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TRANSLATES OF FUNCTIONS OF TWO VARIABLES

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0. Introduction. For $f \in L^1(\mathbf{R}^2)$ and $x \in \mathbf{R}^2$, introduce the translation operator

$$T_x f(t) = f(t - x), \qquad t \in \mathbf{R}^2.$$

Let $\mathbf{R}_+ = (0, \infty)$ and $\mathbf{R}_+^2 = \mathbf{R}_+ \times \mathbf{R}_+$, and regard $L^1(\mathbf{R}_+^2)$ as a closed subspace of $L^1(\mathbf{R}^2)$ by extending the functions to vanish on $\mathbf{R}^2 \backslash \mathbf{R}_+^2$. If f is a function in $L^1(\mathbf{R}_+^2)$, let $I(f)$ be the closure of the linear span of the combined right and upper translates $T_x f$, $x \in \mathbf{R}_+^2$, of f. The aim of the present paper is to attempt to solve the following problem, which was raised by B. Ya. Levin in the late 1950s, according to Boris Korenblum, and recently appeared in [Lev]:

LEVIN'S PROBLEM. *Describe the cyclic vectors of $L^1(\mathbf{R}_+^2)$, that is, characterize those functions $f \in L^1(\mathbf{R}_+^2)$ for which $I(f) = L^1(\mathbf{R}_+^2)$.*

The one-dimensional analog of this problem was solved by Bertil Nyman in his 1950 thesis [Nym] and later independently by V. P. Gurariĭ and B. Ya. Levin [GuL]: the right translates $T_x f$, $x \in \mathbf{R}_+$, of a function $f \in L^1(\mathbf{R}_+)$ span a dense subspace of $L^1(\mathbf{R}_+)$ if and only if

(a) $\hat{f}(z) = \int_0^\infty e^{-tz} f(t)\,dt \neq 0$ for all $z \in \overline{\Pi}_+ = \{w \in \mathbf{C}: \operatorname{Re} w \geqslant 0\}$, and
(b) f does not vanish almost anywhere on any interval $(0, \alpha)$, $\alpha > 0$.

Since both of the above references are somewhat inaccessible, we refer the interested reader to Garth Dales's survey article [Dal], where a proof is given (pp. 196–201), and Gurariĭ's monograph [Gur]. Judging from Nyman's result, one might guess that the right condition in our two-dimensional situation is that

(a′) $\hat{f}(z_1, z_2) = \int_0^\infty \int_0^\infty e^{-t_1 z_1 - t_2 z_2} f(t_1, t_2)\,dt_1\,dt_2 \neq 0$ for all $(z_1, z_2) \in \overline{\Pi}_+^2 = \overline{\Pi}_+ \times \overline{\Pi}_+$, and
(b′) f does not vanish almost everywhere in any neighborhood of the origin.

Clearly (a′) and (b′) are necessary. However, they are far from sufficient. Namely, there are other types of conditions that remain invariant under right and upper translations, too; for instance,

$$\int_0^\infty f(t_1, t_2)\,dt_1 = 0 \quad \text{for almost all} \quad t_2 \in (0, 1),$$

is one.

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Nyman's method of proof, which also was the one used later by Gurariĭ and Levin, was to take an arbitrary $\phi \in L^\infty(\mathbf{R}_+) = L^1(\mathbf{R}_+)^*$ such that $\phi \perp T_x f$ for all $x \in \mathbf{R}_+$, and study its Laplace transform

$$\mathscr{L}[\phi](z) = \int_0^\infty e^{tz}\phi(t)\,dt,$$

which is well defined and analytic in the open left half plane $\Pi_- = \{w \in \mathbf{C}\colon \operatorname{Re} w < 0\}$. He showed that $\mathscr{L}[\phi]$ extends analytically to the set $\mathbf{C}\backslash Z(\hat{f})$, where

$$Z(\hat{f}) = \{w \in \overline{\Pi}_+ \colon \hat{f}(w) = 0\},$$

and that the extension is given by a concrete formula in the open right half plane. In particular, if f meets (a), $\mathscr{L}[\phi]$ extends to an entire function. Finally, Nyman demonstrated that if f also satisfies (b), then that will force $\mathscr{L}[\phi]$ to vanish identically, and so $\phi = 0$; thus $I(f) = L^1(\mathbf{R}_+)$, by the Hahn-Banach theorem.

In our two-dimensional situation, we would be inclined to consider for $\phi \in L^\infty(\mathbf{R}_+^2)$ its Laplace transform

$$\mathscr{L}[\phi](z_1, z_2) = \int_0^\infty \int_0^\infty e^{t_1 z_1 + t_2 z_2}\phi(t_1, t_2)\,dt_1\,dt_2, \qquad (z_1, z_2) \in \Pi_-^2 = \Pi_- \times \Pi_-,$$

and hope that if $\phi \perp T_x f$ for all $x \in \mathbf{R}_+^2$, where f satisfies (a′), then $\mathscr{L}[\phi]$ would extend to an entire function, this being the case in one dimension. Unfortunately, this is not true. Take, for instance, $\phi(t_1, t_2) = 1$ for $(t_1, t_2) \in (0, 1) \times \mathbf{R}_+$ and $\phi(t_1, t_2) = 0$ elsewhere. Then $\phi \perp T_x f$ for all $x \in \mathbf{R}_+^2$ if f is the $L^1(\mathbf{R}_+^2)$ function defined by the relations $f(t_1, t_2) = e^{-t_1 - t_2}$ for $(t_1, t_2) \in [1, \infty) \times \mathbf{R}_+$ and $f(t_1, t_2) = 0$ elsewhere. The Fourier transform $\hat{f}$ of f does not vanish anywhere on $\overline{\Pi}_+^2$, and yet

$$\mathscr{L}[\phi](z_1, z_2) = (e^{z_1} - 1)/(z_1 z_2), \qquad (z_1, z_2) \in \Pi_-^2,$$

does not extend to an entire function.

A crucial property of the Laplace transform in one variable is that the image of a function in $L^\infty(\mathbf{R}_+)$ with compact support is an entire function of finite exponential type. Likewise, we should expect a "good" transformation on $L^\infty(\mathbf{R}_+^2)$ to turn a $\phi \in L^\infty(\mathbf{R}_+^2)$ with support contained in a region

$$R(x_1, x_2) = \{(t_1, t_2) \in \overline{\mathbf{R}}_+^2 \colon t_1 \leqslant x_1 \text{ or } t_2 \leqslant x_2\},$$

for $x_1, x_2 > 0$, into an entire function, and hopefully one of finite exponential type. Clearly, the Laplace transform is unsuitable for this purpose. In this respect, the transformation

$$\mathscr{C}[\phi](\lambda) = \int_0^\infty \int_0^\infty H_\lambda(t_1, t_2)\phi(t_1, t_2)\,dt_1\,dt_2, \qquad \lambda \in \Pi_-,$$

where

$$H_\lambda(t_1, t_2) = \pi^{-1/2}(t_1 + t_2)^{-3/2}((4 + \lambda)t_1 t_2/(t_1 + t_2) + 1/2)$$
$$\cdot \exp((4 + \lambda)t_1 t_2/(t_1 + t_2) - t_1 - t_2),$$

is much more appropriate. Later, we shall see that if $\phi \perp I(f)$ and f meets (a′), then $\mathscr{C}[\phi]$ does indeed extend analytically to the whole complex plane.

The space $L^1(\mathbf{R}_+^2)$ is a commutative Banach algebra (without unit) when equipped with convolution multiplication:

$$f * g(x_1, x_2) = \int_0^{x_2} \int_0^{x_1} f(x_1 - t_1, x_2 - t_2) g(t_1, t_2)\, dt_1\, dt_2, \qquad x_1, x_2 > 0,$$

for $f, g \in L^1(\mathbf{R}_+^2)$. Denote by $A_0(\Pi_+^2)$ the Banach algebra of holomorphic functions on Π_+^2 (Π_+ is the open right half plane) that extend continuously to $\overline{\Pi}_+^2 \cup \{\infty\}$ and have value 0 at ∞. The Fourier transform

$$\hat{f}(z_1, z_2) = \int_0^\infty \int_0^\infty e^{-t_1 z_1 - t_2 z_2} f(t_1, t_2)\, dt_1\, dt_2, \qquad z_1, z_2 \in \overline{\Pi}_+,$$

$f \in L^1(\mathbf{R}_+^2)$, defines a continuous monomorphism (injective homomorphism) $L^1(\mathbf{R}_+^2) \to A_0(\Pi_+^2)$ with dense range. Since a closed (right and upper) translation invariant subspace of $L^1(\mathbf{R}_+^2)$ is the same as a closed ideal, a uniform norm version of Levin's problem would ask the following:

UNIFORM LEVIN PROBLEM. *Which functions $f \in A_0(\Pi_+^2)$ generate an ideal that is dense in $A_0(\Pi_+^2)$?*

In the sections to follow, we shall show that if $f \in A_0(\Pi_+^2)$ has no zeros on $\overline{\Pi}_+^2$ and satisfies the additional condition

$$\log 1/|f(z)| = o(|z|) \tag{0.1}$$

as $|z| \to \infty$ with $z \in \overline{\Pi}_+^2$, then f generates a dense ideal in $A_0(\Pi_+^2)$. We will also show that (0.1) is necessary as $|z| \to \infty$ if z stays within a domain $\Omega_\alpha^2 = \Omega_\alpha \times \Omega_\alpha$ for some $\alpha < \pi/2$, where

$$\Omega_\alpha = \{w \in \mathbf{C}\colon |w| > 1 \text{ and } |\arg w| < \alpha\}.$$

Our precise result, as formulated in Corollary 1.7, the remark thereafter, and Theorem 3.4, narrows the discrepancy between necessary and sufficient conditions even further.

