacle problem arguments.

We turn to the precise statements.

Proposition 8.2. *Assume that the metric disk $B(0,r)$ is precompact in Ω for a given $r>0$. Then, for any $t\in]0,r^2[$, the domain $D(t)$ defined by* (2.1) *for the obstacle problem corresponding to $B(0,r)$ is precompactly contained in $B(0,r)$. In particular, $D(t)\subset B(0,\sqrt{t})$ provided that $B(0,\sqrt{t})$ is precompact in Ω.*

Proof. The proof of the proposition is based on the following lemma, which was communicated to the authors by Boris Korenblum.

Lemma 8.3. *Assume that a positive function ω is subharmonic in the unit disk $\mathbb{D}$ and has the reproducing property*

$$\int_{\mathbb{D}} h(z)\omega(z)\,\mathrm{d}\Sigma(z)=h(0)$$

for all harmonic polynomials h. Then

$$\int_0^1 \omega(r)\,\mathrm{d}r \leqslant 1.$$

Proof. An application of Fatou's lemma yields

$$\int_{\mathbb{D}} h(z)\omega(z)\,\mathrm{d}\Sigma(z) \leqslant h(0)$$

for any function h harmonic and positive in $\mathbb{D}$. In particular,

$$\int_{\mathbb{D}} \frac{1-|z|^2}{|1-z|^2}\omega(z)\,\mathrm{d}\Sigma(z) \leqslant 1. \tag{8.3}$$

By subharmonicity of ω, we have

$$\omega(r) \leqslant \frac{1}{(1-r)^2}\int_{\mathbb{D}(r,1-r)} \omega(z)\,\mathrm{d}\Sigma(z) = \int_{\mathbb{D}} \frac{1}{(1-r)^2} 1_{\mathbb{D}(r,1-r)}(z)\omega(z)\,\mathrm{d}\Sigma(z)$$

for any $r \in]0,1[$, and consequently,

$$\int_0^1 \omega(r)\,\mathrm{d}r \leqslant \int_{\mathbb{D}} \left[\int_0^1 \frac{1}{(1-r)^2} 1_{\mathbb{D}(r,1-r)}(z)\,\mathrm{d}r\right]\omega(z)\,\mathrm{d}\Sigma(z).$$

An explicit calculation shows that

$$\int_0^1 \frac{1}{(1-r)^2} 1_{\mathbb{D}(r,1-r)}(z)\,\mathrm{d}r = \frac{1-|z|^2}{|1-z|^2},$$

which – together with (8.3) – proves the lemma. □

In view of the Cauchy–Bunyakovskiĭ–Schwarz inequality and rotation invariance, we also have, under the conditions of the last lemma,

$$\int_0^1 \sqrt{\omega(r\zeta)}\,\mathrm{d}r \leqslant 1, \quad \zeta \in \mathbb{T}.$$

In geometric language, this inequality means that if the unit disk $\mathbb{D}$ is an ω-mean-value disk $D(1)$, then the length of any radius in the metric (1.1) is at most 1. In other words, the metric distance from the origin to any point of $\partial D(1)$ is at most 1. By conformal invariance, this property holds for flow domains $D(1)$ of arbitrary shape, and we obtain, by similarity arguments, that the metric distance from the origin to any point of $\partial D(t)$ is at most $\sqrt{t}$.

To conclude the proof of the proposition, assume that $B(0,r)$ is precompactly contained in Ω. Then the distance from the origin to any point of $\partial B(0,r)$ is r. If we define

$$T_r = T\big(B(0,r)\big) = \sup\big\{t:\ D(t) \Subset B(0,r)\big\},$$

then the ω-mean-value disk $D(T_r)$ is well-defined (since $T_r < T$) and its boundary $\partial D(T_r)$ touches the boundary $\partial B(0,r)$ at least at one point. Since the distance from the origin to the boundary $\partial D(T_r)$ is at most $\sqrt{T_r}$, we obtain $r \leqslant \sqrt{T_r}$. The proof is complete. □

Thus, in the case of a complete metric (1.1), the ω-mean-value disks $D(t)$ are well-defined for all $t \in]0,+\infty[$. It remains to check that the ω-mean value disks $D(t)$ cover all of Ω as $t \to +\infty$, provided $T = +\infty$. To this end, let Ω' be a precompact Jordan subdomain of Ω. We consider the obstacle problem corresponding to this subdomain, and define for all $t \in]0,+\infty[$ generalized Hele–Shaw domains $D'(t)$ as noncoincidence sets for the obstacle problem, of the form (2.1). By Proposition 2.9, we have $D'(t) \subset D(t)$ for any $t \in]0,+\infty[$, since for each t, the Hele–Shaw flow domain $D(t)$ can be obtained as the noncoincidence set for the obstacle problem corresponding to some larger subdomain $\Omega'' \supset \Omega'$. On the other hand, by Proposition 2.8(b), we have $D'(t) = \Omega'$ for sufficiently large positive t, which shows that

$$\Omega' \subset \bigcup \{D(t)\colon t \in]0, +\infty[\}.$$

The proof of our main Theorems 1.1 and 1.2 is finally complete.

