Outer Functions of Several Complex Variables

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We generalize to function algebras $A(W)$, $W \subset \mathbb{C}^n$, the familiar notion of outer functions on the unit disc. For a class of domains $W$, we show that a function $f(z)$ in $A(W)$ is generalized outer if it does not decrease too fast as $z$ tends to the boundary $\partial W$.

Let $W$ be a bounded domain in $\mathbb{C}^n$, and denote by $A(W)$ the space of continuous functions on $W$ that are holomorphic on $W$. Equipped with the supremum norm and pointwise multiplication, $A(W)$ is a Banach algebra. Let us, for reasons of convenience, restrict our attention to domains $W$ for which the maximal ideal space of $A(W)$ can be identified with $W$, so as to avoid the Hartogs phenomenon and other difficulties. For a function $f \in A(W)$, let

$$Z(f) = \{ z \in \overline{W} : f(z) = 0 \}$$

be its zero set, and denote by $I(f)$ the closure of the principal ideal generated by $f$. For $E \subset \overline{W}$, introduce the notation

$$\mathcal{J}(E) = \{ f \in A(W) : f = 0 \text{ on } E \}.$$ 

Consider the following problem.

**Problem.** Assume $f \in A(W)$ and $Z(f) \subset \partial W$. When does $I(f) = \mathcal{J}(Z(f))$?

By the Beurling–Rudin theorem [Hof, pp. 82–89], the answer to this

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problem when \( W = D \), the open unit disc, is that \( I(f) = \mathcal{J}(Z(f)) \) if and only if \( f \in A(D) \) is an outer function in the sense that

\[
\log |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{i\theta})| \, d\theta. \tag{0.1}
\]

Motivated by this, let us say that a function \( f \in A(W) \) is BR-outer if \( Z(f) \subset \partial W \) and \( I(f) = \mathcal{J}(Z(f)) \). For polydiscs \( D^n, n > 1 \), the natural extension of (0.1) fails to characterize the BR-outer functions (see [Rud, pp. 70–78]); it is necessary, but unfortunately far from being sufficient. In Sections 2 and 3 of this paper, we present a method that, for some domains \( W \subset \mathbb{C}^n \), including the bidisc \( D^2 \) and the unit ball of \( \mathbb{C}^n \), will show that a function \( f \in A(W) \) with \( Z(f) \subset \partial W \) is BR-outer if \( f(z) \) does not decrease too fast as \( z \to Z(f) \); our precise statement is contained in Section 3. Necessary conditions for \( f \in A(W) \) to be BR-outer stem from the fact that \( f \) must be outer for many analytic discs \( D + P \). In [Hed], the author showed, among other things, that a function \( f \in A(D^2) \) with \( Z(f) = \{(1, 1)\} \) is BR-outer if and only if the functions \( f(1, \cdot) \) and \( f(\cdot, 1) \) are both outer.

Let \( p \) be a peaking function (see [Gam, p. 56]) in \( A(W) \), peaking at the set \( E \subset \partial W \). If \( \varphi \in A(D) \) is an outer function with \( Z(\varphi) = \{1\} \), then it is easy to see that \( \varphi \circ p \) is BR-outer in \( A(W) \). If \( f \in A(W) \) has

\[
|f(z)| \geq |\varphi(p(z))|, \quad z \in W, \tag{1.1}
\]

then \((1 - p^n)(\varphi \circ p)f\) is in \( A(W) \) for \( n \geq 1 \), and

\[
((1 - p^n)(\varphi \circ p)f) \cdot f = (1 - p^n) \cdot \varphi \circ p - \varphi \circ p \quad \text{as} \quad n \to \infty,
\]

that is, \( \varphi \circ p \in I(f) \). Hence \( f \) is BR-outer if \( Z(f) = E \). This provides us with a fairly wide class of BR-outer functions in \( A(W) \). In Section 3 we will show that condition (1.1) can be weakened substantially, to require only that

\[
\log 1/|f(z)| = o(1/(1 - |p(z)|)) \quad \text{as} \quad W \ni z \to E,
\]

to ensure that \( f \) is BR-outer.

