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Spectral Properties of Invariant Subspaces in the Bergman Space

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We study the relationship between two types of spectra associated with invariant subspaces of the Bergman space $L_a^2(\mathbb{D})$ and the function theoretic properties of the invariant subspaces themselves. For instance, we prove that if an invariant subspace J contains a function that is bounded away from 0 on some neighborhood of a point λ on the unit circle $\mathbb{T}$, then the spectrum of $z[J]$, multiplication by z, when regarded as operating on the quotient space $L_a^2(\mathbb{D})/J$, does not contain the point λ. A consequence of this result is that the spectrum associated with the invariant subspace of all functions vanishing on a prescribed Bergman space zero sequence coincides with the closure of the sequence.

INTRODUCTION

Let $L_a^2(\mathbb{D})$ denote the standard Bergman space of all holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ that satisfy the integrability condition

$$\|f\|_{L^2} = \left(\int_{\mathbb{D}} |f(z)|^2 \, dS(z) \right)^{1/2} < \infty.$$

Here, dS denotes area measure in $\mathbb{C}$, normalized by a constant factor:

$$dS(z) = dx\, dy/\pi, \qquad z = x + iy.$$

A closed subspace J of $L_a^2(\mathbb{D})$ is said to be z-invariant, or just invariant, provided the product zf belongs to J whenever $f \in J$. Here, we use the standard notation z for the coordinate function:

$$z(\lambda) = \lambda, \qquad \lambda \in \mathbb{D}.$$

The structure of the lattice of invariant subspaces in $L_a^2(\mathbb{D})$ has attracted

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a lot of attention from operator theorists as well as function theorists, but most results have been disappointing, in the sense that one realizes that no simple characterization such as is known for the Hardy space $H^2(\mathbb{D})$ is possible for the Bergman space. The famous theorem on the invariant subspaces of $H^2(\mathbb{D})$ is due to Arne Beurling [1], and it asserts that every z-invariant subspace J of $H^2(\mathbb{D})$, analogously defined as for the Bergman space, is either trivial, that is, $J=\{0\}$, or has the form $J=uH^2(\mathbb{D})$, where u is an inner function, that is, a bounded analytic function on $\mathbb{D}$ with non-tangential boundary values having modulus 1 almost everywhere.

Given an invariant subspace J of the Bergman space $L^2_a(\mathbb{D})$, consider the operator $z[J] : L^2_a(\mathbb{D})/J \to L^2_a(\mathbb{D})/J$ defined by the relation

$$z[J](f+J)=zf+J, \qquad f \in L^2_a(\mathbb{D}).$$

We write $\sigma(z[J])$ for the spectrum of the operator $z[J]$, which consists of those $\lambda \in \mathbb{C}$ for which the operator $\lambda - z[J]$, acting on $L^2_a(\mathbb{D})/J$, is not invertible. It is well known that the spectrum $\sigma(z[J])$ is a compact subset of the closed unit disk $\bar{\mathbb{D}}$. Because the lattice of invariant subspaces of the Bergman space is very rich, it is appropriate to also consider another spectral notion (we may call it the weak spectrum): let $\sigma'(z[J])$ denote the collection of all $\lambda \in \mathbb{C}$ for which the operator $\lambda - z[J] : L^2_a(\mathbb{D})/J \to L^2_a(\mathbb{D})/J$ is not onto. What can go wrong is that $\lambda - z[J]$ need not be one-to-one even if it is onto; this occurs precisely (for $\lambda \in \mathbb{D}$) when the invariant subspace fails to have what Richter [5] calls the codimension 1 property. It is not difficult to see that the set $\sigma'(z[J])$ is also a compact subset of $\bar{\mathbb{D}}$, and clearly we have the inclusion $\sigma'(z[J]) \subset \sigma(z[J])$. There are cases when $\sigma(z[J]) = \bar{\mathbb{D}}$ and $\sigma'(z[J]) = \mathbb{T}$; for an example, see [3].

