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ON THE MAKAROV LAW OF THE ITERATED LOGARITHM

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Abstract. We obtain considerable improvement of Makarov's estimate of the boundary behavior of a general conformal mapping from the unit disk to a simply connected domain in the complex plane. We apply the result to improve Makarov's comparison of harmonic measure with Hausdorff measure on simply connected domains.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Suppose that a function φ is analytic in the disk $\mathbb{D}$. Nikolai Makarov [8] proved that there exists a positive absolute constant C_1 such that

$$\limsup_{r\to 1^-} \frac{|\varphi(re^{i\theta})|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \le C_1 ||\varphi||_{\mathcal{B}} \tag{1.1}$$

for almost all $\zeta \in \partial\mathbb{D}$. Here,

$$\|\varphi\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} (1-|z|^2)|\varphi'(z)|$$

is the usual Bloch seminorm. Later, Christian Pommerenke ([9], p. 186) showed that the inequality (1.1) is true for $C_1 = 1$, and that there exists a function φ analytic in disk $\mathbb{D}$ such that the inequality (1.1) fails for $C_1 \le 0.685$.

Suppose that a function f is analytic and univalent in the disk $\mathbb{D}$. It is known that ([9], p. 9) that $\|\log f'\|_{\mathcal{B}} \le 6$. On the other hand, by Becker's univalent criterion ([9], p. 16), any function g analytic in $\mathbb{D}$ must be univalent if $||\log g'||_{\mathcal{B}} \le 1$. Therefore, we see that the Makarov law of the iterated logarithm amounts to the inequality

$$\limsup_{r\to 1^-} \frac{|\log f'(re^{i\theta})|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \le C_2 \tag{1.2}$$

for almost all $\zeta \in \partial\mathbb{D}$, where C_2 is an absolute constant and f is a function analytic and univalent in the disk $\mathbb{D}$.

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Let $C_{\mathrm{M}} = \inf\{C_2 : \text{(1.2) holds}\}$. Pommerenke's estimates lead to the following inequalities ([9], p. 188):

$$0.685 < C_{\mathrm{M}} \le 6.$$

Recently, in [6], it was shown that $C_{\mathrm{M}} \le 2\sqrt{3}$. In this paper, we shall demonstrate that

$$0.91 \le C_{\mathrm{M}} \le 2\sqrt{\frac{\sqrt{24}-3}{5}} = 1.2326\dots. \tag{1.3}$$

On the one hand, to obtain the upper estimate in (1.3), we combine the following two results.

Result 1. In the paper [2], it was shown that

$$\limsup_{t\to 0} \frac{\beta_f(t)}{|t|^2} \le \frac{\sqrt{24}-3}{5} = 0.3798\dots, \tag{1.4}$$

where

$$\beta_f(t) = \limsup_{r\to 1^-} \frac{\log \int_{-\pi}^{\pi} \left|f'(re^{i\theta})^t\right| \frac{\mathrm{d}\theta}{2\pi}}{\log \frac{1}{1-r}}$$

is the integral means spectrum.

Result 2. In the paper [5], it was established that for all positive δ, the following inequality holds:

$$\limsup_{r\to 1-} \frac{|\log f'(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \le 2 \limsup_{t\to 0} \frac{\sqrt{\beta_{f,\delta}(t)}}{|t|} \tag{1.5}$$

for almost all ζ on $|\zeta| = 1$. Here, we use the notation

$$\beta_{f,\delta}(t) = \sup_{r\in]0,1[} \frac{\log\left\{\delta \int_{|z|=r} \left|f'(z)^t\right| \frac{\mathrm{d}\theta}{2\pi}\right\}}{\log \frac{1}{1-r}}.$$

We remark that $\beta_f(t) \le \beta_{f,\delta}(t)$, and that

$$\beta_{f,\delta}(t) \to \beta_f(t) \quad \text{as} \quad \delta \to 0.$$

This allows us to apply the results (suitably modified) of the above-mentioned paper [2].

On the other hand, to prove the lower estimate in (1.3), we use the example which was constructed in the paper [7], combined with an iterative process introduced by Pommerenke.

We apply the upper estimate in (1.3) to the following classical problem. Let Ω be a simply connected domain on the complex plane with Jordan boundary and let f be the Riemann mapping from the disk $\mathbb{D}$ onto Ω. By the Carathéodory extension theorem, then, f extends to a homeomorphism of (the closure of) $\bar{\mathbb{D}}$ onto $\bar{\Omega}$.

Problem. *Suppose that A is a Borel set of positive linear measure on $\partial\mathbb{D}$. What can we say about the metric properties of $f(A)$?*

The classical Riesz-Privalov theorem states that if the domain $\Omega = f(\mathbb{D})$ has a rectifiable boundary and $A \subset \partial\mathbb{D}$ has positive linear measure, then the linear measure of $f(A)$ is also positive. Lavrentiev later showed that the rectifiability assumption is essential here.

