Invariant Subspaces in Banach Spaces of Analytic Functions

HÅKAN HEDENMALM & ALLEN SHIELDS

Introduction

Let $\mathbf{D}$ denote the open unit disc $\{z \in \mathbf{C}: |z| < 1\}$, let $\mathbf{T}$ denote the boundary $\{z \in \mathbf{C}: |z| = 1\}$, and let $\mathbf{D}^-$ denote the closure of $\mathbf{D}$. By $A = A(\mathbf{D})$ we denote the disc algebra, consisting of those continuous functions on $\mathbf{D}^-$ that are analytic in $\mathbf{D}$. We introduce also the Fréchet space $\mathcal{O}(\mathbf{D})$ of all holomorphic functions on $\mathbf{D}$, with the topology of uniform convergence on compact subsets, and the space $\mathcal{O}(\mathbf{D}^-)$ of (germs of) functions holomorphic on neighborhoods of $\mathbf{D}^-$, with the inductive limit topology. Strictly speaking the elements of $\mathcal{O}(\mathbf{D}^-)$ are equivalence classes, but we shall often regard them as individual functions. A *sequence* $\{f_k\}$ in $\mathcal{O}(\mathbf{D}^-)$ converges to a function f if and only if all the functions are analytic in some fixed open set U containing $\mathbf{D}^-$, with $f_k \to f$ uniformly on compact subsets of U.

Let X be a Banach subspace of $\mathcal{O}(\mathbf{D})$; more precisely, this means that X is a vector subspace of $\mathcal{O}(\mathbf{D})$ and X has a norm with respect to which it is a Banach space. We assume further that the injection map $X \to \mathcal{O}(\mathbf{D})$ is continuous, and that X contains $\mathcal{O}(\mathbf{D}^-)$ as a dense subspace. (The injection map, $\mathcal{O}(\mathbf{D}^-) \to X$, is automatically sequentially continuous; see the remarks following Lemma 1 below.) Since the point evaluation functionals at the points of $\mathbf{D}$ are continuous functionals on $\mathcal{O}(\mathbf{D})$ it follows that they are also continuous on X. (In fact, the continuity of these functionals is equivalent to the continuity of the injection map of X into $\mathcal{O}(\mathbf{D})$; see Proposition 1 of [6].)

Note that X must be separable. Indeed, every function in $\mathcal{O}(\mathbf{D}^-)$ is the limit, in the topology of $\mathcal{O}(\mathbf{D}^-)$, of a sequence of polynomials with rational coefficients.

Let $M(X)$ denote the space of multipliers of X, that is, the set of all those functions $\varphi \in \mathcal{O}(\mathbf{D})$ such that $\varphi X \subset X$. Using the closed graph theorem, one shows that if $\varphi \in M(X)$ then multiplication by φ is a bounded linear transformation on X. Also, by [10, Prop. 11] one has $M(X) \subset H^\infty$, and the supremum norm is less than or equal to the operator norm. Here H^∞ denotes the space of bounded analytic functions in $\mathbf{D}$ with the supremum norm. Using

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this, one shows that $M(X)$ is complete in the operator norm and thus is a commutative Banach algebra with identity. If X contains the constant function 1, then $M(X) \subset X$.

We shall assume that $M(X) \supset \mathcal{O}(\mathbf{D}^-)$. A closed subspace I of X is called *invariant* (for the operator of multiplication by z) if $zI \subset I$. This is equivalent to the condition $\mathcal{O}(\mathbf{D}^-)I \subset I$; we obtain this as a corollary to the following result, which asserts the automatic (sequential) continuity of certain injection maps.

LEMMA 1. *Let Y be a vector subspace of $\mathcal{O}(\mathbf{D})$ with a complete, translation invariant metric, such that the point evaluation functionals at the points of $\mathbf{D}$ are continuous on Y. If $\mathcal{O}(\mathbf{D}^-) \subset Y$, then the injection map is sequentially continuous.*

Proof. Let $f_k \to f$ in $\mathcal{O}(\mathbf{D}^-)$. Thus all the functions $\{f_k\}$ and f are analytic in some fixed open disc containing $\mathbf{D}^-$. Let $A_r = A(r\mathbf{D})$ $(r > 1)$ denote the space of continuous functions on $(r\mathbf{D})^-$ that are analytic in $r\mathbf{D}$, with the supremum norm. By restricting to the disc $r\mathbf{D}$ we may regard all the functions $\{f_k\}$ and f as being in A_r, with $f_k \to f$ in the norm of A_r, for each r sufficiently close to 1. Fix such an r. We claim that $\{f_k\}$ also converges to f in the metric of Y. Indeed, by restricting to $\mathbf{D}$ we may regard A_r as being a Banach space of analytic functions on $\mathbf{D}$ with bounded point evaluations, and with $A_r \subset Y$. The closed graph theorem shows that the injection map of A_r into Y is continuous, and thus $f_k \to f$ in the metric of Y, as asserted. (See Proposition 6 of [6].) □

Returning to our previous situation, we see that the injection maps of $\mathcal{O}(\mathbf{D}^-)$ into X and into $M(X)$ are both sequentially continuous.

COROLLARY. *If I is a closed subspace of X and if $zI \subset I$, then $\mathcal{O}(\mathbf{D}^-)I \subset I$.*

Proof. If p is a polynomial then $pI \subset I$. Let now φ be analytic in a neighborhood of $\mathbf{D}^-$; we must show that $\varphi I \subset I$. Choose a sequence of polynomials $\{p_k\}$ that converges to φ uniformly in some neighborhood of $\mathbf{D}^-$, and therefore in the topology of $\mathcal{O}(\mathbf{D}^-)$. By the lemma, with $Y = M(X)$, we have $p_k \to \varphi$ in the norm of $M(X)$, and the result follows. □

A basic problem in the theory is the following.

PROBLEM 1. Characterize the invariant subspaces of X.

An interesting subproblem is the following.

