



On the asymptotic free boundary for the American put option problem

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Abstract

In practical work with American put options, it is important to be able to know when to exercise the option, and when not to do so. In computer simulation based on the standard theory of geometric Brownian motion for simulating stock price movements, this problem is fairly easy to handle for options with a short lifespan, by analyzing binomial trees. It is considerably more challenging to make the decision for American put options with long lifespan. In order to provide a satisfactory analysis, we look at the corresponding free boundary problem, and show that the free boundary—which is the curve that separates the two decisions, to exercise or not to—has an asymptotic expansion, where the coefficient of the main term is expressed as an integral in terms of the free boundary. This raises the perspective that one could use numerical simulation to approximate the integral and thus get an effective way to make correct decisions for long life options.

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1. Introduction

1.1. General background

Initial remarks. The standard model of stock price movement, as proposed by Samuelson, is geometric Brownian motion. On the basis of this model, it is possible to analyze prices of derivative securities such as options, by use of the arbitrage principle, which postulates that a riskless portfolio must, in the absence of transaction costs, earn the risk-free interest rate. In this paper we shall be concerned with the American put option, which is a contract that allows the holder to sell a stock at a fixed price—independent of market movements—at any moment during the duration of the contract. It should be pointed out that the academic community is not unilaterally in favor of the geometric Brownian motion model (see, for instance, [10]).

A general background on the analysis of various types of options can be found in, e.g., the books of Hull [8] and of Jastrow and Turnbull [9].

The payoff function; geometric Brownian motion. Essentially, the presentation here can be found in, e.g., Myneni's paper [14]. Let t be a time parameter, *which expresses the time remaining until the option expires*. We note that t flows backwards with time, so that t decreases as real time passes by. Typically, we are interested in an interval $0 \leq t \leq T$, where $t = T$ corresponds to the time when the option is issued and $t = 0$ is the time of expiration. *It is convenient to express all money in terms of its equivalent value at the deadline $t = 0$.* Let $s(t)$ denote the stock price at time t . If r is the risk-free continuously compounded interest rate, assumed to be constant, then $s(t) \exp(-rt)$ is the nominal value (that is, the dollar amount we would see on the screen) of the stock at (remaining) time t . For reasons of convenience, *we normalize the nominal exercise price of the put option to equal 1.* When we think of the stock price as a free parameter, we write s in place of $s(t)$. The reward function is

$$V_r(t, s) = \max\{0, \exp(rt) - s\}; \quad (1.1)$$

it expresses the payoff earned by exercising the option contract at the point (t, s) . We consider put options of the *American* type, which means that the holder is at liberty to exercise at any time from purchase at $t = T$ to expiration $t = 0$. Let $\hat{V}_r(t, s)$ denote the correct price of the option at (remaining) time t . It is well known from the Black–Scholes analysis that $\hat{V}_r(t, s)$ is obtained by optimizing—over all stopping strategies—the expected value of the reward function over all stock paths starting at $s(t) = s$, under the assumption of risk neutrality. Risk neutrality means that the expected growth of a risky asset like $s(t)$ is postulated to equal that of a riskless one (and since all monetary values are discounted to their equivalents at time $t = 0$, the expected growth is 0). The assumption of geometric Brownian motion leads to the infinitesimal equation

$$ds(t) = \lambda s(t) dt + \sigma s(t) d\mathbf{w}(-t),$$

where σ^2 is the variance per unit of time, λ a drift rate (the intrinsic growth rate of the stock), and $\mathbf{w}(-t)$ is the unit Brownian motion (we write $-t$ to indicate that our time

parameter flows backwards). The postulate of risk neutrality translates into $\lambda = 0$. By Ito’s formula, this then leads to

$$d(\log s(t)) = \frac{\sigma^2}{2} dt + \sigma d\mathbf{w}(-t). \tag{1.2}$$

1.2. The obstacle problem

Shift to ordinary Brownian motion. It is more convenient to work with ordinary Brownian motion rather than geometric Brownian motion, and so we introduce the stochastic process

$$\mathbf{x}(t) = \frac{\sigma}{\sqrt{2}} t - \frac{\sqrt{2}}{\sigma} \log s(t),$$

which follows a standard Wiener process, as can be seen from (1.2):

$$d\mathbf{x}(t) = -\sqrt{2} d\mathbf{w}(-t).$$

Likewise, we switch from the pair (t, s) to (t, x) as our basic coordinates, where

$$x = \frac{\sigma}{\sqrt{2}} t - \frac{\sqrt{2}}{\sigma} \log s$$

is the free variable corresponding to $\mathbf{x}(t)$. In the new coordinate system, the reward function (1.1) takes the form

$$V_{r,\sigma}(t, x) = \max\{0, e^{rt} - e^{\sigma^2 t/2 - \sigma x/\sqrt{2}}\}. \tag{1.3}$$

Let

$$\boxplus = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

be the heat operator. We say that a function f of the variables (t, x) is *caloric* if $\boxplus f = 0$, *subcaloric* if it is real-valued and $\boxplus f \leq 0$, and *supercaloric* if it is real-valued and $\boxplus f \geq 0$. We need to introduce the upper half-plane (or positive time plane)

$$\mathbb{R}_+^2 = \{(t, x): t \in \mathbb{R}_+, x \in \mathbb{R}\},$$

where \mathbb{R} is the real line, and $\mathbb{R}_+ = \{t \in \mathbb{R}: t > 0\}$. The following statement is rather well-known, and may be taken as the formal definition of the envelope function \hat{V} .

Theorem 1.1. *The envelope function $\hat{V}_{r,\sigma}$ is supercaloric in \mathbb{R}_+^2 . Moreover, it equals the infimum of all C^2 -smooth supercaloric functions that majorize $V_{r,\sigma}(t, x)$ on \mathbb{R}_+^2 .*

There is a rather sizeable literature on parabolic obstacle problems; see, e.g., Friedman’s book [6] as well as the papers of Friedman [5,7], Grigelionis and Shiryaev [3], Cannon, Henry, and Kotlow [3]. Interesting related papers are McKean [11] and van Moerbeke [12, 13].