We recall some basic definitions. Suppose that $\varphi : [0,+\infty) \to [0,+\infty)$ is continuous and increasing, with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Let E be a Borel set in the plane $\mathbb{C}$. The φ-Hausdorff measure of the set E is defined by

$$\Lambda_\varphi(E) = \lim_{\varepsilon\to 0^+} \inf_{\{B_k\}_k} \sum \varphi(\operatorname{diam} B_k),$$

where the infimum is taken over all countable coverings $\{B_k\}_k$ of E such that $\operatorname{diam} B_k < \varepsilon$ for all k. In the case $\varphi(t) = t^\alpha$ for some positive real parameter α, we write Λ_α in place of Λ_{t^α}. In [1], Lennart Carleson proved that there exists $\varepsilon > 0$ such that $0 < \Lambda_{1/2+\varepsilon}(f(A)) \le +\infty$. Later, Makarov [8] showed that $0 < \Lambda_\alpha(f(A)) \le +\infty$ for any α with $0 < \alpha < 1$. Indeed, he proved a substantially stronger assertion: There exists an absolute positive constant C such that $0 < \Lambda_\varphi(f(A)) \le +\infty$, where

$$\varphi(t) = t\,\exp\left\{C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right\}, \qquad 0 < t < 10^{-7}.$$

Later, Pommerenke and Steffen Rohde [9] showed that one can actually take $C = 30$. We use the upper estimate in (1.2) and the Makarov-Pommerenke-Rohde scheme to reduce this constant to any C with

$$C > 6\sqrt{\frac{\sqrt{24}-3}{5}} = 3.6977\ldots;$$

for instance, $C = 37/10$ will do.

2. The estimate from above

We need the standard weighted Bergman (Hilbert) spaces $\mathcal{H}_\alpha(\mathbb{D})$, the elements of which are complex-valued holomorphic functions g in $\mathbb{D}$, subject to the norm boundedness condition

$$\|g\|_\alpha^2 = \int_{\mathbb{D}} |g(z)|^2\, \mathrm{d}A_\alpha(z) < +\infty.$$

The real parameter α is confined to $-1 < \alpha < +\infty$, and $\mathrm{d}A_\alpha$ is the Borel probability measure

$$\mathrm{d}A_\alpha(z) = (\alpha+1)\left(1-|z|^2\right)^\alpha \mathrm{d}A(z),$$

with

$$\mathrm{d}A(z) = \frac{\mathrm{d}x\mathrm{d}y}{\pi}, \qquad z = x + \mathrm{i}y \in \mathbb{C}.$$

Lemma 2.1. *For g analytic in $\mathbb{D}$ and positive real α, we have*

$$\int_{-\pi}^{\pi} \left|g(re^{\mathrm{i}\theta})\right|^2 \frac{\mathrm{d}\theta}{2\pi} \le \frac{\|g\|_{\alpha-1}^2}{(1-r^2)^\alpha}, \qquad 0 < r < 1.$$

Proof. We compute that

$$\int_{-\pi}^{\pi} \left|g(re^{\mathrm{i}\theta})\right|^2 \frac{\mathrm{d}\theta}{2\pi} = \sum_{n=0}^{+\infty} \left|\widehat{g}(n)\right|^2 r^{2n},$$

while

$$\|g\|_\alpha^2 = \sum_{n=0}^{+\infty} \frac{n!}{(\alpha+1)_n} \left|\widehat{g}(n)\right|^2.$$

Here, $(\beta)_k$ is the standard Pochhammer symbol, and $\widehat{g}(n)$ are the Taylor coefficients of g. It follows that we have the sharp estimate

$$\int_{-\pi}^{\pi} \left|g(re^{\mathrm{i}\theta})\right|^2 \frac{\mathrm{d}\theta}{2\pi} \le A(r,\alpha)\,\|g\|_\alpha^2,$$

where

$$A(r,\alpha) = \max_n \left\{ \frac{(\alpha+1)_n}{n!}\, r^{2n} \right\}.$$

Next, we consider $r^2 = e^{-\varrho}$, with $0<\varrho<+\infty$, and take logarithms:

$$\log A(r,\alpha) = \max_n \left\{ -n\varrho + \sum_{k=1}^{n} \log\left(1+\frac{\alpha}{k}\right) \right\}.$$

Clearly,

$$\sum_{k=1}^{n} \log\left(1+\frac{\alpha}{k}\right) \le \alpha \sum_{k=1}^{n} \frac{1}{k},$$

while for $n=1,2,3,\ldots$, we have

$$\sum_{k=1}^{n} \frac{1}{k} \le 1+\log n,$$

so that if

$$B(r,\alpha) = \max_{n\in\{1,2,3\ldots\}} \big\{ -n\varrho + \alpha + \alpha\log n \big\},$$

then

$$\log A(r,\alpha) \le \max\big\{0, B(r,\alpha)\big\}.$$

We quickly see that

$$B(r,\alpha) \le \alpha\log\frac{1}{\varrho} = \alpha\log\frac{1}{\log\frac{1}{r^2}} \le \alpha\log\frac{1}{1-r^2},$$

so that

$$A(r,\alpha) \le \frac{1}{(1-r^2)^\alpha}.$$

The assertion is immediate. □

Let f be a univalent function of the class $\mathcal{S}$, and consider

$$g_t(z) = \big[f'(z)\big]^{t/2}, \qquad z\in\mathbb{D},$$

for complex t. The power is defined by selecting the holomorphic logarithm of f' with value 0 at the origin.

