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A COMPUTATION OF GREEN FUNCTIONS FOR THE WEIGHTED BIHARMONIC OPERATORS $\Delta|z|^{-2\alpha}\Delta$, WITH $\alpha > -1$

PER JAN HÅKAN HEDENMALM

0. Introduction. Let Ω be a simply connected bounded domain in the complex plane $\mathbb{C}$, with smooth boundary $\partial\Omega$. The Green function for the Laplace operator Δ is the solution $\Gamma(z, \zeta; \Omega)$, with parameter values $z, \zeta \in \bar{\Omega}$, to the Poisson equation

$$\begin{cases} \Delta\Gamma(\cdot, \zeta; \Omega) = \delta_\zeta & \text{on } \Omega, \\ \Gamma(z, \zeta; \Omega) = 0, & z \in \partial\Omega, \end{cases}$$

where δ_ζ denotes the unit point mass at the interior point $\zeta \in \Omega$; it is well known and easily checked that this function is symmetric in its arguments: $\Gamma(z, \zeta; \Omega) = \Gamma(\zeta, z; \Omega)$. A fundamental fact in the potential theory of the region Ω is the fact that the Green function has constant sign: $\Gamma(z, \zeta; \Omega) < 0$ on $\Omega^2 = \Omega \times \Omega$. This has the physical interpretation that a membrane always follows the direction of the force, no matter where it is applied. In his 1908 memorial [9, pages 541–543], [10, pages 1298–1299], Jacques Hadamard mentions a conjecture, which he ascribes to Tommaso Boggio [4], stating that the Green function $U(z, \zeta; \Omega)$ for the squared operator Δ^2, which solves

$$\begin{cases} \Delta^2 U(\cdot, \zeta; \Omega) = \delta_\zeta & \text{on } \Omega, \\ U(z, \zeta; \Omega) = 0, & z \in \partial\Omega, \\ \nabla_z U(z, \zeta; \Omega) = 0, & z \in \partial\Omega, \end{cases}$$

where ∇_z denotes the gradient taken with respect to the z variable, should also have constant sign, in this case positive, throughout Ω^2. Hadamard also adds the comment that he considers this very likely for convex regions Ω. That it is so if Ω is the unit disk $\mathbb{D}$ was well known before Hadamard wrote his paper, although I do not really know who first noticed this fact. Still, I should like to point out the 1901 papers [4], [19] by Boggio and John Henry Michell as possible sources. It deserves to be mentioned that there is an 1862 book on elasticity theory by Alfred Clebsch, and papers very close to those of Boggio and Michell by Emilio Almansi (1896) and Giuseppe Lauricella (1896). The solution for the disk has the explicit

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form

$$U(z,\zeta;\mathbb{D}) = |z-\zeta|^2\Gamma(z,\zeta;\mathbb{D}) + (1-|z|^2)(1-|\zeta|^2), \qquad (z,\zeta)\in\mathbb{D}^2,$$

where

$$\Gamma(z,\zeta;\mathbb{D}) = \log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2, \qquad (z,\zeta)\in\mathbb{D}^2,$$

and in Section 2, we verify, for the sake of completeness, the positivity of the function $U(z,\zeta;\mathbb{D})$. In an appendix to the 1908 paper, Hadamard computes the Green function for the Pascal limaçon D_α [10, page 637], which is the image of the unit disk $\mathbb{D}$ under the conformal mapping

$$\varphi_\alpha(z) = z + \alpha z^2,$$

for parameter values $|\alpha| < 1/2$, and claimed that it is positive for such α [10, page 1299]. The formula he obtains is, in the coordinates of the disk,

$$U(\varphi_\alpha(z),\varphi_\alpha(\zeta);D_\alpha) = \left|\frac{\varphi_\alpha(z)-\varphi_\alpha(\zeta)}{z-\zeta}\right|^2 U(z,\zeta;\mathbb{D})$$
$$- \frac{|\alpha|^4}{1-2|\alpha|^2}(1-|z|^2)^2(1-|\zeta|^2)^2, \qquad (z,\zeta)\in\mathbb{D}^2.$$

Hadamard refers to an earlier paper by Almansi [1], where this formula, or at least the solution to the related Dirichlet problem, was originally obtained.

Replacing the operator Δ by its square Δ^2 has the physical interpretation that we replace the membrane by an (infinitesimally) thin elastic plate, spread over Ω, and clamped at $\partial\Omega$ (see [5, pages 250–252], [2, pages 232–239]), and the issue that Boggio and Hadamard were concerned with is whether the deflection, resulting from a downward point load at an arbitrary point in Ω, is always directed downward with the load. This Boggio-Hadamard conjecture, as I have decided to call it, was later disproved by Richard Duffin [6] for an infinite strip, by Charles Loewner [18] and Gabor Szegö [21] for certain nonconvex regions, and by Paul Garabedian [8] for a sufficiently eccentric ellipse. Garabedian's example is interesting, and deserves some attention. He takes the ellipse

$$x^2/a^2 + y^2 < 1,$$

with $a > 1$, and places a point load near the boundary point $(x, y) = (a, 0)$. His computations then show, more or less, that there exist two absolute constants a_0 and a_1, with $1 < a_0 < 2 < a_1$, such that, for a_0, $1 < a < a_0$, the deflection of the plate near the boundary point $(x, y) = (-a, 0)$ is in the same direction as the point load, but for parameter values $a_0 < a < a_1$, it goes the other way. Torbjörn Lundh has assisted me with the computation of a_0 and a_1; they have the approximate values $a_0 \approx 1.5933$ and $a_1 \approx 2.4716$. Garabedian also calculates the deflection near the boundary point (0, 1), keeping the same point load as before, and shows that a switch of direction occurs, but for a higher critical value of the parameter a. I believe the situation first described is the critical one for the ellipse, and that in fact, the Boggio-Hardamard conjecture does hold for ellipses of the above type, provided that $1 < a < a_0$. It should be mentioned that there is a much earlier paper by Boggio [3] dealing with the Dirichlet problem for Δ^2 on ellipses.

In later work associated with the Boggio-Hadamard conjecture, the focus seems to have been on finding counterexamples, which is exemplified by the more recent papers by Mitsuru Nakai and Leo Sario [20], and by Vladimir Kozlov, Vladimir Kondrat'ev, and Vladimir Maz'ya [17]. In [11], however, Walter Hayman and Boris Korenblum show that the Green function for $(-\Delta)^m$ on the unit ball of $\mathbb{R}^n$, with Dirichlet data, is always positive.

In this paper we shall compute explicitly the Green functions for the singularly weighted biharmonic operators $\Delta|z|^{-2n}\Delta$ on the unit disk $\mathbb{D}$; this corresponds to obtaining the Green function for Δ^2 on the Riemann surface which is the image of $\mathbb{D}$ under the mapping $z \mapsto z^{n+1}$. We shall in fact study the Green function $U_\alpha(z, \zeta)$ for the weighted biharmonic operators $\Delta|z|^{-2\alpha}\Delta$ on the unit disk $\mathbb{D}$, for parameter values $\alpha > -1$, and consider integers $\alpha = n$ a special case. In Section 3, we discuss the domain of definition of the partial differential operator $\Delta|z|^{-2\alpha}\Delta$, and the uniqueness of the associated Dirichlet problem. In Section 4, we carry out the explicit computation of the functions $U_n(z, \zeta)$ for $n = 0, 1, 2, 3, \ldots$, and obtain an integral representation for the more general $U_\alpha(z, \zeta)$. The formulas obtained may be useful for calculating other Green functions, such as those associated with the operators $\Delta(1 - |z|^2)^{-\alpha}\Delta$. In Section 5, it is demonstrated that the functions $U_\alpha(z, \zeta)$ may be expressed in terms of a certain function $E_\infty(z, \zeta)$. The function $E_\infty(z, \zeta)$ is shown to be positive, which, by the nature of the relationship between $U_\alpha(z, \zeta)$ and $E_\infty(z, \zeta)$ immediately leads to the positivity of $U_\alpha(z, \zeta)$.

As before, let Ω be a simply connected bounded domain in the complex plane, but this time it should be star-shaped and have real analytic boundary. In Section 6, an ingenious idea due to Hadamard [9], [10] is exploited to show that a clamped plate the shape of Ω bends (everywhere) in the direction of a point load, no matter where applied, once this is true at application points "infinitesimally" close to the boundary $\partial\Omega$. The argument is sufficiently flexible to apply to weighted problems as well, and thus leads to a different proof of the positivity of $U_\alpha(z, \zeta)$. It is possible to generalize the method to higher dimensions and higher-order elliptic partial differential operators. This then yields a simpler proof of the positivity of the Green function for the operator $(-\Delta)^m$ on the unit ball of $\mathbb{R}^n$ than the one produced by Hayman and Korenblum [11].

The positivity of $U_\alpha(z, \zeta)$ is applied in Section 7 to produce new factoring theorems for the Bergman L^p spaces on the disk (see [7], [12], [13], [14], [15] for related developments).

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1. Some words on notation. If we write $z = x + iy$, with x, y real, and $i^2 = -1$, we define the partial differential operators

$$\partial_z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y),$$

$$\bar{\partial}_z = \frac{1}{2}(\partial/\partial x + i\partial/\partial y),$$

$$\Delta_z = \partial_z\bar{\partial}_z = \frac{1}{4}(\partial^2/\partial x^2 + \partial^2/\partial y^2);$$

we shall frequently omit the subscript z if in the context there can be no misunderstanding regarding what variable we are differentiating with respect to. The definition of the Laplace operator Δ is somewhat nonstandard; it usually is defined to be 4 times bigger than here. The advantage with our notation is that if f is a holomorphic function, we get the beautiful identity $\Delta|f|^2 = |f'|^2$. Another nonstandard feature in this paper is that given a domain Ω in the complex plane $\mathbb{C}$, we consider locally integrable (dm_2) functions u on Ω as distributions via the duality relation

$$\langle \phi, u \rangle = \int_\Omega u(z)\phi(z)dm_2(z)/\pi, \qquad \phi \in C_0^\infty(\Omega),$$

where dm_2 denotes the usual planar area measure. This, together with the unusual definition of the Laplacian, makes our Green functions for Δ and Δ^2 bigger than usual, by the factors 4π and 16π, respectively. We frequently write dS for dm_2/π.

In this paper, we shall use the symbols Re and Im to indicate the operations of taking real and imaginary parts of a complex number.