Another way of stating the definition of the weak spectrum $\sigma'(z[J])$ is as follows: if λ is a complex number, we have $\lambda \in \mathbb{C} \setminus \sigma'(z[J])$ if and only if

$$(\lambda - z)\, L^2_a(\mathbb{D}) + J = L^2_a(\mathbb{D}).$$

Earlier work on spectra associated with invariant subspaces in the Bergman space can be found in [4, 5, 6].

1. Results

An invariant subspace J in the Bergman space $L^2_a(\mathbb{D})$ is said to have the codimension 1 property if the invariant subspace zJ has codimension 1 in J. The following basic result on the spectral notions $\sigma(z[J])$ and $\sigma'(z[J])$ can be found in [5].

THEOREM 1.1 (Richter). *Let J be an invariant subspace of $L^2_a(\mathbb{D})$, other than the trivial subspace $\{0\}$. If J has the codimension 1 property, $\sigma(z[J]) = \sigma'(z[J])$. If, on the other hand, J does not have the codimension 1 property, then $\sigma(z[J]) = \bar{\mathbb{D}}$, and $\sigma'(z[J]) \supset \mathbb{T}$.*

The next lemma explains that at points in $\bar{\mathbb{D}} \backslash \sigma(z[J])$, there are functions in J bounded away from 0.

LEMMA 1.2. *Every invariant subspace J of the Bergman space $L^2_a(\mathbb{D})$, other than $\{0\}$, contains a non-identically vanishing function G_J, which extends to a holomorphic function on the region*

$$\{z \in \mathbb{C} : 1/\bar{z} \notin \sigma(z[J])\},$$

and has $|G_J(z)| \geq 1$ on the union of arcs $\mathbb{T} \backslash \sigma(z[J])$.

Proof. The assertion is void if $\sigma(z[J]) = \bar{\mathbb{D}}$, so we may as well assume that J has the codimension 1 property, by Theorem 1.1. Then the subspace zJ also has the codimension 1 property [5], so by Theorem 1.1, $\sigma(z[zJ]) = \sigma'(z[zJ])$. We show that

$$\sigma(z[zJ]) = \sigma(z[J]) \cup \{0\}. \tag{1.1}$$

It is sufficient to prove this equality with the $\sigma(\cdot)$'s replaced by $\sigma'(\cdot)$'s. By definition, if I is an invariant subspace, $\lambda \in \mathbb{C} \backslash \sigma'(z[I])$ if and only if

$$(\lambda - z)\, L^2_a(\mathbb{D}) + I = L^2_a(\mathbb{D}).$$

Clearly, the weak spectrum has the monotonicity property that $\sigma'(z[I']) \supset \sigma'(z[I])$ if I' is another invariant subspace with $I' \subset I$. From this we see that $\sigma'(z[zJ]) \supset \sigma'(z[J])$, and it is not difficult to see that $0 \in \sigma'(z[zJ])$ directly from the definition. For these reasons, to verify (1.1) we just need to show that $\sigma'(z[zJ]) \subset \sigma'(z[J]) \cup \{0\}$. To this end, let us take a $\lambda \in \mathbb{C} \backslash \sigma'(z[J]) \backslash \{0\}$, and try to show that $\lambda \in \mathbb{C} \backslash \sigma'(z[zJ])$. By the definition of the spectrum, we have that

$$(\lambda - z)\, L^2_a(\mathbb{D}) + J = L^2_a(\mathbb{D}),$$

so by multiplying both sides by z, we have in particular

$$(\lambda - z)\, L^2_a(\mathbb{D}) + zJ \supset zL^2_a(\mathbb{D}).$$

There are functions in $(\lambda - z)\, L^2_a(\mathbb{D})$ that do not vanish at 0, for instance the function $\lambda - z$ itself, so that since $zL_a^2(\mathbb{D})$ has codimension 1 in $L_a^2(\mathbb{D})$, we must in fact have

$$(\lambda - z)\, L_a^2(\mathbb{D}) + zJ = L_a^2(\mathbb{D}).$$

This shows that $\lambda \in \mathbb{C} \setminus \sigma'(z[zJ])$, as asserted.