The proof of the theorem below is based on the recent method of Hedenmalm and Shimorin [2] (see also [3]).

Theorem 2.2. *Let $f\in\mathcal{S}$, and assume that $\beta\in\mathbb{R}$ has $B|t|^2\le\beta\le 1$ and $\beta\ne 0$, for some fixed positive B with*

$$\frac{\sqrt{24}-3}{5} < B \le 1.$$

Then, if the complex parameter t is sufficiently close to 0, we have that

$$\big\|[f']^{t/2}\big\|_{\beta-1} = O_B(1),$$

where the O_B bound depends only on B.

Proof. We first observe that an inspection of the proof of Proposition 4.7 of [2] reveals that the constant $C_3(\alpha,\nu)$ appearing there can be assumed to be an absolute constant if $-1<\alpha\le 0$ and $0<\nu\le 1$.

Next, we look at the estimate in Lemma 4.4 of [2], where a constant $C_1=C_1(\alpha,\theta)$ appears. This constant may in fact be chosen independently of θ and α, at least with α confined to a bounded interval and θ with $0<\theta<1$.

We use the above two observations to conclude that the constant $C_6(\alpha,\theta)$ of Theorem 4.9 of [2] can be assumed to depend only on θ.

Finally, we look at the estimates (6.8) and (6.11) of [2], which involve the first two terms of the norm expansion, as well as the estimate involving the first three terms in the addendum paper [3]. By using the improved norm control of error terms in the method that follows from the above two observations, the asserted norm control follows. The details are tedious but straightforward, and therefore left to the interested reader. □

In view of Lemma 2.1, we have the following.

Corollary 2.3. *Let $f\in\mathcal{S}$. Assume that $\beta\in\mathbb{R}$ has $B|t|^2\le\beta\le 1$ and $\beta\ne 0$, for some fixed positive B with*

$$\frac{\sqrt{24}-3}{5}<B\le 1.$$

Then, if the complex parameter t is sufficiently close to 0, we have that

$$\int_{-\pi}^{\pi}\left|\left[f'(re^{\mathrm{i}\theta})\right]^t\right|\frac{\mathrm{d}\theta}{2\pi}=O_B\left(\frac{1}{(1-r^2)^\beta}\right),$$

where the O_B bound depends only on B.

We get the following consequence of the above corollary.

Corollary 2.4. *Let $f\in\mathcal{S}$. Then, for $0<\delta\le 1$, we have*

$$\beta_{f,\delta}(t)\le\left(\frac{\sqrt{24}-3}{5}+o(1)\right)|t|^2,\quad \textit{as}\quad |t|\to 0.$$

Proof. We note that from Corollary 2.3, we have

$$\log\left\{\delta\int_{-\pi}^{\pi}\left|f'(re^{i\theta})^t\right|\frac{\mathrm{d}\theta}{2\pi}\right\}\le\log\int_{-\pi}^{\pi}\left|f'(re^{i\theta})^t\right|\frac{\mathrm{d}\theta}{2\pi}\le B|t|^2\log\frac{1}{1-r^2}+O_B(1),$$

provided

$$\frac{\sqrt{24}-3}{5}<B\le 1$$

and t is sufficiently close to 0. The assertion is now immediate from the definition of $\beta_{f,\delta}(t)$. □

Corollary 2.5. *For $f\in\mathcal{S}$, we have*

$$\frac{\left|\log f'(re^{i\theta})\right|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}}\le 2\sqrt{\frac{\sqrt{24}-3}{5}}=1.2326\dots.$$

Proof. This follows from a combination of Corollary 2.4 and the results of Kayumov [5]. □

For a Borel measurable subset $\mathcal{A}$ of the unit circle $\mathbb{T}$, let $|\mathcal{A}|_{\mathfrak{s}}$ denote its normalized length (the normalization is such that $\mathbb{T}$ gets total length 1). We have a weak type corollary of Corollary 2.3.