In this section, we shall describe a method to study the closed ideals in function algebras on bounded domains in \( \mathbb{C}^n \), generalizing a well-known
one-dimensional technique known as the Beurling–Carleman transform, used by many authors, for instance, I. M. Gelfand [Gel], T. Carleman [Car], A. Beurling in the proof of his famous invariant subspace theorem [Beu], B. Nyman [Nym], B. Korenblum [Kor], Y. Domar [Dom], V. P. Gurarii [Gur], and A. B. Aleksandrov [Ale]. It is hard to pinpoint precisely who invented this method; certainly, Gelfand's paper is the earliest reference known to the author.

Let $f \in A(W)$ satisfy $Z(f) \subset \partial W$. Pick a function $a \in A(W)$ such that $Z(f) = Z(a) \subset \partial W$; we wish to find conditions on $f$ that will ensure that $a \in I(f)$. If the function $a$ can be chosen so that it generates $\mathcal{S}(Z(f))$ after closure, then $f$ is BR-outer if and only if $a \in I(f)$. This is the case if $Z(f)$ is a peak set with peaking function $p$ (see [Gam, p. 56]), because then $a = 1 - p$ will generate $\mathcal{S}(Z(f))$, by, for instance, Lemma 3.1 in [GHM].

Pick an arbitrary functional $\phi \in I(f) = (A(W) / I(f))^*$, and consider the function

$$\Phi(\lambda) = \langle (\lambda - a + I(f))^{-1}, \phi \rangle,$$

which is well-defined and analytic for $\lambda \in \mathbb{C} \setminus \{0\}$, by general commutative Banach algebra theory. Here we used our assumption that the maximal ideal space of $A(W)$ is $\overline{W}$. Our plan is to gain some insight into whether $a \in I(f)$ by estimating the growth of $\Phi(\lambda)$ as $\lambda \rightarrow 0$. For $\lambda \notin K \equiv a(\overline{W})$,

$$(\lambda - a + I(f))^{-1} = (\lambda - a)^{-1} + I(f),$$

and so

$$|\Phi(\lambda)| \lesssim \frac{\|\phi\|}{d(\lambda, K)}, \quad \lambda \in \mathbb{C} \setminus K,$$

(2.1)

where $d$ is the Euclidean metric in $\mathbb{C}$. To estimate $\Phi(\lambda)$ on $K$, we need to find elements of the cosets $(\lambda - a + I(f))^{-1}$, $\lambda \in K \setminus \{0\}$. Let $\psi_\lambda \in C^1(K)$ be such that $0 \leq \psi_\lambda \leq 1$ on $K$, $\psi_\lambda(z) = 1$ near $\lambda$, and $\psi_\lambda(z) = 0$ near 0, to be specified in greater detail later in Section 3. Assume $a \in A^1(W)$, that is, that its partial derivatives of order one extend continuously to $\overline{W}$. Then the function

$$\chi_\lambda(z) \equiv \psi_\lambda(a(z)), \quad z \in \overline{W},$$

satisfies the estimate $\|\partial \chi_\lambda\| \lesssim \|\partial \psi_\lambda / \partial z\|_{C(K)} \cdot \|\partial a\|$. Here we use the convenient notation

$$\partial \varphi = \sum_{j=1}^n (\partial \varphi / \partial z_j) \, dz_j$$

and

$$\bar{\partial} \varphi = \sum_{j=1}^n (\partial \varphi / \partial \bar{z}_j) \, d\bar{z}_j.$$
For function $\varphi$ on $W$, $\|\varphi\|$ is the supremum norm of $\varphi$ on $W$. The norm of a 1-form

$$\omega = \sum_{j=1}^{n} (\omega_j \, dz_j + \omega_j' \, d\bar{z}_j)$$

is

$$\|\omega\| = \max\{\|\omega_1\|, \|\omega_1'\|, \ldots, \|\omega_n\|, \|\omega_n'\|\}.$$