The scaling properties of the heat operator \boxplus permit us to reduce the number of parameters by 1. We introduce the real parameter ϱ , $-1 < \varrho < 1$, and let V_ϱ denote $V_{r,\sigma}$, with the parameter settings

$$r = 1 - \varrho^2, \quad \sigma = \sqrt{2}(1 + \varrho);$$

$$V_\varrho(t, x) = \max\{0, e^{(1-\varrho^2)t} - e^{(1+\varrho)^2t - (1+\varrho)x}\}. \tag{1.4}$$

Likewise, \hat{V}_ϱ denotes the function $\hat{V}_{r,\sigma}$ with the same settings. If, however, σ and r are given, we pick α, ϱ according to

$$\alpha = \frac{\sigma^2 + 2r}{2\sqrt{2}\sigma}, \quad \varrho = \frac{\sigma^2 - 2r}{\sigma^2 + 2r},$$

and recover the function $V_{r,\sigma}$ from the formula

$$V_{r,\sigma}(t, x) = V_\varrho(\alpha^2t, \alpha x); \tag{1.5}$$

by the scaling properties of \boxplus , we then recover the envelope as well:

$$\hat{V}_{r,\sigma}(t, x) = \hat{V}_\varrho(\alpha^2t, \alpha x). \tag{1.6}$$

In the sequel, we shall only be concerned with the function V_ϱ ; moreover, we shall drop the subscript ϱ whenever this does not lead to confusion.

An affine shift of coordinates. If a function $f(t, x)$ is caloric, then so is the transformed function

$$e^{\beta^2t + \beta x} f(t, x + 2\beta t); \tag{1.7}$$

here, β is a real parameter. The calculation that shows this *also reveals that the supercaloric functions are preserved under the transformation*. Actually, it is possible to give a complete characterization of the transformations of this type that preserve the caloric functions.

Choose $\beta = \varrho$ in the substitution (1.7), and introduce the function

$$V'_\varrho(t, x) = e^{\varrho^2t + \varrho x} V_\varrho(t, x + 2\varrho t).$$

Then the least supercaloric majorant to V'_ϱ , denoted by \hat{V}'_ϱ , is related to \hat{V}_ϱ in a straightforward fashion:

$$\hat{V}'_\varrho(t, x) = e^{\varrho^2t + \varrho x} \hat{V}_\varrho(t, x + 2\varrho t).$$

In other words, we may as well replace V_ϱ by V'_ϱ in our considerations. The function V'_ϱ is simpler-looking:

$$V'_\varrho(t, x) = \max\{0, e^{t+\varrho x} - e^{t-x}\}, \quad (t, x) \in \mathbb{R}_+^2.$$

In the sequel, we shall consider only the transformed function V'_ϱ , and write V_ϱ for it.

Introduction of a new parameter. We introduce the parameter θ , confined to $0 < \theta < +\infty$, and let $V_{\varrho,\theta}$ be the function

$$V_{\varrho,\theta}(t, x) = e^{t+\varrho x} - \frac{1}{2}(1 + \theta - \varrho + \theta\varrho)e^{t-x} - \frac{1}{2}(1 - \theta)(1 + \varrho)e^{t+x},$$

$$(t, x) \in \Pi_+ = \{(t, x) \in \mathbb{R}_+^2 : x > 0\},$$

extended to the whole positive time half-plane \mathbb{R}_+^2 by

$$V_{\varrho,\theta}(t, x) = 0, \quad (t, x) \in \mathbb{R}_+^2 \setminus \Pi_+.$$

We readily calculate that on \mathbb{R}_+^2 ,

$$\boxplus V_{\varrho,\theta}(t, x) = -\theta(1 + \varrho)e^t \delta_0(x) + (1 - \varrho^2)e^{t+\varrho x} 1_{\Pi_+}(t, x), \tag{1.8}$$

where 1_E denotes the characteristic function of the set E , and δ_0 is the unit Dirac mass at 0.

From this, it is evident that the role of θ is to scale the mass distribution along the t -axis. The value $\theta = 1$ corresponds to the put option problem in the introduction. We have the corresponding envelope function $\hat{V}_{\varrho,\theta}$, and Theorem 1.1 generalizes to the new setting.

1.3. The free boundary

The continuation region. The region

$$\mathcal{D}(\varrho, \theta) = \{(t, x) \in \mathbb{R}_+^2 : V_{\varrho,\theta}(t, x) < \hat{V}_{\varrho,\theta}(t, x)\}$$

is called the *continuation region*, while the boundary curve

$$\Gamma(\varrho, \theta) = \partial\mathcal{D}(\varrho, \theta) \cap \mathbb{R}_+^2$$

is the *free boundary*, or *decision boundary*. Note that

$$\mathbb{R}_+^2 \setminus \Pi_+ \subset \mathcal{D}(\varrho, \theta).$$

We parametrize $\Gamma(\varrho, \theta)$ by

$$x = \phi_{\varrho,\theta}(t), \quad 0 < t < +\infty,$$

and note that it is well-known that $\phi_{\varrho,\theta}(t)$ is an increasing function, with $\phi_{\varrho,\theta}(0) = 0$ and

$$\lim_{t \rightarrow +\infty} \phi_{\varrho,\theta}(t) = \mu(\varrho, \theta) = \frac{1}{1 + \varrho} \log\left(1 + \theta \frac{1 + \varrho}{1 - \varrho}\right), \tag{1.9}$$

where the equation is taken to define the constant $\mu = \mu(\varrho, \theta)$. Moreover, at least for $\theta = 1$, it is known that the function $x = \phi_{\varrho,\theta}(t)$ is continuous and concave (see, for instance, [4]). In financial terms, in the continuation region, we keep the options contract, while at the decision boundary, we exercise it. The behavior of the free boundary near $t = 0$ has been studied extensively (see [1]):

$$\phi_{\varrho,1}(t) \sim (1 + \varrho)\sqrt{2t \log \frac{1}{t}}, \quad t \rightarrow 0 \text{ (for } \theta = 1\text{)}.$$