Corollary 2.6. *For $0 < r < 1$ and $0 < \alpha < +\infty$, let $\mathcal{M}_f(r, \alpha)$ denote the set*

$$\mathcal{M}_f(r, \alpha) = \big\{\zeta \in \mathbb{T} : |f'(r\zeta)| \le e^{-\alpha}\big\}.$$

Fix a real parameter B such that

$$\frac{\sqrt{24} - 3}{5} < B \le 1.$$

Then we have the following estimate for $f \in \mathcal{S}$:

$$\big|\mathcal{M}_f(r, \alpha)\big|_{\mathfrak{s}} \le C_B\, e^{-\frac{\alpha^2}{4B \log \frac{1}{1-r}}},$$

provided r is close enough to 1, as specified by

$$1 - e^{-\alpha/(2Bt_0)} \le r < 1.$$

Here, t_0 is a small positive absolute constant, and the positive constant C_B only depends on B.

Proof. By Corollary 2.3, we have

$$\int_{-\pi}^{\pi} \big|\big[f'(re^{\mathrm{i}\theta})\big]^t\big| \frac{\mathrm{d}\theta}{2\pi} \le \frac{C_B}{(1 - r)^{B|t|^2}},$$

for all complex t with $|t| \le t_0$, where t_0 is some positive absolute constant. We shall apply this estimate to negative t. Switching the sign of t, we get from the weak type estimate of Kolmogorov that

$$\big|\mathcal{M}_f(r, \alpha)\big|_{\mathfrak{s}} \le C_B\, e^{-\alpha t}\, (1 - r)^{-B|t|^2},$$

where on the right hand side we are free to optimize the right hand side over t with $0 < t < t_0$. The optimal t is

$$t = \frac{\alpha}{2B \log \frac{1}{1-r}},$$

which belongs to the interval $]0, t_0[$ provided r is in the indicated interval. Plugging in this value of t, we get the indicated estimate. □

3. An application: comparison of harmonic measure with Hausdorff measure

We get an improvement of the constant in Makarov's distortion theorem; the size of the constant is important, because different constants give rise to nonequivalent Hausdorff measures.

Corollary 3.1. *Suppose that A is Borel set of positive linear measure on $\partial\mathbb{D}$, and that f is a conformal mapping from $\mathbb{D}$ onto a simply connected domain in the complex plane. Suppose the constant C has*

$$C > 6\sqrt{\frac{\sqrt{24} - 3}{5}} = 3.6977\ldots.$$

Then $0 < \Lambda_\varphi(f(A)) \le +\infty$, *where*

$$\varphi(t) = t\, \exp\left\{ C \sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}} \right\}, \qquad 0 < t < 10^{-7}.$$

Proof. Careful consideration of the proof of Theorem 10.6 from the book [9] (p. 229) shows that Makarov's distortion theorem holds with constant C, provided C is greater than 3 times the constant appearing on the right hand side of the displayed equation in Corollary 2.5. The assertion is immediate. □

Remark 3.2. We do not know if the factor 3 of the above proof is an artefact of the proof, or a genuine obstacle.

4. The estimate from below

To prove a lower estimate we use a construction which generates fractal type mappings. Let f be a conformal mapping from $\mathbb{D}$ into $\mathbb{D}$ with $f(0) = 0$, such that $f''/f' \in H^\infty(\mathbb{D})$; here $H^\infty(\mathbb{D})$ is the usual space of bounded analytic functions in $\mathbb{D}$. We write F for the related function

$$F(z) = \log\frac{zf'(z)}{f(z)}, \tag{4.1}$$

which is the holomorphic logarithm with $F(0) = 0$. We consider the associated functions

$$f_m(z) = \big\{f(z^m)\big\}^{1/m}, \quad m = 1, 2, \dots,$$

which also map $\mathbb{D}$ into $\mathbb{D}$. For $q = 2, 3, 4, \dots$ and $n = 1, 2, 3, \dots$, we define

$$g_{q,n}(z) = f_q \circ f_{q^2} \circ f_{q^3} \circ \cdots \circ f_{q^n}(z), \tag{4.2}$$

which then map $\mathbb{D}$ into $\mathbb{D}$. We also consider the limit as $n \to +\infty$:

$$g_{q,\infty}(z) = \lim_{n\to+\infty} g_{q,n}(z). \tag{4.3}$$

It is easy to see that

$$f_{q^j}\left(z^q\right) = f_{q^{j+1}}(z)^q;$$

a consequence of this is that

$$g_{q,n+1}(z) = \big\{f \circ g_{q,n}(z^q)\big\}^{1/q},$$

so that in the limit as $n \to +\infty$, we get

$$g_{q,\infty}(z) = \big\{f \circ g_{q,\infty}(z^q)\big\}^{1/q}. \tag{4.4}$$

We remark that a construction of this type was first used by Pommerenke to produce a bounded univalent function in $\mathbb{D}$ whose Taylor coefficients do not decrease as fast as $O(1/n^{0.83})$. We need the concept of asymptotic variance:

$$\big\{\sigma_h\big\}^2 = \limsup_{r\to1^-} \frac{\int_{-\pi}^{\pi} \left|h(re^{i\theta})\right|^2 \frac{d\theta}{2\pi}}{\log\frac{1}{1-r}},$$

where h is assumed analytic in $\mathbb{D}$.

