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AN OFF-DIAGONAL ESTIMATE OF BERGMAN KERNELS

Håkan HEDENMALM

Lund University, Mathematics, Box 118, S-22100 Lund, Sweden

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ABSTRACT. – For weights on the unit disk that are logarithmically subharmonic and reproducing for the origin, an estimate from above and below is obtained for the Bergman kernel associated with the weight. In particular, that kernel is zero free, and bounded from above by twice the unweighted Bergman kernel. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Reproducing kernel functions play an important role for our understanding of the function and operator theories of Bergman spaces. Here we shall be concerned with weighted Bergman spaces on the unit disk $\mathbb{D}$. Let $d\Sigma$ be area measure, normalized so that the area of $\mathbb{D}$ is 1. For a summable function $\omega : \mathbb{D} \to [0, +\infty[$, which is positive on a set of positive area-measure, we let $L^2(\mathbb{D}, \omega)$ be the Hilbert space of complex-valued functions on $\mathbb{D}$ which are square summable with respect to the measure $\omega\, d\Sigma$: the norm is expressed by

$$\|f\|_\omega = \left(\int_{\mathbb{D}} |f|^2 \omega\, d\Sigma\right)^{1/2}, \quad f \in L^2(\mathbb{D}, \omega).$$

As a Hilbert space, $L^2(\mathbb{D}, \omega)$ is equipped with a sesquilinear form:

$$\langle f, g\rangle_\omega = \int_{\mathbb{D}} f\, \bar{g}\, \omega\, d\Sigma, \quad f, g \in L^2(\mathbb{D}, \omega).$$

We introduce the space $A^2(\mathbb{D}, \omega)$ as the closure of the (analytic) polynomials in the norm of $L^2(\mathbb{D}, \omega)$, and say that it is a Bergman space provided ω is a *Bergman weight*, which requires of ω that

$$\left|p(z)\right| \leqslant C(X)\, \|p\|_\omega, \quad z \in X,$$

holds for all polynomials and all compact subsets X of $\mathbb{D}$, where $C(X)$ is some positive constant which depends on X. In this case, $A^2(\mathbb{D}, \omega)$ is a Hilbert space of analytic functions on $\mathbb{D}$, with bounded point evaluations at all points of $\mathbb{D}$. By the representation theorem for bounded linear functionals on a Hilbert space, to each $\lambda \in \mathbb{D}$, there is a unique element $K_\omega(\cdot, \lambda)$ in $A^2(\mathbb{D}, \omega)$, such that

$$f(\lambda) = \left\langle f, K_\omega(\cdot, \lambda)\right\rangle_\omega, \quad f \in A^2(\mathbb{D}, \omega).$$

The function $K_\omega(z,\zeta)$, with $(z,\zeta)\in\mathbb{D}\times\mathbb{D}$, is called the *Bergman kernel function for the weight* ω. The space $A^2(\mathbb{D},\omega)$ is separable, and hence has a countable orthonormal basis $\varphi_1,\varphi_2,\varphi_3,\ldots$. One shows that the Bergman kernel function has the representation

$$K_\omega(z,\zeta)=\sum_{n=0}^{\infty}\varphi_n(z)\,\bar\varphi_n(\zeta),\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

whence it follows that the complex conjugate of $K_\omega(z,\zeta)$ equals $K_\omega(\zeta,z)$. It follows from an application of the Cauchy–Schwarz inequality that

$$\bigl|K_\omega(z,\zeta)\bigr|\leqslant K_\omega(z,z)^{1/2}K_\omega(\zeta,\zeta)^{1/2},\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

and we call this the *trivial estimate* of the kernel function. The quantity $K_\omega(z,z)$ expresses the square of the norm of the functional of point evaluation at z, and one shows that it tends to $+\infty$ whenever $|z|\to 1$. As a consequence, the trivial estimate gives little information about the behavior of the weighted Bergman kernel function away from the diagonal. Let us contrast it with the behavior of the Bergman kernel for the weight $\omega(z)\equiv 1$,

$$K(z,\zeta)=\sum_{n=0}^{\infty}(n+1)\,z^n\bar\zeta^n=\frac{1}{(1-z\bar\zeta)^2},\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