Here, we shall focus on the behavior of the free boundary as $t \rightarrow +\infty$. We shall demonstrate that $\phi_{\varrho,\theta}(t)$ has the asymptotic expansion

$$\phi_{\varrho,\theta}(t) = \mu(\varrho, \theta) - \beta_1 t^{-3/2} e^{-t} - \dots - \beta_N t^{-N-1/2} e^{-t} + O_{L^p}(t^{-N-1} e^{-t}),$$

where $\beta_j = \beta_j(\varrho, \theta)$ are certain real-valued coefficients. The term

$$O_{L^p}(t^{-N-1} e^{-t})$$

stands for an $L^p(\mathbb{R}_+)$ -function times $t^{-N-1} e^{-t}$, and we are free to choose p with $2 < p < +\infty$. We also show how to express β_j in terms of an integral, which should permit numerical algorithms to yield good approximations of β_j . In particular,

$$\beta_1 = \frac{e^{-\varrho\mu}}{2\sqrt{\pi}} \int_0^{+\infty} \phi_{\varrho,\theta}(t) e^{\varrho\phi_{\varrho,\theta}(t)} \phi'_{\varrho,\theta}(t) e^t dt,$$

which we may—by integration by parts—also write in the form

$$\beta_1 = \frac{1}{2\sqrt{\pi}} \left\{ \frac{\theta\mu e^{-\varrho\mu}}{1-\varrho} + \frac{(1+\varrho\mu)e^{-\varrho\mu}-1}{\varrho^2} + \frac{1}{\varrho^2} \int_0^{+\infty} [(\varrho\mu - \varrho\phi_{\varrho,\theta}(t) + 1)e^{\varrho(\phi_{\varrho,\theta}(t)-\mu)} - 1] e^t dt \right\},$$

since it is a consequence of our analysis that

$$\int_0^{+\infty} \phi'_{\varrho,\theta}(t) e^{\varrho\phi_{\varrho,\theta}(t)} e^t dt = \frac{\theta}{1-\varrho}.$$

1.4. Asymptotic estimation of the free boundary

Estimation of $\hat{V}_{\varrho,\theta}$ from above and below. We need to get a grasp of the asymptotic behavior of the function $\phi_{\varrho,\theta}(t)$ as $t \rightarrow +\infty$. Let $\tilde{V}_{\varrho,\theta}$ be the solution to the heat equation with boundary values equal to $V_{\varrho,\theta}$ on the two half-lines

$$\begin{aligned} &\{(t, x) \in \mathbb{R}^2: t = 0, -\infty < x < \mu\} \quad \text{and} \\ &\{(t, x) \in \mathbb{R}^2: 0 < t < +\infty, x = \mu\}; \end{aligned}$$

it is unique under mild growth restrictions. This function arises from the stopping strategy of stopping on the two given lines, and hence we must have

$$\tilde{V}_{\varrho,\theta}(t, x) \leq \hat{V}_{\varrho,\theta}(t, x), \quad (t, x) \in \Pi_\mu,$$

where

$$\Pi_\mu = \{(t, x) \in \mathbb{R}^2: 0 < t < +\infty, -\infty < x < \mu\}, \tag{1.10}$$

and $\mu = \mu(\varrho, \theta)$ is as in (1.9).

Let \hat{V}_∞ be the function

$$\hat{V}_\infty(t, x) = \eta e^{t+x}, \quad (t, x) \in \mathbb{R}_+^2,$$

where η is the constant

$$\eta = \frac{1}{2} (1 + \varrho) [e^{(\varrho-1)\mu} - 1 + \theta];$$

then

$$\hat{V}_{\varrho,\theta}(t, x) \leq \hat{V}_\infty(t, x), \quad (t, x) \in \Pi_\mu.$$

A calculation reveals that

$$\hat{V}_\infty(t, x) - V_{\varrho,\theta}(t, x) = \frac{1}{2} (1 - \varrho^2) e^{\varrho\mu} e^t (x - \mu)^2 + O(e^t (x - \mu)^3), \quad (1.11)$$

near the line $x = \mu$. Let V_1 denote the function

$$V_1(t, x) = \hat{V}_\infty(t, x) - \tilde{V}_{\varrho,\theta}(t, x), \quad (t, x) \in \Pi_\mu,$$

and note that it is caloric and that it vanishes along the boundary vertical line $x = \mu$. By the reflection principle for caloric functions [16, pp. 115–116], then, V_1 extends to a caloric function throughout the positive time half-plane, with

$$V_1(t, x) \equiv -V_1(t, 2\mu - x), \quad (t, x) \in \mathbb{R}_+^2.$$

The boundary values of V_1 along the x -axis are given by

$$V_1(0, x) = \hat{V}_\infty(0, x) - \tilde{V}_{\varrho,\theta}(0, x) = \hat{V}_\infty(0, x) - V_{\varrho,\theta}(0, x), \quad x \in]-\infty, \mu].$$

This is not as explicit as desired, so we decompose

$$V_1(t, x) = V_2(t, x) - V_3(t, x),$$

where V_2 and V_3 are caloric in the positive time half-plane, with boundary values

$$V_2(0, x) = \eta \operatorname{sgn}(\mu - x) \min\{e^x, e^{2\mu-x}\},$$

and

$$V_3(0, x) = V(x) = e^{\varrho x} - \frac{1}{2}(1 - \varrho)e^{\varrho\mu} e^{\mu-x} - \frac{1}{2}(1 - \theta)(1 + \varrho)e^x, \quad x \in [0, \mu],$$

with $V_3(0, x) = 0$ for $x \in \mathbb{R} \setminus [0, 2\mu]$, and $V_3(x) = -V_3(2\mu - x)$ for $x \in [\mu, 2\mu]$. The first boundary moment of V_2 is

$$\int_{-\infty}^{+\infty} (x - \mu) V_2(0, x) dx = -2\eta,$$

and that of V_3 is

$$\begin{aligned} \int_{-\infty}^{+\infty} (x - \mu) V_3(0, x) dx &= -2 \frac{e^{\varrho\mu} - 1 - \varrho\mu}{\varrho^2} + (1 - \varrho)e^{(\varrho+1)\mu}(e^{-\mu} - 1 + \mu) \\ &\quad + (1 - \theta)(1 + \varrho)(e^\mu - 1 - \mu), \end{aligned}$$

so that

$$\int_{-\infty}^{+\infty} (x - \mu) V_1(0, x) dx = -2\eta + 2 \frac{e^{\varrho\mu} - 1 - \varrho\mu}{\varrho^2} - (1 - \varrho)e^{(\varrho+1)\mu}(e^{-\mu} - 1 + \mu) - (1 - \theta)(1 + \varrho)(e^\mu - 1 - \mu) < 0; \tag{1.12}$$

after all, the function $V_1(0, x)$ is positive for $x < \mu$ and negative for $x > \mu$. We need the following observation.