PROBLEM 2. Let $\mathfrak{F}$ be a collection of functions in X, and let $I(\mathfrak{F}, X)$ be the smallest invariant subspace of X containing $\mathfrak{F}$. When is $I(\mathfrak{F}, X) = X$?

In other words, when is $\mathcal{O}(\mathbf{D}^-)\mathfrak{F}$ dense in X? For some spaces X a solution to Problem 2 leads to a solution to Problem 1. This is the case for $X = H^2$, and this is how Beurling [4] proved his invariant subspace theorem. Next we assume that $\mathfrak{F}$ contains only one function.

PROBLEM 3. Let $f \in X$, and let $I(f, X)$ be the smallest invariant subspace of X containing f. When is $I(f, X) = X$?

It is customary to say that f is *cyclic* (for the operator of multiplication by z) if $I(f, X) = X$. Clearly a cyclic function cannot vanish anywhere in $\mathbf{D}$. By Beurling's invariant subspace theorem a function $f \in H^2$ is cyclic in H^2 if and only if it is an outer function.

We are especially interested in the case $X = D$, the *Dirichlet space*, consisting of those analytic functions $f = \sum a_n z^n$ in $\mathbf{D}$ such that

$$\|f\|_D^2 = \sum_0^\infty (n+1)|a_n|^2 = \sum_0^\infty |a_n|^2 + \int_{\mathbf{D}} |f'|^2 \, dA,$$

where dA denotes normalized area measure on $\mathbf{D}$. We recall the related spaces D_α, $0 \leq \alpha < 1$, defined by the condition: $\sum (n+1)^\alpha |a_n|^2 < \infty$. They were studied by Carleson in this thesis [7].

Let $L^2(\mathbf{D})$ denote the Hilbert space of square integrable functions on $\mathbf{D}$ with respect to area measure, and let $L_a^2(\mathbf{D})$ be its analytic subspace $\mathcal{O}(\mathbf{D}) \cap L^2(\mathbf{D})$; $L_a^2(\mathbf{D})$ is known as the *Bergman space* on $\mathbf{D}$. A function $f \in \mathcal{O}(\mathbf{D})$ is in the Dirichlet space D if and only if $f' \in L_a^2(\mathbf{D})$.

Let $\mathbf{D}_e$ be the exterior disc $(\mathbf{C} \cup \{\infty\}) \setminus \mathbf{D}^-$, and let $L^2(\mathbf{D}_e)$ be the Hilbert space of square integrable functions on $\mathbf{D}_e$ with respect to area measure on the Riemann sphere (note that this is a finite measure). Also, let $L_a^2(\mathbf{D}_e) = \mathcal{O}(\mathbf{D}_e) \cap L^2(\mathbf{D}_e)$. Then $\phi = \sum_0^\infty a_n z^{-n} \in L_a^2(\mathbf{D}_e)$ if and only if ϕ is square integrable in a neighborhood of the boundary. A calculation shows that $\varphi \in L_a^2(\mathbf{D}_e)$ if and only if $\sum_1^\infty |a_n|^2/n < \infty$.

The *Smirnov space*, $N_+(\mathbf{D})$, consists of those analytic functions in $\mathbf{D}$ representable as f/g, where $f, g \in H^\infty$ and g is an outer function.

We call a closed subset $E \subset \mathbf{T}$ a *Bergman–Smirnov exceptional set* if every function in $\mathcal{O}((\mathbf{C} \cup \{\infty\}) \setminus E) \cap N_+(\mathbf{D}) \cap L_a^2(\mathbf{D}_e)$ is constant. For information about logarithmic capacity see [15, Chap. III].

LEMMA 2. *If E is a Bergman–Smirnov exceptional set, then E has logarithmic capacity zero.*

Proof. Assume that $E \subset \mathbf{T}$ is a closed set of positive capacity; we shall show that E is not a Bergman–Smirnov exceptional set. Because E has positive capacity, there is a Borel probability measure μ supported on E such that

$$\infty > I(\mu) \overset{\text{def}}{=} \iint \log|z-w|^{-1} \, d\mu(z) \, d\mu(w) = \sum_1^\infty |\hat{\mu}(n)|^2/n. \tag{1}$$

For the calculation establishing the second equality see [6, p. 294]. (Here $\hat{\mu}(n) = \int w^{-n} \, d\mu(w)$, $-\infty < n < \infty$; $w = \exp(i\vartheta)$.) Now let

$$g(z) = \int (1 - \bar{w}z)^{-1} \, d\mu(w).$$

Note that g is holomorphic in $\mathbf{C} \cup \{\infty\} \setminus E$. For $z \in \mathbf{D}$ we have (see [6, (20), p. 294]): $g(z) = \sum_0^\infty \hat{\mu}(n) z^n$. This is the "analytic projection" of the Fourier-Stieltjes series for $d\mu$, and hence is in H^p for all $p < 1$. In particular, it is in $N_+(\mathbf{D})$ (see [9, Thms. 4.2 and 1.1]).

For $|z| > 1$ a calculation shows that $g(z) = -\sum_1^\infty \hat{\mu}(-n) z^{-n}$. Thus by (1), $g \in L_a^2(\mathbf{D}_e)$ (recall that $\hat{\mu}(-n) = \hat{\mu}(n)^*$ since μ is a real measure; here $*$ denotes complex conjugation).

If g is not constant, then E is not a Bergman–Smirnov exceptional set and the proof is complete. If g is constant, then $\hat{\mu}(n) = 0$ for all $n \neq 0$, and thus μ is normalized Lebesgue measure on $\mathbf{T}$. But this means that $E = \mathbf{T}$. Now $\mathbf{T}$ is not a Bergman–Smirnov exceptional set; indeed, since $\mathbf{T}$ separates the Riemann sphere we may define a function analytic in the complement by taking any function holomorphic in $\mathbf{D}$, in particular a function in $N_+(\mathbf{D})$, and any function holomorphic in $\mathbf{D}_e$, in particular a nonconstant Bergman function. This completes the proof. $\square$

As regards Problem 3, in this paper we show that if f is an outer function in the Dirichlet space and in the disc algebra ($f \in D \cap A$), and if the boundary zero set of f is a Bergman–Smirnov exceptional set, then f is cyclic in the Dirichlet space. We have a similar result concerning Problem 2. We also show that every closed countable subset of $\mathbf{T}$ is a Bergman–Smirnov exceptional set. In [6, Thm. 5, p. 293] it is shown that if $f \in \mathbf{D}$ is cyclic in D, then f is outer and the radial limit boundary function, $\lim f(re^{i\vartheta})$ $(r \uparrow 1)$, can vanish only on a set of logarithmic capacity zero.