We are now in a position to prove the assertion of the lemma. Let $G_J \in J \ominus zJ$ have norm 1. Then the kernel representation formula

$$G_J(\lambda) = \langle G_J, (1 - \bar{\lambda} z)^{-2} \rangle_{L^2}, \qquad \lambda \in \mathbb{D},$$

generalizes to

$$G_J(\lambda) = \langle G_J + zJ, (1 - \bar{\lambda} z[zJ])^{-2}\, (1 + zJ) \rangle_{L_a^2/zJ},$$

where $1 + zJ$ denotes the coset containing the constant function 1 in the quotient space $L_a^2(\mathbb{D})/zJ$, and we see that the expression on the right-hand side is a well-defined holomorphic function in the variable λ on the set

$$\{z \in \mathbb{C} : 1/\bar{z} \notin \sigma(z[zJ])\},$$

which coincides with

$$\{z \in \mathbb{C} : 1/\bar{z} \notin \sigma(z[J])\},$$

because the additional point 0 in (1.1) now corresponds to the point at infinity. The functions G_J were studied *in extenso* in [2]; for instance, by Corollary 4.3 [2] and the proof of Proposition 1.3 [2], it is clear that G_J has modulus $\geqslant 1$ at every boundary point to which it extends continuously. This concludes the proof of the lemma. ∎

Given a function $f \in L_a^2(\mathbb{D})$, its lower zero set (or liminf zero set), written $Z_*(f)$, consists of all actual zeros of f inside the open unit disk $\mathbb{D}$, and of all points λ on the unit circle $\mathbb{T}$ for which

$$\liminf_{\mathbb{D} \ni z \to \lambda} |f(z)| = 0.$$

Extend this notion to collections of functions $\mathscr{F}$ in $L_a^2(\mathbb{D})$ by declaring

$$Z_*(\mathscr{F}) = \bigcap \{Z_*(f) : f \in \mathscr{F}\}.$$

We are finally in a position to state our main result.

THEOREM 1.3. *Let J be an invariant subspace of $L_a^2(\mathbb{D})$. Then $\sigma'(z[J]) = Z_*(J)$.*

For the proof, we need the following lemma.

LEMMA 1.4. *Let $f \in L_a^2(\mathbb{D})$ be such that on an open disk $D(z_0, \rho)$, centered at $z_0 \in \mathbb{T}$, with radius $\rho > 0$, we have*

$$|f(z)| > \varepsilon, \qquad z \in \mathbb{D} \cap D(z_0, \rho),$$

for some constant $\varepsilon > 0$. Then there exists a bounded analytic function g on $\mathbb{D}$ such that

$$1/2 < |f(z)\, g(z)| < 2, \qquad z \in \mathbb{D} \cap D(z_0, \rho'),$$

for some smaller radius ρ', $0 < \rho' < \rho$.

Proof. Consider the function $1/f$, which is homomorphic, zero-free, and bounded on $\mathbb{D} \cap D(z_0, \rho)$, and meromorphic in the whole unit disk $\mathbb{D}$. On the region $\mathbb{D} \cap D(z_0, \rho)$, we are now in a situation where we may apply the standard Nevanlinna theory, to show that the harmonic function $\log |\varepsilon/f|$ has boundary values in the sense of distribution theory on $\mathbb{D} \cap D(z_0, \rho)$, and these boundary values form a negative Borel measure μ. We may then pick a slightly smaller radius ρ'', $0 < \rho'' < \rho$, and let φ be the Poisson extension to the whole disk $\mathbb{D}$ corresponding to the part of the measure μ that falls upon the arc $\mathbb{T} \cap D(z_0, \rho'')$. The negative measure μ is finite on that arc, because we can map $\mathbb{D} \cap D(z_0, \rho)$ conformally onto $\mathbb{D}$, and on $\mathbb{D}$, and the mapped measure on $\mathbb{T}$ corresponding to μ must be bounded; the rest is an exercise in conformal mapping. We now find a bounded holomorphic function g on $\mathbb{D}$ having $|g| = \varepsilon^{-1} \exp(\varphi)$ on $\mathbb{D}$, and by construction and the Schwarz reflection principle, fg extends holomorphically across the arc $\mathbb{T} \cap D(z_0, \rho'')$, and has modulus 1 on it. The function fg clearly meets the assertion, for some small radius ρ'. ∎