2. The Green functions $\Gamma(z, \zeta)$ and $U(z, \zeta)$ for the operators Δ and Δ^2. It is well known that the Green function for the Laplacian Δ on the unit disk $\mathbb{D}$ is given by the formula

$$\Gamma(z, \zeta) = \log\left|\frac{z - \zeta}{1 - \bar{\zeta}z}\right|^2, \qquad (z, \zeta) \in \mathbb{D}^2.$$

It is also known, albeit not so well, that the Green function for the bi-Laplacian

Δ^2 on the unit disk $\mathbb{D}$ has the form

$$U(z,\zeta) = |z-\zeta|^2\Gamma(z,\zeta) + (1-|z|^2)(1-|\zeta|^2), \qquad (z,\zeta) \in \mathbb{D}^2.$$

As indicated above, we frequently suppress the $\mathbb{D}$'s in the expressions $\Gamma(z, \zeta; \mathbb{D})$ and $U(z, \zeta; \mathbb{D})$ when it is clear that we are dealing with the unit disk. One can show that $U(z, \zeta) > 0$ for all $(z, \zeta) \in \mathbb{D}^2$. In fact, by the elementary inequality

$$\log x > 1 - 1/x, \qquad 0 < x < 1,$$

we have

$$\Gamma(z,\zeta) > -\frac{(1-|z|^2)(1-|\zeta|^2)}{|z-\zeta|^2}, \qquad (z,\zeta) \in \mathbb{D}^2, z \neq \zeta,$$

from which the assertion $U(z, \zeta) > 0$ immediately follows for $(z, \zeta) \in \mathbb{D}^2$.

3. Domain of definition for $\Delta|z|^{-2\alpha}\Delta$; uniqueness of the Dirichlet problem. We have to clarify upon which classes of functions the singularly weighted partial differential operator $\Delta|z|^{-2\alpha}\Delta$ is well defined, and in what sense we expect the boundary conditions to determine the function $U_\alpha(z, \zeta)$ uniquely. Since the function $|z|^{-2\alpha}$ is C^∞ on $\mathbb{D}\backslash\{0\}$, we can define the operator $\Delta|z|^{-2\alpha}\Delta$ unambiguously as an operator acting on the space $\mathscr{D}'(\mathbb{D}\backslash\{0\})$ of distributions on $\mathbb{D}\backslash\{0\}$. We have trouble defining the operator on the space $\mathscr{D}'(\mathbb{D})$ of distributions on $\mathbb{D}$, because it may be impossible to make sense out of the product of the function $|z|^{-2\alpha}$ with an arbitrary distribution on $\mathbb{D}$. Some distributions, however, coincide with $L^1(dS)$ functions near 0; denote by $\mathscr{D}'_0(\mathbb{D})$ the space of such distributions. If $u \in \mathscr{D}'_0(\mathbb{D})$ is such that Δu belongs to $\mathscr{D}'_0(\mathbb{D})$ too, we may safely multiply Δu by $|z|^{-2\alpha}$ to get a distribution on $\mathbb{D}\backslash\{0\}$ and a measurable function near 0; if, moreover, this measurable function belongs to $L^1(dS)$, then $|z|^{-2\alpha}\Delta u$ can be regarded as an element of $\mathscr{D}'_0(\mathbb{D})$, and we have no difficulty applying another Laplacian. We consider the space of such distributions u the domain of definition for the operator $\Delta|z|^{-2\alpha}\Delta$, and denote it by $\mathscr{D}'_{0,\alpha}(\mathbb{D})$.

We should now like to say a word or two about uniqueness of the Dirichlet problem. To this end, the following Almansi-type representation formula shall prove useful.

LEMMA 3.1 ($-1 < \alpha$). *Suppose $u \in \mathscr{D}'_{0,\alpha}(\mathbb{D})$. Then u has $\Delta|z|^{-2\alpha}\Delta u = 0$ on $\mathbb{D}$ if and only if u is of the form*

$$u(z) = v(z) + (1-|z|^{2\alpha+2})w(z), \qquad z \in \mathbb{D},$$

where v and w are harmonic on $\mathbb{D}$. Moreover, if u is given by the above expression, then

$$\Delta u(z) = -(\alpha+1)|z|^{2\alpha}\big((\alpha+1)w(z) + z\partial w(z) + \bar{z}\bar{\partial} w(z)\big), \qquad z \in \mathbb{D}\backslash\{0\}.$$

Proof. We first check that if u is of the indicated form, then it solves the differential equation. A computation reveals that Δu is given by the above formula, so that $|z|^{-2\alpha}\Delta u$ is in $\mathscr{D}_0(\mathbb{D})$, and

$$\Delta|z|^{-2\alpha}\Delta u(z) = -(\alpha+1)\Delta((\alpha+1)w(z) + z\partial w(z) + \bar{z}\bar{\partial}w(z)) = 0, \qquad z \in \mathbb{D}.$$

To see that the middle expression equals 0, we argue as follows. The function w is harmonic, thus ∂w is holomorphic, and $z\partial w$ is holomorphic, too. The conclusion that $z\partial w(z) + \bar{z}\bar{\partial}w(z)$ is harmonic is immediate.

We proceed to check that any solution u to $\Delta|z|^{-2\alpha}\Delta u = 0$ has the prescribed form. One obtains $|z|^{-2\alpha}\Delta u = h$, where h is harmonic, that is, $\Delta u(z) = |z|^{2\alpha}h(z)$. It is convenient to write $h = f + \bar{g}$, where f and g are holomorphic, and $g(0) = 0$. These functions have power series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad z \in \mathbb{D},$$

$$g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad z \in \mathbb{D};$$

associate with them the holomorphic functions

$$F_\alpha(z) = -\sum_{n=0}^{\infty} \frac{a_n z^n}{(\alpha+1)(\alpha+n+1)}, \qquad z \in \mathbb{D},$$

$$G_\alpha(z) = -\sum_{n=1}^{\infty} \frac{b_n z^n}{(\alpha+1)(\alpha+n+1)}, \qquad z \in \mathbb{D}.$$

The function

$$H_\alpha(z) = (1 - |z|^{2\alpha+2})(F_\alpha(z) + \bar{G}_\alpha(z)), \qquad z \in \mathbb{D},$$

then solves $\Delta H_\alpha(z) = |z|^{2\alpha}h(z)$, and therefore $u - H_\alpha$ is harmonic. The proof is complete. □

The uniqueness result for the Dirichlet problem is as follows.

LEMMA 3.2 ($-1 < \alpha$). *Suppose $u \in \mathscr{D}'_{0,\alpha}(\mathbb{D})$ solves $\Delta|z|^{-2\alpha}\Delta u = 0$ on $\mathbb{D}$, and that u is of class C^1 on $\bar{\mathbb{D}}\backslash\{0\}$. If both u and ∇u vanish on the boundary $\mathbb{T}$, then $u(z) \equiv 0$ throughout $\mathbb{D}$.*

Proof. By Lemma 3.1, u has the form

$$u(z) = v(z) + (1 - |z|^{2\alpha+2})w(z), \qquad z \in \mathbb{D},$$

where v and w are harmonic on $\mathbb{D}$. By radial symmetry, the functions

$$u_n(z, e^{i\theta}) = e^{-in\theta}u(e^{i\theta}z), \qquad z \in \mathbb{D},$$

$$v_n(z, e^{i\theta}) = e^{-in\theta}v(e^{i\theta}z), \qquad z \in \mathbb{D},$$

$$w_n(z, e^{i\theta}) = e^{-in\theta}w(e^{i\theta}z), \qquad z \in \mathbb{D},$$

retain the properties of u, v, and w, for integers $n \in \mathbb{Z}$, and real θ, and the same can be said for their averages over θ,

$$\hat{u}_n(z) = \int_{-\pi}^{\pi} u_n(z, e^{i\theta})\, d\theta/2\pi, \qquad z \in \mathbb{D},$$

$$\hat{v}_n(z) = \int_{-\pi}^{\pi} v_n(z, e^{i\theta})\, d\theta/2\pi, \qquad z \in \mathbb{D},$$

$$\hat{w}_n(z) = \int_{-\pi}^{\pi} w_n(z, e^{i\theta})\, d\theta/2\pi, \qquad z \in \mathbb{D}.$$

These averages satisfy $\hat{u}_n(\gamma z) = \gamma^n \hat{u}_n(z)$, $\hat{v}_n(\gamma z) = \gamma^n \hat{v}_n(z)$, $\hat{w}_n(\gamma z) = \gamma^n \hat{w}_n(z)$, for $|\gamma| = 1$ so that, being harmonic, the functions $\hat{v}_n(z)$ and $\hat{w}_n(z)$ are of the form constant $\times$ z^n for $n = 0, 1, 2, \ldots$, and of the form constant $\times$ $\bar{z}^{-n}$ for $n = -1, -2, -3, \ldots$. The relation between u, v, and w translates to $\hat{u}_n(z) = \hat{v}_n(z) + (1 - |z|^{2\alpha+2})\hat{w}_n(z)$, and the vanishing of u along with its gradient on $\mathbb{T}$ entails that the same holds for $\hat{u}_n$. Hence the functions $\hat{v}_n$ and $\hat{w}_n$ both vanish identically. But then v and w must vanish identically, too, because $\hat{v}_n$ and $\hat{w}_n$ may be interpreted as their Fourier coefficients. The conclusion follows. □

4. Definition of the Green function $U_\alpha(z, \zeta)$ for the operator $\Delta|z|^{-2\alpha}\Delta$. Fix the real parameter $\alpha > -1$. The Green function $U_\alpha(z, \zeta)$ solves, for a fixed $\zeta \in \mathbb{D}$, the partial differential equation boundary value problem

$$\begin{cases} \Delta_z |z|^{-2\alpha} \Delta_z U_\alpha(z, \zeta) = \delta_\zeta(z), & z \in \mathbb{D}, \\ U_\alpha(z, \zeta) = 0, & z \in \mathbb{T}, \\ \nabla_z U_\alpha(z, \zeta) = 0, & z \in \mathbb{T}. \end{cases}$$

The singular partial differential operator $\Delta|z|^{-2\alpha}\Delta$ was defined and discussed in the previous section. We understand as implicit in the boundary condition that $U_\alpha(\cdot, \zeta)$ should extend as a C^1 function up to the boundary $\mathbb{T}$. This determines the Green function $U_\alpha(z, \zeta)$ uniquely, as we saw in Section 3. If $U_\alpha(z, \zeta)$ solves the above partial differential equation on $\mathbb{D}$, then clearly

$$|z|^{-2\alpha}\Delta_z U_\alpha(z,\zeta) = \Gamma(z,\zeta) + H_\alpha(z,\zeta), \qquad (z,\zeta) \in \mathbb{D}^2,$$

that is,

$$\Delta_z U_\alpha(z,\zeta) = |z|^{2\alpha}(\Gamma(z,\zeta) + H_\alpha(z,\zeta)), \qquad (z,\zeta) \in \mathbb{D}^2, \tag{4.1}$$

for some harmonic function $H_\alpha(\cdot,\zeta)$ on $\mathbb{D}$. As a consequence, we see that $U_\alpha(\cdot,\zeta)$ is smooth (in fact, real analytic) on $\mathbb{D}\setminus\{0,\zeta\}$.