which stays bounded off any given neighborhood of the boundary diagonal $\{(z,z)\colon z\in\mathbb{T}\}$. Here, $\mathbb{T}$ denotes the unit circle. It is also of some interest to have control from below on the kernel function; for instance, the standard Bergman kernel has modulus at least $1/4$. It is not possible to obtain such information for general weights, as it is known that they may exhibit zeros. However, this possibility can be ruled out for *logarithmically subharmonic* weights ω, by a result of Hedenmalm (see [2]). The term logarithmically subharmonic is taken to mean the requirement that $\log\omega$ is subharmonic. A logarithmically subharmonic area-summable weight can be shown to be a Bergman weight in the above sense. Given a general logarithmically subharmonic weight ω, the associated weight

$$\nu(z)=K_\omega(0,0)^{-1/2}\bigl|K_\omega(z,0)\bigr|^2\omega(z),\qquad z\in\mathbb{D},$$

is also logarithmically subharmonic, and it has the additional property of being *reproducing for the origin*, in the sense that:

$$h(0)=\int_{\mathbb{D}}h(z)\,\nu(z)\,d\Sigma(z)$$

holds for all bounded harmonic functions h on $\mathbb{D}$. We may then study ν in place of ω, and uncover information about ω through this venue. For this reason, we shall assume that the weight ω we started with meets the following two conditions:

- ω is logarithmically subharmonic on $\mathbb{D}$, and
- ω is reproducing for the origin.

The first interesting off-diagonal estimate of the weighted Bergman kernel function in this setting was obtained recently by Shimorin [4]. He showed that if $\omega=|\varphi|^2$, where φ is holomorphic, then $|K_\omega(z,\zeta)|$ can be bounded from above by an expression of the type $\min(\Phi(z),\Phi(\zeta))$, where $\Phi(z)$ is a positive continuous function on $\mathbb{D}$, which tends to $+\infty$ as

$|z| \to 1$. The actual function obtained was of the order of magnitude

$$\Phi(z) \asymp \left(1 - |z|^2\right)^{-9/2}, \quad z \in \mathbb{D}.$$

Still, this yields no information when both parameters are at the boundary, where one might suspect that that the kernel is bounded nevertheless, away from the diagonal, of course.

2. Results

The purpose of this paper is to obtain the following result:

THEOREM 2.1. – *Under the above conditions on ω, the associated Bergman kernel enjoys the estimates*:

$$\frac{1}{2}\, K_\omega(\zeta,\zeta)\, \frac{(1-|\zeta|^2)^2}{|1-z\bar{\zeta}|^2} \leqslant \left|K_\omega(z,\zeta)\right| \leqslant \frac{2}{|1-z\bar{\zeta}|^2}, \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

It should be pointed out that $H^2(\mathbb{D}) \subset A^2(\mathbb{D},\omega) \subset A^2(\mathbb{D})$, where $H^2(\mathbb{D})$ and $A^2(\mathbb{D})$ denote the standard Hardy and Bergman spaces, respectively, and that each inclusion is a contractive operation (in case $\omega = |\varphi|^2$, with φ holomorphic, this is done in [3]), which leads to the following information about the kernel function along the diagonal:

$$\frac{1}{1-|z|^2} \leqslant K_\omega(z,z) \leqslant \frac{1}{(1-|z|^2)^2}, \quad z \in \mathbb{D}.$$

The factor 2 on the right hand side of the estimate in the theorem cannot be replaced by any smaller constant. This can be seen by comparing with the Hardy kernel $(1 - z\bar{\zeta})^{-1}$, which is a limit case of kernels for the spaces $A^2(\mathbb{D},\omega)$: the value at the point $(z,-z)$ tends to $1/2$ as $|z| \to 1$, whereas the estimate yields $2/4$, which hence cannot be improved.

By a theorem due to Hedenmalm [2], the kernel function K_ω is zero-free on $\mathbb{D}\times\mathbb{D}$ as soon as the weight ω is logarithmically subharmonic (actually, it suffices to assume that $\log\omega(z) - \log(1-|z|^2)$ is subharmonic on $\mathbb{D}$). The additional information that the weight is reproducing then leads to the quantitative control from below in Theorem 2.1. When the weight is of the form $\omega = |\phi'|^2$, with ϕ holomorphic, the kernel K_ω is equivalent—after a change of conformal coordinates—to the unweighted Bergman kernel on $\phi(\mathbb{D})$, if we think of the latter as a Riemann surface sheeted over the complex plane. The result should therefore be compared with the theorem of Suita and Yamada on the Lu Qi-Keng conjecture [5].