Lemma 1.2. *Suppose f is a continuous odd complex-valued function on the real line, which decays exponentially rapidly at infinity. Then its caloric extension $f(t, x)$ to the positive time half plane, as given by the formula*

$$f(t, x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4t}} f(\xi) d\xi,$$

has the asymptotics

$$f(t, x) = \frac{x}{4\sqrt{\pi t^{3/2}}} \int_{-\infty}^{+\infty} f(\xi)\xi d\xi + O(x t^{-5/2})$$

as $t \rightarrow +\infty$ and x is kept inside a compact interval of the real line.

The proof is left as an exercise to the reader.

In view of the lemma and (1.11), we obtain

$$\begin{aligned} \tilde{V}_{\varrho,\theta}(t, x) - V_{\varrho,\theta}(t, x) &= \hat{V}_\infty(t, x) - V_{\varrho,\theta}(t, x) - V_1(t, x) \\ &= \frac{1}{2}(1 - \varrho^2) e^{\varrho\mu} e^t (x - \mu)^2 \\ &\quad - \frac{x - \mu}{4\sqrt{\pi} t^{3/2}} \int_{-\infty}^{+\infty} V_1(0, \xi)(\xi - \mu) d\xi \\ &\quad + O(e^t (x - \mu)^3) + O((x - \mu) t^{-5/2}), \end{aligned}$$

so that $V_{\varrho,\theta}(t, x) < \tilde{V}_{\varrho,\theta}(t, x)$ holds in a domain of the type

$$x - \mu < -B_1 t^{-3/2} e^{-t} + O(t^{-5/2} e^{-t}),$$

where the positive constant B_1 is given by

$$B_1 = \frac{e^{-\varrho\mu}}{2\sqrt{\pi}(1 - \varrho^2)} \int_{-\infty}^{+\infty} (\mu - \xi) V_1(0, \xi) d\xi, \tag{1.13}$$

which is evaluated in Eq. (1.12). We have obtained the following statement.

Lemma 1.3. *The function $\phi_{\varrho,\theta}$ describing the decision boundary enjoys the following estimate:*

$$\mu - B_1 t^{-3/2} e^{-t} + O(t^{-5/2} e^{-t}) \leq \phi_{\varrho,\theta}(t) < \mu, \quad 0 < t < \infty,$$

where the positive constant B_1 is given by (1.13).

Note that this proves the assertion that the vertical line $x = \mu$ is an asymptote for the decision boundary.

2. The balayage equation

2.1. Derivation of the balayage equation

Integration by parts and balayage. We introduce the function

$$U_{\varrho,\theta}(t, x) = \hat{V}_{\varrho,\theta}(t, x) - V_{\varrho,\theta}(t, x);$$

in the continuation region $\mathcal{D}(\varrho, \theta)$, it solves the overdetermined problem

$$\begin{cases} \boxplus U_{\varrho,\theta} = - \boxplus V_{\varrho,\theta} & \text{on } \mathcal{D}(\varrho, \theta), \\ U_{\varrho,\theta} = 0 & \text{on } \partial\mathcal{D}(\varrho, \theta), \\ \partial_x U_{\varrho,\theta} = 0 & \text{on } \Gamma(\varrho, \theta). \end{cases}$$

In the complement \mathbb{R}_+^2 , we have

$$\hat{V}_{\varrho,\theta}(t, x) = V_{\varrho,\theta}(t, x), \quad (t, x) \in \mathbb{R}_+^2 \setminus \mathcal{D}(\varrho, \theta),$$

which makes $U_{\varrho,\theta} = 0$ there. Also, as $U_{\varrho,\theta} = 0$ on $\partial\mathcal{D}(\varrho, \theta)$, we can extend the function $U_{\varrho,\theta}$ continuously to all of \mathbb{R}^2 by declaring $V = 0$ throughout $\mathbb{R}^2 \setminus \mathcal{D}(\varrho, \theta)$. Since $U_{\varrho,\theta}$ vanishes together with its gradient along $\Gamma(\varrho, \theta)$, it actually solves

$$\begin{cases} \boxplus U_{\varrho,\theta} = -1_{\mathcal{D}(\varrho,\theta)} \boxplus V_{\varrho,\theta} & \text{on } \mathbb{R}^2, \\ U_{\varrho,\theta} = 0 & \text{on } \mathbb{R}^2 \setminus \mathcal{D}(\varrho, \theta). \end{cases} \tag{2.1}$$

Written out more explicitly using (1.8), (2.1) assumes the following form:

$$\boxplus U_{\varrho,\theta} = [\theta(1 + \varrho)\delta_0(x) - (1 - \varrho^2)1_{\Pi_+ \cap \mathcal{D}(\varrho,\theta)}(t, x)]e^{t+\varrho x}, \tag{2.2}$$

on \mathbb{R}_+^2 , while

$$U_{\varrho,\theta}(t, x) = 0, \quad (t, x) \in \mathbb{R}^2 \setminus \mathcal{D}(\varrho, \theta). \tag{2.3}$$

By integration by parts, if h is a compactly supported C^∞ -smooth function in \mathbb{R}^2 , then

$$\int_{\mathbb{R}^2} \boxplus U_{\varrho,\theta}(t, x)h(t, x) dt dx = \int_{\mathbb{R}^2} U_{\varrho,\theta}(t, x) \boxplus^* h(t, x) dt dx, \tag{2.4}$$

where

$$\boxplus^* = -\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

is the adjoint heat operator. We would like to plug the functions

$$h(t, x) = h_z(t, x) = e^{-tz^2+xz}$$

into (2.4), for $z \in \mathbb{C}$, because they are all \boxplus^* -caloric. This function, unfortunately, is never compactly supported; however, it might be possible to approximate it by compactly supported functions so that (2.4) holds for $h = h_z$ in the limit. As we proceed in this manner, taking into account the known growth properties of $V_{\varrho, \theta}$, we find that we should restrict the complex parameter z to

$$1 < \operatorname{Re} z \quad \text{and} \quad 1 < \operatorname{Re}(z^2). \tag{2.5}$$