By general ellipticity theory, we can expect the Green function $U_\alpha(\cdot,\zeta)$ to extend real analytically across the boundary $\mathbb{T}$, if $\zeta \in \mathbb{D}$ is kept fixed. In particular, it is not a serious restriction to decide to look only for a function $H_\alpha(\cdot,\zeta)$ satisfying the regularity condition that it extends continuously up to the boundary $\mathbb{T}$. By Green's theorem and the boundary conditions on $U_\alpha(z,\zeta)$, we have, again for fixed $\zeta \in \mathbb{D}$, the identity

$$\int_{\mathbb{D}} \Delta_z U_\alpha(z,\zeta)\varphi(z)\,dS(z) = \int_{\mathbb{D}} U_\alpha(z,\zeta)\Delta_z\varphi(z)\,dS(z), \tag{4.2}$$

valid for C^2 functions φ on $\overline{\mathbb{D}}$. If we specialize to harmonic φ, we get

$$\int_{\mathbb{D}} \Delta_z U_\alpha(z,\zeta)\varphi(z)\,dS(z) = 0,$$

and consequently,

$$\int_{\mathbb{D}} H_\alpha(z,\zeta)\varphi(z)|z|^{2\alpha}\,dS(z) = -\int_{\mathbb{D}} \Gamma(z,\zeta)\varphi(z)|z|^{2\alpha}\,dS(z). \tag{4.3}$$

It deserves to be pointed out that if $H_\alpha(z,\zeta)$ is any nice harmonic function on $\mathbb{D}$ in the z variable, and we simply define $U_\alpha(z,\zeta)$ as the solution to the Poisson equation (4.1) with zero boundary values, then (4.3) is equivalent to requiring $\nabla_z U_\alpha(z,\zeta) = 0$ on $\mathbb{T}$. If we apply the operator Δ_ζ to both sides of equation (4.3), we obtain

$$\int_{\mathbb{D}} \Delta_\zeta H_\alpha(z,\zeta)\varphi(z)|z|^{2\alpha}\,dS(z) = -\varphi(\zeta)|\zeta|^{2\alpha}, \tag{4.4}$$

for all harmonic φ that extend to C^2 functions on $\overline{\mathbb{D}}$. An approximation argument shows that the above relation holds for all $\varphi \in L_h^2(\mathbb{D},\alpha)$, where $L_h^2(\mathbb{D},\alpha)$ is the Hilbert space of all harmonic square integrable functions on $\mathbb{D}$, supplied with the norm

$$\left(\int_{\mathbb{D}} |\varphi(z)|^2 |z|^{2\alpha}\, dS(z)\right)^{1/2}.$$

We can interpret (4.4) as saying that $\Delta_\zeta H_\alpha(z, \zeta)$ equals $-|\zeta|^{2\alpha}$ times the harmonic reproducing kernel function for the space $L_h^2(\mathbb{D}, \alpha)$. Relation (4.3) determines the function $H_\alpha(\cdot, \zeta)$ uniquely; in fact, we see that $H_\alpha(\cdot, \zeta)$ coincides with the orthogonal projection $L^2(\mathbb{D}, \alpha) \to L_h^2(\mathbb{D}, \alpha)$ of the function $\Gamma(\cdot, \zeta)$. Here, of course, $L^2(\mathbb{D}, \alpha)$ denotes the Hilbert space of all square integrable functions with respect to the positive finite Borel measure $|z|^{2\alpha}\, dS(z)$; the expression for the norm is the same as the above one concerning the subspace $L_h^2(\mathbb{D}, \alpha)$.

Before we continue our computations, it is necessary to note the following fact, which will be used but not be explicitly mentioned in the sequel: if $\lambda > 0$, and j, k are nonnegative integer, then

$$\Delta(|z|^{2\lambda} z^j \bar{z}^k) = (j + \lambda)(k + \lambda)|z|^{2\lambda - 2} z^j \bar{z}^k.$$

THEOREM 4.1. *The function H_α has the representation*

$$H_\alpha(z, \zeta) = (\alpha + 1)^{-1}(1 - |\zeta|^{2\alpha+2}) \frac{1 - |\zeta z|^2}{|1 - \bar{\zeta} z|^2}, \qquad (z, \zeta) \in \mathbb{D}^2.$$

Remark. (a) Note that the function $H_\alpha(z, \zeta)$ is positive on $\mathbb{D}^2$. If it were negative at some point, one could show that the Green function $U_\alpha(z, \zeta)$ would have to attain a negative value somewhere in $\mathbb{D}^2$.

(b) The resemblance between H_α and the usual Poisson kernel is not accidental. By inspection, the function $\zeta \mapsto H_\alpha(z, \zeta)$ solves, for fixed $z \in \mathbb{T}$, the partial differential equation boundary value problem

$$\begin{cases} \Delta_\zeta |\zeta|^{-2\alpha} \Delta_\zeta H_\alpha(z, \zeta) = 0, & \zeta \in \mathbb{D}, \\ H_\alpha(z, \zeta) = 0, & \zeta \in \mathbb{T}, \\ \partial/\partial n(\zeta) H_\alpha(z, \zeta) = \delta_z(\zeta), & \zeta \in \mathbb{T}, \end{cases}$$

where $\partial/\partial n$ denotes differentiation in the inward normal direction. This fact can actually be argued without knowing the explicit expression for $H_\alpha(z, \zeta)$, and then the formula for $H_\alpha(z, \zeta)$ follows (more or less) from Lemma 3.1, for $z \in \mathbb{T}$. The extension to the bidisk is then obtained by observing that the function is harmonic in the z variable.

Proof of Theorem 4.1. First, observe that for $j = 0, 1, 2, \ldots,$

(4.5)

$$\int_{\mathbb{D}} z^j \Gamma(z, \zeta) |z|^{2\alpha}\, dS(z) = -(\alpha + 1)^{-1}(j + \alpha + 1)^{-1}(1 - |\zeta|^{2\alpha+2})\zeta^j, \qquad \zeta \in \mathbb{D}.$$

We also have, for $j = 0, 1, 2, 3, \ldots$,

$$\int_{\mathbb{D}} (z^j(1-\bar{\zeta}z)^{-1} - z^j)|z|^{2\alpha}\, dS(z) = 0,$$

$$\int_{\mathbb{D}} z^j(1-\zeta\bar{z})^{-1}|z|^{2\alpha}\, dS(z) = \zeta^j/(\alpha + j + 1),$$

and considering that

$$\frac{1-|\zeta z|^2}{|1-\bar{\zeta}z|^2} = 2\,\mathrm{Re}\,\frac{1}{1-\bar{\zeta}z} - 1,$$

we see that

$$\int_{\mathbb{D}} \frac{1-|\zeta z|^2}{|1-\bar{\zeta}z|^2} z^j|z|^{2\alpha}\, dS(z) = \zeta^j/(\alpha + j + 1).$$

It follows that, if H_α is given by the formula

$$H_\alpha(z,\zeta) = (\alpha+1)^{-1}(1-|\zeta|^{2\alpha+2})\frac{1-|\zeta z|^2}{|1-\bar{\zeta}z|^2}, \qquad (z,\zeta) \in \mathbb{D}^2,$$

then we have

$$\int_{\mathbb{D}} H_\alpha(z,\zeta)z^j|z|^{2\alpha}\, dS(z) = (\alpha+1)^{-1}(j+\alpha+1)^{-1}(1-|\zeta|^{2\alpha+2})\zeta^j, \qquad \zeta \in \mathbb{D},$$

so that, by (4.5), we have

$$\int_{\mathbb{D}} H_\alpha(z,\zeta)p(z)|z|^{2\alpha}\, dS(z) = -\int_{\mathbb{D}} \Gamma(z,\zeta)p(z)|z|^{2\alpha}\, dS(z),$$

for all analytic polynomials p. Taking complex conjugates, we obtain (4.3) by a simple approximation argument. □

COROLLARY 4.2. *If $U_\alpha(z,\zeta)$ is the Green function for the operator $\Delta|z|^{-2\alpha}\Delta$, then*

$$\Delta_z U_\alpha(z,\zeta) = |z|^{2\alpha}(\Gamma(z,\zeta) + H_\alpha(z,\zeta)), \qquad (z,\zeta) \in \mathbb{D}^2.$$

COROLLARY 4.3. *The following identity holds:*

$$U_\alpha(z,\zeta) = \int_{\mathbb{D}} \Gamma(z,\xi)(\Gamma(\xi,\zeta) + H_\alpha(\xi,\zeta))|\xi|^{2\alpha}\, dS(\xi), \qquad (z,\zeta) \in \mathbb{D}^2.$$

PROPOSITION 4.4. *We have the formula*

$$\int_{\mathbb{D}} \Gamma(z, \xi) H_\alpha(\xi, \zeta) |\xi|^{2\alpha} \, dS(\xi)$$

$$= -(\alpha + 1)^{-2}(1 - |z|^{2\alpha+2})(1 - |\zeta|^{2\alpha+2})$$

$$\times \left(2 \operatorname{Re} \sum_{j=0}^{\infty} \frac{\bar{\zeta}^j z^j}{\alpha + j + 1} - (\alpha + 1)^{-1}\right), \qquad (z, \zeta) \in \mathbb{D}^2.$$

Proof. Note that the expression on the right-hand side vanishes for $z \in \mathbb{T}$, and if we apply the operator Δ_z to the right-hand side, we obtain

$$(\alpha + 1)^{-1} |z|^{2\alpha}(1 - |\zeta|^{2\alpha+2})\left(2 \operatorname{Re} \sum_{j=0}^{\infty} \bar{\zeta}^j z^j - 1\right)$$

$$= (\alpha + 1)^{-1} |z|^{2\alpha}(1 - |\zeta|^{2\alpha+2}) \frac{1 - |\zeta z|^2}{|1 - \bar{\zeta} z|^2},$$

which coincides with $|z|^{2\alpha} H_\alpha(z, \zeta)$. The assertion is now evident. □

COROLLARY 4.5. *We have the formula*

$$U_\alpha(z, \zeta) = \int_{\mathbb{D}} \Gamma(z, \xi)\Gamma(\xi, \zeta) |\xi|^{2\alpha} \, dS(\xi) - (\alpha + 1)^{-2}(1 - |z|^{2\alpha+2})(1 - |\zeta|^{2\alpha+2})$$

$$\times \left(2 \operatorname{Re} \sum_{j=0}^{\infty} \frac{\bar{\zeta}^j z^j}{\alpha + j + 1} - (\alpha + 1)^{-1}\right).$$

It is rather difficult to evaluate the integral expression appearing in Corollary 4.5; in view of later results, we may turn things around and consider Corollary 4.5 as a convenient vehicle for evaluating this very integral, which pops up in different circumstances as well; see Corollary 4.7.