Similar results hold for the algebra $L^1(\mathbf{R}_+^2)$: if $f \in L^1(\mathbf{R}_+^2)$ has a Fourier transform $\hat{f}$ that does not vanish anywhere on $\overline{\Pi}_+^2$, $\hat{f}$ has bounded derivatives of order $\leqslant 2$ on some region $\overline{\Pi}^2_+ \setminus K$, where K is a compact subset of $\overline{\Pi}^2_+$, and

$$\log 1/|\hat{f}(z)| = o(|z|) \tag{0.2}$$

as $|z| \to \infty$ with $z \in \overline{\Pi}^2_+$, then f generates a dense ideal in $L^1(\mathbf{R}^2_+)$. Conversely, as for $A_0(\Pi^2_+)$, (0.2) is necessary as $|z| \to \infty$ for $z \in \Omega^2_\alpha$ if $\alpha < \pi/2$. Our sharpest results are formulated in Theorems 1.8 and 3.1.

In section 4, we study the two-dimensional Volterra algebra $L^1([0, 1]^2)$, with restricted convolution as multiplication. Observe that the radical Banach algebra $L^1([0, 1]^2)$ is isomorphic to $L^1(\mathbf{R}^2_+)/\mathbf{J}$, where $\mathbf{J}$ is the closed ideal

$$\mathbf{J} = \{f \in L^1(\mathbf{R}^2_+): f = 0 \text{ almost everywhere on } [0, 1]^2\}.$$

In [Str], Elizabeth Strouse was interested in the following problem, which is closely related to Levin's problem:

STROUSE'S PROBLEM. *Which functions $f \in L^1([0, 1]^2)$ are cyclic, that is, generate an ideal that is dense in $L^1([0, 1]^2)$?*

In section 4, we will show that this is the case if

$$\log 1/|\hat{f}(z)| = o(|z|)$$

as z approaches the point (∞, ∞) along certain cones inside Π^2_+. The precise result is contained in Theorems 4.6 and 4.7; necessary conditions are stated in Corollary 4.5. Here, $\hat{f}$ denotes the Fourier transform of f:

$$\hat{f}(z) = \int_0^1 \int_0^1 e^{-t_1 z_1 - t_2 z_2} f(t_1, t_2)\, dt_1\, dt_2 .$$

It should be observed that the Fourier transform is not a homomorphism on $L^1([0, 1]^2)$.

1. Necessary conditions. In the sequel, we let $\mathbf{D}$ denote the open unit disc $\{z \in \mathbf{C}: |z| < 1\}$, and denote by $A(\mathbf{D}^n)$ the polydisc algebra, which consists of all continuous functions on $\overline{\mathbf{D}}^n$ that are holomorphic in $\mathbf{D}^n$; when equipped with the supremum norm and pointwise multiplication, $A(\mathbf{D}^n)$ is a Banach algebra with maximal ideal space $\overline{\mathbf{D}}^n$. The space $A(\mathbf{D})$ is called the *disc algebra*, and $A(\mathbf{D}^2)$ is usually referred to as the *bidisc algebra*. For a function $f \in A(\mathbf{D}^n)$, let

$$Z(f) = \{z \in \overline{\mathbf{D}}^n: f(z) = 0\}$$

be its zero set, and denote by $I(f)$ the closure of the principal ideal generated by f.

For a set $E \subset \overline{\mathbf{D}}^n$, introduce the notation

$$\mathscr{I}(E) = \{f \in A(\mathbf{D}^n) : f = 0 \text{ on } E\}.$$

After a Möbius transformation in each variable, we may identify the algebra $A_0(\Pi_+^2)$, as defined in the introduction, with the closed $A(\mathbf{D}^2)$-ideal $\mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\}))$. Then the uniform Levin problem takes the following form:

PROBLEM 1.1. *For which functions* $f \in \mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\}))$ *does* $I(f) = \mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\}))$?

When $(\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$ is replaced by the set $\{1\} \times \overline{\mathbf{D}}$, or a subset thereof, this problem was solved by the author in [Hed2]. In [Hed3], the author introduced the following terminology: A function $f \in A(\mathbf{D}^2)$ is called *BR-outer* if $Z(f) \subset \partial(\mathbf{D}^2)$ and $I(f) = \mathscr{I}(Z(f))$. Problem 1.1 asks for a concrete description of those BR-outer functions $f \in A(\mathbf{D}^2)$ for which $Z(f) = \{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$.

We will need a few preparatory results. The following lemma is an easy consequence of the Herglotz representation.

LEMMA 1.2. *If* $f \in A(\mathbf{D})$ *has* $Z(f) = \{1\}$, *the following conditions are equivalent*:

(a) f *is an outer function.*
(b) $\lim_{t \to 1^-} (1 - t) \log|f(t)| = 0$.
(c) $\log 1/|f(z)| = o(1/(1 - |z|))$ *as* $|z| \to 1^-$.

The next lemma is a simple modification of Lemma 1.1 in [Hed2].

LEMMA 1.3. *Let* $\{f_n\}_0^\infty$ *be a bounded sequence of functions in* $H^\infty(\mathbf{D})$ *that are zero-free on* $\mathbf{D}$, *which converges uniformly on compact subsets of* $\mathbf{D}$ *to a function* $f \in H^\infty(\mathbf{D})$, *also zero-free on* $\mathbf{D}$. *If, for some sequence* $\{\alpha_n\}_0^\infty \subset \mathbf{D}$, $\alpha_n \to 1$,

$$\lim_{n\to\infty} (1 - |\alpha_n|)\log|f_n(\alpha_n)| = -\beta \in [-\infty, 0],$$

then

$$\lim_{t\to 1^-} (1 - t)\log|f(t)| \leqslant -\beta.$$

In particular, β *cannot be infinite.*

PROPOSITION 1.4. *Let* $f \in C(\overline{\mathbf{D}} \times K)$, *where* K *is a compact subset of a finite-dimensional vector space over* $\mathbf{R}$ *or* $\mathbf{C}$, *and suppose* $f(\cdot, \xi) \in A(\mathbf{D})$ *for all* $\xi \in K$. *If* f *is zero-free on* $\mathbf{D} \times K$, *then the following are equivalent*:

(a) $\log 1/|f(z, \xi)| = o(1/(1 - |z|))$ *as* $|z| \to 1$ *for all* $\xi \in K$.
(b) $\sup_{\xi \in K} \log 1/|f(z, \xi)| = o(1/(1 - |z|))$ *as* $|z| \to 1$.

Proof. The implication (b)$\Rightarrow$(a) is trivial. To attack the other one, assume (b) does not hold, that is, there is a sequence $\{z_n\}_0^\infty \subset \mathbf{D}$, $|z_n| \to 1$, such that

$$\lim_{n\to\infty} (1 - |z_n|) \sup_{\xi \in K} \log 1/|f(z_n, \xi)| = \alpha \in (0, \infty].$$

After taking a subsequence and rotating, we may assume without loss of generality that $z_n \to 1$. Since K was compact and f was continuous, there is a sequence $\{\xi_n\}_0^\infty \subset K$ such that

$$\log 1/|f(z_n, \xi_n)| = \sup_{\xi \in K} \log 1/|f(z_n, \xi)|.$$

Replacing $\{\xi_n\}_0^\infty$ by a subsequence, we may assume that ξ_n converges to some $\eta \in K$ as $n \to \infty$. If we apply Lemma 1.3 to the functions $f_n(z) = f(z, \xi_n)$ and $f(z) = f(z, \eta)$, we conclude that

$$\lim_{t\to 1^-} (1 - t)\log 1/|f(t, \eta)| \geqslant \alpha > 0,$$

so (a) cannot hold either. The proof of the proposition is complete.