3. The proof

Let Δ stand for a quarter of the standard Laplacian. The Green function Γ for the bi-Laplacian Δ^2 and the Dirichlet problem on the unit disk $\mathbb{D}$ is given explicitly by the formula

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2 + \left(1-|z|^2\right)\left(1-|\zeta|^2\right), \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

and it is well known that it is strictly positive on $\mathbb{D}\times\mathbb{D}$. It follows from the arguments applied in [1], slightly modified to suit the present setting, that $A^2(\mathbb{D},\omega)$ consists of those holomorphic functions f in $\mathbb{D}$ for which

$$\|f\|_\omega^2 = \int_{\mathbb{D}} |f|^2\, d\Sigma + \int_{\mathbb{D}\times\mathbb{D}} \Gamma(z,\zeta)\left|f'(z)\right|^2 \Delta\omega(\zeta)\, d\Sigma(z)\, d\Sigma(\zeta) < +\infty. \tag{3.1}$$

The function ω is subharmonic because $\log\omega$ is—by Jensen's inequality—which makes the factor $\Delta\omega$ positive.

For $\lambda \in \mathbb{D}$, let $\phi = \phi_\lambda$ denote the involutive Mœbius automorphism of the unit disk:

$$\phi_\lambda(z) = \frac{\lambda - z}{1 - z\bar\lambda}, \quad z \in \mathbb{D}.$$

Let $\Gamma_{|\phi'|^2}(z,\zeta)$ denote the Green function for the weighted biharmonic operator $\Delta|\phi'|^{-2}\Delta$, given explicitly by:

$$\Gamma_{|\phi'|^2}(z,\zeta) = \Gamma\big(\phi(z),\phi(\zeta)\big) = \left|\phi'(z)\phi'(\zeta)\right|\Gamma(z,\zeta), \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

If $\nu : \mathbb{D} \to [0,+\infty[$ is in $L^1(\mathbb{D})$, $\log\nu$ is subharmonic, and $|\phi'|^2\nu$ is reproducing for λ, that is:

$$\int_{\mathbb{D}} h\,|\phi'|^2 \nu\, d\Sigma = h(\lambda)$$

for all bounded harmonic functions h on $\mathbb{D}$, then the appropriate analog of (3.1) reads

$$\|f\phi'\|_\nu^2 = \int_{\mathbb{D}} |f\phi'|^2\, d\Sigma + \int_{\mathbb{D}\times\mathbb{D}} \Gamma_{|\phi'|^2}(z,\zeta)\left|f'(z)\right|^2 \Delta\nu(\zeta)\, d\Sigma(z)\, d\Sigma(\zeta) < +\infty, \tag{3.2}$$

meaning both that the norm on $A^2(\nu)$ can be expressed in this fashion, and that a function f, holomorphic on $\mathbb{D}$, is in $A^2(\nu)$ if and only if (3.2) is fulfilled.

The Green function $\Gamma_{\bar\phi'}(z,\zeta)$ for the complex-weighted biharmonic operator $\Delta(\bar\phi')^{-1}\Delta$ takes the form

$$\Gamma_{\bar\phi'}(z,\zeta) = -\frac{1-|\lambda|^2}{(1-\lambda\bar z)(1-\lambda\bar\zeta)}\,\Gamma(z,\zeta), \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D}. \tag{3.3}$$

An important property that it has is the identity

$$\left|\Gamma_{\bar\phi'}(z,\zeta)\right|^2 = \Gamma(z,\zeta)\,\Gamma_{|\phi'|^2}(z,\zeta), \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D}. \tag{3.4}$$

We need to find an analog of (3.1) and (3.2) which works for $\Gamma_{\bar\phi'}(z,\zeta)$. If u, v are two complex-valued functions of class C^2 on the closed unit disk, and

$$\int_{\mathbb{D}} uh\bar\phi'\, d\Sigma = 0 \tag{3.5}$$

holds for all bounded harmonic functions h on $\mathbb{D}$, then

$$\int_{\mathbb{D}} u\,v\,\bar\phi'\, d\Sigma = \int_{\mathbb{D}\times\mathbb{D}} \Gamma_{\bar\phi'}(z,\zeta)\,\Delta u(z)\,\Delta v(\zeta)\, d\Sigma(z)\, d\Sigma(\zeta). \tag{3.6}$$