We now present a formula giving $U_n(z, \zeta)$ for integers $n = 0, 1, 2, \ldots$; the positivity of this function, however, is not completely obvious from the representation.

THEOREM 4.6. *For $n = 0, 1, 2, \ldots$, we have the formula*

$$(n + 1)^2 U_n(z, \zeta) = |z^{n+1} - \zeta^{n+1}|^2 \Gamma(z, \zeta)$$

$$+ 2 \sum_{j=1}^{n} j^{-1}(1 - |z|^{2j})(1 - |\zeta|^{2j}) \operatorname{Re}\{(\bar{\zeta} z)^{n+1-j}\}$$

$$+ (n + 1)^{-1}(1 - |\zeta|^{2n+2})(1 - |z|^{2n+2}).$$

Proof. Let us decide to write $V_n(z, \zeta)$ for the expression on the right-hand side. Since $V_n(\cdot, \zeta)$ vanishes on the unit circle $\mathbb{T}$, it suffices to verify that, if we apply Δ_z to it, then we obtain $(n+1)^2|z|^{2n}(\Gamma(z,\zeta) + H_n(z,\zeta))$. If we apply Δ_z to the expression $|z^{n+1} - \zeta^{n+1}|^2\Gamma(z,\zeta)$, we get, after some computation,

$$\Delta_z\{|z^{n+1} - \zeta^{n+1}|^2\Gamma(z,\zeta)\} = (n+1)^2|z|^{2n}\Gamma(z,\zeta) + 2(n+1)(1-|\zeta|^2)\operatorname{Re}\left\{\frac{\bar{z}^n(z^{n+1}-\zeta^{n+1})}{(1-\bar{\zeta}z)(z-\zeta)}\right\}.$$

We also have the identity

$$\Delta_z\{(1-|z|^{2j})(1-|\zeta|^{2j})\operatorname{Re}((\bar{\zeta}z)^{n+1-j})\} = -j(n+1)|z|^{2j-2}(1-|\zeta|^{2j})\operatorname{Re}\{(\bar{\zeta}z)^{n+1-j}\},$$

valid for $j = 0, \dots, n+1$. Consequently, we obtain the formula

$$\Delta_z V_n(z,\zeta) = (n+1)|z|^{2n}\Bigg((n+1)\Gamma(z,\zeta) + 2(1-|\zeta|^2)\operatorname{Re}\left\{\frac{z^{n+1}-\zeta^{n+1}}{z^n(1-\bar{\zeta}z)(z-\zeta)}\right\} - 2\sum_{j=1}^{n}(1-|\zeta|^{2j})\operatorname{Re}\{(\zeta/z)^{n+1-j}\} - (1-|\zeta|^{2n+2})\Bigg).$$

The problem now reduces to showing that

$$2(1-|\zeta|^2)\operatorname{Re}\left\{\frac{1-(\zeta/z)^{n+1}}{(1-\bar{\zeta}z)(1-\zeta/z)}\right\} - 2\sum_{j=1}^{n}(1-|\zeta|^{2j})\operatorname{Re}\{(\zeta/z)^{n+1-j}\} - (1-|\zeta|^{2n+2}) = (1-|\zeta|^{2n+2})\frac{1-|\zeta z|^2}{|1-\bar{\zeta}z|^2},$$

or, if we change the order of summation,

$$2(1-|\zeta|^2)\operatorname{Re}\left\{(1-\bar{\zeta}z)^{-1}\sum_{j=0}^{n}(\zeta/z)^j\right\} - 2\sum_{j=1}^{n}(1-|\zeta|^{2n+2-2j})\operatorname{Re}\{(\zeta/z)^j\} = 2(1-|\zeta|^{2n+2})\operatorname{Re}\frac{1}{1-\bar{\zeta}z}.$$

We will do this by proving that

$$(1-|\zeta|^2)\sum_{j=0}^{n}(\zeta/z)^j - (1-\bar{\zeta}z)\sum_{j=1}^{n}(1-|\zeta|^{2n+2-2j})(\zeta/z)^j = 1-|\zeta|^{2n+2};$$

if we divide both sides by $1 - \bar{\zeta}z$ and take real parts, the desired assertion then follows. The left-hand side of this new expression simplifies,

$$(1 - |\zeta|^2) \sum_{j=0}^{n} (\zeta/z)^j - (1 - \bar{\zeta}z) \sum_{j=1}^{n} (1 - |\zeta|^{2n+2-2j})(\zeta/z)^j$$

$$= (1 - |\zeta|^2) \sum_{j=0}^{n} (\zeta/z)^j - \sum_{j=1}^{n} (1 - |\zeta|^{2n+2-2j})(\zeta/z)^j + \sum_{j=0}^{n-1} (|\zeta|^2 - |\zeta|^{2n+2-2j})(\zeta/z)^j$$

$$= (1 - |\zeta|^2)(1 + \zeta/z) + (1 - |\zeta|^2) \sum_{j=1}^{n-1} (\zeta/z)^j - (1 - |\zeta|^2)(\zeta/z)^n$$

$$- \sum_{j=1}^{n-1} (1 - |\zeta|^{2n+2-2j})(\zeta/z)^j + |\zeta|^2(1 - |\zeta|^{2n}) + \sum_{j=1}^{n-1} (|\zeta|^2 - |\zeta|^{2n+2-2j})(\zeta/z)^j$$

$$= 1 - |\zeta|^2 + (1 - |\zeta|^2) \sum_{j=1}^{n-1} (\zeta/z)^j + \sum_{j=1}^{n-1} (|\zeta|^2 - 1)(\zeta/z)^j + |\zeta|^2 - |\zeta|^{2n+2}$$

$$= 1 - |\zeta|^{2n+2},$$

which does it. □

Corollary 4.5 and Theorem 4.6 have the following consequence.

COROLLARY 4.7. *For $n = 0, 1, 2, \ldots$, we have the formula*

$$(n + 1)^2 \int_{\mathbb{D}} \Gamma(z, \xi)\Gamma(\xi, \zeta)|\xi|^{2n}\, dS(\xi)$$

$$= |z^{n+1} - \zeta^{n+1}|^2\Gamma(z, \zeta) + 2 \sum_{j=1}^{n} j^{-1}(1 - |z|^{2j})(1 - |\zeta|^{2j}) \times \operatorname{Re}\{(\bar{\zeta}z)^{n+1-j}\}$$

$$+ 2(1 - |z|^{2n+2})(1 - |\zeta|^{2n+2}) \times \operatorname{Re} \sum_{j=0}^{\infty} \frac{\bar{\zeta}^j z^j}{n + j + 1}.$$

5. The Green function for the operator $\Delta \exp(2 \operatorname{Im} z)\Delta$ on the upper half plane. Let $\mathbb{C}_+$ denote the open upper half plane,

$$\mathbb{C}_+ = \{z \in \mathbb{C}: \operatorname{Im} z > 0\}.$$

The Green function $\Gamma(z, \zeta; \mathbb{C}_+)$ for the Laplace operator Δ on $\mathbb{C}_+$ has the form

$$\Gamma(z, \zeta; \mathbb{C}_+) = \log\left|\frac{z - \zeta}{z - \bar{\zeta}}\right|^2, \qquad z, \zeta \in \mathbb{C}_+ \cup \mathbb{R};$$

for a fixed $\zeta \in \mathbb{C}_+$, it solves the partial differential equation boundary value problem

$$\begin{cases} \Delta_z \Gamma(z, \zeta; \mathbb{C}_+) = \delta_\zeta(z) & \text{on } \mathbb{C}_+, \\ \Gamma(z, \zeta; \mathbb{C}_+) = 0, & z \in \mathbb{R}. \end{cases}$$

From the elementary estimate

$$1 - 1/t \leqslant \log t \leqslant t - 1, \qquad t > 0,$$

with equal signs if and only if $t = 1$, and the identity

$$\left|\frac{z - \zeta}{z - \bar{\zeta}}\right|^2 = 1 - 4\frac{\operatorname{Im} z \operatorname{Im} \zeta}{|z - \bar{\zeta}|^2},$$

we see that

$$-4\frac{\operatorname{Im} z \operatorname{Im} \zeta}{|z - \zeta|^2} \leqslant \Gamma(z, \zeta; \mathbb{C}_+) \leqslant -4\frac{\operatorname{Im} z \operatorname{Im} \zeta}{|z - \bar{\zeta}|^2}, \qquad z, \zeta \in \mathbb{C}_+ \cup \mathbb{R},$$

from which we derive the inequalities

$$0 \leqslant |z - \zeta|^2 \Gamma(z, \zeta; \mathbb{C}_+) + 4 \operatorname{Im} z \operatorname{Im} \zeta \leqslant 16\frac{(\operatorname{Im} z \operatorname{Im} \zeta)^2}{|z - \bar{\zeta}|^2}, \qquad z, \zeta \in \mathbb{C}_+ \cup \mathbb{R}.$$

In the notation

$$U(z, \zeta; \mathbb{C}_+) = |z - \zeta|^2 \Gamma(z, \zeta; \mathbb{C}_+) + 4 \operatorname{Im} z \operatorname{Im} \zeta, \qquad z, \zeta \in \mathbb{C}_+ \cup \mathbb{R},$$

we get

$$0 \leqslant U(z, \zeta; \mathbb{C}_+) \leqslant 16\frac{(\operatorname{Im} z \operatorname{Im} \zeta)^2}{|z - \bar{\zeta}|^2}, \qquad z, \zeta \in \mathbb{C}_+ \cup \mathbb{R}, \tag{5.1}$$

with strict inequality for $z, \zeta \in \mathbb{C}_+$. The reason why we call this function $U(z, \zeta; \mathbb{C}_+)$ is that it happens to be the Green function for the operator Δ^2 on $\mathbb{C}_+$, that is, it solves (for a fixed ζ) the partial differential equation boundary value problem

$$\begin{cases} \Delta_z^2 U(\cdot, \zeta; \mathbb{C}_+) = \delta_\zeta & \text{on } \mathbb{C}_+, \\ U(z, \zeta; \mathbb{C}_+) = 0, & z \in \mathbb{R}, \\ \nabla_z U(z, \zeta; \mathbb{C}_+) = 0, & z \in \mathbb{R}. \end{cases}$$

For our purposes, however, it shall be sufficient to know that if $\zeta \in \mathbb{C}_+$ is kept fixed, $U(z, \zeta, \mathbb{C}_+)$ and $\nabla_z U(z, \zeta, \mathbb{C}_+)$ both vanish for all $z \in \mathbb{R}$, and that $U(z, \zeta, \mathbb{C}_+) > 0$ on $\mathbb{C}_+$.

