JOURNAL OF FUNCTIONAL ANALYSIS 80, 9–15 (1988)

Outer Functions of Several Complex Variables

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Communicated by Donald Sarason

Received April 3, 1987

We generalize to function algebras $A(W)$, $W \subset \mathbf{C}^n$, the familiar notion of outer functions on the unit disc. For a class of domains W, we show that a function $f(z)$ in $A(W)$ is generalized outer if it does not decrease too fast as z tends to the boundary ∂W.

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Let W be a bounded domain in $\mathbf{C}^n$, and denote by $A(W)$ the space of continuous functions on $\bar{W}$ that are holomorphic on W. Equipped with the supremum norm and pointwise multiplication, $A(W)$ is a Banach algebra. Let us, for reasons of convenience, restrict our attention to domains W for which the maximal ideal space of $A(W)$ can be identified with $\bar{W}$, so as to avoid the Hartogs phenomenon and other difficulties. For a function $f \in A(W)$, let

$$Z(f) = \{z \in \bar{W}: f(z) = 0\}$$

be its zero set, and denote by $I(f)$ the closure of the principal ideal generated by f. For $E \subset \bar{W}$, introduce the notation

$$\mathscr{I}(E) = \{f \in A(W): f = 0 \text{ on } E\}.$$

Consider the following problem.

PROBLEM. Assume $f \in A(W)$ and $Z(f) \subset \partial W$. When does $I(f) = \mathscr{I}(Z(f))$?

By the Beurling–Rudin theorem [Hof, pp. 82–89], the answer to this

* This research was supported by the Swedish Natural Science Research Council (NFR) and the National Science Foundation (NSF).

0022-1236/88 $3.00

problem when $W = \mathbf{D}$, the open unit disc, is that $I(f) = \mathscr{I}(Z(f))$ if and only if $f \in A(\mathbf{D})$ is an outer function in the sense that

$$\log |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{i\theta})| \, d\theta. \tag{0.1}$$

Motivated by this, let us say that a function $f \in A(W)$ is BR-outer if $Z(f) \subset \partial W$ and $I(f) = \mathscr{I}(Z(f))$. For polydiscs $\mathbf{D}^n$, $n > 1$, the natural extension of (0.1) fails to characterize the BR-outer functions (see [Rud, pp. 70–78]); it is necessary, but unfortunately far from being sufficient. In Sections 2 and 3 of this paper, we present a method that, for some domains $W \subset \mathbf{C}^n$, including the bidisc $\mathbf{D}^2$ and the unit ball of $\mathbf{C}^n$, will show that a function $f \in A(W)$ with $Z(f) \subset \partial W$ is BR-outer if $f(z)$ does not decrease too fast as $z \to Z(f)$; our precise statement is contained in Section 3. Necessary conditions for $f \in A(W)$ to be BR-outer stem from the fact that $f \circ L$ must be outer for many analytic discs $L: \mathbf{D} \to \bar{W}$. In [Hed], the author showed, among other things, that a function $f \in A(\mathbf{D}^2)$ with $Z(f) = \{(1, 1)\}$ is BR-outer if and only if the functions $f(1, \cdot)$ and $f(\cdot, 1)$ are both outer.

1

Let p be a peaking function (see [Gam, p. 56]) in $A(W)$, peaking at the set $E \subset \partial W$. If $\varphi \in A(\mathbf{D})$ is an outer function with $Z(\varphi) = \{1\}$, then it is easy to see that $\varphi \circ p$ is BR-outer in $A(W)$. If $f \in A(W)$ has

$$|f(z)| \geqslant |\varphi(p(z))|, \qquad z \in W, \tag{1.1}$$

then $(1 - p^n)(\varphi \circ p)/f$ is in $A(W)$ for $n \geqslant 1$, and

$$((1 - p^n)(\varphi \circ p)/f) \cdot f = (1 - p^n) \cdot \varphi \circ p \to \varphi \circ p \qquad \text{as} \quad n \to \infty,$$

that is, $\varphi \circ p \in I(f)$. Hence f is BR-outer if $Z(f) = E$. This provides us with a fairly wide class of BR-outer functions in $A(W)$. In Section 3 we will show that condition (1.1) can be weakened substantially, to require only that

$$\log 1/|f(z)| = o(1/(1 - |p(z)|)) \qquad \text{as} \quad W \ni z \to E,$$

to ensure that f is BR-outer.

2

In this section, we shall describe a method to study the closed ideals in function algebras on bounded domains in $\mathbf{C}^n$, generalizing a well-known one-dimensional technique known as the Beurling–Carleman transform, used by many authors, for instance, I. M. Gelfand [Gel], T. Carleman [Car], A. Beurling in the proof of his famous invariant subspace theorem [Beu], B. Nyman [Nym], B. Korenblum [Kor], Y. Domar [Dom], V. P. Gurariĭ [Gur], and A. B. Aleksandrov [Ale]. It is hard to pinpoint precisely who invented this method; certainly, Gelfand's paper is the earliest reference known to the author.