We compute

$$\int_{\mathbb{R}_+^2} \theta(1 + \varrho)\delta_0(x) e^{t+\varrho x} e^{-tz^2+xz} dt dx = \frac{\theta(1 + \varrho)}{z^2 - 1},$$

while

$$\begin{aligned} & \int_{\mathbb{R}_+^2} (1 - \varrho^2) 1_{\Pi_+ \cap \mathcal{D}(\varrho, \theta)(t, x)} e^{t+\varrho x} e^{-tz^2+xz} dt dx \\ &= \frac{1 - \varrho^2}{\varrho - z} \int_0^{+\infty} e^{(\varrho+z)\phi_{\varrho, \theta}(t)} e^{(1-z^2)t} dt - \frac{1 - \varrho^2}{(\varrho + z)(z^2 - 1)}. \end{aligned}$$

It now follows from (1.8) and (2.4) that

$$\int_0^{+\infty} e^{\phi_{\varrho, \theta}(t)[\varrho+z]} e^{-[z^2-1]t} dt = \frac{1 - \varrho + \theta(\varrho + z)}{(1 - \varrho)(z^2 - 1)}. \tag{2.6}$$

We shall call this the *balayage equation* for the free boundary equation $x = \phi_{\varrho, \theta}(t)$. The reason is that the positive mass concentrated on the half-line

$$\{(t, x) \in \mathbb{R}_+^2 : x = 0\}$$

should be counterbalanced by a corresponding negative mass spread evenly over

$$\Pi_+ \cap \mathcal{D}(\varrho, \theta) = \{(t, x) \in \mathbb{R}_+^2 : 0 < x < \phi_{\varrho, \theta}(t)\}.$$

It is possible to show that, under reasonable restrictions, the balayage equation characterizes the free boundary. We shall, however, not pursue this matter here.

2.2. Equivalent formulations of the balayage equation

Rewriting the balayage equation. In the context of the balayage equation (2.6), we introduce the complex variable $s = z^2$. We use the principal branch of the square root to

define \sqrt{s} , so that if s has positive real part, then so does \sqrt{s} . In particular, if z meets (2.5), then $\sqrt{s} = z$. It follows that we may rewrite (2.6) in the form

$$\int_0^{+\infty} e^{\phi_{\varrho,\theta}(t)[\varrho+\sqrt{s}]} e^{-[s-1]t} dt = \frac{1 - \varrho + \theta(\varrho + \sqrt{s})}{(1 - \varrho)(s - 1)}, \tag{2.7}$$

for all $s \in \mathbb{C}$ with $\text{Re } s > 1$.

We introduce the function $\varphi_{\varrho,\theta}$,

$$\varphi_{\varrho,\theta}(t) = \mu - \phi_{\varrho,\theta}(t), \quad t \in [0, +\infty[,$$

where $\mu = \mu(\varrho, \theta)$ is as before. The function $\varphi_{\varrho,\theta}$ is decreasing, with

$$\varphi_{\varrho,\theta}(0) = \mu$$

and

$$\varphi_{\varrho,\theta}(t) = O(t^{-3/2}e^{-t}) \quad \text{as } t \rightarrow +\infty, \tag{2.8}$$

in view of Lemma 1.3. In terms of this function, the balayage equation (2.7) assumes the form

$$\int_0^{+\infty} e^{-\varphi_{\varrho,\theta}(t)[\varrho+\sqrt{s}]} e^{-(s-1)t} dt = \frac{e^{-\mu\varrho}}{1 - \varrho} \frac{1 - \varrho + \theta[\varrho + \sqrt{s}]}{s - 1} e^{-\mu\sqrt{s}}, \tag{2.9}$$

for $\text{Re } s > 1$. *This is the version we shall use many times in the sequel.* By the way, an integration by parts manoeuvre applied to (2.7) results in the simpler-looking identity

$$\int_0^{+\infty} \phi'_{\varrho,\theta}(t) e^{\phi_{\varrho,\theta}(t)[\varrho+\sqrt{s}]} e^{-(s-1)t} dt = \frac{\theta}{1 - \varrho},$$

valid for $\text{Re } s > 1$. In terms of the function $\varphi_{\varrho,\theta}$, the relationship reads

$$\int_0^{+\infty} |\varphi'_{\varrho,\theta}(t)| e^{-\varphi_{\varrho,\theta}(t)[\varrho+\sqrt{s}]} e^{-(s-1)t} dt = \frac{\theta e^{-\mu\varrho}}{1 - \varrho} e^{-\mu\sqrt{s}}, \tag{2.10}$$

where, again, $\text{Re } s > 1$.

2.3. A general scheme for analyzing the balayage equation

Taylor’s formula for the exponential function. The exponential function has a Taylor expansion about the origin:

$$e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^N}{N!} + E_{N+1}(z),$$

where the remainder term $E_{N+1}(z)$ can be expressed in the form

$$E_{N+1}(z) = \frac{1}{N!} \int_0^z (z - \xi)^N e^{\xi} d\xi = \frac{z^{N+1}}{N!} \int_0^1 (1 - t)^N e^{tz} dt,$$

which represents an entire function, and enjoys the estimate

$$|E_{N+1}(z)| \leq \frac{|z|^{N+1}}{(N+1)!} \max\{1, e^{\operatorname{Re} z}\}. \tag{2.11}$$

As we apply the above formula to the balayage equation (2.9), the result is

$$\begin{aligned} & \frac{1}{s-1} - [\varrho + \sqrt{s}] \mathcal{L}[\varphi_{\varrho, \theta}](s-1) + \frac{1}{2!} [\varrho + \sqrt{s}]^2 \mathcal{L}[\varphi_{\varrho, \theta}^2](s-1) - \dots \\ & + \frac{(-1)^N}{N!} [\varrho + \sqrt{s}]^N \mathcal{L}[\varphi_{\varrho, \theta}^N](s-1) \\ & + \int_0^{+\infty} E_{N+1}(-[\varrho + \sqrt{s}]\varphi_{\varrho, \theta}(t)) e^{-(s-1)t} dt \\ & = \frac{e^{-\mu\varrho}}{1-\varrho} \frac{1-\varrho + \theta[\varrho + \sqrt{s}]}{s-1} e^{-\mu\sqrt{s}}, \end{aligned} \tag{2.12}$$

for $\operatorname{Re} s > 1$. Here, \mathcal{L} denotes the Fourier–Laplace transform, as defined by

$$\mathcal{L}[f](z) = \int_0^{+\infty} e^{-tz} f(t) dt,$$

wherever the integral converges. The intention is to use the identity (2.12) to successively obtain more information regarding the function $\varphi_{\varrho, \theta}$. Indeed, in view of Lemma 1.3, we have some input to initiate the iterative process.