For $\beta > 0$, write

$$E_\beta(z, \zeta) = \beta^2 U_{\beta-1}(e^{iz/\beta}, e^{i\zeta/\beta}), \qquad z, \zeta \in \mathbb{C}_+ \cup \mathbb{R},$$

and note that this function is periodic, with period $2\beta\pi$, in the two coordinates z and ζ. From the change of variables formula for the Laplace operator, together with Theorem 4.1 and Corollary 4.2, we see that

$$\begin{aligned} \Delta_z E_\beta(z, \zeta) &= e^{-2\,\mathrm{Im}\,z}\big(\Gamma(e^{iz/\beta}, e^{i\zeta/\beta}; \mathbb{D}) + H_{\beta-1}(e^{iz/\beta}, e^{i\zeta/\beta})\big) \\ &= e^{-2\,\mathrm{Im}\,z}\big(\Gamma(e^{iz/\beta}, e^{i\zeta/\beta}; \mathbb{D}) + (1 - e^{-2\,\mathrm{Im}\,\zeta})P_\beta(z, \zeta)\big), \end{aligned} \tag{5.2}$$

where

$$P_\beta(z, \zeta) = \beta^{-1}\frac{1 - e^{-2(\mathrm{Im}\,z + \mathrm{Im}\,\zeta)/\beta}}{|1 - e^{i(z-\bar{\zeta})/\beta}|^2},$$

so that if we multiply by $e^{2\,\mathrm{Im}\,z}$ and apply another Laplacian, we get

(5.3)
$$\Delta_z e^{2\,\mathrm{Im}\,z}\Delta_z E_\beta(\cdot, \zeta) = \Delta_z \Gamma(e^{iz/\beta}, e^{i\zeta/\beta}; \mathbb{D}) = \sum_{n=-\infty}^{\infty} \delta_0(z - \zeta + 2n\beta\pi), \qquad z \in \mathbb{C}_+,$$

where δ_0 denotes the Dirac measure at the point 0 in the complex plane $\mathbb{C}$. For fixed $\zeta \in \mathbb{C}_+$, the function $E_\beta(z, \zeta)$ vanishes on $z \in \mathbb{R}$ together with its normal derivative, so (5.3) suggests that we should have

$$E_\beta(z, \zeta) = \sum_{n=-\infty}^{\infty} E_\infty(z, \zeta + 2n\beta\pi), \tag{5.4}$$

where $E_\infty(z, \zeta)$ is the Green function associated with the operator $\Delta \exp(2\,\mathrm{Im}\,z)\Delta$, and clamped boundary values:

$$\begin{cases} \Delta_z e^{2\,\mathrm{Im}\,z}\Delta_z E_\infty(\cdot, \zeta) = \delta_\zeta & \text{on } \mathbb{C}_+, \\ E_\infty(z, \zeta) = 0, & z \in \mathbb{R}, \\ \nabla_z E_\infty(z, \zeta) = 0, & z \in \mathbb{R}. \end{cases} \tag{5.5}$$

To get uniqueness for the solution to this problem, one needs to impose additional growth restrictions near infinity; however, it is not necessary here to go into detail on this matter.

Our next job is to find an explicit expression for $E_\infty(z, \zeta)$ and verify that (5.4) holds. As the notation suggests, we get $E_\infty(z, \zeta)$ by letting the integer n tend to

$+\infty$ in the identity

$$E_n(z,\zeta) = |e^{iz} - e^{i\zeta}|^2 \Gamma(e^{iz/n}, e^{i\zeta/n}; \mathbb{D})$$
$$+ \frac{2}{n} \sum_{j=1}^{n-1} (j/n)^{-1} (1 - e^{-2(j/n)\operatorname{Im} z})(1 - e^{-2(j/n)\operatorname{Im}\zeta}) \operatorname{Re}\{e^{i(1-j/n)(z-\bar{\zeta})}\}$$
$$+ n^{-1}(1 - e^{-2\operatorname{Im} z})(1 - e^{-2\operatorname{Im}\zeta}),$$

valid by Theorem 4.6. As $n \to +\infty$, $\Gamma(e^{iz/n}, e^{i\zeta/n}; \mathbb{D})$ tends to $\Gamma(z, \zeta; \mathbb{C}_+)$, and the sum

$$\frac{2}{n} \sum_{j=1}^{n-1} (j/n)^{-1} (1 - e^{-2(j/n)\operatorname{Im} z})(1 - e^{-2(j/n)\operatorname{Im}\zeta}) \operatorname{Re}\{e^{i(1-j/n)(z-\bar{\zeta})}\}$$
$$+ n^{-1}(1 - e^{-2\operatorname{Im} z})(1 - e^{-2\operatorname{Im}\zeta})$$

converges to the integral

$$2 \int_0^1 (1 - e^{-2t\operatorname{Im} z})(1 - e^{-2t\operatorname{Im}\zeta}) \operatorname{Re}\{e^{i(1-t)(z-\bar{\zeta})}\}\, dt/t$$
$$= 2e^{-\operatorname{Im} z - \operatorname{Im}\zeta} \int_0^1 (e^{t\operatorname{Im} z} - e^{-t\operatorname{Im} z})(e^{t\operatorname{Im}\zeta} - e^{-t\operatorname{Im}\zeta}) \cos\big((1-t)(\operatorname{Re} z - \operatorname{Re}\zeta)\big)\, dt/t.$$

We thus *define*

$$E_\infty(z,\zeta) = |e^{iz} - e^{i\zeta}|^2 \Gamma(z, \zeta; \mathbb{C}_+) \tag{5.6}$$
$$+ 2e^{-\operatorname{Im} z - \operatorname{Im}\zeta} \int_0^1 (e^{t\operatorname{Im} z} - e^{-t\operatorname{Im} z})(e^{t\operatorname{Im}\zeta} - e^{-t\operatorname{Im}\zeta})$$
$$\times \cos\big((1-t)(\operatorname{Re} z - \operatorname{Re}\zeta)\big)\, dt/t,$$

and note that this function solves (5.5). The convergence $E_n(z,\zeta) \to E_\infty(z,\zeta)$ is such that $\Delta_z E_n(z,\zeta) \to \Delta_z E_\infty(z,\zeta)$ as $n \to +\infty$, at least in the sense of distributions, so by (5.2),

$$\Delta_z E_\infty(z,\zeta) = e^{-2\operatorname{Im} z}\big(\Gamma(z,\zeta;\mathbb{C}_+) + (1 - e^{-2\operatorname{Im}\zeta}) P_\infty(z,\zeta)\big),$$

where

$$P_\infty(z,\zeta) = 2\frac{\operatorname{Im} z + \operatorname{Im}\zeta}{|z - \bar{\zeta}|^2},$$

because as $n \to +\infty$, $\Gamma(e^{iz/n}, e^{i\zeta/n}; \mathbb{D})$ tends to $\Gamma(z, \zeta; \mathbb{C}_+)$, and $P_n(z, \zeta)$ tends to $P_\infty(z, \zeta)$.

The functions $P_\beta(z, \zeta)$ and $P_\infty(z, \zeta)$ play the rôle of the Poisson kernel on $\mathbb{C}_+/\Sigma_\beta$ and $\mathbb{C}_+$, respectively. Here, Σ_β stands for the group of horizontal translations $z \mapsto z + 2n\beta\pi$, $n \in \mathbb{Z}$.

The next step toward obtaining (5.4) is to show that

$$\Delta_z E_\beta(z, \zeta) = \sum_{n=-\infty}^{\infty} \Delta_z E_\infty(z, \zeta + 2n\beta\pi).$$

This is actually a simple consequence of the basic periodicity identities

$$\Gamma(e^{iz/\beta}, e^{i\zeta/\beta}; \mathbb{D}) = \sum_{n=-\infty}^{\infty} \Gamma(z, \zeta + 2n\beta\pi; \mathbb{C}_+), \qquad z, \zeta \in \mathbb{C}_+,$$

and

$$P_\beta(z, \zeta) = \sum_{n=-\infty}^{\infty} P_\infty(z, \zeta + 2n\beta\pi), \qquad z, \zeta \in \mathbb{C}_+.$$

For a fixed $\zeta \in \mathbb{C}_+$, consider the difference function

$$Y(z) = E_\beta(z, \zeta) - \sum_{n=-\infty}^{\infty} E_\infty(z, \zeta + 2n\beta\pi),$$

which is harmonic on $\mathbb{C}_+$, and extends real-analytically across the real line $\mathbb{R}$, because the involved functions $E_\beta(z, \zeta)$ and $E_\infty(z, \zeta)$ do (since $\Delta_z E_\beta(z, \zeta)$ and $\Delta_z E_\beta(z, \zeta)$ are nice), and because the summation process converges comfortably. Thus $\Delta Y(z) = 0$ in a neighborhood of $\mathbb{R}$, and in view of the boundary conditions on $E_\beta(z, \zeta)$ and $E_\infty(z, \zeta)$, $Y(z) = 0$ and $\nabla Y(z) = 0$ both hold on $\mathbb{R}$. The uniqueness principle for the Cauchy problem (the Cauchy-Kovalevskaya theorem, or Holmgren's theorem) asserts that $Y(z) \equiv 0$ throughout the region where Y is real analytic, in particular on $\mathbb{C}_+$; hence (5.4) holds. To emphasize this fact, we formulate it as a theorem.