We want to apply this to the functions:

$$u(z) = \frac{1}{\bar{\phi}'(z)} K_\omega(\lambda, z)\,\omega(z) - \frac{1}{\phi'(0)}, \tag{3.7}$$

and $v(z) = p(z)\bar{q}(z)$, where initially p and q are polynomials. We note that

$$\phi'(z) = -\frac{1-|\lambda|^2}{(1-\bar{\lambda}z)^2}, \quad z \in \mathbb{D},$$

so that $\phi'(0)$ is real, and

$$\frac{\phi'(z)}{\phi'(0)} = K(z,\lambda) = \frac{1}{(1-\bar{\lambda}z)^2}, \quad z \in \mathbb{D}, \tag{3.8}$$

is the Bergman kernel function. We first check that (3.5) is fulfilled. The function u is in $L^1(\mathbb{D})$, because for $f \in A^2(\mathbb{D},\omega)$, $f\omega \in L^1(\mathbb{D})$, by the Cauchy–Schwarz inequality, so the integral in (3.5) makes sense for bounded h. For bounded holomorphic h,

$$\begin{aligned}\int_{\mathbb{D}} u\, h\, \bar{\phi}'\, d\Sigma &= \int_{\mathbb{D}} \left(K_\omega(\lambda,z)\,\omega(z) - \frac{\bar{\phi}'(z)}{\phi'(0)} \right) h(z)\, d\Sigma(z) \\ &= h(\lambda) - \int_{\mathbb{D}} K(\lambda,z)\, h(z)\, d\Sigma(z) = h(\lambda) - h(\lambda) = 0,\end{aligned}$$

and for bounded antiholomorphic h with $h(0) = 0$,

$$\begin{aligned}\int_{\mathbb{D}} uh\bar{\phi}'\, d\Sigma &= \int_{\mathbb{D}} \left(K_\omega(\lambda,z)\,\omega(z) - \frac{\bar{\phi}'(z)}{\phi'(0)} \right) h(z)\, d\Sigma(z) \\ &= K_\omega(\lambda,0)h(0) - \int_{\mathbb{D}} K(\lambda,z)h(z)\, d\Sigma(z) = K_\omega(\lambda,0)\, h(0) - h(0) = 0,\end{aligned}$$

where we have used the reproducing property of the weight, and the fact that it applies not just to bounded harmonic functions, but also to functions that can be appropriately approximated by such functions. Approximating bounded harmonic functions by sums of bounded holomorphic and antiholomorphic functions, we see that (3.5) follows.

So how do we extend the validity of (3.6) beyond the setting of u being of class C^2 up to the boundary of $\mathbb{D}$? To begin with, we notice that for $w \in C^2(\overline{\mathbb{D}})$:

$$\int_{\mathbb{D}} \Gamma_{\bar{\phi}'}(z,\zeta)\, \Delta w(z)\, d\Sigma(z) = \int_{\mathbb{D}} \Delta_z \Gamma_{\bar{\phi}'}(z,\zeta)\, w(z)\, d\Sigma(z), \quad \zeta \in \mathbb{D}, \tag{3.9}$$

by Green's theorem. The Laplacian of the weighted Green function has the following form:

$$\Delta_z \Gamma_{\bar{\phi}'}(z,\zeta) = \bar{\phi}'(z)\big(G(z,\zeta) + H_{\bar{\phi}'}(z,\zeta)\big), \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D}, \tag{3.10}$$

where

$$G(z,\zeta) = \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2, \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

is the Green function for the Laplacian Δ, and $H_{\bar\phi'}(z,\zeta)$ is a function which, for fixed $\zeta\in\mathbb{D}$, is harmonic on some neighborhood of $\overline{\mathbb{D}}$. Also for fixed $\zeta\in\mathbb{D}$,

$$\Gamma_{\bar\phi'}(z,\zeta)=\mathrm{O}\big(\big(1-|z|^2\big)\big)\quad\text{as } |z|\to 1.$$

We now see that (3.9) extends to all w in the norm closure of $C^2(\overline{\mathbb{D}})$ in the Sobolev-type space $W^1_\Delta(\mathbb{D})$ of locally area-integrable functions w with likewise locally area-integrable Laplacian Δw (interpreted in the sense of distribution theory), subject to the restraints

$$\int_{\mathbb{D}}\big|\omega(z)\big|\,d\Sigma(z)<+\infty,\qquad \int_{\mathbb{D}}\big(1-|z|^2\big)^2\big|\Delta\omega(z)\big|\,d\Sigma(z)<+\infty.$$