The pseudohyperbolic metric ρ on $\mathbf{D}$ is given by the formula

$$\rho(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right|, \qquad z, w \in \mathbf{D}.$$

Extend it to $\overline{\mathbf{D}}$ by saying that if either z or w has modulus 1, z say, then $\rho(z, w) = 1$ if $z \neq w$ and $\rho(z, w) = 0$ if $z = w$. For $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\overline{\mathbf{D}}^2$, let the formula

$$\rho(z, w) = \max\{\rho(z_1, w_1), \rho(z_2, w_2)\}$$

define the pseudohyperbolic metric on the bidisc. It enjoys the property (see [Jan, pp. 10–13]) that

$$\rho(z, w) = \sup\{|f(z)|: f \in A(\mathbf{D}^2), \|f\|_\infty \leqslant 1, f(w) = 0\}, \qquad z, w \in \overline{\mathbf{D}}^2.$$

The equivalence relation $\rho(z, w) < 1$ defines the Gleason parts of $\overline{\mathbf{D}}^2$.

Let $\mathscr{U}$ be the collection of all continuous mappings $L: \overline{\mathbf{D}} \to \overline{\mathbf{D}}^2$ such that $f \circ L \in A(\mathbf{D})$ whenever $f \in A(\mathbf{D}^2)$, and $L(z) \in (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$ if and only if $z = 1$. Also, introduce the subclass $\mathscr{L}$ consisting of the analytic discs $L(z) = (z, z)$, $z \in \overline{\mathbf{D}}$, $L(z) = (z, \alpha)$, $z \in \overline{\mathbf{D}}$, and $L(z) = (\alpha, z)$, $z \in \overline{\mathbf{D}}$, where $\alpha \in \overline{\mathbf{D}} \setminus \{1\}$.

LEMMA 1.5. *If $f \in A(\mathbf{D}^2)$ has $I(f) = \mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\}))$, then $f \circ L$ is outer for all $L \in \mathscr{U}$.*

Proof. Let us first show that the $A(\mathbf{D})$-ideal generated by $\mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})) \circ L$ is dense in $\mathscr{I}(\{1\})$. Consider the function

$$g(z_1, z_2) = (1 - z_1)(1 - z_2), \qquad (z_1, z_2) \in \overline{\mathbf{D}}^2,$$

which belongs to $\mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\}))$. Then

$$g \circ L(z) = (1 - L_1(z))(1 - L_2(z)), \qquad z \in \overline{\mathbf{D}},$$

if

$$L(z) = (L_1(z), L_2(z)), \qquad z \in \overline{\mathbf{D}}.$$

Observe that $L_1, L_2 \in A(\mathbf{D})$, and $\|L_1\|_\infty, \|L_2\|_\infty \leqslant 1$. If $|L_1(0)|, |L_2(0)| < 1$, then by Schwarz's lemma,

$$\rho(L(z), L(0)) \leqslant |z|, \qquad z \in \overline{\mathbf{D}},$$

so that

$$|L_j(z)| \leqslant \frac{\rho(L_j(z), L_j(0)) + \rho(L_j(0), 0)}{1 + \rho(L_j(z), L_j(0))\rho(L_j(0), 0)} \leqslant \frac{|L_j(0)| + |z|}{1 + |L_j(0)z|}, \qquad z \in \mathbf{D}, j = 1, 2.$$

Hence

(1.1)
$$1 - |L_j(z)| \geqslant \frac{1 + |L_j(0)z| - |L_j(0)| - |z|}{1 + |L_j(0)z|}$$

$$= (1 - |L_j(0)|)\frac{1 - |z|}{1 + |L_j(0)z|} \geqslant \frac{1}{2}(1 - |L_j(0)|)(1 - |z|), \qquad z \in \mathbf{D}, j = 1, 2,$$

and since $g \circ L$ only vanishes at the point 1, the above estimate forces it to be an outer function, by Lemma 1.1. If either $|L_1(0)$ or $|L_2(0)|$ is 1, $|L_1(0)| = 1$ say, then L_1 is identically constant by the maximum principle, and $L_1(z) \not\equiv 1$. In this case, too, $g \circ L$ is outer by a similar argument. We conclude that $g \circ L$ is outer for all $L \in \mathscr{U}$, so by the Beurling-Rudin theorem [Hof, pp. 82–89], the $A(\mathbf{D})$-ideal generated by $\mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})) \circ L$ is dense in $\mathscr{I}(\{1\})$. If $I(f) = \mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\}))$, then $f \circ L$ and $\mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})) \circ L$ should generate the same closed ideals in $A(\mathbf{D})$, and therefore $f \circ L$ must be outer, again by the Beurling-Rudin theorem, for all $L \in \mathscr{U}$.

THEOREM 1.6. *Let $f \in A(\mathbf{D}^2)$ have $Z(f) = (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$. Then the following three conditions* (a)–(c) *are equivalent*:

(a) *$f \circ L$ is outer for all $L \in \mathscr{L}$.*
(b) *$f \circ L$ is outer for all $L \in \mathscr{U}$.*
(c) *For every compact $K \subset \overline{\mathbf{D}} \setminus \{1\}$,*
  (i) $\log 1/|f(z)| = o(1/(1-|z_1|))$ *as* $z \to \{1\} \times K$,
  (ii) $\log 1/|f(z)| = o(1/(1-|z_2|))$ *as* $z \to K \times \{1\}$, *and*
  (iii) $\log 1/|f(z)| = o(1/(1-|z_1|) + 1/(1-|z_2|))$ *as* $z \to (1, 1)$.

Proof. Clearly, (b) implies (a), because $\mathscr{L} \subset \mathscr{U}$. Let us proceed to check that (c) implies (b). We can distinguish the following three cases:

Case 1: $L(0) \in \mathbf{D}^2$, *that is*, $|L_1(0)|, |L_2(0)| < 1$. By Lemma 1.1 it suffices to show that

$$\log 1/|f \circ L(z)| = o(1/(1-|z|)) \quad \text{as} \quad z \to 1.$$

By (i)–(iii), (iii) holds as $z \to Z(f)$. If we combine this with (1.1), we get

$$\log 1/|f(L_1(z), L_2(z))| = o(1/(1-|L_1(z)|) + 1/(1-|L_2(z)|))$$

$$= o\left(\frac{2}{(1-|L_1(0)|)(1-|z|)} + \frac{2}{(1-|L_2(0)|)(1-|z|)}\right) = o\left(\frac{1}{1-|z|}\right) \text{ as } z - 1,$$

so the assertion follows.

Case 2: $L_1(0) \in \mathbf{T} \setminus \{1\}$, $|L_2(0)| < 1$. Then $L_1(z) \equiv L_1(0)$, so by (ii) and (1.1),

$$\log 1/|f \circ L(z)| = o(2/(1-|L_2(z)|)) = o(1/(1-|z|)) \quad \text{as} \quad z \to 1,$$

so by Lemma 1.1, $f \circ L$ is an outer function.

Case 3: $|L_1(0)| < 1$, $L_2(0) \in \mathbf{T} \setminus \{1\}$. This is dealt with in the same way as case 2.

It remains to show that (c) follows from (a). By Proposition 1.4, the assumption that $f(\cdot, \alpha)$ and $f(\alpha, \cdot)$ are outer for all $\alpha \in \overline{\mathbf{D}} \setminus \{1\}$ implies (i) and (ii). For $(\alpha, \beta) \in (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$, let

$$L_{\alpha,\beta}(z) = (\alpha z, \beta z), \qquad z \in \overline{\mathbf{D}};$$

clearly, $L_{\alpha,\beta} \in \mathscr{U}$. By (a), $f \circ L_{1,1}$ is outer, and by (i) and (ii), $f \circ L_{\alpha,\beta}$ is outer for all other (α, β) in $(\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$. Proposition 1.4 together with Lemma 1.2 now shows that

$$\sup_{(\alpha,\beta) \in (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})} \log 1/|f \circ L_{\alpha,\beta}(z)| = o(1/(1-|z|)), \qquad |z| \to 1,$$

which in turn implies that

$$\sup_{z \in r\mathbf{D}^2} \log 1/|f(z)| = o(1/(1-r)), \qquad r \to 1.$$

This is equivalent to

$$\log 1/|f(z)| = o(1/(1-|z_1|) + 1/(1-|z_2|)), \qquad z \to \partial(\mathbf{D}^2),$$

and (iii) follows. The proof of the theorem is complete.

Remark. In [Hed2], the author characterized which functions $f \in A(\mathbf{D}^2)$ have $I(f) = \mathscr{I}(\{1\} \times \overline{\mathbf{D}})$. If $f \in A(\mathbf{D}^2)$, $Z(f) = (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$, and $|f(z)| \geqslant |g(z) \cdot h(z)|$, $z \in \mathbf{D}^2$, where g and h are functions in $A(\mathbf{D}^2)$ for which $I(g) = \mathscr{I}(\{1\} \times \overline{\mathbf{D}})$ and $I(h) = \mathscr{I}(\overline{\mathbf{D}} \times \{1\})$, it is not hard to show that $I(f) = \mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\}))$. It is not clear which functions f can be dominated from below in this fashion.