PROBLEM 4. Is every closed subset of $\mathbf{T}$ of logarithmic capacity zero a Bergman–Smirnov exceptional set?

If the answer is affirmative then, combining this with the discussion above, we would have the following result. An outer function $f \in D \cap A$ is cyclic in D if and only if its boundary zero set has logarithmic capacity zero. This would provide an affirmative answer (for f in $D \cap A$) to the conjecture in [6, p. 296]: an outer function in D is cyclic if and only if the radial limit function vanishes only on a set of capacity zero.

In this connection we note that Carleson has proved [8, Chap. VI, Thm. 1(a), p. 73] that a compact subset E of $\mathbf{C}$ has logarithmic capacity zero if and only if every function analytic and square integrable (with respect to area measure on the Riemann sphere) in $(\mathbf{C} \cup \{\infty\}) \setminus E$ is constant.

Duality

The bilinear form linking a Banach space and its dual space will always be denoted by $\langle \cdot, \cdot \rangle$. Let X^* be the dual Banach space to X. For $\lambda \in \mathbf{C} \setminus \mathbf{D}^-$ we define the function f_λ by $f_\lambda(w)=(\lambda-w)^{-1}$, $w\in\mathbf{D}$. We also define $f_\infty=0$; thus $f_\lambda\in\mathcal{O}(\mathbf{D}^-)$ and so $f_\lambda\in X$ for all $\lambda\in\mathbf{D}_e$. The map $\lambda\to f_\lambda$ is an analytic function on $\mathbf{D}_e$, with values in X. To every functional $\phi\in X^*$ we associate the analytic function

$$\tilde{\phi}(\lambda)=\langle f_\lambda,\phi\rangle,\quad \lambda\in\mathbf{D}_e, \tag{2}$$

which vanishes at ∞. If ϕ_w ($w\in\mathbf{D}$) denotes the functional of evaluation at w, then $\tilde{\phi}_w(\lambda)=(\lambda-w)^{-1}$.

Since the functions f_λ, $\lambda\in\mathbf{D}_e$, span a dense subspace of $\mathcal{O}(\mathbf{D}^-)$, and since $\mathcal{O}(\mathbf{D}^-)$ is dense in X, we see that ϕ is uniquely determined by $\tilde{\phi}$. This allows us to identify the dual space X^* with the space of analytic functions $\tilde{\phi}$ on $\mathbf{D}_e$, where $\phi\in X^*$. We shall make this identification, and we shall write ϕ instead of $\tilde{\phi}$.

Finally we note that if $\lambda\in\mathbf{D}_e$ then, as is obvious from (2), point evaluation at λ is a weak-* continuous linear functional on X^*.

Let $\mathcal{O}_0(\mathbf{D}_e^-)$ denote the space of functions analytic on $\mathbf{D}_e^-$ (the closure of $\mathbf{D}_e$) that vanish at ∞. The next lemma shows that $\mathcal{O}_0(\mathbf{D}_e^-)\subset X^*$. Since the functionals ϕ_w ($w\in\mathbf{D}$) are in $\mathcal{O}_0(\mathbf{D}_e^-)$, we see that $\mathcal{O}_0(\mathbf{D}_e^-)$ is weak-* dense in X^*.

NOTATIONS. (i) If $f\in\mathcal{O}(\mathbf{D})$ and $\psi\in\mathcal{O}(\mathbf{D}_e)$, then f_r and ψ_r denote the functions $f_r(w)=f(rw)$ and $\psi_r(\lambda)=\psi(\lambda/r)$, $0<r<1$.

(ii) If $f\in\mathcal{O}(\mathbf{D})$ then $\hat{f}$ denotes the sequence of power series coefficients; if $\phi\in\mathcal{O}_0(\mathbf{D}_e)$ then $\hat{\phi}$ denotes the sequence of power series coefficients about ∞. Thus

$$f(w)=\sum_0^\infty \hat{f}(n)w^n,\quad w\in\mathbf{D};\qquad \phi(\lambda)=\sum_1^\infty \hat{\phi}(n)\lambda^{-n},\quad \lambda\in\mathbf{D}_e.$$

Let $f\in X$ and $\phi\in X^*$. We shall show that the following formula is valid if either $f\in\mathcal{O}(\mathbf{D}^-)$ or $\phi\in\mathcal{O}_0(\mathbf{D}_e^-)$:

$$\langle f,\phi\rangle=\sum_0^\infty \hat{f}(n)\hat{\phi}(n+1). \tag{3}$$

LEMMA 3.

(a) *If* $f\in\mathcal{O}(\mathbf{D}^-)$ *then* (3) *is valid for all* $\phi\in X^*$.

(b) *If* $\phi\in\mathcal{O}_0(\mathbf{D}_e^-)$ *then* $\phi\in X^*$, *and* (3) is valid for all $f\in X$.

(c) *If* $f\in X$ *and if* $f_r\to f$ *in norm, then*

$$\begin{aligned}\langle f,\phi\rangle&=\lim_{r\uparrow 1}\langle f_r,\phi\rangle=\lim_r\sum_0^\infty \hat{f}(n)\hat{\phi}(n+1)r^n\\ &=\lim\langle f,\phi_r\rangle=\lim\langle f_r,\phi_r\rangle\end{aligned}$$

for all $\phi\in X^*$.