Lemma 4.1. *Suppose that $g_{q,\infty}$ as given by (4.3) is a Hölder continuous mapping. Then, for almost all $\theta \in [-\pi, \pi]$, the following equality holds:*

$$\limsup_{r\to 1-} \frac{\left|\log g'_{q,\infty}(re^{i\theta})\right|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} = \sigma_{\log g'_{q,\infty}}. \tag{4.5}$$

Remark 4.2. It may be possible to obtain this result by analyzing the methods of the paper [10] by Przytycki, Urbanski, and Zdunik (see also [11]). We prefer, however, to supply a direct proof.

Proof. We begin with the identity

$$f_{q^{k+1}}\circ f_{q^{k+2}}\circ\cdots\circ f_{q^{k+n}}(z) = \left\{f_q\circ f_{q^2}\circ\cdots\circ f_{q^n}\left(z^{q^k}\right)\right\}^{1/q^k} = \left\{g_{q,n}\left(z^{q^k}\right)\right\}^{1/q^k}, \tag{4.6}$$

valid for $k = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$. By letting $n \to +\infty$ in this identity, we see that

$$\lim_{n\to+\infty} f_{q^{k+1}}\circ f_{q^{k+2}}\circ\cdots\circ f_{q^{k+n}}(z) = \left\{g_{q,\infty}\left(z^{q^k}\right)\right\}^{1/q^k}. \tag{4.7}$$

By the chain rule, we have

$$\log g'_{q,n}(z) = \sum_{k=1}^{n} \log\left\{f'_{q^k}\left(f_{q^{k+1}}\circ\cdots\circ f_{q^n}(z)\right)\right\};$$

here, the term with $k = n$ is to be interpreted as $\log f'_{q^n}(z)$. We check that

$$\log f'_{q^k}(w) = \log\frac{w^{q^k} f'\left(w^{q^k}\right)}{f\left(w^{q^k}\right)} + \frac{1}{q^k}\log\frac{f\left(w^{q^k}\right)}{w^{q^k}},$$

so that by (4.6),

$$\begin{aligned}\log f'_{q^k}\left(f_{q^{k+1}}\circ\cdots\circ f_{q^n}(z)\right) = {}& \log\frac{g_{q,n-k}\left(z^{q^k}\right) f'\left(g_{q,n-k}\left(z^{q^k}\right)\right)}{f\left(g_{q,n-k}\left(z^{q^k}\right)\right)} \\ &+ \frac{1}{q^k}\log\frac{f\left(g_{q,n-k}\left(z^{q^k}\right)\right)}{g\left(z^{q^k}\right)}.\end{aligned}$$

Thus,

$$\log g'_{q,n}(z) = \sum_{k=1}^{n}\log\frac{g_{q,n-k}\left(z^{q^k}\right) f'\left(g_{q,n-k}\left(z^{q^k}\right)\right)}{f\left(g_{q,n-k}\left(z^{q^k}\right)\right)} + \sum_{k=1}^{n}\frac{1}{q^k}\log\frac{f\left(g_{q,n-k}\left(z^{q^k}\right)\right)}{g_{q,n-k}\left(z^{q^k}\right)},$$

which implies

$$\left|\log g'_{q,n}(z) - \sum_{k=1}^{n} F\circ g_{q,n-k}\left(z^{q^k}\right)\right| \le C\sum_{k=1}^{n}\frac{1}{q^k} \le \frac{Cq}{q-1},$$

where C is the positive constant such that

$$\left|\log\frac{f(w)}{w}\right| \le C, \qquad w\in\mathbb{D};$$

the function F is given by (4.1). We shall be concerned with the limit as $n\to+\infty$ in the above identities. We get that

$$\left|\log g'_{q,\infty}(z) - \sum_{k=1}^{+\infty} F\circ g_{q,\infty}\left(z^{q^k}\right)\right| \le \frac{Cq}{q-1}. \tag{4.8}$$

Let $\{\widehat{g}_{q,\infty}(k)\}_k$ denote the sequence of Taylor coefficients of the function $g_{q,\infty}$. Since $g_{q,\infty}$ is assumed to be a Hölder map, we know from the work of Smith and Stegenga [12] that

$$\sum_{k=1}^{\infty} k^{1+\varepsilon} \big|\widehat{g}_{q,\infty}(k)\big|^2 < +\infty \tag{4.9}$$

holds for some $\varepsilon > 0$. Since (4.9) is equivalent to having

$$\int_{\mathbb{D}} \frac{|g'_{q,\infty}(z)|^2}{(1-|z|^2)^{\varepsilon}} \, \mathrm{d}A(z) < +\infty,$$

we see that (by using that F' is bounded, which follows from the condition $f''/f' \in H^{\infty}(\mathbb{D})$) the same is true for the Taylor coefficients of the function $F \circ g_{q,\infty}(z)$:

$$\sum_{k=1}^{\infty} k^{1+\varepsilon} \Big|\widehat{F \circ g}_{q,\infty}(k)\Big|^2 < +\infty. \tag{4.10}$$