Let us translate Theorem 1.6 to the algebra $A_0(\Pi_+^2)$. The compactification of $\overline{\Pi}_+^2$ that we will use is $(\overline{\Pi}_+ \cup \{\infty\})^2$. The class $\mathscr{U}_*$ will consist of all continuous mappings $L: \overline{\Pi}_+ \cup \{\infty\} \to (\overline{\Pi}_+ \cup \{\infty\})^2$ such that $f \circ L \in A_0(\Pi_+)$ whenever $f \in A_0(\Pi_+^2)$, and $L(z) \in \overline{\Pi}_+^2$ if and only if $z \neq \infty$. Moreover, the subclass $\mathscr{L}_*$ will consist of the "analytic half-planes" $L(z) = (z, z)$, $z \in \overline{\Pi}_+ \cup \{\infty\}$, $L(z) = (z, \alpha)$, $z \in \overline{\Pi}_+ \cup \{\infty\}$, and $L(z) = (\alpha, z)$, $z \in \overline{\Pi}_+ \cup \{\infty\}$, where $\alpha \in \overline{\Pi}_+$.

COROLLARY 1.7. *Let $f \in A_0(\Pi_+^2)$ be zero-free on $\overline{\Pi}_+^2$. Then the following three conditions* (a)–(c) *are equivalent*:

(a) *$f \circ L$ is outer for all $L \in \mathscr{L}_*$.*
(b) *$f \circ L$ is outer for all $L \in \mathscr{U}_*$.*
(c) *For every compact $K \subset \overline{\Pi}_+$,*
  (i) $\log 1/|f(z)| = o(|z_1|^2/\mathrm{Re}\, z_1)$ *as $z \to \{\infty\} \times K$,*
  (ii) $\log 1/|f(z)| = o(|z_2|^2/\mathrm{Re}\, z_2)$ *as $z \to K \times \{\infty\}$, and*
  (iii) $\log 1/|f(z)| = o(|z_1|^2/\mathrm{Re}\, z_1 + |z_2|^2/\mathrm{Re}\, z_2)$ *as $z \to (\infty, \infty)$.*

Remark. By Lemma 1.5, each of the three equivalent conditions in Corollary 1.7 is necessary for f to generate an ideal that is dense in $A_0(\Pi_+^2)$.

Let us try to translate condition (a) of Corollary 1.7 to $L^1(\mathbf{R}_+^2)$. To do this we need to introduce some notation. If $f \in L^1_{loc}(\mathbf{R}^n)$, let supp f be its essential support, that is, the smallest closed set $E \subset \mathbf{R}^n$ such that f vanishes almost everywhere off E. If $f \in L^1(\mathbf{R}_+^2)$, let

$$\mathscr{F}_1 f(z_1, t_2) = \int_0^\infty e^{-t_1 z_1} f(t_1, t_2)\, dt_1 \qquad z_1 \in \overline{\Pi}_+, t_2 \in \mathbf{R}_+$$

be its partial Fourier transform with respect to the first coordinate, and let

$$\mathscr{F}_2 f(t_1, z_2) = \int_0^\infty e^{-t_2 z_2} f(t_1, t_2)\, dt_2, \qquad t_1 \in \mathbf{R}_+, z_2 \in \overline{\Pi}_+,$$

be its partial Fourier transform with respect to the second. If $f \in L^1(\mathbf{R}_+^2)$ is such that $\hat{f}$ is zero-free on $\overline{\Pi}_+^2$, and $\alpha \in \overline{\Pi}_+$, the condition that $\hat{f}(\cdot, \alpha)$ is an outer function on Π_+ is equivalent to $0 \in \operatorname{supp} \mathscr{F}_2 f(\cdot, \alpha)$ (see [Koo], pp. 183–184), and similarly, $\hat{f}(\alpha, \cdot)$ is outer if and only if $0 \in \operatorname{supp} \mathscr{F}_1 f(\alpha, \cdot)$; the condition that $\hat{f} \circ L$ is outer for the diagonal $L(z) = (z, z)$ is equivalent to $0 \in \operatorname{supp} g$, where

$$g(x) = \int_0^z f(t, x-t)\, dt, \qquad x > 0.$$

Let us collect these observations in a theorem.

THEOREM 1.8. *If $f \in L^1(\mathbf{R}_+^2)$ is cyclic, then*

(a) $\hat{f}(z) \neq 0$ *for all* $z \in \overline{\Pi}_+^2$,
(b) $0 \in \operatorname{supp} \mathscr{F}_1 f(\alpha, \cdot)$ *for all* $\alpha \in \overline{\Pi}_+$,
(c) $0 \in \operatorname{supp} \mathscr{F}_2 f(\cdot, \alpha)$ *for all* $\alpha \in \overline{\Pi}_+$,
(d) $0 \in \operatorname{supp} g$, *where*

$$g(x) = \int_0^x f(t, x-t)\, dt, \qquad x > 0.$$

Moreover, if a function $f \in L^1(\mathbf{R}_+^2)$ satisfies condition (a), *then it meets* (b)–(d) *if and only if $\hat{f}$ satisfies one of the equivalent conditions* (a)–(c) *of Corollary* 1.7.

There are other conditions, similar to (d) of Theorem 1.8, which are necessary for a function $f \in L^1(\mathbf{R}_+^2)$ to be cyclic. Let $g_{\lambda,\xi}$ be the $L^1(\mathbf{R}_+)$ function

$$g_{\lambda,\xi}(x) = \int_0^x f(\lambda t, x-t) e^{-\xi t}\, dt, \qquad x > 0$$

for $\lambda > 0$ and $\xi \in \overline{\Pi}_+$. It is not hard to see that 0 must be in the essential support of $g_{\lambda,\xi}$ if f is to be cyclic in $L^1(\mathbf{R}_+^2)$. The strength of Theorem 1.8 is that it asserts that this condition is satisfied if (a)–(d) are met. This is so because if f meets (a), then $0 \in \operatorname{supp} g_{\lambda,\xi}$ if and only if $\hat{f} \circ L_{\lambda,\xi}$ is outer (compute the Fourier transform of $g_{\lambda,\xi}$, compare it with $\hat{f} \circ L_{\lambda,\xi}$, and use [Koo, pp. 183–184]), where

$$L_{\lambda,\xi}(z) = (z, \lambda z + \xi), \qquad z \in \overline{\Pi}_+,$$

and clearly, $L_{\lambda,\xi} \in \mathscr{U}_*$.

Question 1.9. Is the necessary condition obtained in Lemma 1.5 also sufficient, that is, if $f \in A(\mathbf{D}^2)$ has $Z(f) = (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$ and $f \circ L$ is outer for all $L \in \mathscr{U}$, does it follow that $I(f) = \mathscr{I}(Z(f))$?

Question 1.10. If $f \in L^1(\mathbf{R}_+^2)$ meets (a)–(d) of Theorem 1.8, does it follow that f is cyclic in $L^1(\mathbf{R}_+^2)$?

Clearly these questions are strongly interrelated. If the answer to Question 1.9 is negative, then most likely the answer to Question 1.10 is negative as well.

The proof of Lemma 1.5 can be modified to show that if $f \in A(\mathbf{D}^2)$ has $I(f) = \mathscr{I}((\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\}))$, then $f \circ L$ is outer for all continuous mappings $L = (L_1, L_2)\colon \overline{\mathbf{D}} \to \overline{\mathbf{D}}^2$ such that $L_1, L_2 \in A(\mathbf{D})$ and $L(0) \in \mathbf{D}^2$. It is not clear whether this follows from any one of the equivalent conditions (a)–(c) of Theorem 1.6.

In section 3, we shall see that if $f \in A(\mathbf{D}^2)$ has $Z(f) = (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$, then $I(f) = \mathscr{I}(Z(f))$ holds if

$$\log 1/|f(z)| = o(1/d(z)) \quad \text{as} \quad \mathbf{D}^2 \ni z \to Z(f),$$

where $d(z)$ is the Euclidean distance between z and $Z(f)$.

2. An analytic semigroup. Let

$$a^\alpha(t_1, t_2) = \frac{\alpha}{2\sqrt{\pi}\,\Gamma((\alpha+1)/2)} \cdot (t_1 t_2)^{(\alpha-1)/2} \cdot (t_1 + t_2)^{-\alpha/2-1}, \qquad t_1, t_2 > 0$$

for $\operatorname{Re} \alpha > 0$, which is a locally integrable function on $\mathbf{R}^2_+$. Then the function

$$A^\alpha(t_1, t_2) = a^\alpha(t_1, t_2) e^{-t_1 - t_2}, \qquad t_1, t_2 > 0$$

is in $L^1(\mathbf{R}^2_+)$, and we shall see that A^α is an analytic semigroup in $L^1(\mathbf{R}^2_+)$ over Π_+. An extensive theory of analytic semigroups in convolution Banach algebras has been developed in recent years (see [Sin]). For instance, Jean Esterle [Est] has shown that Wiener's Tauberian theorem follows from the existence of a certain analytic semigroup in $L^1(\mathbf{R})$.