9. Wrapped Hele–Shaw flow

Let us consider Hele–Shaw flow on a compact real-analytic surface **S**. We have an injection point $z_0 \in \mathbf{S}$, and at time $t = 0$, there is no fluid in the surface. We get a small circular-like blob about z_0 for small positive t, but eventually, as t grows, topological restrictions apply, and two parts of the boundary will meet, and we may develop a hole in the fluid; for instance, this will happen if the surface looks like the surface on a cigar. The end result is that the whole surface is filled at a time that corresponds to the area of **S**, and then the flow comes to an end. This means that such a question as long term behavior of geodesics which makes perfect sense for metric flow cannot be asked in the Hele–Shaw flow context.

To address the issue, we introduce another notion of flow, *wrapped Hele–Shaw flow*. The understanding is best achieved by considering first a concrete example. So, let **S** be the unit sphere with the standard Riemannian metric, and z_0 the north pole. As we inject fluid into the north pole, we cover increasingly bigger spherical caps, until we reach the critical moment when the whole sphere is covered, except for the south pole. At that point, we allow the fluid to grow a second layer, so that in fact, the fluid flow is taking place on a sheeted surface over the sphere, the south pole being the point where the two sheets meet. We then continue the flow, having covered the entire sphere once, and a spherical cap at the south pole once more. The next obstruction occurs at the north pole, and we proceed just as we did at the south pole. As a result, we can trace the trajectories of the particles injected initially at time $t = 0$ indefinitely, and in this example they are the great circles passing through the north pole. It is natural to call these trajectories *wrapped Hele–Shaw geodesics*. Let us see what to do if **S** is a more general compact real-analytic surface. To clarify the idea, let us agree that *we want the wrapped Hele–Shaw geodesics to be the maximal real-analytic continuations of the ordinary Hele–Shaw geodesics emanating from the injection point*. As in the case of the sphere, we need to permit the flow to take place on some other surface sheeted over **S**. The discussion is simplified if we consider the universal covering surface $\boldsymbol{\Omega}$ of **S**, which is either the sphere, the plane, or the disk. We focus on hyperbolic compact real-analytic surfaces **S**. Although the sphere cannot be supplied with a hyperbolic metric, there are plenty of other compact Riemann surfaces of higher genus (number of “handles”) of this type. Lift the Riemannian metric of **S** to the covering surface $\boldsymbol{\Omega}$. The universal covering surface of a hyperbolic surface is either the unit disk $\mathbb{D}$ or the whole complex plane $\mathbb{C}$. The complex plane appears only in rather degenerate cases, such as when **S** is the torus with flat metric, and the generic situation is that of $\boldsymbol{\Omega} = \mathbb{D}$. We have a canonical projection $\boldsymbol{\pi} : \boldsymbol{\Omega} \to \mathbf{S}$ and a discrete group Γ of conformal automorphisms of $\boldsymbol{\Omega}$ such that the projection induces the identification $\mathbf{S} = \boldsymbol{\Omega}/\Gamma$. Pick a point $z_0^* \in \boldsymbol{\Omega}$ such that $\boldsymbol{\pi}(z_0^*) = z_0$. The ordinary Hele–Shaw flow starting at z_0^* in $\boldsymbol{\Omega}$ falls under the auspices of our main theorem, so that the corresponding flow domains are always real-analytic Jordan domains. *The image of that flow under the projection $\boldsymbol{\pi}$ then equals the wrapped Hele–Shaw flow on* **S**.

The ergodic or chaotic properties of geodesic flow on hyperbolic compact has attracted considerable attention [2]. It would be nice to to have a similar study of the wrapped Hele–Shaw geodesics.

Without the hyperbolicity assumption on **S**, the flow on the covering surface $\boldsymbol{\Omega}$ may develop contact points or cusps. The contact points are not a serious problem, because we may allow them to continue to flow on different sheets on some surface sheeted over $\boldsymbol{\Omega}$. But cusps are more serious, and we do not fully understand what happens they are present.

10. Questions

It is desirable to obtain further geometric properties of the Hele–Shaw flow domains $\mathbf{D}(t)$ on a simply connected hyperbolic real-analytically smooth surface $\boldsymbol{\Omega}$. A particularly natural question is whether they are all (geodesically) star-shaped with respect to the injection point z_0. We do not know the answer. One reason for wanting information of this type is that it would make it easier to extend our main theorem to C^∞-smooth surfaces. It may then be possible to find an appropriate approximation process to show that the Hele–Shaw flow domains $\mathbf{D}(t)$ are all C^∞-smooth Jordan domains under the weaker assumption, and that would more or less do it. In the case of planar Hele–Shaw flow, this is of course so, the flow domains being circular disks about z_0. What is perhaps less obvious is that even if we start with some irregular blob which happens to be star-shaped with respect to z_0, then all the domains we get from injecting more fluid into the surface are also star-shaped with respect to z_0 (see [7, Theorem 4.1] and [22, Theorem 1]). Convexity, however, is generally *not* preserved by such Hele–Shaw flow.

In the introduction, we mentioned the concept of Hele–Shaw completeness. A natural question is whether it depends on the position of the injection point z_0.

Higher-dimensional analogues exist for the classical Hadamard theorem, involving sectional curvature to replace Gaussian curvature in the two-dimensional setting. We do not know whether such conditions are relevant for possible analogues of our main theorem on Hele–Shaw flow.

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