If $\varphi_\lambda$ is a solution (in the sense of distributions) in $C(\overline{W})$ to the equation

$$\bar{\partial} \varphi_\lambda = \frac{\bar{\partial} \chi_\lambda}{(\lambda - a) f}, \quad (2.2)$$

it is easy to check that the function

$$q_\lambda = \frac{1 - \chi_\lambda}{\lambda - a} + f \cdot \varphi_\lambda$$

is in $A(W)$ and that it is an element of the coset $(\lambda - a + I(f))^{-1}$; in fact

$$\bar{\partial} q_\lambda = -\bar{\partial} \chi_\lambda/(\lambda - a) + f \cdot \bar{\partial} \varphi_\lambda = 0$$

and

$$(\lambda - a) q_\lambda - 1 = f \cdot g_\lambda,$$

where $g_\lambda$ is the $A(W)$ function

$$g_\lambda = -\chi_\lambda/f + (\lambda - a) \varphi_\lambda.$$

At this point we assume that $W$ is such that the $\bar{\partial}$ equation $\bar{\partial} u = \omega$ has a solution $u \in C(\overline{W})$ with $\|u\| \leq C \|\omega\|$, $C$ depending only on the domain $W$, whenever $\omega = \sum \omega_j \, d\bar{z}_j$ is a $\bar{\partial}$-closed $(0, 1)$-form with $\omega_j \in C(\overline{W})$, $j = 1, \ldots, n$. There is a sizeable literature on the $\bar{\partial}$ problem; see, for instance, [Ran], [HeC], and [Cha]. By [HeC, pp. 672, 676], the bidisc $D^2$ and all strictly pseudoconvex bounded domains $W$ have the above-mentioned property. Clearly, the right-hand side of (2.2) is a $\bar{\partial}$-closed $(0, 1)$-form, so then we can find $\varphi_\lambda \in C(\overline{W})$ with

$$\|\varphi_\lambda\| \leq C \frac{\|\bar{\partial} \chi_\lambda\|}{(\lambda - a) f} \leq C \cdot \|\bar{\partial} \chi_\lambda\| \cdot \|(\lambda - a)^{-1}\| \cdot \|1/f\|_{L^\infty(\Omega(\lambda))} \cdot \|1/f\|_{L^\infty(\Omega(\lambda))},$$
where $\Omega(\lambda)$ denotes the support of $\tilde{\chi}_{\lambda}$. This together with the estimate

$$
\|q_{\lambda}\| \leq \left|\frac{1 - \chi_{\lambda}}{\lambda - a}\right| + \|f\| \cdot \|\phi_{\lambda}\|
$$

shows that

$$
|\Phi(\lambda)| \leq \|\phi\| \cdot \left(\left|\frac{1 - \chi_{\lambda}}{\lambda - a}\right| + C \cdot \|f\| \cdot \|\tilde{\chi}_{\lambda}\|_{L^\infty(\Omega(\lambda))}
\right)
\cdot \|1/(\lambda - a)\|_{L^\infty(\Omega(\lambda))} \cdot \|1/f\|_{L^\infty(\Omega(\lambda))}
\), \quad \lambda \in C \setminus \{0\}, (2.3)
$$

because $\Phi(\lambda) = \langle q_{\lambda}, \phi \rangle$. In certain cases (see Section 3), (2.1) and (2.3) together with the Phragmén–Lindelöf principle force $\Phi$ to have the form

$$
\Phi(\lambda) = A/\lambda, \quad \lambda \in C \setminus \{0\},
$$

for some constant $A$. If this is the case, and $\gamma$ is a circle around the origin,

$$
\langle a, \phi \rangle = \langle (2\pi i)^{-1} \int_{\gamma} \lambda(\lambda - a + I(\lambda))^{-1} d\lambda, \phi \rangle = (2\pi i)^{-1} \int_{\gamma} \lambda \Phi(\lambda) d\lambda = 0,
$$

and since $\phi \perp I(\lambda)$ was arbitrary, we obtain $a \in I(\lambda)$, which was our desired conclusion.