THEOREM 2.1. *The functions A^α form an analytic semigroup in $L^1(\mathbf{R}^2_+)$ over Π_+, that is,*

(a) *the mapping $\alpha \mapsto A^\alpha$, $\alpha \in \Pi_+$, is analytic, and*
(b) *$A^{\alpha+\beta} = A^\alpha * A^\beta$ for all $\alpha, \beta \in \Pi_+$.*

Moreover,

$$\widehat{a^\alpha}(z_1, z_2) = (\sqrt{z_1} + \sqrt{z_2})^{-\alpha}, \qquad (z_1, z_2) \in \Pi^2_+,$$

and

$$\widehat{A^\alpha}(z_1, z_2) = (\sqrt{z_1 + 1} + \sqrt{z_2 + 1})^{-\alpha}, \qquad (z_1, z_2) \in \overline{\Pi}^2_+,$$

Remark. It follows that a^α satisfies all reasonable criteria for being an analytic semigroup in $L^1_{loc}(\mathbf{R}^2_+)$ over Π_+.

Proof of Theorem 2.1. Clearly, $A^\alpha(t_1, t_2)$ is an analytic function of α when $(t_1, t_2) \in \mathbf{R}^2_+$ is fixed, and $\|A^\alpha\|$ varies continuously with $\alpha \in \Pi_+$, so by Lemma 2.7 in [Sin, p. 16], (a) follows.

We will concentrate on showing that

$$\widehat{a^\alpha}(z_1, z_2) = \int_0^\infty \int_0^\infty e^{-t_1 z_1 - t_2 z_2} a^\alpha(t_1, t_2)\, dt_1\, dt_2$$

$$= (\sqrt{z_1} + \sqrt{z_2})^{-\alpha}, \qquad (z_1, z_2) \in \Pi_+^2,$$

because the other statements are an easy consequence of this. By formula 3.383.7 in [GrR, p. 319],

$$\int_0^\infty x^\nu (x + \beta)^{-\nu - 1/2} e^{-\mu x}\, dx = 2^\nu \Gamma(\nu) \beta^{-1/2} e^{\beta\mu/2} \cdot D_{-2\nu}(\sqrt{2\beta\mu}),$$

where $D_{-2\nu}$ is defined on p. 1064 [GrR], so if we plug in $x = t_1$, $\mu = z_1$, $\beta = t_2$, and $\nu = (\alpha + 1)/2$, we get

$$\int_0^\infty t_1^{(\alpha-1)/2} (t_1 + t_2)^{-\alpha/2 - 1} e^{-t_1 z_1}\, dt_1$$

$$= 2^{(\alpha+1)/2} \Gamma((\alpha + 1)/2) t_2^{-1/2} e^{t_2 z_1/2} D_{-\alpha-1}(\sqrt{2 t_2 z_1}),$$

so that

$$\int_0^\infty a^\alpha(t_1, t_2) e^{-t_1 z_1}\, dt_1 = 2^{(\alpha-1)/2} \cdot \frac{\alpha}{\sqrt{\pi}} \cdot t_2^{\alpha/2 - 1} e^{t_2 z_1/2} D_{-\alpha-1}(\sqrt{2 t_2 z_1}).$$

According to formula 7.725.6 in [GrR, p. 887],

$$\int_0^\infty t_2^{\alpha/2 - 1} e^{t_2 z_1/2} D_{-\alpha-1}(\sqrt{2 t_2 z_1}) e^{-t_2 z_2}\, dt_2$$

$$= 2^{1/2 - 3\alpha/2} \cdot \frac{\sqrt{\pi}}{\alpha} z_2^{-\alpha/2} \cdot F((\alpha + 1)/2, \alpha/2; \alpha + 1; 1 - z_1/z_2),$$

if $\mathrm{Re}(z_2/z_1) > 1/2$, and so

$$\widehat{a^\alpha}(z_1, z_2) = 2^{-\alpha} z_2^{-\alpha/2} \cdot F((\alpha + 1)/2, \alpha/2; \alpha + 1; 1 - z_1/z_2).$$

According to formula 9.132.1 in [GrR, p. 1043],

$$F((\alpha + 1)/2, \alpha/2; \alpha + 1; 1 - z_1/z_2)$$

$$= (z_2/z_1)^{(\alpha+1)/2} \cdot (-\alpha 2^\alpha) \cdot F((\alpha + 1)/2, \alpha/2 + 1; 3/2; z_2/z_1)$$

$$+ (z_2/z_1)^{\alpha/2} \cdot 2^\alpha \cdot F(\alpha/2, (\alpha + 1)/2; 1/2; z_2/z_1),$$

and by formulas 9.121.2 and 9.121.4 in [GrR, p. 1040],

$$F((\alpha+1)/2, \alpha/2+1; 3/2; z_2/z_1)$$
$$= -(2\alpha)^{-1} \cdot z_2^{-1/2} z_1^{(\alpha+1)/2}((\sqrt{z_1}+\sqrt{z_2})^{-\alpha} - (\sqrt{z_1}-\sqrt{z_2})^{-\alpha}),$$

and

$$F(\alpha/2, (\alpha+1)/2; 1/2; z_2/z_1) = (1/2) z_1^{\alpha/2}((\sqrt{z_1}+\sqrt{z_2})^{-\alpha} + (\sqrt{z_1}-\sqrt{z_2})^{-\alpha}).$$

It follows that

$$\widehat{a^\alpha}(z_1, z_2) = 2^{-\alpha} z_2^{-\alpha/2} \cdot z_2^{\alpha/2} \cdot 2^{\alpha-1}((\sqrt{z_1}+\sqrt{z_2})^{-\alpha} - (\sqrt{z_1}-\sqrt{z_2})^{-\alpha}$$
$$+ (\sqrt{z_1}+\sqrt{z_2})^{-\alpha} + (\sqrt{z_1}-\sqrt{z_2})^{-\alpha}) = (\sqrt{z_1}+\sqrt{z_2})^{-\alpha},$$

when $\operatorname{Re}(z_2/z_1) > 1/2$, and since $\widehat{a^\alpha}$ is holomorphic on Π_+^2, the above equality holds on all of Π_+^2, and the assertion follows.

There are many other interesting semigroups in $L^1(\mathbf{R}_+^2)$. One is the fractional integration semigroup (see [Sin])

$$I^\alpha(t_1, t_2) = (\Gamma(\alpha))^{-2} \cdot (t_1 t_2)^{\alpha-1} e^{-t_1-t_2}, \qquad t_1, t_2 > 0, \quad \operatorname{Re}\alpha > 0.$$

Its Fourier transform is

$$\widehat{I^\alpha}(z_1, z_2) = (z_1+1)^{-\alpha}(z_2+1)^{-\alpha}, \qquad z_1, z_2 \in \overline{\Pi}_+.$$

The reason why we prefer to use the analytic semigroup A^α instead of I^α is that the Fourier transform $\widehat{A^\alpha}(z_1, z_2)$ of A^α decreases at the same rate in all directions as $|(z_1, z_2)| \to \infty$, whereas $\widehat{I^\alpha}(z_1, z_2)$ decreases faster along the diagonal than when one of the variables is fixed as $|(z_1, z_2)| \to \infty$. Why this behavior is preferable will become clearer in the next section (see the remark after Corollary 3.3).

PROPOSITION 2.2. *The function*

$$H_\lambda(t_1, t_2) = \pi^{-1/2}(t_1+t_2)^{-3/2} \cdot ((4+\lambda)t_1 t_2/(t_1+t_2) + 1/2)$$
$$\cdot \exp((4+\lambda)t_1 t_2/(t_1+t_2) - t_1 - t_2), \qquad t_1, t_2 > 0,$$

is in $L^1(\mathbf{R}_+^2)$ *for* $\operatorname{Re}\lambda < 0$ *and has the Fourier transform*

$$\widehat{H_\lambda}(z_1, z_2) = \frac{\sqrt{z_1+1}+\sqrt{z_2+1}}{(\sqrt{z_1+1}+\sqrt{z_2+1})^2 - 4 - \lambda}, \qquad z_1, z_2 \in \overline{\Pi}_+.$$

Moreover, the $L^1(\mathbf{R}_+^2)$ *norm of* H_λ *satisfies*

$$\|H_\lambda\| \leqslant 25(5 + |\lambda|)(1 + |\operatorname{Re} \lambda|^{-2}), \qquad \operatorname{Re} \lambda < 0. \tag{2.1}$$

Proof. Let us first show that $H_\lambda \in L^1(\mathbf{R}_+^2)$ for $\operatorname{Re} \lambda < 0$ and that (2.1) is satisfied. Introduce the regions

$$\Omega_0 = \{(t_1, t_2) \in \mathbf{R}_+^2 : t_1^2 + t_2^2 < 1\},$$

$$\Omega_1 = \{(t_1, t_2) \in \mathbf{R}_+^2 : t_1^2 + t_2^2 > 1, t_2 < t_1/2\},$$

and

$$\Omega_2 = \{(t_1, t_2) \in \mathbf{R}_+^2 : t_1^2 + t_2^2 > 1, t_1/2 < t_2 < t_1\};$$

by symmetry,

(2.2)
$$\int_0^\infty \int_0^\infty |H_\lambda(t_1, t_2)| dt_1\, dt_2 = \iint_{\Omega_0} |H_\lambda(t_1, t_2)|\, dt_1\, dt_2 + 2 \iint_{\Omega_1} |H_\lambda(t_1, t_2)|\, dt_1\, dt_2 + 2 \iint_{\Omega_2} |H_\lambda(t_1, t_2)|\, dt_1\, dt_2 .$$