Let $f \in A(W)$ satisfy $Z(f) \subset \partial W$. Pick a function $a \in A(W)$ such that $Z(f) \subset Z(a) \subset \partial W$; we wish to find conditions on f that will ensure that $a \in I(f)$. If the function a can be chosen so that it generates $\mathscr{I}(Z(f))$ after closure, then f is BR-outer if and only if $a \in I(f)$. This is the case if $Z(f)$ is a peak set with peaking function p (see [Gam, p. 56]), because then $a = 1 - p$ will generate $\mathscr{I}(Z(f))$, by, for instance, Lemma 3.1 in [GHM].

Pick an arbitrary functional $\phi \in I(f)^{\perp} = (A(W)/I(f))^*$, and consider the function

$$\Phi(\lambda) = \langle (\lambda - a + I(f))^{-1}, \phi \rangle,$$

which is well-defined and analytic for $\lambda \in \mathbf{C} \setminus \{0\}$, by general commutative Banach algebra theory. Here we used our assumption that the maximal ideal space of $A(W)$ is $\bar{W}$. Our plan is to gain some insight into whether $a \in I(f)$ by estimating the growth of $\Phi(\lambda)$ as $\lambda \to 0$. For $\lambda \notin K \equiv a(\bar{W})$,

$$(\lambda - a + I(f))^{-1} = (\lambda - a)^{-1} + I(f),$$

and so

$$|\Phi(\lambda)| \leqslant \frac{\|\phi\|}{d(\lambda, K)}, \qquad \lambda \in \mathbf{C} \setminus K, \tag{2.1}$$

where d is the Euclidean metric in $\mathbf{C}$. To estimate $\Phi(\lambda)$ on K, we need to find elements of the cosets $(\lambda - a + I(f))^{-1}$, $\lambda \in K \setminus \{0\}$. Let $\psi_\lambda \in C^1(K)$ be such that $0 \leqslant \psi_\lambda \leqslant 1$ on K, $\psi_\lambda(z) = 1$ near λ, and $\psi_\lambda(z) = 0$ near 0, to be specified in greater detail later in Section 3. Assume $a \in A^1(W)$, that is, that its partial derivatives of order one extend continuously to $\bar{W}$. Then the function

$$\chi_\lambda(z) \equiv \psi_\lambda(a(z)). \qquad z \in \bar{W},$$

satisfies the estimate $\|\bar{\partial}\chi_\lambda\| \leqslant \|\partial\psi_\lambda/\partial\bar{z}\|_{C(K)} \cdot \|\partial a\|$. Here we use the convenient notation

$$\partial\varphi = \sum_{j=1}^{n} (\partial\varphi/\partial z_j)\, dz_j$$

and

$$\bar{\partial}\varphi = \sum_{j=1}^{n} (\partial\varphi/\partial\bar{z}_j)\, d\bar{z}_j.$$

For function φ on W, $\|\varphi\|$ is the supremum norm of φ on W. The norm of a 1-form

$$\omega = \sum_{j=1}^{n} (\omega_j \, dz_j + \omega_j' \, d\bar{z}_j)$$

is

$$\|\omega\| = \max\{\|\omega_1\|, \|\omega_1'\|, ..., \|\omega_n\|, \|\omega_n'\|\}.$$

If φ_λ is a solution (in the sense of distributions) in $C(\bar{W})$ to the $\bar{\partial}$ equation

$$\bar{\partial}\varphi_\lambda = \frac{\bar{\partial}\chi_\lambda}{(\lambda - a) f}, \tag{2.2}$$

it is easy to check that the function

$$q_\lambda = \frac{1 - \chi_\lambda}{\lambda - a} + f \cdot \varphi_\lambda$$

is in $A(W)$ and that it is en element of the coset $(\lambda - a + I(f))^{-1}$; in fact

$$\bar{\partial} q_\lambda = -\bar{\partial}\chi_\lambda/(\lambda - a) + f \cdot \bar{\partial}\varphi_\lambda = 0$$