Due to the logarithmic singularity in $\Delta\Gamma_{\bar\phi'}(\cdot,\zeta)$ at the point ζ, it is conceivable that the right hand side of (3.9) might be undefined on a set of area measure 0 for a given function $w\in W^1_\Delta(\mathbb{D})$. One shows that $C^2(\overline{\mathbb{D}})$ is in fact dense in $W^1_\Delta(\mathbb{D})$ in the following sense. Let $w\in W^1_\Delta(\mathbb{D})$ be arbitrary. The dilatation $w_r(z)=w(rz)$, $0<r<1$, is in the space $W^1_\Delta(\mathbb{D})$ as well, and tends to w in norm as $r\to 1$. The dilated function w_r is in $L^1(\mathbb{D})$, and its Laplacian Δw_r is in $L^1(\mathbb{D})$, as well. The function w_r splits as a sum $w_r=w_1+w_2$, where w_1 is harmonic on a disk $\varrho\mathbb{D}$ of radius $1<\varrho<1/r$ centered at the origin, and w_2 is the logarithmic potential

$$w_2(z)=\int_{\varrho\mathbb{D}}\log|z-\zeta|^2\Delta w_r(\zeta)\,d\Sigma(\zeta),\quad z\in\mathbb{D},$$

where we think of w_r as extended to the disk $\varrho\mathbb{D}$ by its defining expression $w_r(z)=w(rz)$. The function w_1 is already of class C^2 on the closed unit disk, so we need only approximate the logarithmic potential w_2. This is done by convolution of Δw_r, restricted to $\varrho\mathbb{D}$, with a smooth positive test function with small support about the origin, having integral 1. The logarithmic potential associated with that smooth mass distribution then approximates w_2, and it is sufficiently smooth on $\overline{\mathbb{D}}$. It follows that (3.9) holds for all $w\in W^1_\Delta(\mathbb{D})$.

Let us see what consequences this has for obtaining (3.6) for the function u given by (3.7) and for $v\in C^2(\overline{\mathbb{D}})$. We should first check that $u\in W^1_\Delta(\mathbb{D})$. To this end, the following lemma is helpful. Let us first introduce some terminology: a function is said to be *logarithmically subharmonic* if its logarithm is subharmonic.

LEMMA 3.1. – *Let $\mu:\mathbb{D}\to[0,+\infty[$ be logarithmically subharmonic. Then, for holomorphic f, $|f|^2\mu$ is subharmonic, and*

$$\big|\Delta(f\mu)\big|^2\leqslant(\Delta\mu)\Delta\big(|f|^2\mu\big).$$

Also, if f does not vanish identically, $|f|^2\mu$ is logarithmically subharmonic.

Proof. – This is an elementary computation. □

To obtain that $u\in W^1_\Delta(\mathbb{D})$, it suffices to check that

$$\int_{\mathbb{D}}\big|\Delta(f\omega)\big|\big(1-|z|^2\big)^2\,d\Sigma(z)<+\infty,$$

whenever $f\in A^2(\mathbb{D},\omega)$. By the lemma, we have:

$$\int_{\mathbb{D}} \left|\Delta(f\omega)(z)\right|\left(1-|z|^2\right)^2 d\Sigma(z)$$
$$\leqslant \left(\int_{\mathbb{D}} \Delta\omega(z)\left(1-|z|^2\right)^2 d\Sigma(z)\right)^{1/2}\left(\int_{\mathbb{D}} \Delta\left(|f(z)|^2\omega(z)\right)\left(1-|z|^2\right)^2 d\Sigma(z)\right)^{1/2}.$$

For logarithmically subharmonic $\mu:\mathbb{D}\to[0,+\infty[$, with $\mu\in L^1(\mathbb{D})$, one shows that

$$\int_{\mathbb{D}} \Delta\mu(z)\left(1-|z|^2\right)^2 d\Sigma(z) + 2\int_{\mathbb{D}} \mu(z)\,d\Sigma(z) = 4\int_{\mathbb{D}} |z|^2\mu(z)\,d\Sigma(z) < +\infty,$$

whence the assertion $u\in W^1_\Delta(\mathbb{D})$ follows. Applying (3.9) and (3.10) to the function $w=u$, and using property (3.5), we see that:

$$\begin{aligned}
&\int_{\mathbb{D}} \Gamma_{\bar\phi'}(z,\zeta)\,\Delta u(z)\,d\Sigma(z)\\
&\quad=\int_{\mathbb{D}} \Delta_z\Gamma_{\bar\phi'}(z,\zeta)\,u(z)\,d\Sigma(z)\\
&\quad=\int_{\mathbb{D}} \left(G(z,\zeta)+H_{\bar\phi'}(z,\zeta)\right)u(z)\bar\phi'(z)\,d\Sigma(z)\\
&\quad=\int_{\mathbb{D}} G(z,\zeta)\,u(z)\,\bar\phi'(z)\,d\Sigma(z).
\end{aligned}$$