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Let us concentrate on the special case when $Z(f)$ is a peak set and $a = 1 - p$, where $p$ peaks at $Z(f)$. Then $K \subset \{z \in C: |z - 1| \leq 1\}$. Choose the function $\psi_{\lambda}$ of Section 2 such that $\psi_{\lambda}(z) = 1$ when $|z - \lambda| \leq (1 - |1 - \lambda|)/3$ and $\psi_{\lambda}(z) = 0$ when $|z - \lambda| \geq 2(1 - |1 - \lambda|)/3$; this can be done so that $\|\partial \psi_{\lambda}/\partial \bar{z}\|_{C(K)} \leq 5/(1 - |1 - \lambda|)$. We then get the estimates

$$
\|\tilde{\chi}_{\lambda}\| \leq 5 \|\partial a\|/(1 - |1 - \lambda|), \quad |1 - \lambda| < 1,
$$

$$
\left|\frac{1 - \chi_{\lambda}}{\lambda - a}\right| \leq 3/(1 - |1 - \lambda|), \quad |1 - \lambda| < 1,
$$

and

$$
\|1/(\lambda - a)\|_{L^\infty(\Omega(\lambda))} \leq 3/(1 - |1 - \lambda|), \quad |1 - \lambda| < 1.
$$

If

$$
\log^+ \|1/f\|_{L^\infty(\Omega(\lambda))} = o(1/(1 - |1 - \lambda|)) \quad \text{as} \quad \lambda \to 0, \quad (3.1)
$$
the argument used in the proof of Theorem 2.1 in [Hed] shows that (2.1) and (2.3) do indeed force $\Phi$ to have the form

$$\Phi(\lambda) = A/\lambda, \quad \lambda \in \mathbb{C} \setminus \{0\},$$

for some constant $A$ depending on $a$, and this for all $\phi \perp I(f)$, and so $a \in I(f)$. By Lemma 3.1 in [GHM], it follows that $I(f) = \mathcal{F}(Z(f))$, that is BR-outer. It is easy to see that (3.1) is equivalent to

$$\log 1/|f(z)| = o(1/(1 - |p(z)|)) \quad \text{as} \quad W \ni z \to Z(f).$$

Before we formulate this as a theorem, let us recall our assumptions. We assume that $W$ is a bounded domain with the properties that the maximal ideal space of $A(W)$ is $\overline{W}$, and that the $\overline{\partial}$ problem $\overline{\partial}u = \omega$ has a solution $u \in C(\overline{W})$ with \( \|u\| \leq C \|\omega\| \) whenever $\omega$ is a $\overline{\partial}$-closed $(0, 1)$-form with coefficients in $C(\overline{W})$. Moreover, we assume that there is a peaking function $p \in A(W)$ for the set $Z(f)$ which is in the space $A^1(W) = C^1(\overline{W}) \cap A(W)$.

**Theorem.** Let $f \in A(W)$ have $Z(f) \subset \partial W$, and assume that there is a peaking function $p$ as above. Then $f$ is BR-outer if

$$\log 1/|f(z)| = o(1/(1 - |p(z)|)) \quad \text{as} \quad W \ni z \to Z(f).$$

**Remarks.** (a) The method developed in Sections 2 and 3 may produce different conditions for $f$ to be BR-outer depending on the choice of the function $a$.

(b) It is not hard to show that if $f$ is a BR-outer function in $A(D^n)$, then $f$ is a cyclic vector in the space $H^2(D^n)$ with respect to multiplication by the coordinate functions $z_1, \ldots, z_n$, and $f$ is an exterior function in $H^\infty(D^n)$, in the sense of Rubel and Shields [RuS].

(c) The above theorem remains true if the condition that $p$ is a peaking function for $Z(f)$ is relaxed to assuming only that $\|p\| = 1$ and that $p(z) = 1$ if and only if $z \in Z(f)$.

**References**


[Car] T. Carleman, L’intégrale de Fourier et questions qui s’y rattachent, Uppsala, Sweden, 1944.

OUTER FUNCTIONS