2.4. Asymptotics of the free boundary

Analytic continuation of the Laplace transform. We recall the definition of the function φ ,

$$\phi_{\varrho, \theta}(t) = \mu - \varphi_{\varrho, \theta}(t), \quad 0 < t < +\infty,$$

which is a decreasing function with $\varphi_{\varrho, \theta}(0) = \mu$ and the asymptotic bound (2.8). We use the representation (2.12) with $z = \sqrt{s}$ and $N = 1$ in the form

$$\begin{aligned} \mathcal{L}[\varphi_{\varrho, \theta}](z^2 - 1) &= \frac{1}{z^2 - 1} - \frac{e^{-\mu\varrho}}{1-\varrho} \frac{1-\varrho + \theta(z + \varrho)}{z^2 - 1} e^{-\mu z} \\ &\quad - \int_0^{+\infty} E_2(-[\varrho + z]\varphi_{\varrho, \theta}(t)) e^{-(z^2-1)t} dt. \end{aligned} \tag{2.13}$$

It is immediate from the definition of $\mu = \mu(\varrho, \theta)$ that the function

$$G(z) = \frac{1}{z^2 - 1} - \frac{e^{-\mu\varrho}}{1-\varrho} \frac{1-\varrho + \theta(z + \varrho)}{z^2 - 1} e^{-\mu z}$$

extends analytically to $\mathbb{C} \setminus \{-1\}$, and that the singularity at $z = -1$ is a simple pole. Moreover, by Lemma 1.3, the left-hand side converges to a holomorphic function for z with $\operatorname{Re}[z^2] > 0$. Furthermore, by (2.11), the function

$$\mathcal{E}_2(z) = \int_0^{+\infty} E_2(-[\varrho + z]\varphi_{\varrho,\theta}(t))e^{-(z^2-1)t} dt$$

expresses an even analytic function in the simply connected domain $\operatorname{Re}[z^2] > -1$, which extends continuously to the closed region $\operatorname{Re}[z^2] \geq -1$. It follows that the right-hand side of (2.13) expresses an analytic function in $\operatorname{Re}[z^2] > -1$ with the exception of a simple pole at $z = -1$.

We introduce the notation

$$\Phi(z) = \mathcal{L}[\varphi_{\varrho,\theta}](z^2 - 1)$$

for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and $\operatorname{Re}[z^2] > 0$, and let Φ denote the possible analytic continuation of this function beyond its initial domain of definition. Then (2.13) reads

$$\begin{aligned} \Phi(z) &= \frac{1}{z^2 - 1} - \frac{e^{-\mu\varrho}}{1 - \varrho} \frac{1 - \varrho + \theta(z + \varrho)}{z^2 - 1} e^{-\mu z} \\ &\quad - \int_0^{+\infty} E_2(-[\varrho + z]\varphi_{\varrho,\theta}(t))e^{-(z^2-1)t} dt, \end{aligned} \tag{2.14}$$

for all $z \in \mathbb{C}$ with $\operatorname{Re}[z^2] > -1$ and $z \neq -1$, and $\Phi(z)$ is holomorphic in $\operatorname{Re}[z^2] > -1$ with the exception of a simple pole at $z = -1$, and continuous up to the boundary.

Estimate of the Laplace transform. We need some size control of $\Phi(z)$, in order to invoke inverse Laplace transformation and obtain information about $\varphi_{\varrho,\theta}$. By integration by parts,

$$\begin{aligned} &\frac{1}{z + \varrho} \int_0^{+\infty} E_2(-[\varrho + z]\varphi_{\varrho,\theta}(t))e^{-[z^2-1]t} dt \\ &= \frac{1}{z^2 - 1} \left[\mu - \frac{1 - e^{-\mu(z+\varrho)}}{z + \varrho} \right] \\ &\quad - \frac{1}{z^2 - 1} \int_0^{+\infty} \varphi'_{\varrho,\theta}(t) E_1(-\varphi_{\varrho,\theta}(t)[z + \varrho])e^{-[z^2-1]t} dt. \end{aligned} \tag{2.15}$$

Combining (2.13) and (2.15), we arrive at

$$\begin{aligned} \Phi(z) &= \frac{1}{z^2 - 1} \left[\mu - \frac{\theta e^{-\mu(z+\varrho)}}{1 - \varrho} \right] \\ &\quad - \frac{1}{z^2 - 1} \int_0^{+\infty} \varphi'_{\varrho,\theta}(t) E_1(-\varphi_{\varrho,\theta}(t)[z + \varrho])e^{-[z^2-1]t} dt. \end{aligned} \tag{2.16}$$

Let us, as a matter of convenience, introduce also the function

$$\Psi(z) = \mu - [z^2 - 1] \Phi(z),$$

which is a Fourier–Laplace transform as well:

$$\Psi(z) = \mathcal{L}[-\varphi'_{\varrho,\theta}](z^2 - 1),$$

for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and $\operatorname{Re}[z^2] > 0$. In terms of $\Psi(z)$, (2.16) reads

$$\Psi(z) = \frac{\theta e^{-\mu(z+\varrho)}}{1-\varrho} + \int_0^{+\infty} \varphi'(t) E_1(-\varphi_{\varrho,\theta}(t)[z+\varrho]) e^{-[z^2-1]t} dt, \tag{2.17}$$

for $z \in \mathbb{C}$ with $\operatorname{Re}[z^2] > -1$. By letting $z \rightarrow 0$ along the positive real axis, we obtain from (2.17) that

$$\Psi(0) = \int_0^{+\infty} |\varphi'_{\varrho,\theta}(t)| e^t dt = \frac{\theta e^{-\mu\varrho}}{1-\varrho} - \int_0^{+\infty} |\varphi'_{\varrho,\theta}(t)| E_1(-\varphi_{\varrho,\theta}(t)\varrho) e^t dt,$$

which simplifies to

$$\int_0^{+\infty} |\varphi'_{\varrho,\theta}(t)| e^{-\varphi_{\varrho,\theta}(t)\varrho} e^t dt = \frac{\theta e^{-\mu\varrho}}{1-\varrho}. \tag{2.18}$$