Proof of Theorem 1.3. Richter [5] has shown that

$$\sigma'(z[J]) \cap \mathbb{D} = Z_*(J) \cap \mathbb{D}.$$

By Lemma 1.2, $Z_*(J) \cap \mathbb{T}$ is contained within $\sigma(z[J]) \cap \mathbb{T}$. This entails that $Z_*(J) \cap \mathbb{T} \subset \sigma'(z[J]) \cap \mathbb{T}$, for the following reasons. If J fails to have the codimension 1 property, then by Theorem 1.1, $\sigma'(z[J]) \supset \mathbb{T}$, which makes the assertion trivial. If, on the other hand, J does have the codimension 1 property, then $\sigma(z[J]) = \sigma'(z[J])$, and all is well.

The rest of the proof is devoted to obtaining the reverse inclusion

$$Z_*(J) \cap \mathbb{T} \supset \sigma'(z[J]) \cap \mathbb{T}.$$

Let f be a function in J, and suppose there exists a point $\lambda \in \mathbb{T}$ such that for some disk centered at λ with radius $R > 0$,

$$D(\lambda, R) = \{z \in \mathbb{C} : |z - \lambda| < R\},$$

we have

$$1/2 < |f(z)| < 2, \qquad z \in D(\lambda, R) \cap \mathbb{D};$$

such a function f exists in J if and only if $\lambda \in \mathbb{T} \setminus Z_*(J)$, by Lemma 1.4. We need to show that $\lambda \notin \sigma'(z[J])$; this amounts to proving that

$$(\lambda - z)\, L_a^2(\mathbb{D}) + J = L_a^2(\mathbb{D}).$$

In other words, we need to show that for every $g \in L_a^2(\mathbb{D})$, an $h \in L_a^2(\mathbb{D})$ can be found such that

$$(\lambda - z)h - g \in J.$$

Fix three real parameters r_1, r_2, r_3 with $0 < r_1 < r_2 < r_3 < R$, and let

$$D(\lambda, r_j) = \{z \in \mathbb{C} : |z - \lambda| < r_j\}, \qquad j = 1, 2, 3,$$

be the disk around λ with radius r_j. Let χ_λ be an infinitely differentiable compactly supported function on $\mathbb{C}$ with values between 0 and 1, which vanishes off the disk $D(\lambda, r_2)$ and has value 1 on the smaller disk $D(\lambda, r_1)$. Let the function q_λ solve the $\bar{\partial}$-problem

$$\bar{\partial} q_\lambda(z) = \frac{g(z)\, \bar{\partial}\chi_\lambda(z)}{(\lambda - z)\, f(z)}, \qquad z \in \mathbb{D}; \tag{1.2}$$

just put

$$q_\lambda(z) = \int_{\mathbb{D}} \frac{g(\zeta)\, \bar{\partial}\chi_\lambda(\zeta)}{(\lambda - \zeta)(z - \zeta)\, f(\zeta)}\, dS(\zeta), \qquad z \in \mathbb{C}. \tag{1.3}$$

Note that since the right-hand side of (1.2) is in $L^2(\mathbb{D}, dS)$, and since we are in fact considering the convolution of that $L^2(\mathbb{D}, dS)$ function with the $\bar{\partial}$-kernel $(\pi z)^{-1}$, which locally belongs to L^q for every $q < 2$, we see that q_λ, as defined by (1.3), belongs to $L^p(\mathbb{D}, dS)$ for all $p < \infty$. One more thing that is immediate is that q_λ is holomorphic off the closure of $D(\lambda, r_2) \cap \mathbb{D}$, and in particular bounded on $\mathbb{C} \setminus D(\lambda, r_3)$ ($q_\lambda(z)$ tends to 0 as $|z| \to \infty$). We consider the function

$$p_\lambda(z) = -g(z)\, \chi_\lambda(z)/f(z) + (\lambda - z)\, q_\lambda(z), \qquad z \in \mathbb{D},$$

which belongs to $L^2(\mathbb{D}, dS)$, because f is bounded away from 0 on the support of χ_λ. Moreover, p_λ is holomorphic on $\mathbb{D}$, since

$$\bar{\partial} p_\lambda(z) = -g(z)\, \bar{\partial}\chi_\lambda(z)/f(z) + (\lambda - z)\, \bar{\partial} q_\lambda(z) = 0, \qquad z \in \mathbb{D}.$$