Proof. (a) Since $f\in\mathcal{O}(\mathbf{D}^-)$, the power series for f converges to f in the topology of $\mathcal{O}(\mathbf{D}^-)$ and hence in the norm of X. Thus

$$\langle f,\phi\rangle = \left\langle \sum_0^\infty \hat{f}(n)w^n, \phi\right\rangle = \sum_0^\infty \hat{f}(n)\langle w^n,\phi\rangle. \tag{4}$$

This is valid for all $f\in \mathcal{O}(\mathbf{D}^-)$ and $\phi\in X^*$. If we take $f=f_\lambda$, so that $\hat{f}_\lambda(n)=\lambda^{-(n+1)}$, then from (2) and (4) we obtain

$$\tilde{\phi}(\lambda)=\langle f_\lambda,\phi\rangle=\sum_1^\infty \lambda^{-k}\langle w^{k-1},\phi\rangle.$$

Thus $\hat{\phi}(k)=\langle w^{k-1},\phi\rangle$, and (3) follows from (4).

(b) Let $\phi\in\mathcal{O}_0(\mathbf{D}_e^-)$ be given. To show that $\phi\in X^*$ we must produce a continuous linear functional ψ on X such that $\phi(\lambda)=\langle f_\lambda,\psi\rangle$ for all $\lambda\in\mathbf{D}_e$. The right side of (3) defines a pairing between $\mathcal{O}(\mathbf{D})$ and $\mathcal{O}_0(\mathbf{D}_e^-)$. With this pairing the elements of $\mathcal{O}_0(\mathbf{D}_e^-)$ become continuous linear functionals on $\mathcal{O}(\mathbf{D})$ and hence on X (see, e.g., [12]). We define ψ to be the functional on X defined by ϕ in this manner:

$$\langle f,\psi\rangle=\sum_0^\infty \hat{f}(n)\hat{\phi}(n+1),\quad f\in X. \tag{5}$$

If we put $f=f_\lambda$ in this formula we obtain $\langle f_\lambda,\psi\rangle=\phi(\lambda)$, as desired. Thus $\phi\in X^*$, and we may identify the linear functional ψ with the analytic function ϕ. Formula (3) now follows from (5).

(c) Let $f\in X$ be such that $f_r\to f$ in norm. The first equality follows from the continuity of ϕ; the second follows from part (a) since $f_r\in\mathcal{O}(\mathbf{D}^-)$ and $(f_r)\hat{}\,(n)=\hat{f}(n)r^n$. For the third equality we replace the factor r^n in the summation by the factor r^{n+1}, which does not affect the validity of the formula. We then group r^{n+1} with $\hat{\phi}(n+1)$ to obtain $(\phi_r)\hat{}\,(n+1)$, and apply part (b). To obtain the last equality we replace r^n by r^{2n+1}, grouping r^n with $\hat{f}(n)$ and r^{n+1} with $\hat{\phi}(n+1)$. □

Note that if both $f\in\mathcal{O}(\mathbf{D}^-)$ and $\phi\in\mathcal{O}_0(\mathbf{D}_e^-)$, then (3) may be rewritten as:

$$\langle f,\phi\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\vartheta})\phi(e^{i\vartheta})e^{i\vartheta}\,d\vartheta.$$

We mention one more problem. Here we assume that X_1 and X_2 are two Banach subspaces of $\mathcal{O}(\mathbf{D})$, with bounded point evaluations. Also, each contains $\mathcal{O}(\mathbf{D}^-)$ as a dense subspace. We represent the dual spaces X_1^*, X_2^* as spaces of analytic functions on $\mathbf{D}_e$, as above.

PROBLEM 5. If $X_1^*\subset X_2^*$, then must we have $X_2\subset X_1$?

The special case when X_1 is the predual of H^∞ and X_2 is the predual of the Bloch space is proved in [2, Prop. 1].

The Principal Results

We now want to study the invariant subspace $I(f,X)$ generated by a function f in X. One way to do this is to identify the set of $\phi\in X^*$ such that

$\phi \perp I(f, X)$. For example, f is cyclic if and only if the only such function is $\phi = 0$. Let both $f \in X$ and $\phi \perp I(f, X)$ be fixed in the following discussion.

Let $M(X^*) = \{\varphi \in \mathcal{O}(\mathbf{D}_e) : \varphi X^* \subset X^*\}$ denote the space of multipliers of X^*. We introduce an element $h \in M(X^*)^*$ by the relation

$$\langle g, h \rangle = \langle f, g\phi \rangle, \quad g \in M(X^*). \tag{6}$$

From now on we shall assume that $\mathcal{O}(\mathbf{D}_e^-) \subset M(X^*)$. For $w \in \mathbf{D}$ let $g_w \in \mathcal{O}_0(\mathbf{D}_e^-)$ be defined by $g_w(\lambda) = (\lambda - w)^{-1}$, $\lambda \in \mathbf{D}_e$. We associate with h the analytic function

$$\tilde{h}(w) = \langle g_w, h \rangle, \quad w \in \mathbf{D}. \tag{7}$$

The functions $\{g_w\}$, $w \in \mathbf{D}$, span $\mathcal{O}_0(\mathbf{D}_e^-)$. Thus the analytic function $\tilde{h}$ determines, and is determined by, the restriction of the functional h to the closure of $\mathcal{O}_0(\mathbf{D}_e^-)$ in $M(X^*)$.

To get some feeling for $\tilde{h}$ we consider a special case. Let $X = H^2$. Then X^* is identified (as above) with $H_0^2(\mathbf{D}_e)$ (the H^2 functions in the exterior disc that vanish at infinity), and $M(X^*) = H^\infty(\mathbf{D}_e)$. Let f and ϕ in X and X^* be given. Initially we do not assume that $\phi \perp I(f, X)$. Since both f and ϕ have boundary values on $\mathbf{T}$ we may consider the product $\psi = f\phi \in L^1(\mathbf{T})$. We have

$$\tilde{h}(w) = \langle f, g_w \phi \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(e^{i\vartheta}) g_w(e^{i\vartheta}) e^{i\vartheta}\, d\vartheta = \sum_0^\infty \hat{\psi}(n) w^n.$$

Thus $\tilde{h}$ is the "analytic projection" of ψ, that is, the projection of the Fourier series for ψ onto its powers series part.