Next, we consider the polynomial

$$G_{q,N}(z) = \sum_{k=1}^{N} \widehat{F \circ g}_{q,\infty}(k)\, z^k,$$

and form the series

$$H_{q,N}(z) = \sum_{k=1}^{+\infty} G_{q,N}\big(z^{q^k}\big).$$

It is easy to see that the Taylor series for $H_{q,N}$ is a lacunary, with Hadamard gaps $\lambda = 1 + 1/N$. For such a gap series, Mary Weiss [13] proved a law for the iterated logarithm:

$$\limsup_{r\to 1-} \frac{|H_{q,N}(re^{i\theta})|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} = \sigma_{H_{q,N}}. \tag{4.11}$$

Our next step is to consider the difference function

(4.12)

$$\begin{aligned} R_{q,N}(z) = \sum_{k=1}^{+\infty} F \circ g_{q,\infty}\big(z^{q^k}\big) - H_{q,N}(z) &= \sum_{k=1}^{+\infty} \Big\{ F \circ g_{q,\infty}\big(z^{q^k}\big) - G_{q,N}\big(z^{q^k}\big) \Big\} \\ &= \sum_{j=N+1}^{+\infty} \sum_{k=1}^{+\infty} \widehat{F \circ g}_{q,\infty}(k)\, z^{jq^k}. \end{aligned}$$

We shall argue that

$$\|R_{q,N}\|_{\mathcal{B}} \to 0 \text{ as } N \to +\infty. \tag{4.13}$$

Suppose for the moment that (4.13) has been established. Then, by Makarov's theorem,

$$\limsup_{r\to 1^-} \frac{|R_{q,N}(re^{i\theta})|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \le C_1 \|R_{q,N}\|_{\mathcal{B}} \to 0 \quad \text{as} \quad N \to +\infty,$$

so that in view of (4.11) and (4.8), we have

$$\limsup_{r\to 1^-} \frac{\left|\log g'_{q,\infty}(re^{i\theta})\right|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} = \lim_{N\to+\infty}\sigma_{H_{q,N}}.$$

It follows from (4.13) that $\sigma_{R_{q,N}} \to 0$ as $N\to+\infty$, which implies that

$$\sigma_{H_{q,N}} \to \sigma_{\log g'_{q,\infty}} \quad \text{as} \quad N\to+\infty.$$

So, once (4.13) is established, the assertion of the lemma follows.

We now turn to the proof of (4.13). By the Cauchy-Schwarz inequality and (4.12),

$$\begin{aligned}
\left|z\,R'_{q,N}(z)\right| &= \left|\sum_{j=N+1}^{+\infty}\sum_{k=1}^{+\infty} jq^k\,\widehat{F\circ g}_{q,\infty}(k)\,z^{jq^k}\right| \\
&\le \sum_{k=1}^{+\infty}\sum_{j=N+1}^{+\infty} jq^k\left|\widehat{F\circ g}_{q,\infty}(k)\right||z|^{jq^k} \\
&\le \sum_{k=1}^{+\infty} q^k\left\{\sum_{j=N+1}^{+\infty} j^{1+\varepsilon}\left|\widehat{F\circ g}_{q,\infty}(j)\right|^2\right\}^{1/2}\left\{\sum_{j=N+1}^{+\infty} j^{1-\varepsilon}|z|^{2jq^k}\right\}^{1/2},
\end{aligned}$$

and as a consequence of (4.10), we have

$$\sum_{j=N+1}^{+\infty} j^{1+\varepsilon}\left|\widehat{F\circ g}_{q,\infty}(j)\right|^2 \to 0 \quad \text{as} \quad N\to+\infty.$$

It remains to verify that

$$\sum_{k=1}^{+\infty} q^k\left\{\sum_{j=N+1}^{+\infty} j^{1-\varepsilon}|z|^{2jq^k}\right\}^{1/2} \le \frac{C}{1-|z|}, \qquad z\in\mathbb{D},$$

for some positive constant C. It is easy to show that

$$\sum_{j=1}^{+\infty} j^{1-\varepsilon}x^{2j} \le \frac{C}{(1-x)^{2-\varepsilon}}, \qquad 0<x<1,$$

for some other positive constant C, at least for $0<\varepsilon<1$. It follows that