As we integrate against the function Δv, with $v\in C^2(\overline{\mathbb{D}})$, we find that:

$$\begin{aligned}
&\int_{\mathbb{D}\times\mathbb{D}} \Gamma_{\bar\phi'}(z,\zeta)\Delta u(z)\Delta v(\zeta)\,d\Sigma(z)\\
&\quad=\int_{\mathbb{D}}\left(\int_{\mathbb{D}} G(z,\zeta)\Delta v(\zeta)\,d\Sigma(\zeta)\right)u(z)\bar\phi'(z)\,d\Sigma(z)\\
&\quad=\int_{\mathbb{D}} v(z)u(z)\bar\phi'(z)\,d\Sigma(z),
\end{aligned}$$

where we again have used property (3.5). In other words, we have finally verified the validity of (3.6) for the given choice of u and v:

$$\begin{aligned}
&\int_{\mathbb{D}}\left(K_\omega(\lambda,z)\omega(z)-\frac{\bar\phi'(z)}{\phi'(0)}\right)p(z)\bar q(z)\,d\Sigma(z)\\
&\quad=\int_{\mathbb{D}\times\mathbb{D}} \Gamma_{\bar\phi'}(z,\zeta)\Delta_z\left(\frac{K_\omega(\lambda,z)}{\bar\phi'(z)}\omega(z)\right)p'(\zeta)\bar q'(\zeta)\,d\Sigma(z)\,d\Sigma(\zeta).
\end{aligned}\tag{3.11}$$

We now use the Cauchy–Schwarz inequality, the lemma, and (3.4), to get that:

$$\begin{aligned}
&\left|\int\limits_{\mathbb{D}\times\mathbb{D}} \Gamma_{\bar\phi'}(z,\zeta)\Delta_z\left(\frac{K_\omega(\lambda,z)}{\bar\phi'(z)}\omega(z)\right)p'(\zeta)\bar q'(\zeta)\,d\Sigma(z)\,d\Sigma(\zeta)\right| \\
&\quad\leqslant \int\limits_{\mathbb{D}\times\mathbb{D}} \left|\Gamma_{\bar\phi'}(z,\zeta)\Delta_z\big(K_\omega(\lambda,z)\,\omega(z)\big)p'(\zeta)\bar q'(\zeta)\right|\,d\Sigma(z)\,d\Sigma(\zeta) \\
&\quad\leqslant \left(\int\limits_{\mathbb{D}\times\mathbb{D}} \Gamma(z,\zeta)\Delta\omega(z)\left|p'(\zeta)\right|^2\,d\Sigma(z)\,d\Sigma(\zeta)\right)^{1/2} \\
&\qquad\times\left(\int\limits_{\mathbb{D}\times\mathbb{D}} \Gamma_{|\phi'|^2}(z,\zeta)\Delta_z\left(\frac{|K_\omega(z,\lambda)|^2}{|\phi'(z)|^2}\omega(z)\right)|q'(\zeta)|^2\,d\Sigma(z)\,d\Sigma(\zeta)\right)^{1/2}.
\end{aligned} \tag{3.12}$$

In view of (3.1),

$$\begin{aligned}
&\int\limits_{\mathbb{D}\times\mathbb{D}} \Gamma(z,\zeta)\,\Delta\omega(z)\left|p'(\zeta)\right|^2\,d\Sigma(z)\,d\Sigma(\zeta) \\
&\quad=\int\limits_{\mathbb{D}}\left|p(z)\right|^2\omega(z)\,d\Sigma(z)-\int\limits_{\mathbb{D}}\left|p(z)\right|^2\,d\Sigma(z),
\end{aligned} \tag{3.13}$$

and in view of (3.2), with

$$\nu(z)=\frac{|K_\omega(z,\lambda)|^2}{K_\omega(\lambda,\lambda)}\,\frac{\omega(z)}{|\phi'(z)|^2},\qquad z\in\mathbb{D},$$

we have

$$\begin{aligned}
&\int\limits_{\mathbb{D}\times\mathbb{D}} \Gamma_{|\phi'|^2}(z,\zeta)\Delta_z\left(\frac{|K_\omega(z,\lambda)|^2}{|\phi'(z)|^2}\omega(z)\right)\left|q'(\zeta)\right|^2\,d\Sigma(z)\,d\Sigma(\zeta) \\
&\quad=\int\limits_{\mathbb{D}}\left|K_\omega(z,\lambda)\right|^2\left|q(z)\right|^2\omega(z)\,d\Sigma(z)-K_\omega(\lambda,\lambda)\int\limits_{\mathbb{D}}\left|q(z)\right|^2\left|\phi'(z)\right|^2\,d\Sigma(z).
\end{aligned} \tag{3.14}$$