THEOREM 5.1. *Let the function $E_\infty(z, \zeta)$ be given by relation (5.6). Then, for $\beta > 0$, the following identity holds:*

$$E_\beta(z, \zeta) = \sum_{n=-\infty}^{\infty} E_\infty(z, \zeta + 2n\beta\pi).$$

In view of the definition of the functions $E_\beta(z, \zeta)$, the following consequence is immediate.

COROLLARY 5.2. *For $\alpha > -1$ and $k = 1, 2, 3, \ldots$, it is true that*

$$U_{(\alpha+1)/k-1}(z^k, \zeta^k) = k^2 \sum_{j=0}^{k-1} U_\alpha(z, e_j(k)\zeta),$$

where

$$e_j(k) = \exp(2j\pi i/k), \qquad j = 0, \ldots, k-1,$$

are the k different roots to the equation $z^k = 1$.

In this section, it will be demonstrated that the Green function $E_\infty(z, \zeta)$ is positive throughout $\mathbb{C}_+ \times \mathbb{C}_+$, whence that $U_\alpha(z, \zeta)$ is positive on $\mathbb{D}^2$, for $\alpha > -1$.

In terms of the function

$$\varphi(x) = (e^x - e^{-x})/(2x) = \sum_{j=0}^{\infty} x^{2j}/(2j+1)!, \qquad x \in \mathbb{R}, \tag{5.7}$$

the expression (5.6) for $E_\infty(z, \zeta)$ takes the form

$$\begin{aligned} E_\infty(z, \zeta) &= |e^{iz} - e^{i\zeta}|^2 \Gamma(z, \zeta; \mathbb{C}_+) \\ &\quad + 8 \operatorname{Im} z \operatorname{Im} \zeta \, e^{-\operatorname{Im} z - \operatorname{Im} \zeta} \int_0^1 \varphi(t \operatorname{Im} z) \varphi(t \operatorname{Im} \zeta) \\ &\quad \times \cos\big((1-t)(\operatorname{Re} z - \operatorname{Re} \zeta)\big) \, t \, dt, \end{aligned}$$

or, if we use the function $U(z, \zeta; \mathbb{C}_+)$ instead of $\Gamma(z, \zeta; \mathbb{C}_+)$, we get

$$\begin{aligned} E_\infty(z, \zeta) &= \left| \frac{e^{iz} - e^{i\zeta}}{z - \zeta} \right|^2 U(z, \zeta; \mathbb{C}_+) + 4 \operatorname{Im} z \operatorname{Im} \zeta \\ &\quad \times \Bigg(2e^{-\operatorname{Im} z - \operatorname{Im} \zeta} \int_0^1 \varphi(t \operatorname{Im} z) \varphi(t \operatorname{Im} \zeta) \\ &\quad \times \cos\big((1-t)(\operatorname{Re} z - \operatorname{Re} \zeta)\big) \, t \, dt - \left| \frac{e^{iz} - e^{i\zeta}}{z - \zeta} \right|^2 \Bigg). \end{aligned} \tag{5.8}$$

The next lemma enables us to rewrite (5.8).

LEMMA 5.3. *If φ is as in (5.7), we have for $z, \zeta \in \mathbb{C}_+$ the identity*

$$\left| \frac{e^{iz} - e^{i\zeta}}{z - \zeta} \right|^2 = 2e^{-\operatorname{Im} z - \operatorname{Im} \zeta} \int_0^1 \varphi\big(t(\operatorname{Im} z - \operatorname{Im} \zeta)\big) \cos\big((1-t)(\operatorname{Re} z - \operatorname{Re} \zeta)\big) \, t \, dt.$$

Proof. Recall the hyperbolic functions

$$\cosh(x) = (e^x + e^{-x})/2, \qquad x \in \mathbb{R},$$

$$\sinh(x) = (e^x - e^{-x})/2, \qquad x \in \mathbb{R},$$

and note that, for $w \in \mathbb{C}$, we have the identity

$$|1 - e^{iw}|^2 = 2e^{-\operatorname{Im} w}\big(\cosh(\operatorname{Im} w) - \cos(\operatorname{Re} w)\big).$$

If w is replaced by $\zeta - z$ on both sides, and the equality is multiplied by $e^{-2 \operatorname{Im} z}$, this results in

$$|e^{iz} - e^{i\zeta}|^2 = 2e^{-\operatorname{Im} z - \operatorname{Im} \zeta}\big(\cosh(\operatorname{Im} z - \operatorname{Im} \zeta) - \cos(\operatorname{Re} z - \operatorname{Re} \zeta)\big), \qquad z, \zeta \in \mathbb{C}.$$

Consider, for real parameters ξ, η, the definite integral

$$I(\xi, \eta) = \int_0^1 \cos((1 - t)\xi)\varphi(t\eta)\, t\, dt;$$

by the previous identity, our job is now to check that

$$I(\xi, \eta) = (\xi^2 + \eta^2)^{-1}(\cosh \eta - \cos \xi).$$

By continuity, we may assume, without loss of generality, that both ξ and η differ from 0. From the definition of the function φ, we have

$$I(\xi, \eta) = \int_0^1 \cos((1 - t)\xi) \sinh(t\eta)\, dt/\eta,$$

so that, if we apply integration by parts once, we get

$$I(\xi, \eta) = \int_0^1 \sin((1 - t)\xi) \cosh(t\eta)\, dt/\xi,$$

and if we do it once more, we obtain

$$I(\xi, \eta) = \xi^{-2}(\cosh \eta - \cos \xi) - \eta^2\xi^{-2}I(\xi, \eta),$$

from which the desired assertion easily follows. □

In view of (5.8) and Lemma 5.3, we may express $E_\infty(z, \zeta)$ in the form

$$(5.9)\qquad E_\infty(z, \zeta) = \left|\frac{e^{iz} - e^{i\zeta}}{z - \zeta}\right|^2 U(z, \zeta; \mathbb{C}_+) + 8 \operatorname{Im} z \operatorname{Im} \zeta\, e^{-\operatorname{Im} z - \operatorname{Im} \zeta}$$
$$\times \int_0^1 \big(\varphi(t \operatorname{Im} z)\varphi(t \operatorname{Im} \zeta) - \varphi(t(\operatorname{Im} z - \operatorname{Im} \zeta))\big)$$
$$\times \cos\big((1 - t)(\operatorname{Re} z - \operatorname{Re} \zeta)\big)\, t\, dt.$$

We are now ready to prove the fundamental positivity result.

THEOREM 5.4. *We have $E_\infty(z, \zeta) > 0$ for all $z, \zeta \in \mathbb{C}_+$.*

Proof. In fact, we shall prove that

$$\int_0^1 \big(\varphi(t \operatorname{Im} z)\varphi(t \operatorname{Im} \zeta) - \varphi(t(\operatorname{Im} z - \operatorname{Im} \zeta))\big) \cos\big((1 - t)(\operatorname{Re} z - \operatorname{Re} \zeta)\big)\, t\, dt > 0,$$

for all $z, \zeta \in \mathbb{C}_+$, making the assertion a consequence of (5.9) and the positivity fact (5.1). To this end, let us write $y = \operatorname{Im} z$, $\eta = \operatorname{Im} \zeta$, $x = \operatorname{Re} z - \operatorname{Re} \zeta$, and

$$\Phi(t; y, \eta) = t\big(\varphi(ty)\varphi(t\eta) - \varphi(t(y - \eta))\big),$$

so that what we wish to show is that

$$(5.10)\qquad \Psi(x, y, \eta) = \int_0^1 \Phi(t; y, \eta) \cos((1 - t)x)\, dt > 0$$

holds for all $x \in \mathbb{R}$, $y, \eta > 0$. It will be instrumental in this aim to know that $\Phi(t; y, \eta)$ is a convex function of t on the interval $[0, 1]$. Considering that $\Phi(t; y, \eta)$ has the form

$$\Phi(t; y, \eta) = \sum_{n=1}^{\infty} A(n; y, \eta) t^{n+1},$$

where

$$A(n; y, \eta) = \sum_{j=0}^{n} \frac{y^j \eta^{n-j}}{(2j + 1)!(2n - 2j + 1)!} - \frac{(y - \eta)^n}{(2n + 1)!} > \frac{y^n + \eta^n}{(2n + 1)!} - \frac{(y - \eta)^n}{(2n + 1)!} \geqslant 0,$$

so that

$$(\partial^2/\partial t^2)\Phi(t; y, \eta) = \sum_{n=1}^{\infty} n(n + 1)A(n; y, \eta)t^{n-1} \geqslant 0, \qquad t \geqslant 0,$$

with strict inequality for $t > 0$, we see that, as a function of t, $\Phi(t; y, \eta)$ is positive on the interval $]0, +\infty[$, has $\Phi(0; y, \eta) = \partial/\partial t\Phi(0; y, \eta) = 0$, and is convex on the interval $[0, +\infty[$, and thus in particular on $[0, 1]$. If the integration by parts formula is applied twice to the definition (5.10) of $\Psi(x, y, \eta)$, and the data that $\Phi(t; y, \eta)$ vanishes along with its derivative at $t = 0$ are used, the result is

$$\Psi(x, y, \eta) = \frac{1}{x^2}\int_0^1 (1 - \cos(x(1-t)))\frac{\partial^2\Phi}{\partial t^2}(t; y, \eta)\, dt,$$

which is clearly positive. The proof is complete. □

Remark. The last trick involving the integration by parts is known as *Pólya's lemma.*

By Theorem 5.1, Theorem 5.4 has the following immediate consequence.

COROLLARY 5.5. *Given* $\beta > 0$, $E_\beta(z, \zeta) > 0$ *holds for all* $z, \zeta \in \mathbb{C}_+$.

This result, in its turn, has the following consequence, in view of the definition of the function $E_\beta(z, \zeta)$.

COROLLARY 5.6. *Given* $\alpha > -1$, $U_\alpha(z, \zeta) > 0$ *holds for all* $z, \zeta \in \mathbb{D}$.

6. Another approach, based on an idea of Hadamard. In retrospect, we can say that the essential tacit ingredient in the previous section is the fact that the operator $\Delta \exp(2 \operatorname{Im} z)\Delta$ is invariant under the Moebius subgroup of horizontal translates of the upper half plane. It is of course also of importance that it was possible to identify the upper half plane modulo a discrete subgroup of the horizontal translations as the unit disk, and that the thus transformed operator $\Delta \exp(2 \operatorname{Im} z)\Delta$ coincided with $\Delta|z|^{-2\alpha}\Delta$ on the disk. It is possible to generalize this idea, but unfortunately, the scope is rather limited, because in contast with the Laplacian itself, (weighted) bi-Laplacians are usually invariant only with respect to small subsets of the Moebius group.