Since

$$(t_1 + t_2)^{-3/2}(|4 + \lambda| t_1 t_2/(t_1 + t_2) + 1/2) \leqslant (|\lambda| + 4)(t_1^2 + t_2^2)^{-1/4} + \tfrac{1}{2}(t_1^2 + t_2^2)^{-3/4}, \qquad (t_1, t_2) \in \mathbf{R}_+^2,$$

and

$$(\lambda + 4)t_1 t_2/(t_1 + t_2) - t_1 - t_2 = \lambda t_1 t_2/(t_1 + t_2) - (t_1 - t_2)^2/(t_1 + t_2),$$

it follows that for $\operatorname{Re} \lambda < 0$,

$$|H_\lambda(t_1, t_2) \leqslant \pi^{-1/2}((|\lambda| + 4)(t_1^2 + t_2^2)^{-1/4} + \tfrac{1}{2}(t_1^2 + t_2^2)^{-3/4}), \qquad (t_1, t_2) \in \mathbf{R}_+^2, \tag{2.3}$$

and

$$|H_\lambda(t_1, t_2) \leqslant \pi^{-1/2}(|\lambda| + 5)\exp(\lambda t_1 t_2/(t_1 + t_2) - (t_1 - t_2)^2/(t_1 + t_2)), \qquad (t_1, t_2) \in \mathbf{R}_+^2 \setminus \Omega_0. \tag{2.4}$$

From (2.3) we obtain, by shifting to polar coordinates, that

$$\iint_{\Omega_0} |H_\lambda(t_1, t_2)|\, dt_1\, dt_2 \leqslant \frac{\pi^{1/2}}{2}\left((|\lambda| + 4)\int_0^1 r^{1/2}\, dr + \frac{1}{2}\int_0^1 r^{-1/2}\, dr\right)$$

$$= \frac{\pi^{1/2}}{2}(2|\lambda|/3 + 11/3) \leqslant |\lambda| + 4.$$

On the region Ω_1, $t_1 - t_2 \geqslant t_1/2$ and $t_1 + t_2 \leqslant 3t_1/2$, so that for $\operatorname{Re}\lambda < 0$,

$$\iint_{\Omega_1} |H_\lambda(t_1, t_2)|\, dt_1\, dt_2 \leqslant \pi^{-1/2}(|\lambda| + 5)\int_0^\infty e^{-t_1/6}(t_1/2)\, dt_1 \leqslant 12(|\lambda| + 5),$$

by (2.4). On Ω_2, $t_1 t_2 \geqslant t_1^2/2$ and $t_1 + t_2 \leqslant 2t_1$, so that for $\operatorname{Re}\lambda < 0$,

$$\iint_{\Omega_2} |H_\lambda(t_1, t_2)|\, dt_1\, dt_2$$

$$\leqslant \pi^{-1/2}(|\lambda| + 5)\int_0^\infty e^{-|\operatorname{Re}\lambda| t_1/4}\cdot(t_1/2)\, dt_1 \leqslant 5(|\lambda| + 5)\cdot|\operatorname{Re}\lambda|^{-2},$$

again by (2.4). The assertion (2.1) now follows from (2.2).

Let us show that H_λ has the prescribed Fourier transform. Observe that

$$H_\lambda = \sum_{n=0}^\infty (4 + \lambda)^n A^{2n+1}, \qquad |4 + \lambda| < 4.$$

Hence

$$\widehat{H_\lambda}(z_1, z_2) = \sum_{n=0}^\infty (4 + \lambda)^n (\widehat{A^1}(z_1, z_2))^{2n+1}$$

$$= \widehat{A^1}(z_1, z_2)(1 - (4 + \lambda)\widehat{A^2}(z_1, z_2))^{-1}$$

$$= \frac{\sqrt{z_1 + 1} + \sqrt{z_2 + 1}}{(\sqrt{z_1 + 1} + \sqrt{z_2 + 1})^2 - 4 - \lambda}, \qquad z_1, z_2 \in \overline{\Pi}_+,$$

for $|4 + \lambda| < 4$, and since $\widehat{H_\lambda}$ is analytic in $\lambda \in \Pi_-$, the assertion follows.

For an ideal I in $L^1(\mathbf{R}_+^2)$, let

$$Z(I) = \bigcap_{f \in I} Z(\hat{f}),$$

where

$$Z(\hat{f}) = \{z \in \overline{\Pi}^2_+ : \hat{f}(z) = 0\}.$$

THEOREM 2.3. *Let I be a closed ideal in $L^1(\mathbf{R}^2_+)$ such that $Z(I) = \varnothing$, and let $\phi \in L^\infty(\mathbf{R}^2_+)$ annihilate I. Then the function*

$$\mathscr{C}[\phi](\lambda) = \langle H_\lambda, \phi\rangle = \int_0^\infty \int_0^\infty H_\lambda(t_1, t_2)\phi(t_1, t_2)\, dt_1\, dt_2, \qquad \operatorname{Re}\lambda < 0,$$

where H_λ is as in Proposition 2.2, extends to an entire function. Furthermore, if $d^2/d\lambda^2\, \mathscr{C}[\phi] \equiv 0$ for every $\phi \in L^\infty(\mathbf{R}^2_+)$ that annihilates I, then $I = L^1(\mathbf{R}^2_+)$.

Proof. Let $L^1_e(\mathbf{R}^2_+)$ denote the unitization of $L^1(\mathbf{R}^2_+)$, where we identify the unit with the Dirac measure δ_0 at the origin (0, 0). The maximal ideal space of $L^1_e(\mathbf{R}^2_+)$ can be identified with $\overline{\Pi}^2_+ \cup \{\infty\}$ in a natural way. Extend ϕ to $L^1_e(\mathbf{R}^2_+)$ by defining $\langle \delta_0, \phi\rangle = 0$. The element $\delta_0 - (4 + \lambda)A^2 + I$ of the quotient Banach algebra $L^1_e(\mathbf{R}^2_+)/I$ is invertible for all $\lambda \in \mathbf{C}$, because it is not contained in the unique maximal ideal (corresponding to the point at infinity) of $L^1_e(\mathbf{R}^2_+)/I$. By checking Fourier transforms, it is easy to see that

$$H_\lambda = A^1 * (\delta_0 - (4 + \lambda)A^2)^{*-1}, \qquad \operatorname{Re}\lambda < 0,$$

and

$$\partial^2 H_\lambda/\partial\lambda^2 = 2A^5 * (\delta_0 - (4 + \lambda)A^2)^{*-3}, \qquad \operatorname{Re}\lambda < 0;$$

it follows that $\partial^2 H_\lambda/\partial\lambda^2 \in L^1(\mathbf{R}^2_+)$ for $\lambda \in \Pi_-$. The formula

$$\mathscr{C}[\phi](\lambda) = \langle (A^1 + I) * (\delta_0 - (4 + \lambda)A^2 + I)^{*-1}, \phi\rangle, \qquad \lambda \in \mathbf{C},$$

defines the analytic extension of $\mathscr{C}[\phi]$ to the whole complex plane.

If $d^2/d\lambda^2\, \mathscr{C}[\phi] \equiv 0$ for every $\phi \in L^\infty(\mathbf{R}^2_+)$ that annihilates I, then $\partial^2 H_\lambda/\partial\lambda^2 \in I$ for all $\lambda \in \Pi_-$, in particular for $\lambda = -4$, so that $A^5 \in I$. By Lemma 2.8 in [Sin, p. 17], $A^5 * L^1(\mathbf{R}^2_+)$ is dense in $L^1(\mathbf{R}^2_+)$, and hence $I = L^1(\mathbf{R}^2_+)$.

3. Sufficient conditions. In this section, we shall obtain some sufficient conditions for Levin's problem and the uniform Levin problem, mainly in terms of the decrease of the Fourier transform of the given function. Our proofs will follow the strategy outlined in [Hed3], where a general method to attack problems of this type was developed.

For the algebra $L^1(\mathbf{R}^2_+)$, the best result we have been able to obtain is the following.