It follows from (3.12)–(3.15) that

$$\begin{aligned}
&\left|\int\limits_{\mathbb{D}}\left(K_\omega(\lambda,z)\omega(z)-\frac{\bar\phi'(z)}{\phi'(0)}\right)p(z)\bar q(z)\,d\Sigma(z)\right| \\
&\quad\leqslant\left(\int\limits_{\mathbb{D}}\left|p(z)\right|^2\omega(z)\,d\Sigma(z)-\int\limits_{\mathbb{D}}\left|p(z)\right|^2\,d\Sigma(z)\right)^{1/2} \\
&\qquad\times\left(\int\limits_{\mathbb{D}}\left|K_\omega(z,\lambda)\right|^2\left|q(z)\right|^2\omega(z)\,d\Sigma(z)-K_\omega(\lambda,\lambda)\int\limits_{\mathbb{D}}\left|q(z)\right|^2\left|\phi'(z)\right|^2\,d\Sigma(z)\right)^{1/2}.
\end{aligned}$$

An approximation argument allows us to take $p\in A^2(\mathbb{D},\omega)$ as long as we keep q a polynomial in the above estimate. We choose $p(z)=K_\omega(z,\lambda)\,q(z)$, and consider the worst-case scenario:

$$\begin{aligned}
&\int_{\mathbb{D}} \left|K_\omega(\lambda,z)\right|^2 \left|q(z)\right|^2 \omega(z)\,d\Sigma(z) - \left|\int_{\mathbb{D}} \frac{\bar{\phi}'(z)}{\phi'(0)} K_\omega(z,\lambda)\left|q(z)\right|^2 d\Sigma(z)\right| \\
&\leqslant \int_{\mathbb{D}} \left|K_\omega(z,\lambda)\right|^2 \left|q(z)\right|^2 \omega(z)\,d\Sigma(z)\left(1 - \frac{\int_{\mathbb{D}} |K_\omega(z,\lambda)|^2 |q(z)|^2 d\Sigma(z)}{\int_{\mathbb{D}} |K_\omega(z,\lambda)|^2 |q(z)|^2 \omega(z)\,d\Sigma(z)}\right)^{1/2} \qquad (3.15)\\
&\quad \times \left(1 - K_\omega(\lambda,\lambda) \frac{\int_{\mathbb{D}} |q(z)|^2 |\phi'(z)|^2 d\Sigma(z)}{\int_{\mathbb{D}} |K_\omega(z,\lambda)|^2 |q(z)|^2 \omega(z)\,d\Sigma(z)}\right)^{1/2}.
\end{aligned}$$

As we drop the last factor (after all, it is between 0 and 1), and use the concavity of the square root function, or more explicitly, that

$$\sqrt{1+t} \leqslant 1 + \tfrac{1}{2}t, \quad t \in [-1, +\infty[,$$

we find that

$$\begin{aligned}
&\int_{\mathbb{D}} \left|K_\omega(\lambda,z)\right|^2 \left|q(z)\right|^2 \omega(z)\,d\Sigma(z) - \left|\int_{\mathbb{D}} \frac{\bar{\phi}'(z)}{\phi'(0)} K_\omega(z,\lambda)\left|q(z)\right|^2 d\Sigma(z)\right| \\
&\quad \leqslant \int_{\mathbb{D}} \left|K_\omega(z,\lambda)\right|^2 \left|q(z)\right|^2 \omega(z)\,d\Sigma(z) - \frac{1}{2}\int_{\mathbb{D}} \left|K_\omega(z,\lambda)\right|^2 \left|q(z)\right|^2 d\Sigma(z),
\end{aligned}$$

so that in view of (3.8), we have:

$$\begin{aligned}
\int_{\mathbb{D}} \left|K_\omega(z,\lambda) q(z)\right|^2 d\Sigma(z) &\leqslant 2\left|\int_{\mathbb{D}} \frac{\bar{\phi}'(z)}{\phi'(0)} K_\omega(z,\lambda)\left|q(z)\right|^2 d\Sigma(z)\right| \\
&= 2\left|\int_{\mathbb{D}} K(\lambda,z) K_\omega(z,\lambda)\left|q(z)\right|^2 d\Sigma(z)\right|.
\end{aligned}$$