From this and the fact that the Laplace transform of a positive function is dominated by its behavior along the real line, we get, by integration by parts, that

$$|\mathcal{L}[\varphi_{\varrho,\theta}](s)| = O\left(\frac{1}{|s|}\right) \text{ as } |s| \rightarrow +\infty, \operatorname{Re} s > -1. \tag{2.19}$$

This leads to an estimate of $\Phi(z)$ in the quarter-plane $\operatorname{Re} z > 0, \operatorname{Re}[z^2] > 0$. We need size control of $\Phi(z)$ in the bigger region $\operatorname{Re}[z^2] > -1$. From (2.17) and the estimate (2.11), we obtain

$$\begin{aligned} |\Psi(z)| &\leq \frac{\theta e^{-\mu(\varrho+\operatorname{Re} z)}}{1-\varrho} \\ &\quad + |z+\varrho| \int_0^{+\infty} |\varphi'_{\varrho,\theta}(t)| \varphi_{\varrho,\theta}(t) \max\{1, e^{-\varphi_{\varrho,\theta}(t)[\varrho+\operatorname{Re} z]}\} e^{2t} dt. \end{aligned}$$

In view of the derived integrability properties of φ and its derivative, this gives the following growth estimate for Φ :

$$|\Phi(z)| = O\left(\frac{1}{|z|} \max\{1, e^{-\mu \operatorname{Re} z}\}\right), \text{ as } |z| \rightarrow +\infty, \operatorname{Re}[z^2] > -1. \tag{2.20}$$

The Laplace transform and a slit domain. It follows from the derived properties of $\Phi(z)$ that the Laplace transform $\mathcal{L}[\varphi_{\varrho,\theta}](s)$ extends analytically to the region $\text{Re } s > -2$ minus the slit $]-2, -1]$ on the real line, and it extends continuously to all boundary points with $\text{Re } s = -2$, except for $s = -2$. At $s = -1$, $\mathcal{L}[\varphi_{\varrho,\theta}](s)$ has a square root branch point, and for real x , $-2 \leq x \leq -1$, we can speak of the functions $\mathcal{L}[\varphi_{\varrho,\theta}](x + i0)$ and $\mathcal{L}[\varphi_{\varrho,\theta}](x - i0)$ as continuous limits from above and below. This makes the following expression well defined:

$$\Lambda(x) = \frac{i}{2\pi} (\mathcal{L}[\varphi_{\varrho,\theta}](-x + i0) - \mathcal{L}[\varphi_{\varrho,\theta}](-x - i0)), \quad 1 \leq x \leq 2.$$

Using the identity (2.16), we can write this function as

$$\Lambda(x) = \frac{1}{\pi x} \left\{ \frac{\theta e^{-\mu\varrho}}{1 - \varrho} \sin[\mu\sqrt{x-1}] - \int_0^{+\infty} |\varphi'_{\varrho,\theta}(\tau)| e^{-\varphi(\tau)\varrho} \sin[\varphi_{\varrho,\theta}(\tau)\sqrt{x-1}] e^{x\tau} d\tau \right\}, \quad 1 \leq x \leq 2, \quad (2.21)$$

from which we see that it is real-valued on the interval in question, and has a square root type singularity at the point $x = 1$. We now extend the function Λ to the whole real line by setting it equal to 0 off the interval $[1, 2]$. We shall need the function φ_Λ , the Laplace transform of Λ :

$$\varphi_\Lambda(t) = \mathcal{L}[\Lambda](t) = \int_1^{+\infty} \Lambda(x) e^{-tx} dx.$$

It is real-valued, and, due to the noted property

$$\Lambda(x) = O(\sqrt{x-1}) \quad \text{as } x \rightarrow 1^+,$$

it has the following asymptotics:

$$\varphi_\Lambda(t) = O(t^{-3/2} e^{-t}) \quad \text{as } t \rightarrow +\infty.$$

The Laplace transform of φ_Λ equals

$$\mathcal{L}[\varphi_\Lambda](s) = \frac{1}{2\pi i} \int_{-2}^{-1} \frac{\mathcal{L}[\varphi_{\varrho,\theta}](x + i0) - \mathcal{L}[\varphi_{\varrho,\theta}](x - i0)}{x - s} dx,$$

which means that the difference

$$\mathcal{L}[\varphi_{\varrho,\theta}](s) - \mathcal{L}[\varphi_\Lambda](s)$$

is holomorphic throughout $\text{Re } s > -2$, with continuous boundary values, except possibly for a logarithmic singularity at $s = -2$. It has the decay rate

$$|\mathcal{L}[\varphi_\Lambda](s)| = O\left(\frac{1}{|s|}\right), \quad |s| \rightarrow +\infty, \quad \text{Re } s > -2,$$

so that according to (2.20), we have

$$|\mathcal{L}[\varphi - \varphi_\Lambda](s)| = O\left(\frac{1}{\sqrt{|s|}}\right) \text{ as } |s| \rightarrow +\infty, \operatorname{Re} s > -2.$$

Using some basic complex analysis, we get the intermediate growth control

$$|\mathcal{L}[\varphi - \varphi_\Lambda](s)| = O\left(\frac{1}{|s|^{(3+\operatorname{Re} s)/2}}\right) \text{ as } |s| \rightarrow +\infty, -2 < \operatorname{Re} s < -1.$$

In particular, by the Plancherel identity, the function

$$t \mapsto e^{(2-\varepsilon)t} (\varphi_{\varrho,\theta}(t) - \varphi_\Lambda(t)) \tag{2.22}$$

is in $L^2(\mathbb{R}_+)$ for each positive ε . Together with

$$\varphi_{\varrho,\theta}(t) - \varphi_\Lambda(t) = O(t^{-3/2}e^{-t}) \text{ as } t \rightarrow +\infty, \tag{2.23}$$

this gives us rather good asymptotic information regarding the behavior of the difference function $\varphi_{\varrho,\theta} - \varphi_\Lambda$.