There is another method to obtain the positivity of a biharmonic Green function, suggested by Hadamard in [9], [10]. The idea is to consider a continuous movement of the boundary, and calculate the change in the biharmonic Green function along the way. It should be pointed out that this idea of Hadamard has been largely neglected in subsequent developments.

We begin with a bounded simply connected domain Ω in the complex plane, having real analytic boundary, for simplicity. The Green function for the biharmonic operator Δ^2 on Ω, given clamped boundary conditions, is denoted by $U(z, \zeta; \Omega)$. It can be checked that it is symmetric in the arguments z, ζ: $U(z, \zeta; \Omega) = U(\zeta, z; \Omega)$ (see [8]). By Green's theorem,

$$\int_\Omega \Delta_z U(z, \zeta; \Omega)\varphi(z)\, dS(z) = \int_\Omega U(z, \zeta; \Omega)\Delta_z\varphi(z)\, dS(z) \tag{6.1}$$

holds for C^2 functions φ on $\bar{\Omega}$. It is easy to see that

$$\Delta_z U(z, \zeta; \Omega) = \Gamma(z, \zeta; \Omega) + H(z, \zeta; \Omega), \tag{6.2}$$

where $\Gamma(z, \zeta; \Omega)$ is the Green function associated with the Laplace operator Δ on Ω, and the function $H(z, \zeta; \Omega)$ is harmonic in the z variable throughout Ω. By elliptic regularity and the smoothness assumption on the boundary $\partial\Omega$, the functions $z \mapsto \Gamma(z, \zeta; \Omega)$ and $z \mapsto U(z, \zeta; \Omega)$ extend, for fixed $\zeta \in \Omega$, harmonically and biharmonically, respectively, across $\partial\Omega$. For fixed $\zeta \in \Omega$, the function $z \mapsto H(z, \zeta; \Omega)$ thus extends harmonically across $\partial\Omega$.

Introduce another region Ω', subject to the same conditions as Ω, which is slightly larger, but only by so much that $z \mapsto U(z, \zeta; \Omega)$ remains biharmonic on $\Omega' \setminus \{\zeta\}$ and C^2 up to the boundary. If we appeal to Green's formula, we obtain, as in (6.1),

$$U(z, \zeta; \Omega) = \int_\Omega \Delta_\xi U(z, \xi; \Omega) \Delta_\xi U(\xi, \zeta; \Omega')\, dS(\xi)$$

and

$$U(z, \zeta; \Omega') = \int_{\Omega'} \Delta_\xi U(z, \xi; \Omega) \Delta_\xi U(\xi, \zeta; \Omega')\, dS(\xi).$$

From the above identities, it is immediate that

$$U(z, \zeta; \Omega') - U(z, \zeta; \Omega) = \int_{\Omega' \setminus \Omega} \Delta_\xi U(z, \xi; \Omega) \Delta_\xi U(\xi, \zeta; \Omega')\, dS(\xi). \tag{6.3}$$

By (6.2), this may be written as

$$\begin{aligned} &U(z, \zeta; \Omega') - U(z, \zeta; \Omega) \\ &\quad = \int_{\Omega' \setminus \Omega} (\Gamma(\xi, z; \Omega) + H(\xi, z; \Omega))(\Gamma(\xi, \zeta; \Omega') + H(\xi, \zeta; \Omega'))\, dS(\xi). \end{aligned} \tag{6.4}$$

At this point, we introduce a family of regions $\Omega(t)$, depending on the parameter $t \in [0, 1]$ in a C^1 fashion; these regions are to expand as t increases. Furthermore, we assume $\Omega(0) = \varnothing$, $\Omega(t) \neq \varnothing$ for $0 < t \leqslant 1$, and that all the regions $\Omega(t)$ (for $0 < t \leqslant 1$) are subject to the requirements imposed earlier on Ω. The condition that $\Omega(t)$ depend on t in a C^1 way has not been spelled out in detail; the reader may use whatever definition he sees fit, and check that the reasoning below is permitted. It is convenient to assume implicit in this requirement that the intersection of all $\Omega(t)$, $0 < t \leqslant 1$, be finite. If, for $0 < t < 1$, we plug $\Omega = \Omega(t)$ and

$\Omega' = \Omega(t+\delta)$ into (6.4), with $0 < \delta \to 0$, the result is

$$\text{(6.5)} \qquad \frac{d}{dt} U(z, \zeta; \Omega(t)) = \lim_{\delta \to 0} \frac{1}{\delta} \int_{\Omega(t+\delta)\setminus\Omega(t)} H(\xi, z; \Omega(t)) H(\xi, \zeta; \Omega(t))\, dS(\xi),$$

$$(z, \zeta) \in \Omega(t)^2 = \Omega(t) \times \Omega(t).$$

The terms in (6.4) involving the Green functions $\Gamma(\xi, z; \Omega(t))$ and $\Gamma(\xi, \zeta; \Omega(t+\delta))$ for fixed $z, \zeta \in \Omega(t)$ disappear because, on $\xi \in \Omega(t+\delta)\setminus\Omega(t)$, the values of these functions tend to 0 as $\delta \to 0$. If $dS_{t,\delta}$ denotes area measure (normalized by the factor π^{-1}, as usual) restricted to the thin band $\Omega(t+\delta)\setminus\Omega(t)$, then, as $0 < \delta \to 0$, $\delta^{-1}\, dS_{t,\delta}$ converges in a weak sense toward a positive finite Borel measure $d\mu_t$, supported on $\partial\Omega(t)$. Relation (6.5) then reduces to

$$\text{(6.6)} \quad \frac{d}{dt} U(z, \zeta; \Omega(t)) = \int_{\partial\Omega(t)} H(\xi, z; \Omega(t)) H(\xi, \zeta; \Omega(t))\, d\mu_t(\xi), \qquad (z, \zeta) \in \Omega(t)^2.$$

The function $U(z, \zeta; \Omega(t))$ vanishes for $(z, \zeta) \in \partial(\Omega(t)^2)$, and thus

(6.7)

$$U(z, \zeta; \Omega(t)) = \int_{\theta(z,\zeta)}^{t} \int_{\partial\Omega(\tau)} H(\xi, z; \Omega(\tau)) H(\xi, \zeta; \Omega(\tau))\, d\mu_\tau(\xi)\, d\tau, \qquad \theta(z, \zeta) \leqslant t \leqslant 1;$$

here $\theta(z, \zeta)$ stands for the infimum over all θ, $0 < \theta \leqslant 1$, for which $z, \zeta \in \Omega(\theta)$ holds. It is clear from (6.7) that $U(z, \zeta; \Omega(1)) \geqslant 0$ holds on $\Omega(1)^2$ provided $H(\xi, \zeta; \Omega(t)) \geqslant 0$ on $\partial\Omega(t) \times \Omega(t)$, for $0 < t < 1$.

Conversely, if $U(z, \zeta; \Omega(1)) \geqslant 0$ on $\Omega(1)^2$, then for a fixed $\zeta \in \Omega(1)$, $\Delta_z U(z, \zeta; \Omega(1)) \geqslant 0$ on $z \in \partial\Omega(1)$, because the Green function has value and normal derivative 0 on the boundary $\partial\Omega(1)$, making its Laplacian equal a quarter of the second normal derivative on the boundary. Thus $U(z, \zeta; \Omega(1)) \geqslant 0$ on $\Omega(1)^2$ implies that $H(\xi, \zeta; \Omega(1)) \geqslant 0$ on $\partial\Omega(1) \times \Omega(1)$.

Given a star-shaped region Ω, "centered" at the origin, a convenient family of expanding domains is $\Omega(t) = t\Omega$, $0 < t \leqslant 1$. And, for geometric reasons, it is easy to see that $H(\xi, \zeta; \Omega(t)) \geqslant 0$ holds on $\partial\Omega(t) \times \Omega(t)$ for all t, $0 < t \leqslant 1$, if and only if this is so for $t = 1$. The above reasoning then leads to the following conclusion.

THEOREM 6.1. *If Ω is star-shaped, with real analytic boundary, then $U(z, \zeta; \Omega) \geqslant 0$ holds on Ω^2 if and only if $H(z, \zeta; \Omega) \geqslant 0$ on $\partial\Omega \times \Omega$.*

We proceed to find a physical interpretation of the function $H(z, \zeta; \Omega)$. Let us agree to equate the expressions $\delta_\zeta(z) = \delta_z(\zeta) = \delta_0(z - \zeta)$, which makes sense in the distribution theory of the region Ω^2. By the symmetry property $U(z, \zeta; \Omega) =$

$U(\zeta, z; \Omega)$, the function $\zeta \mapsto \Delta_z U(z, \zeta; \Omega)$ solves

$$\begin{cases} \Delta_\zeta^2 \Delta_z U(z, \zeta; \Omega) = \Delta_z \delta_0(z - \zeta) & \zeta \in \Omega, \\ \Delta_z U(z, \zeta; \Omega) = 0, & \zeta \in \partial\Omega, \\ \partial/\partial n(\zeta) \Delta_z U(z, \zeta; \Omega) = 0, & \zeta \in \partial\Omega, \end{cases}$$

where $\partial/\partial n$ is differentiation along the inward normal direction, as before. Thus, by (6.2) and the identity $\Delta_\zeta^2 \Gamma(z, \zeta) = \Delta_\zeta \delta_0(z - \zeta) = \Delta_\zeta \delta_0(z - \zeta)$, $H(z, \zeta; \Omega)$ solves (for $z \in \Omega$)

$$\begin{cases} \Delta_\zeta^2 H(z, \zeta; \Omega) = 0 & \zeta \in \Omega, \\ H(z, \zeta; \Omega) = 0, & \zeta \in \partial\Omega, \\ \partial/\partial n(\zeta) H(z, \zeta; \Omega) = -\partial/\partial n(\zeta) \Gamma(z, \zeta; \Omega), & \zeta \in \partial\Omega. \end{cases}$$

The function $P(z, \zeta; \Omega) = -\partial/\partial n(\zeta)\Gamma(z, \zeta; \Omega)$ we recognize as the Poisson kernel on Ω. The Poisson kernel tends to a Dirac point mass as the z variable approaches a boundary point, so that we can say that, for $z \in \partial\Omega$, the function $H(z, \zeta; \Omega)$ solves

$$\begin{cases} \Delta_\zeta^2 H(z, \zeta; \Omega) = 0 & \zeta \in \Omega, \\ H(z, \zeta; \Omega) = 0, & \zeta \in \partial\Omega, \\ \partial/\partial n(\zeta) H(z, \zeta; \Omega) = \delta_z(\zeta), & \zeta \in \partial\Omega. \end{cases}$$

It is natural to think of the function $Q(z, \zeta; \Omega) = H(\zeta, z; \Omega)$ as a biharmonic Poisson kernel; it may be regarded as the limiting case of the Green potential $U(z, \zeta; \Omega)$ as ζ approaches the boundary $\partial\Omega$, provided the potential is rescaled by a suitable positive multiple $C(\zeta)$, which is inversely proportional to the square of the distance between ζ and $\partial\Omega$. One particularly attractive consequence of this way of looking at the function $H(z, \zeta; \Omega)$ is that Theorem 6.1 has the following consequence.