$$\begin{aligned}
\sum_{k=1}^{+\infty} q^k\left\{\sum_{j=N+1}^{+\infty} j^{1-\varepsilon}|z|^{2jq^k}\right\}^{1/2} &= \sum_{k=1}^{+\infty} q^k\,|z|^{q^k}\left\{\sum_{j=N+1}^{+\infty} j^{1-\varepsilon}|z|^{2(j-1)q^k}\right\}^{1/2} \\
&\le C\sum_{k=1}^{+\infty}\frac{q^k|z|^{q^k}}{(1-|z|^{q^k})^{1-\varepsilon/2}} \\
&\le C\sum_{k=1}^{+\infty}\frac{q^k|z|^{q^k}}{\left[\left(q^k|z|^{q^k}\right)(1-|z|)\right]^{1-\varepsilon/2}} \\
&= \frac{C}{(1-|z|)^{1-\varepsilon/2}}\sum_{k=1}^{+\infty} q^{k\varepsilon/2}|z|^{q^k\varepsilon/2}.
\end{aligned}$$

All that needs to be done is to verify that

$$\sum_{k=1}^{+\infty} q^{k\varepsilon/2}|z|^{q^k\varepsilon/2} \le \frac{C}{(1-|z|)^{\varepsilon/2}},$$

with a different positive constant C. If we look at z along the circle $|z| = 1 - q^{-m}$, for some positive integer m, then beyond the first m terms, the decay of the terms is very rapid, and it suffices to only estimate the sum of the first m terms. We get

$$\sum_{k=1}^{m} q^{k\varepsilon/2} = \frac{q^{(m+1)\varepsilon/2} - 1}{q-1},$$

which is comparable to

$$\frac{1}{(q-1)(1-|z|)^{\varepsilon/2}},$$

as desired. The case with intermediate values of $|z|$ is easily treated in a similar fashion. □

We recall that F is given by (4.1).

Proposition 4.3. *Suppose that*

$$\|F\|_{H^\infty(\mathbb{D})} = \sup_{z\in\mathbb{D}} |F(z)| < \log q.$$

Then the function $g_{q,\infty}$ is a Hölder mapping.

Proof. By (4.8), we have

$$\big|\log g'_{q,\infty}(z)\big| \le \frac{Cq}{q-1} + \sum_{k=1}^{+\infty} \big|F \circ g_{q,\infty}\big(z^{q^k}\big)\big|,$$

where $C = \|\log[f(z)/z]\|_{H^\infty(\mathbb{D})}$. As $g_{q,\infty}$ maps $\mathbb{D}$ into $\mathbb{D}$ and $g_{q,\infty}(0) = 0$, we see from the Schwarz lemma that

$$\big|g_{q,\infty}(z)\big| \le |z|, \qquad z \in \mathbb{D}.$$

Similarly, as F is bounded with $F(0) = 0$, we have

$$\big|F(z)\big| \le C'\,|z|, \qquad z \in \mathbb{D},$$

where $C' = \|F\|_{H^\infty(\mathbb{D})}$. Combining the two estimates, we get

$$\big|F \circ g_{q,\infty}\big(z^{q^k}\big)\big| \le C'|z|^{q^k},$$

so that

$$\big|\log g'_{q,\infty}(z)\big| \le \frac{Cq}{q-1} + C' \sum_{k=1}^{+\infty} |z|^{q^k}.$$

It is easy to see that

$$\sum_{k=1}^{+\infty} x^{q^k} \sim \frac{1}{\log q}\,\log\frac{1}{1-x}, \qquad \frac{1}{2} \le x < 1, \tag{4.14}$$

asymptotically as $x \to 1^-$, so that for $C' = \|F\|_{H^\infty(\mathbb{D})} < \log q$, we have

$$\big|\log g'_{q,\infty}(z)\big| \le (1-\varepsilon)\log\frac{1}{1-|z|} + O(1), \qquad z \in \mathbb{D},$$

for some positive ε. The assertion is now immediate, in view of the classical characterization of Hölder maps in terms of the growth of the derivative. □

We can now obtain a lower estimate for Makarov's constant C_{M}.

Proposition 4.4. *We have* $C_{\mathrm{M}} > 0.91$.

Proof. We use the example from the paper [7]:

$$f(z) = \frac{z}{K} \exp \int_0^z \frac{\exp\{(a/b)\sinh(bt)\} - 1}{t}\, \mathrm{d}t \tag{4.15}$$

where $a = 1.906$, $b = 1.246$ and K is the constant

$$K = \exp \int_0^1 \frac{\exp\{(a/b)\sinh(bt)\} - 1}{t}\, \mathrm{d}t = 73.677030\ldots,$$

which guarantees that $f(\mathbb{D}) \subset \mathbb{D}$. Let us remark that in [7], it was established that the function f is univalent in the unit disk for $a = 1.906$ and $b = 1.24$. But the same method shows that this function will be univalent for $b = 1.246$ as well. It is evident that f''/f' is in $H^\infty(\mathbb{D})$. Since

$$F(z) = \log\left[\frac{zf'(z)}{f(z)}\right] = \frac{a}{b}\sinh(bz),$$

we have

$$\sup_{z\in\mathbb{D}} |F(z)| = \frac{a}{b}\sinh(b) = 2.43891\ldots.$$

Hence, by Proposition 4.3, the function $g_{q,\infty}$ defined by (4.7) is a Hölder map for $q \geq 12$. Comparing with Lemma 4.1, we see that we need an estimate from below of the asymptotic variance of $g_{q,\infty}$.