We now make the assumption that $\phi \perp I(f, X)$. This is equivalent to the conditions: $\langle w^n f, \phi \rangle = 0$, $n \geq 0$. One verifies that in this case we have: $\hat{\psi}(n) = 0$, $n < 0$. Thus the analytic projection of ψ coincides with ψ. Therefore the function $\tilde{h}/f$, which is meromorphic in $\mathbf{D}$, should be a continuation of ϕ, in some sense; we shall make this precise in Lemma 5 below. We require the following auxiliary operator.

For $f \in \mathcal{O}(\mathbf{D})$ and $w \in \mathbf{D}$, let

$$(T_w f)(z) = \frac{f(z) - f(w)}{z - w}, \quad z \in \mathbf{D}, \ z \neq w;$$

for $z = w$ let $(T_w f)(w) = f'(w)$. Clearly $T_w f \in \mathcal{O}(\mathbf{D})$. Recall that $g_w(\lambda) = (\lambda - w)^{-1}$, $w \in \mathbf{D}$, $\lambda \neq w$.

LEMMA 4. *For each $w \in \mathbf{D}$, T_w is a bounded linear transformation mapping X into itself; for all $f \in X$, $\phi \in X^*$, and $w \in \mathbf{D}$ we have*

$$\langle T_w f, \phi \rangle = \langle f, g_w \phi \rangle. \tag{8}$$

Proof. Note that the right side of (8) is defined. Indeed, $g_w \in \mathcal{O}(\mathbf{D}_e^-)$ and thus is in $M(X^*)$. Therefore $g_w \phi \in X^*$, and so the pairing with f is defined. We first establish (8) when $f \in \mathcal{O}(\mathbf{D}^-)$; then $T_w f \in \mathcal{O}(\mathbf{D}^-) \subset X$ and so the pairing with ϕ is defined.

Consider the special case $f=f_\lambda$, with $f_\lambda(w)=(\lambda-w)^{-1}$, $|\lambda|>1$. One has that $T_w f_\lambda=(\lambda-w)^{-1}f_\lambda$. Thus from (2) we have

$$\langle T_w f_\lambda, \phi\rangle=(\lambda-w)^{-1}\phi(\lambda).$$

Applying (2) to the right side of (8) we obtain $g_w(\lambda)\phi(\lambda)$, thus establishing (8) for the functions $\{f_\lambda\}$, $\lambda\in \mathbf{D}_e$. It follows that (8) holds for all finite linear combinations of these functions; the set of these linear combinations is sequentially dense in $\mathcal{O}(\mathbf{D}^-)$.

If a sequence of functions converges in $\mathcal{O}(\mathbf{D}^-)$: $f_n\to f$, then $T_w f_n\to T_w f$ in $\mathcal{O}(\mathbf{D}^-)$ for each $w\in\mathbf{D}$. This implies convergence in X. Thus (8) holds for all $f\in\mathcal{O}(\mathbf{D}^-)$. For such f, and for $w\in\mathbf{D}$ and $\phi\in X^*$, we have:

$$|\langle T_w f, \phi\rangle|\le \|f\|_X\|\phi\|_{X^*}\|g_w\|_{M(X^*)}. \tag{9}$$

Now we show that $T_w X\subset X$. Let $f\in X$ be given, and let $\{f_n\}\subset\mathcal{O}(\mathbf{D}^-)$ converge to f in the norm of X. Fix $w\in\mathbf{D}$. From (9) we see that $\{\langle T_w f_n, \phi\rangle\}$ is a Cauchy sequence for all $\phi\in X^*$, and the limit defines a continuous linear functional on X^*. Since $T_w f_n$ is in $\mathcal{O}(\mathbf{D}^-)$ for all n, we see that in X^{**} the limit functional is in the closure of $\mathcal{O}(\mathbf{D}^-)$. But this closure is X (canonically embedded in X^{**}). Thus there exists an element $g\in X$ such that, for all $\phi\in X^*$,

$$\lim\langle T_w f_n, \phi\rangle=\langle g, \phi\rangle.$$

If we choose for ϕ the functional of evaluation at a point $z\in\mathbf{D}$, then we obtain $(T_w f_n)(z)\to g(z)$. On the other hand, convergence in X implies convergence in $\mathcal{O}(\mathbf{D})$, and so $f_n(z)\to f(z)$ uniformly on compact subsets of $\mathbf{D}$. One verifies that $(T_w f_n)(z)\to(T_w f)(z)$ uniformly on compact subsets. Thus $T_w f=g\in X$, and so $T_w X\subset X$ for each $w\in\mathbf{D}$, as asserted. Also, the closed graph theorem shows that T_w is a bounded linear transformation on X. Finally, since (8) holds for each function f_n, it also holds in the limit for f, as required. □

The following lemma is our principal technical result. If $f\in A$, then $Z(f)=\{z\in\mathbf{D}^-: f(z)=0\}$.

LEMMA 5. *Assume that $X\cap A$ is a Banach algebra, containing $\mathcal{O}(\mathbf{D}^-)$ as a dense subalgebra. Let $f\in X\cap A$. If $\phi\in X^*$ and $\phi\perp I(f,X)$, then ϕ has an analytic continuation to $(\mathbf{C}\cup\{\infty\})\backslash Z(f)$. Further, on $\mathbf{D}\backslash Z(f)$ this continuation coincides with $\tilde{h}/f$.*

Proof. First we identify the maximal ideal space of $X\cap A$ with $\mathbf{D}^-$. Indeed, evaluation at each point of $\mathbf{D}^-$ is a multiplicative linear functional on $X\cap A$. On the other hand, let ψ be a given multiplicative functional, and let $\psi(z)=\alpha$. Here z is the identity function $z(w)=w$. Then $\psi(p)=p(\alpha)$ for all polynomials p. We claim that $\alpha\in\mathbf{D}^-$. If not, let $p=z-\alpha$. Then $\psi(p)=0$, and so p is not invertible. This is a contradiction since $1/p\in\mathcal{O}(\mathbf{D}^-)\subset X\cap A$. If q is a polynomial with no zeros in $\mathbf{D}^-$, then from the equation $1=(1/q)q$ we obtain $\psi(1/q)=1/\psi(q)$. Since rational functions with no poles on $\mathbf{D}^-$ are dense in $\mathcal{O}(\mathbf{D}^-)$, which is dense in $X \cap A$, we see that ψ coincides with evaluation at α.