Let $Q(z) = K(z,\lambda)\,q(z)$, and note that the above transforms to

$$\int_{\mathbb{D}} \left|\frac{K_\omega(z,\lambda)}{K(z,\lambda)} Q(z)\right|^2 d\Sigma(z) \leqslant 2\left|\int_{\mathbb{D}} \frac{K_\omega(z,\lambda)}{K(z,\lambda)} \left|Q(z)\right|^2 d\Sigma(z)\right|.$$

Another application of the Cauchy–Schwarz inequality yields:

$$\int_{\mathbb{D}} \left|\frac{K_\omega(z,\lambda)}{K(z,\lambda)} Q(z)\right|^2 d\Sigma(z) \leqslant 2\left(\int_{\mathbb{D}} \left|\frac{K_\omega(z,\lambda)}{K(z,\lambda)} Q(z)\right|^2 d\Sigma(z)\right)^{1/2} \left(\int_{\mathbb{D}} \left|Q(z)\right|^2 d\Sigma(z)\right)^{1/2},$$

so that

$$\left(\int_{\mathbb{D}} \left|\frac{K_\omega(z,\lambda)}{K(z,\lambda)} Q(z)\right|^2 d\Sigma(z)\right)^{1/2} \leqslant 2\left(\int_{\mathbb{D}} \left|Q(z)\right|^2 d\Sigma(z)\right)^{1/2}.$$

This has the interpretation that the function

$$\frac{K_\omega(\cdot,\lambda)}{K(\cdot,\lambda)}$$

extends to a bounded multiplier on A^2, with multiplier norm at most 2. As the multiplier space of A^2 is known to coincide with the space H^∞ of bounded analytic functions on $\mathbb{D}$, with the multiplier norm equal to the uniform norm, we get that

$$\left|K_\omega(z,\lambda)\right| \leqslant 2\left|K(z,\lambda)\right| = \frac{2}{|1-z\bar{\zeta}|^2}, \quad z \in \mathbb{D}.$$

If we instead drop the second last factor in (3.16), we get

$$K_\omega(\lambda,\lambda)\int_{\mathbb{D}}\left|q(z)\phi'(z)\right|^2 d\Sigma(z) \leqslant 2\left|\int_{\mathbb{D}} K(\lambda,z)K_\omega(z,\lambda)\left|q(z)\right|^2 d\Sigma(z)\right|,$$

which leads to

$$(3.16)\quad \left(1-|\lambda|^2\right)^2 K_\omega(\lambda,\lambda)\left(\int_{\mathbb{D}}\left|Q(z)\right|^2 d\Sigma(z)\right)^{1/2} \leqslant 2\left(\int_{\mathbb{D}}\left|\frac{K_\omega(z,\lambda)}{K(z,\lambda)}Q(z)\right|^2 d\Sigma(z)\right)^{1/2},$$

and yields the estimate from below

$$\frac{1}{2}K_\omega(\lambda,\lambda)\,\frac{(1-|\lambda|^2)^2}{|1-z\bar{\zeta}|^2} \leqslant \left|K_\omega(z,\lambda)\right|, \quad z \in \mathbb{D}.$$

To justify the conclusion, we note that the function $F = K_\omega(\cdot,\lambda)$ is a bounded holomorphic function on $\mathbb{D}$, and that it lacks zeros in $\mathbb{D}$, because the weight is logarithmically subharmonic, due to a theorem of Hedenmalm [2]. We would be done if we can show that F is bounded away from 0 in $\mathbb{D}$. As we apply the estimate from below (3.16) to functions $Q = \psi'$, where $\psi : \mathbb{D} \to \mathbb{D}$ is a Mœbius automorphism of the unit disk, we see that $F \circ \psi$ has norm in $A^2(\mathbb{D})$ that is bounded from below by some constant ε, $\varepsilon > 0$, independent of ψ. Assuming that F is not bounded away from 0, we find a sequence of Mœbius automorphisms ψ_n such that $F \circ \psi_n(0) \to 0$ as $n \to +\infty$. But then, as the functions lack zeros, and are uniformly bounded, a normal families argument yields that $F \circ \psi_n$ tends to 0 uniformly on compact subsets of $\mathbb{D}$. Since the functions are uniformly bounded, this violates the known control from below of the $A^2(\mathbb{D})$ norm of $F \circ \psi_n$.

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