2.5. The asymptotic formula

Generalized Taylor series for Λ . The function $\Lambda(x)$ has a convergent series expansion

$$\Lambda(x) = \sqrt{x-1}(\lambda_0 + \lambda_1(x-1) + \lambda_2(x-1)^2 + \dots), \quad 1 \leq x < 2.$$

It is well known in asymptotic analysis (see [2]) that this leads to an asymptotic formula for its Fourier–Laplace transform φ_Λ ,

$$\varphi_\Lambda(t) = \sum_{j=0}^N \lambda_j \Gamma\left(j + \frac{3}{2}\right) t^{-j-3/2} e^{-t} + O(t^{-N-5/2} e^{-t})$$

as $t \rightarrow +\infty$. In view of (2.22) and (2.23), we get the same type of expansion for $\varphi_{\varrho,\theta}$:

$$\varphi_{\varrho,\theta}(t) = \sum_{j=0}^N \lambda_j \Gamma\left(j + \frac{3}{2}\right) t^{-j-3/2} e^{-t} + O_{L^p}(t^{-N-2} e^{-t}), \tag{2.24}$$

where by $O_{L^p}(1)$ we mean an expression with bounded norm in $L^p(\mathbb{R}_+)$, and $2 < p < +\infty$ is arbitrary. The coefficients λ_j may be read off from the identity (2.21); for instance, if we use in addition the identity (2.18), we obtain

$$\begin{aligned} \lambda_0 &= \frac{\theta \mu e^{-\mu \varrho}}{\pi(1-\varrho)} - \frac{1}{\pi} \int_0^{+\infty} |\varphi'_{\varrho,\theta}(t)| \varphi_{\varrho,\theta}(t) e^{-\varphi(t)\varrho} e^t dt \\ &= \frac{1}{\pi} \int_0^{+\infty} |\varphi'_{\varrho,\theta}(t)| (\mu - \varphi_{\varrho,\theta}(t)) e^{-\varphi(t)\varrho} e^t dt. \end{aligned} \tag{2.25}$$

By (2.24) with $N = 1$, this leads to

$$\varphi_{\varrho,\theta}(t) = \beta_1 t^{-3/2} e^{-t} + O_{L^p}(t^{-2} e^{-t}),$$

for any p , $2 < p < +\infty$, with $\beta_1 = \Gamma(3/2) \lambda_0$. This is the formula alluded to in the introduction, Subsection 1.3.

3. Further topics

Maximal meromorphic extensions. It is of interest to apply the general scheme (2.12) not just to $N = 1$, but also $N = 2, 3, 4, \dots$. Unfortunately, this approach seems to run into difficulty after a few steps. In any case, it is possible to show that $\mathcal{L}[\varphi_{\varrho,\theta}](s)$ has an analytic extension to the half-plane $\text{Re } s > -4$ minus the slit $]-4, -1]$ along the real axis. One possible approach to get an explicit expression for $\mathcal{L}[\varphi_{\varrho,\theta}](s)$ would be to obtain the maximal meromorphic continuation to a Riemann surface sheeted over the complex plane. If the structure of the Riemann surface would happen to be simple, it might be possible to express $\mathcal{L}[\varphi_{\varrho,\theta}](s)$ in terms of the poles of the maximal extension. This would then also give an explicit way to compute the free boundary function $\varphi_{\varrho,\theta}(t)$ itself.

Small values of θ . The parameter θ was introduced mainly for the purpose of having the opportunity to choose θ close to 0, since everything is completely understood for $\theta = 0$. It might then, in a second step, be possible to extend the analysis to general values of θ . By varying θ in the original obstacle problem, we realize that we should be able to solve boundary value problems for the heat equation on certain subdomains of \mathbb{R}_+^2 ; the Green function on such subdomains is in principle possible to obtain (see [15, pp. 521–535]).

Here, however, we do the analysis on the basis of the balayage equation, and focus on θ close to 0. As we differentiate the balayage equation (2.9) with respect to the parameter θ , we have

$$\begin{aligned} & \int_0^{+\infty} \frac{d\varphi_{\varrho,\theta}}{d\theta}(t) e^{-\varphi_{\varrho,\theta}(t)[\varrho+\sqrt{s}]} e^{-(s-1)t} dt \\ &= \left[\frac{d\mu}{d\theta} - \frac{1}{1-\varrho} + (\varrho + \sqrt{s}) \frac{\theta}{1-\varrho} \frac{d\mu}{d\theta} \right] \frac{e^{-\mu(\varrho+\sqrt{s})}}{s-1}, \end{aligned}$$

and after another differentiation, we have

$$\begin{aligned} & \int_0^{+\infty} \frac{d^2\varphi_{\varrho,\theta}}{d\theta^2}(t) e^{-\varphi_{\varrho,\theta}(t)[\varrho+\sqrt{s}]} e^{-(s-1)t} dt \\ & - (\varrho + \sqrt{s}) \int_0^{+\infty} \left(\frac{d\varphi}{d\theta}(t) \right)^2 e^{-\varphi_{\varrho,\theta}(t)[\varrho+\sqrt{s}]} e^{-(s-1)t} dt \\ &= \left\{ \frac{d^2\mu}{d\theta^2} + (\varrho + \sqrt{s}) \left[\frac{\theta}{1-\varrho} \frac{d^2\mu}{d\theta^2} + \frac{2}{1-\varrho} \frac{d\mu}{d\theta} - \left(\frac{d\mu}{d\theta} \right)^2 \right] \right. \\ & \left. - (\varrho + \sqrt{s})^2 \frac{\theta}{1-\varrho} \frac{d\mu}{d\theta} \right\} \frac{e^{-\mu(\varrho+\sqrt{s})}}{s-1}. \end{aligned}$$

For $\theta = 0$, $\varphi_{\varrho,\theta}(t) \equiv 0$, and as we apply the first identity, we realize that we also have

$$\left. \frac{d\varphi_{\varrho,\theta}}{d\theta}(t) \right|_{\theta=0} = 0, \quad 0 < t < +\infty.$$

Inserting $\theta = 0$ also into the second, we get

$$\int_0^{+\infty} \frac{d^2\varphi}{d\theta^2}(t) \Big|_{\theta=0} e^{-(s-1)t} dt = \frac{1}{(1-\varrho)^2} \frac{1}{\sqrt{s+1}}, \quad \operatorname{Re} s > 1,$$

and, as a consequence,

$$\frac{d^2\varphi_{\varrho,\theta}}{d\theta^2}(t) \Big|_{\theta=0} = \frac{\pi^{-1/2}}{2(1-\varrho)^2} \int_t^{+\infty} \tau^{-3/2} e^{-\tau} d\tau, \quad 0 < t < +\infty.$$

This gives us some understanding of the behavior of $\varphi_{\varrho,\theta}(t)$ for small θ and fixed ϱ and t .

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