THEOREM 3.1. *Let $f \in L^1(\mathbf{R}^2_+)$ be such that $\hat{f}$ has bounded derivatives of order $\leqslant 2$ on some region $\overline{\Pi}^2_+ \setminus K$, where K is a compact subset of $\overline{\Pi}^2_+$, and let $w(z) = (\sqrt{z_1+1} + \sqrt{z_2+1})^2 - 4$. Moreover, let $M\colon (0, \infty) \to [1, \infty)$ be a continuous decreasing function such that*

$$\int_0^1 \log M(x)\, dx < \infty .$$

Then f is cyclic in $L^1(\mathbf{R}^2_+)$ if

(a) *$\hat{f}(z) \neq 0$ for all $z \in \overline{\Pi}^2_+$,*
(b) *$0 \in \operatorname{supp} \mathscr{F}_1 f(z_1, \cdot)$ for all $z_1 \in \overline{\Pi}_+$,*
(c) *$0 \in \operatorname{supp} \mathscr{F}_2 f(\cdot, z_2)$ for all $z_2 \in \overline{\Pi}_+$,*
(d) *$\log 1/|\hat{f}(z)| = o(|z|^2/\operatorname{Re} w(z))$ for $\operatorname{Re} w(z) \geqslant 1$, as $\overline{\Pi}^2_+ \ni z \to (\infty, \infty)$, and*
(e) *$\log 1/|\hat{f}(z)| = O(\exp(\varepsilon|z|^{1/2}) + M(\operatorname{Re} w(z)))$ as $|z| \to \infty$ with $z \in \overline{\Pi}^2_+$ for all $\varepsilon > 0$.*

Remark. Of these conditions, (a)–(c) are necessary by Theorem 1.8, and (d) is somewhat stronger than condition (d) of Theorem 1.8. The global condition (e) is necessary to make our proof work; it would be interesting to know whether the result remains true without it.

Before we prove Theorem 3.1, let us state two corollaries.

COROLLARY 3.2. *Let $f \in L^1(\mathbf{R}^2_+)$ be such that $\hat{f}$ has bounded derivatives of order $\leqslant 2$ on some region $\overline{\Pi}^2_+ \setminus K$, where K is a compact subset of $\overline{\Pi}^2_+$, and let $w(z) = (\sqrt{z_1 + 1} + \sqrt{z_2 + 1})^2 - 4$. Then f is cyclic in $L^1(\mathbf{R}^2_+)$ if*

(a) *$\hat{f}(z) \neq 0$ for all $z \in \overline{\Pi}^2_+$, and*
(b) *$\log 1/|\hat{f}(z)| = o(|z|^2/\operatorname{Re} w(z))$ as $|z| \to \infty$ with $z \in \overline{\Pi}^2_+$.*

Proof. Condition (d) of Theorem 3.1 is trivially satisfied. Conditions (i)–(iii) of Corollary 1.7 follow from (a) and (b) by the simple estimate

$$|z|^2/\operatorname{Re} w(z) \leqslant |z|^2/(\operatorname{Re} z_1 + \operatorname{Re} z_2), \qquad z \in \overline{\Pi}^2_+,$$

so conditions (b) and (c) of Theorem 3.1 are met by Theorem 1.8. Finally, condition (e) of Theorem 3.1 is satisfied if our choice of M is $M(x) = 1 + x^{-2}$.

COROLLARY 3.3. *Let $f \in L^1(\mathbf{R}^2_+)$ be such that $\hat{f}$ has bounded derivatives of order $\leqslant 2$ on some region $\overline{\Pi}^2_+ \setminus K$, where K is a compact subset of $\overline{\Pi}^2_+$. Then f is cyclic in $L^1(\mathbf{R}^2_+)$ if*

(a) *$\hat{f}(z) \neq 0$ for all $z \in \overline{\Pi}^2_+$, and*
(b) *$\log 1/|\hat{f}(z)| = o(|z|)$ as $|z| \to \infty$ with $z \in \overline{\Pi}^2_+$.*

Remark. Had we used the fractional integration semigroup $\{I^\alpha\}_{\operatorname{Re}\alpha > 0}$, defined in section 2, instead of $\{A^\alpha\}_{\operatorname{Re}\alpha > 0}$, in a way we have not made precise, we would have replaced condition (b) of Corollary 3.3 by the stronger condition

$$\log 1/|\hat{f}(z)| = o(|z_1 + 1|^{1/2}|z_2 + 1|^{1/2}) \quad \text{as} \quad |z| \to \infty \quad \text{with} \quad z \in \overline{\Pi}_+^2,$$

which is fine near the diagonal $z_1 = z_2$, but requires a different order of magnitude when one of the variables remains bounded.

Proof of Theorem 3.1. Observe that $w(\Pi_+^2) = \Pi_+$, so that $\operatorname{Re} w(z) \geqslant 0$ for all $z \in \overline{\Pi}_+^2$. By Theorem 1.8 and the observation that (d) is stronger than condition (d) of that theorem, we have that for every compact $X \subset \overline{\Pi}_+$,

$$\log 1/|\hat{f}(z)| = o(|z_1|^2/\operatorname{Re} z_1) \quad \text{as} \quad z_1 \to \infty \quad \text{and} \quad z_2 \in X, \tag{3.1}$$

and

$$\log 1/|\hat{f}(z)| = o(|z_2|^2/\operatorname{Re} z_2) \quad \text{as} \quad z_2 \to \infty \quad \text{and} \quad z_1 \in X. \tag{3.2}$$

We want to obtain estimates of $\log 1/|\hat{f}(z)|$ in terms of the function $w(z)$. Since $|w(z)|$ is proportional to $|z|$ as $|z| \to \infty$ with $z \in \overline{\Pi}_+^2$, (d) states that

$$\log 1/|\hat{f}(z)| = o(|w(z)|^2/\operatorname{Re} w(z)) \quad \text{for} \quad \operatorname{Re} w(z) \geqslant 1, \quad \text{as} \quad \overline{\Pi}_+^2 \ni z \to (\infty, \infty),$$

or, in other words, given an $\varepsilon > 0$, there is an $R(\varepsilon)$ such that

$$\log 1/|\hat{f}(z)| \leqslant \varepsilon |w(z)|^2/\operatorname{Re} w(z) \tag{3.3}$$

for $z \in \overline{\Pi}_+^2$ with $|z_1|, |z_2| > R(\varepsilon)$ and $\operatorname{Re} w(z) \geqslant 1$. If $|z_2| \leqslant R(\varepsilon)$ and $\delta > 0$, then by (3.1) there is an $S(\delta, \varepsilon)$ such that

$$\log 1/|\hat{f}(z)| \leqslant \delta |z_1|^2/\operatorname{Re} z_1 \quad \text{for} \quad |z_1| \geqslant S(\delta, \varepsilon). \tag{3.4}$$

We will now show that there is a number $A(\varepsilon)$ such that $\operatorname{Re} z_1 \geqslant \operatorname{Re} w(z)/2$ on the set

$$U(\varepsilon) = \{z \in \overline{\Pi}_+^2 : |z_2| \leqslant R(\varepsilon) \text{ and } A(\varepsilon)(1 + |w(z)|^{1/2}) \leqslant \operatorname{Re} w(z)\}.$$

To this end, observe that

$$|\sqrt{z_1 + 1}| = |z_1 + 1|^{1/2} \leqslant |(\sqrt{z_1 + 1} + \sqrt{z_2 + 1})^2|^{1/2} \leqslant |w(z)|^{1/2} + 2, \qquad z \in \overline{\Pi}_+^2,$$

and that

$$|\sqrt{z_2 + 1}| \leqslant R(\varepsilon)^{1/2} + 1, \qquad z \in U(\varepsilon).$$

If $A(\varepsilon)$ is suitably chosen, it now follows that

$$\operatorname{Re} z_2 + 2 \operatorname{Re}(\sqrt{z_1+1}\sqrt{z_2+1} - 1) \leqslant \tfrac{1}{2} \operatorname{Re} w(z), \qquad z \in U(\varepsilon),$$

from which the assertion immediately follows, since

$$\operatorname{Re} w(z) = \operatorname{Re} z_1 + \operatorname{Re} z_2 + 2 \operatorname{Re}(\sqrt{z_1+1}\sqrt{z_2+1} - 1).$$

If $z \in \overline{\Pi}{}^2_+$ and $|w(z)| \geqslant 10$, $|z_1|^2 \leqslant 2|w(z)|^2$, so by (3.4),

$$\log 1/|\hat{f}(z)| \leqslant 4\delta |w(z)|^2/\operatorname{Re} w(z), \tag{3.5}$$

for $z \in U(\varepsilon)$ with $|w(z)| \geqslant 10$ and $|z_1| \geqslant S(\delta, \varepsilon)$. There is a corresponding statement if we switch the roles of z_1 and z_2. This, together with (3.5) and (3.3), shows that if we choose $\delta = \varepsilon/4$, there is an $\mathscr{R}(\varepsilon)$ such that

$$\log 1/|\hat{f}(z)| \leqslant \varepsilon |w(z)|^2/\operatorname{Re} w(z) \tag{3.6}$$

for $z \in \overline{\Pi}{}^2_+$ such that $|w(z)| \geqslant \mathscr{R}(\varepsilon)$ and

$$A(\varepsilon)(1 + |w(z)|^{1/2}) \leqslant \operatorname{Re} w(z),$$

possibly with a bigger $A(\varepsilon)$ than previously. From (e) we get the global condition

$$\log 1/|\hat{f}(z)| = O(\exp(\varepsilon |w(z)|^{1/2}) + M(\operatorname{Re} w(z))) \tag{3.7}$$

as $|w(z)| \to \infty$ with $z \in \overline{\Pi}{}^2_+$ for all $\varepsilon > 0$.