and

$$(\lambda - a) \, q_\lambda - 1 = f \cdot g_\lambda,$$

where g_λ is the $A(W)$ function

$$g_\lambda = -\chi_\lambda/f + (\lambda - a) \, \varphi_\lambda.$$

At this point we assume that W is such that the $\bar{\partial}$ equation $\bar{\partial}u = \omega$ has a solution $u \in C(\bar{W})$ with $\|u\| \leqslant C \, \|\omega\|$, C depending only on the domain W, whenever $\omega = \sum \omega_j \, d\bar{z}_j$ is a $\bar{\partial}$-closed (0, 1)-form with $\omega_j \in C(\bar{W})$, $j = 1, ..., n$. There is a sizeable literature on the $\bar{\partial}$ problem; see, for instance, [Ran], [HeC], and [Cha]. By [HeC, pp. 672, 676], the bidisc $\mathbf{D}^2$ and all strictly pseudoconvex bounded domains W have the above-mentioned property. Clearly, the right-hand side of (2.2) is a $\bar{\partial}$-closed (0, 1)-form, so then we can find $\varphi_\lambda \in C(\bar{W})$ with

$$\|\varphi_\lambda\| \leqslant C \left\| \frac{\bar{\partial}\chi_\lambda}{(\lambda - a) f} \right\| \leqslant C \cdot \|\bar{\partial}\chi_\lambda\| \cdot \|(\lambda - a)^{-1}\|_{L^\infty(\Omega(\lambda))} \cdot \|1/f\|_{L^\infty(\Omega(\lambda))},$$

where $\Omega(\lambda)$ denotes the support of $\bar{\partial}\chi_\lambda$. This together with the estimate

$$\|q_\lambda\| \leqslant \left\| \frac{1-x_\lambda}{\lambda-a} \right\| + \|f\| \cdot \|\varphi_\lambda\|$$

shows that

$$|\Phi(\lambda)| \leqslant \|\phi\| \cdot \left(\left\| \frac{1-\chi_\lambda}{\lambda-a} \right\| + C \cdot \|f\| \cdot \|\bar{\partial}\chi_\lambda\|_{L^\infty(\Omega(\lambda))} \right.$$

$$\left. \cdot \|1/(\lambda-a)\|_{L^\infty(\Omega(\lambda))} \cdot \|1/f\|_{L^\infty(\Omega(\lambda))} \right), \qquad \lambda \in \mathbf{C} \setminus \{0\}, \tag{2.3}$$

because $\Phi(\lambda) = \langle q_\lambda, \phi \rangle$. In certain cases (see Section 3), (2.1) and (2.3) together with the Phragmén–Lindelöf principle force Φ to have the form

$$\Phi(\lambda) = A/\lambda, \qquad \lambda \in \mathbf{C} \setminus \{0\},$$

for some constant A. If this is the case, and γ is a circle around the origin,

$$\langle a, \phi \rangle = \left\langle (2\pi i)^{-1} \int_\gamma \lambda(\lambda - a + I(f))^{-1} \, d\lambda, \phi \right\rangle = (2\pi i)^{-1} \int_\gamma \lambda \Phi(\lambda) \, d\lambda = 0,$$

and since $\phi \perp I(f)$ was arbitrary, we obtain $a \in I(f)$, which was our desired conclusion.

3

Let us concentrate on the special case when $Z(f)$ is a peak set and $a = 1 - p$, where p peaks at $Z(f)$. Then $K \subset \{z \in \mathbf{C}: |z-1| \leqslant 1\}$. Choose the function ψ_λ of Section 2 such that $\psi_\lambda(z) = 1$ when $|z - \lambda| \leqslant (1 - |1-\lambda|)/3$ and $\psi_\lambda(z) = 0$ when $|z-\lambda| \geqslant 2(1-|1-\lambda|)/3$; this can be done so that $\|\partial\psi_\lambda/\partial\bar{z}\|_{C(K)} \leqslant 5/(1-|1-\lambda|)$. We then get the estimates

$$\|\bar{\partial}\chi_\lambda\| \leqslant 5\, \|\partial a\|/(1-|1-\lambda|), \qquad |1-\lambda| < 1,$$

$$\left\| \frac{1-\chi_\lambda}{\lambda-a} \right\| \leqslant 3/(1-|1-\lambda|), \qquad |1-\lambda| < 1,$$

and

$$\|1/(\lambda-a)\|_{L^\infty(\Omega(\lambda))} \leqslant 3/(1-|1-\lambda|), \qquad |1-\lambda| < 1.$$