Let $I(f)=I(f,X)$ and observe that $I(f)\cap A$ is a closed ideal in $X \cap A$. Also,

$$(\lambda - z)+I(f)\cap A$$

is an element of the quotient algebra $(X\cap A)/(I(f)\cap A)$, and is invertible if $\lambda \in \mathbf{C}\setminus Z(f)$. Indeed, a multiplicative linear functional on the quotient algebra induces a multiplicative functional on $X\cap A$, that is, evaluation at a point of $\mathbf{D}^-$. One sees that only the points in $Z(f)$ give rise to functionals on the quotient algebra.

Since $\phi \perp I(f,X)$, ϕ defines a linear functional on the quotient algebra. The analytic continuation of ϕ promised in the statement of the lemma is given by the formula:

$$\phi(\lambda)=\langle(\lambda - z+I(f)\cap A)^{-1},\phi\rangle, \quad \lambda \in \mathbf{C}\setminus Z(f), \tag{10}$$

and $\phi(\infty)=0$. This defines an analytic function on $\mathbf{C}\cup\{\infty\}\setminus Z(f)$. By comparing (10) with (2) we see that $\phi(\lambda)$ coincides with $\tilde{\phi}(\lambda)$ in $\mathbf{D}_e$.

It remains to show that the extension of ϕ given by (10) coincides with $\tilde{h}/f$ in $\mathbf{D}\setminus Z(f)$. From Lemma 4 we see that $T_w f \in X\cap A$, for all $w \in \mathbf{D}$. We claim that if w is in $\mathbf{D}\setminus Z(f)$, then $T_w f/f(w)$ is an element of the coset

$$(w-z+I(f)\cap A)^{-1}.$$

Indeed, one verifies that $1-(w-z)T_w f/f(w)\in I(f)\cap A$, as claimed. Thus, from (10) one has $\phi(w)=\langle T_w f,\phi\rangle/f(w)$ for $w\in \mathbf{D}\setminus Z(f)$. Now from (8), (6), and (7) we have:

$$\phi(w)f(w)=\langle T_w f,\phi\rangle=\langle f, g_w\phi\rangle=\langle g_w, h\rangle=\tilde{h}(w).$$

Thus $\phi=\tilde{h}/f$ in $\mathbf{D}\setminus Z(f)$, as required. □

NOTE. The hypothesis that $X\cap A$ is a Banach algebra containing $\mathcal{O}(\mathbf{D}^-)$ as a dense subalgebra is satisfied when $X=D$, the Dirichlet space, and when $X=D_\alpha$, $0\le\alpha<1$ (see the Introduction for the definition of these spaces).

Lemma 5 gives us a concrete form for the analytic extension of ϕ to

$$\mathbf{C}\cup\{\infty\}\setminus Z(f).$$

We shall seek conditions on f ensuring that ϕ vanishes identically.

We make the further assumption that $M(X^*)=H^\infty(\mathbf{D}_e)$ (recall that we always have $M(X^*)\subset H^\infty(\mathbf{D}_e)$). This assumption is satisfied when X is the Dirichlet space or one of the D_α spaces, $0\le\alpha<1$. If $X=D$, then X^* is the subspace of the Bergman space $L_a^2(\mathbf{D}_e)$ of the exterior disc consisting of those functions that vanish at ∞.

Let $\mathcal{E}(X)$ denote the class of closed subsets $E\subset\mathbf{T}$ of Lebesgue measure zero such that if ϕ is holomorphic on $\mathbf{C}\cup\{\infty\}\setminus E$ and vanishes at ∞, if $\phi\mid\mathbf{D}_e\in X^*$, and if $\phi\mid\mathbf{D}$ is in the Smirnov class $N_+(\mathbf{D})$, then ϕ vanishes identically.

Note that for the Dirichlet space D, $\mathcal{E}(D)$ coincides with the Bergman–Smirnov exceptional sets defined earlier.

In an appendix we list all assumptions made about X.

THEOREM 1. *Let X be as above. Assume that $M(X^*)=H^\infty(\mathbf{D}_e)$, and that $X\cap A$ is a Banach algebra containing $\mathcal{O}(\mathbf{D}^-)$ as a dense subalgebra. If $f\in X\cap A$ is an outer function with $Z(f)\in\mathcal{E}(X)$, then f is cyclic in X.*

Proof. Let $\phi\perp I(f,X)$ be given, and let h be defined by (6); we must show that $\phi=0$. Since $h\in H^\infty(\mathbf{D}_e)^*$, the restriction of h to $A(\mathbf{D}_e)$ (the subalgebra of functions that extend to be continuous on $\mathbf{D}_e^-$) can be represented by integrating against a complex Borel measure μ on $\mathbf{T}$:

$$\langle g,h\rangle=\int g\,d\mu,\quad g\in A(\mathbf{D}_e). \tag{11}$$

Calculating from (6), (7), and (11) we have, for $w\in\mathbf{D}$,

$$\tilde{h}(w)=\sum_0^\infty \hat{\mu}(n+1)w^n. \tag{12}$$

Thus $\tilde{h}$ is in H^p for all $p<1$, and hence is in $N_+(\mathbf{D})$. Since f is an outer function, it follows that $\tilde{h}/f\in N_+(\mathbf{D})$. From Lemma 5, $\phi\,|\,\mathbf{D}$ is in $N_+(\mathbf{D})$, and of course $\phi\,|\,\mathbf{D}_e\in X^*$ since $\phi\in X^*$. Thus ϕ vanishes identically, since $Z(f)\in\mathcal{E}(X)$, which completes the proof. □