COROLLARY 6.2. *Suppose Ω is star-shaped, with real analytic boundary. Let A be the intersection of a neighborhood of $\partial\Omega$ (in $\mathbb{C}$) with Ω. Then $U(z, \zeta; \Omega) \geqslant 0$ on Ω^2 if it holds on the set $\Omega \times A$. This has the physical interpretation that, in order to know that a clamped plate the shape of Ω bends in the direction of a point load everywhere, no matter where the load is applied, you just need to check that this is so when the point load is applied near the boundary.*

The above considerations apply to weighted problems as well. The formula corresponding to (6.7) for the operator $\Delta|z|^{-2\alpha}\Delta$ turns out to be

$$U_\alpha(z, \zeta) = 2 \int_{\max\{|z|,|\zeta|\}}^1 \int_{\tau\mathbb{T}} H_\alpha(\xi/\tau, z/\tau) H_\alpha(\xi/\tau, \zeta/\tau)\, ds(\xi) \tau^{2\alpha}\, d\tau$$

$$= \pi^{-1} \int_{\max\{|z|,|\zeta|\}}^1 \int_{-\pi}^{\pi} H_\alpha(e^{i\theta}, z/\tau) H_\alpha(e^{i\theta}, \zeta/\tau)\, d\theta\, \tau^{2\alpha+1}\, d\tau, \qquad (z, \zeta) \in \mathbb{D}^2,$$

where ds stands for linear measure in $\mathbb{C}$, normalized so that the unit circle $\mathbb{T}$ gets total mass 1. The positivity of the Green potential is immediate from this representation.

7. Applications to the factoring theory of the Bergman space. The positivity of the Green functions $U_\alpha(z, \zeta)$ for the operators $\Delta|z|^{-2\alpha}\Delta$ is important for the factoring theory of the Bergman spaces $L^p_a(\mathbb{D})$, as developed in [12], [13] for the case $p = 2$, and in the paper [7] by Peter Duren, Dmitry Khavinson, Harold Shapiro, and Carl Sundberg for general p, $1 \leqslant p < \infty$. The following theorem contains the essentials of what has been known so far [12], [13], [7]; we formulate it for finite sequences, but it remains valid, *mutatis mutandis*, also for infinite zero sequences. For parameter values $0 < p < \infty$, the Bergman p-space $L^p_a(\mathbb{D})$ consists of all holomorphic functions on $\mathbb{D}$ which also have

$$\|f\|_{L^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dS(z)\right)^{1/p} < \infty .$$

For $1 \leqslant p < \infty$, this is a Banach space, and for $p = 2$, a Hilbert space.

FACTORING THEOREM. *Fix a parameter value p, $1 \leqslant p < \infty$, and suppose $A = \{a_j\}_0^N$ is a finite sequence of points in the open unit disk $\mathbb{D}$. Then there exists a function $G^p_A \in L^p_a(\mathbb{D})$, unique up to multiplication by a unimodular constant, such that*

(a) *G^p_A vanishes precisely on A in the closed unit disk $\bar{\mathbb{D}}$,*

(b) *G^p_A has norm* 1,

(c) *Every $f \in L^p_a(\mathbb{D})$ that vanishes on A has a factoring $f = G^p_A \cdot g$, where $g \in L^p_a(\mathbb{D})$ has $\|g\|_{L^p} \leqslant \|f\|_{L^p}$.*

This function G^p_A has a holomorphic extension across the circle $\mathbb{T}$, $|G^p_A| \geqslant 1$ holds on $\mathbb{T}$, and $|G^p_A|^p \, dS$ is a representing measure for the origin, that is,

$$h(0) = \int_{\mathbb{D}} h(z)|G^p_A(z)|^p \, dS(z)$$

holds for all bounded harmonic functions h on $\mathbb{D}$.

Remark. We may choose to call these functions G^p_A finite zero-based inner functions for the Bergman p-space, or finite Blaschke-type functions for the Bergman p-space.

Obviously, the function $G^p_\varnothing$ equals the constant function 1, so that in Korenblum's terminology, where, given two functions $f, g \in L^p_a(\mathbb{D})$, $f \prec_p g$ provided that

$$\|fh\|_{L^p} \leqslant \|gh\|_{L^p}, \qquad h \in H^\infty(\mathbb{D}),$$

we may regard the essential assertion of the Factoring theorem, that

$$\|G^p_A h\|_{L^p} \geqslant \|h\|_{L^p}, \qquad h \in L^p_a(\mathbb{D}),$$

as saying that $1 = G^p_\varnothing \prec_p G^p_A$. It is natural to ask if this should be thought of as a consequence of the trivial fact that $\varnothing$ is a subset of A. In other words, the issue is whether the following conjecture holds.

CONJECTURE 7.1. *If A is a finite sequence in $\mathbb{D}$ and B is a subset of A, then $G^p_B \prec_p G^p_A$.*

If $O_n = \{0, \dots, 0\}$ denotes the sequence consisting of the point 0 repeated n times, the corresponding Blaschke-type function $G^p_{O_n}$ can be computed explicitly, and it has the form

$$G^p_{O_n}(z) = (1 + np/2)^{1/p} z^n, \qquad z \in \mathbb{D}.$$

Using the fact obtained in this paper that the functions $U_\alpha(z, \zeta)$ are positive (Corollary 5.6), we are now able to state a theorem corroborating the above conjecture in a special situation.

THEOREM 7.2. *If A is a finite sequence in $\mathbb{D}$, which contains O_n as a subsequence, then $G^p_{O_n} \prec_p G^p_A$, that is,*

$$\|G^p_{O_n} f\|_{L^p} \leqslant \|G^p_A f\|_{L^p}, \qquad f \in L^p_a(\mathbb{D}).$$

Proof. For a finite sequence B of points in $\mathbb{D}$, let Φ^p_B denote the solution to the Poisson problem

$$\begin{cases} \Delta\Phi^p_B(z) = |G^p_B(z)|^p - 1, & z \in \mathbb{D}, \\ \Phi^p_B = 0, & z \in \mathbb{T}. \end{cases}$$

We then have the formula [12], [13], [7]

$$\int_{\mathbb{D}} |G^p_B(z) f(z)|^p \, dS(z) = \int_{\mathbb{D}} |f(z)|^p \, dS(z) + \int_{\mathbb{D}} \Phi^p_B(z) \Delta_z |f(z)|^p \, dS(z),$$

for polynomials f in the z variable. Note here that, because f is a polynomial, $|f|^p$ is subharmonic in the whole complex plane, in particular, $\Delta |f|^p$ is a nonnegative integrable function on $\mathbb{D}$. The function Φ^p_B is also known to be nonnegative on $\mathbb{D}$. If we write $\Psi = \Phi^p_A - \Phi^p_{O_n}$, we get

$$\int_{\mathbb{D}} |G^p_A(z) f(z)|^p \, dS(z) = \int_{\mathbb{D}} |G^p_{O_n} f(z)|^p \, dS(z) + \int_{\mathbb{D}} \Psi(z) \Delta |f(z)|^p \, dS(z),$$

again for polynomials f. We plan to show that $\Psi \geqslant 0$ on $\mathbb{D}$; an approximation argument then yields the desired assertion. Clearly, the function Ψ solves the

Poisson equation

$$\begin{cases}\Delta\Psi(z) = |G_A^p(z)|^p - |G_{0_n}^p f(z)|^p, & z \in \mathbb{D},\\ \Psi(z) = 0, & z \in \mathbb{T}.\end{cases}$$

It is known [12], [13], [7] that Φ_A^p and $\Phi_{0_n}^p$ extend real analytically across $\mathbb{T}$, and have $\nabla\Phi_A^p = \nabla\Phi_{0_n}^p = 0$ on $\mathbb{T}$. Consequently, the function Ψ extends real analytically across $\mathbb{T}$, and solves the overdetermined partial differential equation boundary value problem

$$\begin{cases}|z|^{-np}\Delta\Psi(z) = |G_A^p(z)/z^n|^p - (1 + np/2)^{-1} & z \in \mathbb{D},\\ \Psi(z) = 0, & z \in \mathbb{T},\\ \nabla\Psi(z) = 0, & z \in \mathbb{T}.\end{cases}$$

Since we have the extra boundary value information, we may apply a Laplacian to both sides of the differential equation, and still have a uniquely determined solution, according to Section 3. The result is then

$$\begin{cases}\Delta_z|z|^{-np}\Delta_z\Psi(z) = \Delta_z(|G_A^p(z)/z^n|^p) & z \in \mathbb{D},\\ \Psi(z) = 0, & z \in \mathbb{T},\\ \nabla\Psi(z) = 0, & z \in \mathbb{T};\end{cases}$$

note that

$$\Delta_z(|G_A^p(z)/z^n|^p) \geqslant 0, \qquad z \in \mathbb{D},$$

because G_A^p/z^n is holomorphic on $\mathbb{D}$. We may then deduce the identity

$$\Psi(z) = \int_{\mathbb{D}} U_{np/2}(z,\zeta)\Delta_\zeta(|G_A^p(\zeta)/\zeta^n|^p)\, dS(\zeta)(\geqslant 0), \qquad z \in \mathbb{D},$$

simply because the right-hand side solves the same partial differential equation as Ψ, and with a small effort of technical nature, one sees that it also meets the same boundary conditions. □

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Department of Mathematics, Uppsala University, Vox 480, S-75106 Uppsala, Sweden; haakan@tdb.uu.se