Let us for the moment assume that we know that fp_λ belongs to J. We then put

$$h(z) = g(z)\frac{1-\chi_\lambda(z)}{\lambda - z} + f(z)\, q_\lambda(z), \qquad z \in \mathbb{D},$$

and note that f is bounded on $D(\lambda, R) \cap \mathbb{D}$, and q_λ is bounded on $\mathbb{C} \setminus D(\lambda, r_3)$ and belongs to $L^2(dS)$ on $\mathbb{D}$, so that the product fq_λ clearly is in $L^2(\mathbb{D}, dS)$. The function h thus belongs to $L^2(\mathbb{D}, dS)$, and since

$$\bar{\partial} h(z) = -g(z)\, \bar{\partial}\chi_\lambda(z)/(\lambda - z) + f(z)\, \bar{\partial} q_\lambda(z) = 0, \qquad z \in \mathbb{D},$$

h belongs to $L_a^2(\mathbb{D})$. To check that the function h does what we set out for it to do, observe that

$$(\lambda - z)\, h(z) - g(z) = -\chi_\lambda(z)\, g(z) + (\lambda - z)\, f(z)\, q_\lambda(z) = f(z)\, p_\lambda(z),$$

so that the assertion is immediate once we know that fp_λ is in J. The way p_λ is constructed, this function is bounded on $\mathbb{D} \setminus D(\lambda, r_3)$, and $L^2(dS)$ on $\mathbb{D} \cap D(\lambda, r_3)$. The properties of the function f complement those of p_λ: f is bounded on $D(\lambda, R)$, and $L^2(dS)$ on $\mathbb{D} \setminus D(\lambda, R)$. Using this information, it is not difficult to show that

$$f(z)\, p_\lambda(\rho z) \to f(z)\, p_\lambda(z), \qquad \text{as} \quad 1 > \rho \to 1,$$

in the norm of $L_a^2(\mathbb{D})$. Since the functions $f(z)\, p_\lambda(\rho z)$ belong to J, for all ρ with $0 < \rho < 1$, we see that $fp_\lambda \in J$. The proof is now complete. ∎

COROLLARY 1.5. *Let A be a zero sequence in $\mathbb{D}$ for a function in $L_a^2(\mathbb{D})$, and consider the associated invariant subspace*

$$\mathscr{I}(A) = \{ f \in L_a^2(\mathbb{D}) : f = 0 \text{ on } A \},$$

counting multiplicities when necessary. Then $\sigma(z[\mathscr{I}(A)]) = \bar{A}$, the closure of A in $\bar{\mathbb{D}}$.

Proof. Invariant subspaces of the type $\mathscr{I}(A)$ always have the codimension 1 property [5], and consequently $\sigma(z[\mathscr{I}(A)]) = \sigma'(z[\mathscr{I}(A)])$, by Theorem 1.1. So, by Theorem 1.3, all we need to do is show that $Z_*(\mathscr{I}(A)) = \bar{A}$. Clearly, $Z_*(\mathscr{I}(A)) \supset \bar{A}$; to prove the reverse inclusion, note that by Theorem 3.5 [2], there exists a function G_A which vanishes precisely on A in $\mathbb{D}$, extends holomorphically across the set $\mathbb{T} \setminus \bar{A}$, and has modulus $\geqslant 1$ there. The assertion is immediate. ∎

Remark 1.6. Richter [5] obtained Corollary 1.5 under the very restrictive condition on the sequence A that it be interpolating for the space $L_a^2(\mathbb{D})$.

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