NOTE. In case $X=D$, the measure μ in the proof is absolutely continuous. Indeed, let $f\in D$ and $\phi\in D^*$, with $\phi\perp I(f)$. Then ϕ is in $L_a^2(\mathbf{D}_e)$, and $\phi(\infty)=0$. From (6) and Lemma 3(c) we have:

$$\langle g,h\rangle=\langle f,\phi g\rangle=\lim_{r\uparrow 1}\langle f,(\phi g)_r\rangle=\lim\langle f,\phi_r g_r\rangle,\quad g\in H^\infty(\mathbf{D}_e).$$

We claim that the right side can be replaced by $\lim\langle f,\phi g_r\rangle$. Indeed,

$$\langle f,\phi_r g_r\rangle=\langle f,\phi g_r\rangle+\langle f,(\phi_r-\phi)g\rangle+\langle f,(\phi_r-\phi)(g_r-g)\rangle.$$

We have $\|\phi_r-\phi\|_B\to 0$, and so the second term on the right goes to zero. Also,

$$\|(\phi_r-\phi)(g_r-g)\|_B\le\|\phi_r-\phi\|_B\|g_r-g\|_\infty\le 2\|g\|_\infty\|\phi_r-\phi\|_B\to 0,$$

and so the third term goes to zero also. Thus we have, for $g\in H^\infty(\mathbf{D}_e)$,

$$\langle g,h\rangle=\lim\langle f,\phi g_r\rangle=\lim\langle g_r,h\rangle=\lim\int g_r\,d\mu. \tag{13}$$

The last equality follows from (11) since $g_r\in A(\mathbf{D}_e)$. The existence of this last limit for all g in $H^\infty(\mathbf{D}_e)$ (or even for all Blaschke products) implies that μ is absolutely continuous (see [13, Thm. 2]).

In this example, if we multiply the formal Fourier series on $\mathbf{T}$ for f and for ϕ, then all terms of negative index vanish, since $\phi\perp z^n f$ and the terms of positive index coincide with $\hat{h}(n)$, $n\ge 0$. The connection with $\hat{\mu}$ is given in (12).

A slight modification of the proof of Theorem 1 leads to the following improvement of the result. A collection of functions $\mathfrak{F} \subset H^2$ will be said to have inner factor 1 if $I(\mathfrak{F}, H^2) = H^2$. If $\mathfrak{F} \subset A$, let $Z(\mathfrak{F}) = \bigcap\{Z(f): f \in \mathfrak{F}\}$.

THEOREM 2. *Let X and $\mathcal{E}(X)$ be as in Theorem 1. Assume that $M(X^*) = H^\infty(\mathbf{D}_e)$, and that $X \cap A$ is a Banach algebra containing $\mathcal{O}(\mathbf{D}^-)$ as a dense subalgebra. If $\mathfrak{F} \subset X \cap A$ has inner factor* 1, *and if $Z(\mathfrak{F}) \in \mathcal{E}(X)$, then $I(\mathfrak{F}, X) = X$.*

Proof. Let $f \in \mathfrak{F}$, let $\phi \in X^*$ with $\phi \perp I(\mathfrak{F}, X)$, and let h_f be the element of $M(X^*)^*$ given by (6). Since $\phi \perp I(f)$, Lemma 5 tells us that ϕ has an analytic continuation to $\mathbf{C} \cup \{\infty\} \setminus Z(f)$ that coincides (in $\mathbf{D}$) with $\tilde{h}_f/f$. Since this is true for every $f \in \mathfrak{F}$, we see that ϕ extends to be holomorphic in $\mathbf{C} \cup \{\infty\} \setminus Z(\mathfrak{F})$, and that the functions $\tilde{h}_f/f$ $(f \in \mathfrak{F})$ coincide in $\mathbf{D}$. As in the proof of Theorem 1, $\tilde{h}_f \in N_+(\mathbf{D})$. Since $\mathfrak{F}$ has inner factor 1, one can show that $\phi \mid \mathbf{D} \in \mathrm{N}_+(\mathbf{D})$. The proof is completed as before. □

The following is an immediate consequence of Theorem 1.

COROLLARY. *If $f \in D \cap A$ is an outer function, and if $Z(f)$ is a Bergman-Smirnov exceptional set, then f is cyclic in D.*

It is natural to ask what the Bergman–Smirnov exceptional sets look like.

PROPOSITION 1. *If $\{z_0\} \in \mathbf{T}$, then $\{z_0\}$ is a Bergman–Smirnov exceptional set.*

Proof. Let $z_0 = 1$, and let ϕ be holomorphic in $\mathbf{C} \cup \{\infty\} \setminus \{1\}$, with $\phi \mid \mathbf{D} \in N_+(\mathbf{D})$ and $\phi \mid \mathbf{D}_e \in L^2_a(\mathbf{D}_e)$. We assume that $\phi(\infty) = 0$, and we must show that ϕ is identically zero. Let $\phi \mid \mathbf{D} = \varphi/\psi$, where φ, ψ are bounded analytic functions in $\mathbf{D}$ and ψ is an outer function. The fact that $\phi \mid \mathbf{D}_e$ is in the Bergman space of $\mathbf{D}_e$ is equivalent to the convergence of the series $\sum |\hat{\phi}(n)|^2/n$. Applying the Cauchy (–Buniakovski–Schwarz) inequality, we have $\phi(\lambda) = O((|\lambda| - 1)^{-1})$ as $|\lambda| \downarrow 1$. We actually have "o", as one sees by approximating $\phi \mid \mathbf{D}_e$ by polynomials in λ^{-1}.