If

$$\log^+ \|1/f\|_{L^\infty(\Omega(\lambda))} = o(1/(1-|1-\lambda|)) \qquad \text{as} \quad \lambda \to 0, \tag{3.1}$$

the argument used in the proof of Theorem 2.1 in [Hed] shows that (2.1) and (2.3) do indeed force Φ to have the form

$$\Phi(\lambda) = A/\lambda, \qquad \lambda \in \mathbf{C} \setminus \{0\},$$

for some constant A depending on ϕ, and this for all $\phi \perp I(f)$, and so $a \in I(f)$. By Lemma 3.1 in [GHM], it follows that $I(f) = \mathscr{I}(Z(f))$, that is BR-outer. It is easy to see that (3.1) is equivalent to

$$\log 1/|f(z)| = o(1/(1-|p(z)|)) \qquad \text{as} \quad W \ni z \to Z(f).$$

Before we formulate this as a theorem, let us recall our assumptions. We assume that W is a bounded domain with the properties that the maximal ideal space of $A(W)$ is $\bar{W}$, and that the $\bar{\partial}$ problem $\bar{\partial} u = \omega$ has a solution $u \in C(\bar{W})$ with $\|u\| \leqslant C_W \cdot \|\omega\|$ whenever ω is a $\bar{\partial}$-closed (0, 1)-form with coefficients in $C(\bar{W})$. Moreover, we assume that there is a peaking function $p \in A(W)$ for the set $Z(f)$ whch is in the space $A^1(W) = C^1(\bar{W}) \cap A(W)$.

THEOREM. *Let $f \in A(W)$ have $Z(f) \subset \partial W$, and assume that there is a peaking function p as above. Then f is BR-outer if*

$$\log 1/|f(z)| = o(1/(1-|p(z)|)) \qquad \textit{as} \quad W \ni z \to Z(f).$$

Remarks. (a) The method developed in Sections 2 and 3 may produce different conditions for f to be BR-outer depending on the choice of the function a.

(b) It is not hard to show that if f is a BR-outer function in $A(\mathbf{D}^n)$, then f is a cyclic vector in the space $H^2(\mathbf{D}^n)$ with respect to multiplication by the coordinate functions $z_1, ..., z_n$, and f is an exterior function in $H^\infty(\mathbf{D}^n)$, in the sense of Rubel and Shields [RuS].

(c) The above theorem remains true if the condition that p is a peaking function for $Z(f)$ is relaxed to assuming only that $\|p\| = 1$ and that $p(z) = 1$ if and only if $z \in Z(f)$.

REFERENCES

[Ale] A. B. ALEKSANDROV, Invariant subspaces of the shift operator, an axiomatic approach, *J. Soviet Math.* **22** (1983), 1695–1708.

[Beu] A. BEURLING, On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81** (1949), 239–255.

[Car] T. CARLEMAN, L'intégrale de Fourier et questions qui s'y rattachent, Uppsala, Sweden, 1944.

[Cha] P. CHARPENTIER, Formules explicites pour les solutions minimales de l'équation $\bar{\partial} u = f$ dans la boule et dans le polydisque de $\mathbf{C}^n$, *Ann. Inst. Fourier (Grenoble)* **30**(4) (1980), 121–154.

[Dom] Y. Domar, On the analytic transform of bounded linear functionals on certain Banach algebras, *Studia Math.* **53** (1975), 203–224.

[Gam] T. W. Gamelin, "Uniform Algebras," Prentice–Hall, Englewood Cliffs, NJ, 1969; Chelsea, New York, 1984.

[Gel] I. M. Gelfand, Ideale und primäre Ideale in normierten Ringen, *Mat. Sbornik* **51**:9 (1941), 41–47.

[GHM] P. Gorkin, H. Hedenmalm, and R. Mortini, A Beurling–Rudin theorem for H^∞, *Illinois J. Math.* **31** (1987), 629–644.

[Gur] V. P. Gurariĭ, Harmonic analysis in spaces with a weight, *Trudy Moskov. Mat. Obschch.* **35** (1976), 21–76; English translation, *Trans. Moscow Math. Soc.* **35** (1979), 21–75.

[Hed] H. Hedenmalm, Outer functions in function algebras on the bidisc, *Trans. Amer. Math. Soc.*, to appear.

[HeC] G. M. Henkin and E. M. Čirka, Boundary properties of holomorphic functions of several complex variables, *J. Soviet Math.* **5** (1976), 612–687.

[Hof] K. Hoffman, "Banach Spaces of Analytic Functions," Prentice–Hall, Englewood Cliffs, NJ, 1962.

[Kor] B. I. Korenblum, Closed ideals in the ring A^n, *Funct. Anal. Appl.* **6** (1972), 203–214.

[Nym] B. Nyman, "On the one-Dimensional Translation Group and Semigroup in Certain Function Spaces," Thesis, Uppsala, Sweden, 1950.

[Ran] R. M. Range, Holomorphic functions and integral representations in several complex variables, *in* "Graduate Texts in Mathematics, Vol. 108, "Springer-Verlag, New York/Berlin/Heidelberg/Tokyo, 1986.

[RuS] L. A. Rubel and A. L. Shields, The failure of interior–exterior factorization in the polydisc and the ball, *Tôhoku Math. J.* **24** (3) (1972), 409–413.

[Rud] W. Rudin, "Function Theory in Polydisks," Benjamin, New York, 1969.