It follows from (4.4) that

$$g_{q,\infty}(z) = \gamma_1 z + \sum_{j=1}^{\infty} \gamma_{jq+1} z^{jq+1},$$

where

$$\gamma_1 = K^{-1/(q-1)}.$$

It is easy to see that the Taylor coefficients of $g_{q,\infty}(z)$ are all positive (or zero). Indeed, the function

$$F(z) = \log \frac{zf'(z)}{f(z)}$$

has positive Taylor coefficients, so by exponentiation, $zf'(z)/f(z)$ does, as well. By integration, then, $\log[f(z)/z]$ has positive Taylor coefficients, which entails that $\log[f_m(z)/z]$ also has positive Taylor coefficients, and hence each f_m has positive Taylor coefficients. It is immediate that each $g_{q,n}$ has positive Taylor coefficients, whence the assertion that $g_{q,\infty}$ has positive Taylor coefficients follows. We finally note that $F \circ g_{q,\infty}$ also has positive Taylor coefficients.

It follows from the definition of asymptotic variance and from (4.8) that

$$\left\{\sigma_{\log g'_{q,\infty}}\right\}^2 = \limsup_{r\to 1^-} \frac{1}{\log\frac{1}{1-r}} \int_{-\pi}^{\pi} \left| \sum_{k=1}^{+\infty} F \circ g_{q,\infty}\left(r^{q^k} e^{\mathrm{i}\theta q^k}\right)\right|^2 \frac{\mathrm{d}\theta}{2\pi}. \tag{4.16}$$

We see that

$$\sum_{k=1}^{+\infty} F\circ g_{q,\infty}\big(z^{q^k}\big)=\sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty}\widehat{F\circ g}_{q,\infty}(j)\,z^{jq^k}. \tag{4.17}$$

Suppose that h_n is a sequence of analytic in the disk functions with positive Taylor coefficients. Then the following inequality holds:

$$\int_{-\pi}^{\pi}\bigg|\sum_{n=1}^{+\infty}h_n(re^{i\theta})\bigg|^2\frac{\mathrm{d}\theta}{2\pi}\ge\sum_{n=1}^{+\infty}\int_{-\pi}^{\pi}\big|h_n(re^{i\theta})\big|^2\frac{\mathrm{d}\theta}{2\pi}.$$

This inequality is an immediate consequence of Parseval's formula. Hence, by (4.17),

$$\int_{-\pi}^{\pi}\bigg|\sum_{k=1}^{+\infty}F\circ g_{q,\infty}\big(r^{q^k}e^{\mathrm{i}\theta q^k}\big)\bigg|^2\frac{\mathrm{d}\theta}{2\pi}\ge\sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty}\widehat{F\circ g}_{q,\infty}(j)^2r^{2jq^k}.$$

Next, we note that by (4.14),

$$\sum_{k=1}^{+\infty}r^{2jq^k}\sim\frac{1}{\log q}\,\log\frac{1}{1-r^{2j}}\sim\frac{1}{\log q}\,\log\frac{1}{1-r},\quad\text{as } r\to 1,$$

as long as j is fixed. As a result, we obtain from (4.16) that

$$\big\{\sigma_{\log g'_{q,\infty}}\big\}^2\ge\frac{1}{\log q}\sum_{j=1}^{+\infty}\widehat{F\circ g}_{q,\infty}(j)^2.$$

It follows from (4.4) and the positivity of the Taylor coefficients that

$$\widehat{F\circ g}_{q,\infty}(j)\ge\gamma_1^{j/q}\,\widehat{F\circ f_q}(j),\qquad j=1,2,3,\ldots.$$

This implies the inequality

$$\big\{\sigma_{\log g'_{q,\infty}}\big\}^2\ge\frac{1}{\log q}\sum_{j=1}^{\infty}\gamma_1^{2j/q}\,\widehat{F\circ f_q}(j)^2.$$

Straightforward calculations based on this estimate, with the choice $q=34$, yield

$$\sigma_{\log g'_{q,\infty}}>0.910462.$$

The assertion is now immediate. □

A combination of Lemma 4.1 and Proposition 4.4 gives the following.

Theorem 4.5. *We have*

$$0.91<C_{\mathrm{M}}\le 2\sqrt{\frac{\sqrt{24}-3}{5}}=1.2326\ldots.$$

Remark 4.6. It probably follows from the results of Przytycki, Urbański, and Zdunik [10] together with Proposition 4.4 that the optimal constant C in Corollary 3.1 is at least greater than 0.91.

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