Let $\gamma: \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ be the Möbius map $\gamma(z) = (z+i)/(z-i)$, which maps the lower half plane onto $\mathbf{D}$. Thus $\phi \circ \gamma$ is an entire function and, if $y = \operatorname{Im} z$, then

$$(\phi \circ \gamma)(z) = O\left(\frac{|z|^2}{y}\right) \quad \text{as } |z| \to \infty, \text{ with } y > 0. \tag{14}$$

Also, $\psi \circ \gamma$ is bounded in $\{y < 0\}$. We apply Lemma 4.4 of [11] to conclude that $\phi \circ \gamma$ is an entire function of exponential type $(-\tau)$, where

$$\tau = \limsup_{y \to -\infty} \frac{\log|(\psi \circ \gamma)(iy)|}{|y|} = \limsup_{r \uparrow 1} \frac{1}{2}(1-r)\log|\psi(r)|$$

and $r = (y+1)/(y-1)$. Because ψ is bounded, $\tau \leq 0$. Also, since ψ is an outer function, $\log|\psi(r)| = \int (1-r^2)|w-r|^{-2}\, d\sigma(w)$. Here $w = \exp(it)$, and

$d\sigma(w) = \log|\psi(w)|\, dm(w)$, where dm denotes normalized Lebesgue measure on $\mathbf{T}$. In particular, σ has no point masses and so $\sigma(\{1\}) = 0$. Using this, one can show that $(1-r)\log|\psi(r)| \to 0$, and so $\tau = 0$.

Thus $F(z) = (\phi \circ \gamma)(z)/(z-i)$ is an entire function of exponential type zero. By (14), F is bounded on the half line: $x = 0$, $y > 0$. Hence by [5, Thm. 6.2.14, p. 84], F is constant. Therefore $(\phi \circ \gamma)(z) = c(z-i)$, and so $\phi(\lambda) = 2ic/(\lambda - 1)$. Hence $c = 0$; otherwise, $\phi \mid \mathbf{D}_e$ would not be in the Bergman space. Thus $\phi = 0$, as required. □

Note that the proof did not fully use the hypothesis that $\phi \mid \mathbf{D}$ is in the Smirnov class. Indeed, all that was required was that the measure σ put no mass at $\{1\}$.

PROPOSITION 2. *Let E_1 and E_2 be two disjoint closed subsets of $\mathbf{T}$ such that E_1 is a Bergman–Smirnov exceptional set. Then every function ϕ holomorphic on $\mathbf{C} \cup \{\infty\} \setminus (E_1 \cup E_2)$, that lies in the Smirnov class in $\mathbf{D}$ and in the Bergman space in $\mathbf{D}_e$, extends analytically across E_1.*

Proof. Let ϕ be such a function. Using the Cauchy integral we may decompose ϕ as $\phi = \phi_1 + \phi_2$, where ϕ_i is holomorphic on the complement of E_i, $i = 1, 2$. We claim that $\phi_i \mid \mathbf{D}_e \in L_a^2(\mathbf{D}_e)$ and $\phi \mid \mathbf{D} \in N_+(\mathbf{D})$, $i = 1, 2$. For the Bergman space we need only verify square integrability near the boundary $\mathbf{T}$. This is clear for ϕ_1 except near E_1, but there we have $\phi_1 = \phi - \phi_2$, a difference of two functions that are each square integrable near E_1.

As regards the Smirnov class we recall that a holomorphic function f in $\mathbf{D}$ is in N_+ if and only if the family $\{\log^+|f_r|\}$, $0 < r < 1$, is uniformly integrable in $L^1(\mathbf{T})$ (see, e.g., [14, §§3.1, 3.3]). Here $f_r(w) = f(rw)$, $|w| = 1$. Now fix disjoint (linear) neighborhoods U_1, U_2 of E_1 and of E_2 in $\mathbf{T}$. Then the family $\{\log^+|(\phi_1)_r|\}$ is uniformly integrable in $\mathbf{T} \setminus U_1$. Also, this family is the difference of two uniformly integrable families in U_1, and so it is uniformly integrable there. Thus $\phi_1 \in N_+(\mathbf{D})$, as claimed.

Since E_1 is a Bergman–Smirnov exceptional set, ϕ_1 must be constant, and thus ϕ extends analytically across E_1, as claimed. □

COROLLARY. *The union of two disjoint Bergman–Smirnov exceptional sets is a Bergman–Smirnov exceptional set.*

THEOREM 3. *Every countable closed subset of the circle $\mathbf{T}$ is a Bergman–Smirnov exceptional set.*

Proof. Let $E \subset \mathbf{T}$ be a countable closed set and assume that there exists a nonconstant function ϕ holomorphic in the complement of E, with ϕ in the Bergman space in $\mathbf{D}_e$ and in the Smirnov class in $\mathbf{D}$. Let E_1 be the (closed) subset of E consisting of those points across which ϕ cannot be continued analytically. Then E_1 is not empty and, being closed and countable, E_1 must have an isolated point. But, by Propositions 1 and 2, ϕ is continuable across any isolated points. □

The same argument shows that if $E \subset \mathbf{T}$ is not a Bergman–Smirnov exceptional set, then there is a nonempty closed subset E_1 of E, with no isolated points, that is not a Bergman–Smirnov exceptional set. A closed set E is the union of a countable set (not necessarily closed) and a perfect set, called the perfect core of E. (A perfect set is a set that is closed and has no isolated points. If not empty it has the cardinality of the continuum.) Thus, if E is not an exceptional set then neither is its perfect core.

APPENDIX. For the convenience of the reader we list here all the assumptions made at various places in this paper about the space X.

(a) The embedding map of X into $\mathcal{O}(\mathbf{D})$ is continuous; X contains $\mathcal{O}(\mathbf{D}^-)$ as a dense subset.
(b) $\mathcal{O}(\mathbf{D}^-) \subset M(X)$.
(c) $\mathcal{O}(\mathbf{D}_e^-) \subset M(X^*)$.
(d) $X \cap A$ is a Banach algebra containing $\mathcal{O}(\mathbf{D}^-)$ as a dense subset.
(e) $M(X^*) = H^\infty(\mathbf{D}_e)$.

Finally, the reader may consult [1] for further information about invariant subspaces in spaces of analytic functions.

Added in proof. The editor notes with regret the death of Allen Shields on September 16, 1989.

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Håkan Hedenmalm
Department of Mathematics
Uppsala University
Thunbergsvägen 3
S-752 38 Uppsala
Sweden

Allen Shields
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1003