Let $I(f)$ denote the closure of the ideal generated by f. We wish to show that $I(f) = L^1(\mathbf{R}^2_+)$. Let $\phi \in L^\infty(\mathbf{R}^2_+) = L^1(\mathbf{R}^2_+)^*$ be an arbitrary functional that annihilates $I(f)$, and let $\mathscr{C}[\phi]$ be as in Theorem 2.3, which is an entire function by (a). If we can show that $d^2/d\lambda^2\, \mathscr{C}[\phi](\lambda) \equiv 0$, then the assertion $I(f) = L^1(\mathbf{R}^2_+)$ will follow from Theorem 2.3. By Proposition 2.2, we have the estimate

$$|\mathscr{C}[\phi](\lambda)| \leqslant 25\|\phi\|(5 + |\lambda|)(1 + |\operatorname{Re} \lambda|^{-2}), \qquad \operatorname{Re} \lambda < 0. \tag{3.8}$$

We will need to estimate $\mathscr{C}[\phi]$ in the right half plane as well. In order to do so, we need to produce elements of the coset

$$(A^1 + I(f)) * (\delta_0 - (4 + \lambda)A^2 + I(f))^{*-1}$$

for $\operatorname{Re} \lambda > 0$, which will be done by solving a certain $\bar{\partial}$ problem.

Let $\varphi \in C^\infty(\overline{\mathbf{R}}_+)$ be a function such that $\varphi = 1$ on $[0, 1/3]$, $0 \leqslant \varphi \leqslant 1$ on $(1/3, 2/3)$, and $\varphi = 0$ on $[2/3, \infty)$. It is not hard to find a function φ such that, in addition, $|d\varphi/dx| \leqslant 6$, $|d^2\varphi/dx^2| \leqslant 150$, and $|d^3\varphi/dx^3| \leqslant 7{,}500$. In the following, unless explicitly stated otherwise, λ is to be a complex parameter with $\operatorname{Re}\lambda > 0$. Introduce the function

$$\chi_\lambda(z) = \varphi(|w(z) - \lambda|/\operatorname{Re}\lambda), \qquad z \in \overline{\Pi}^2_+,$$

and observe that it is supported on the set

$$\{z \in \overline{\Pi}^2_+ : |w(z) - \lambda| \leqslant \tfrac{2}{3}\operatorname{Re}\lambda\}.$$

For multi-indices $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$ and $\beta = (\beta_1, \beta_2) \in \mathbf{N}^2$, where $\mathbf{N} = \{0, 1, 2, \dots\}$, introduce the partial differential operator

$$D^{(\alpha,\beta)} = \frac{\partial^{|\alpha|+|\beta|}}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}\partial \bar z_1^{\beta_1}\partial \bar z_2^{\beta_2}};$$

here, $|\alpha| = \alpha_1 + \alpha_2$ and $|\beta| = \beta_1 + \beta_2$. If $|\alpha| + |\beta| \geqslant 1$, the function $D^{(\alpha,\beta)}\chi_\lambda$ is supported on the set

$$\Omega(\lambda) = \{z \in \overline{\Pi}^2_+ : \tfrac{1}{3}\operatorname{Re}\lambda \leqslant |w(z) - \lambda| \leqslant \tfrac{2}{3}\operatorname{Re}\lambda\}.$$

Now and onwards, let $\|\cdot\|$ be the supremum norm on $\overline{\Pi}^2_+$, and let $\|\cdot\|_{\Omega(\lambda)}$ be the supremum norm on $\Omega(\lambda)$. If we use the estimates for the derivatives of φ, the chain rule, and the fact that $D^{(\alpha,\beta)}\chi_\lambda$ is supported on $\Omega(\lambda)$ for $|\alpha| + |\beta| \geqslant 1$, we obtain, after quite a few tedious but straightforward calculations, the following estimates:

$$(3.9)\qquad \begin{cases} \|D^{(\alpha,\beta)}\chi_\lambda\| \leqslant 8(|\lambda| + 2)/\operatorname{Re}\lambda & \text{if} \quad |\alpha| + |\beta| = 1, \\ \|D^{(\alpha,\beta)}\chi_\lambda\| \leqslant 150((|\lambda| + 2)/\operatorname{Re}\lambda)^2 & \text{if} \quad |\alpha| + |\beta| = 2, \text{ and} \\ \|D^{(\alpha,\beta)}\chi_\lambda\| \leqslant 50{,}000((|\lambda| + 2)/\operatorname{Re}\lambda)^3 & \text{if} \quad |\alpha| + |\beta| = 3. \end{cases}$$

The $\bar\partial$ problem we wish to solve is

$$(3.10)\qquad \partial u_\lambda/\partial \bar z_j = (1 + z_1)(1 + z_2)\frac{\partial\chi_\lambda/\partial\bar z_j}{(1 - (\lambda + 4)\widehat{A^2}(z))\hat f(z)}, \qquad j = 1, 2,$$

and we want to control the supremum norm of the solution u_λ and its derivatives of order $\leqslant 2$. $\bar\partial$ problems have been studied on the bidisc $\mathbf{D}^2$ rather than on Π^2_+, so let us introduce the coordinates

$$\zeta_j = \frac{1 - z_j}{1 + z_j} \qquad z_j \in \Pi_+, \qquad j = 1, 2;$$

then $\zeta = (\zeta_1, \zeta_2) \in \mathbf{D}^2$ if and only if $z \in \Pi_+^2$. In these coordinates, (3.10) takes the form

$$\text{(3.11)} \quad \partial u_\lambda/\partial\bar{\zeta}_j = \omega_{\lambda,j} \equiv (1 + z_1)(1 + z_2)\frac{\partial\chi_\lambda/\partial\bar{\zeta}_j}{(1 - (\lambda + 4)\widehat{A^2}(z))\hat{f}(z)}, \qquad j = 1, 2.$$

A necessary condition for (3.11) to be soluble is that the (0, 1)-form $\omega_{\lambda,1}\, d\bar{\zeta}_1 + \omega_{\lambda,2}\, d\bar{\zeta}_2$ be $\bar{\partial}$-closed, or in other words, that

$$\partial\omega_{\lambda,1}/\partial\bar{\zeta}_2 = \partial\omega_{\lambda,2}/\partial\bar{\zeta}_1,$$

which clearly holds in our situation. We will use the notation

$$D_\zeta^{(\alpha,\beta)} = \frac{\partial^{|\alpha|+|\beta|}}{\partial\zeta_1^{\alpha_1}\partial\zeta_2^{\alpha_2}\partial\bar{\zeta}_1^{\beta_1}\partial\bar{\zeta}_2^{\beta_2}}.$$

Mario Landucci [Lan] has shown that there exists a solution u_λ to (3.11) such that

$$\text{(3.12)} \qquad \max_{|\alpha|+|\beta|\leqslant 2} \|D_\zeta^{(\alpha,\beta)}u_\lambda\| \leqslant C \max_{|\alpha|+|\beta|\leqslant 2,\, j=1,2} \|D_\zeta^{(\alpha,\beta)}\omega_{\lambda,j}\|.$$

Here and throughout the rest of this proof, C will denote a positive real constant, not necessarily the same at different occurrences. If the constant depends on any of the relevant parameters, we will indicate this with a subscript. If we use the estimates (3.9) for the partial derivatives of χ_λ, the chain rule, the fact that $\omega_{\lambda,j}$ is supported on $\Omega(\lambda)$, and the assumption that the partial derivatives of $\hat{f}$ of order $\leqslant 2$ are bounded outside a compact set, we obtain, after quite a few rather tedious but straightforward computations, which are omitted, the following estimate:

$$\max_{|\alpha|+|\beta|\leqslant 2, j=1,2} \|D_\zeta^{(\alpha,\beta)}\omega_{\lambda,j}\| \leqslant C\frac{|\lambda|^{12}}{(\operatorname{Re}\lambda)^4}\cdot \|1/\hat{f}\|_{\Omega(\lambda)}^3$$

when $|\lambda| \geqslant r$, for some constant $r \geqslant 1$. By (3.12), we get

$$\text{(3.13)} \qquad \max_{|\alpha|+|\beta|\leqslant 2} \|D_\zeta^{(\alpha,\beta)}u_\lambda\| \leqslant C\frac{|\lambda|^{12}}{(\operatorname{Re}\lambda)^4}\cdot \|1/\hat{f}\|_{\Omega(\lambda)}^3,$$

for $|\lambda| \geqslant r$. Let

$$K_\lambda(z) = (1 + z_1)^{-1}(1 + z_2)^{-1}u_\lambda(z), \qquad z \in \overline{\Pi}_+^2.$$

We want to show that on $(i\mathbf{R})^2$, K_λ is the Fourier transform of an $L^1(\mathbf{R}^2)$ function k_λ, and we wish to control the norm of k_λ. Let (iy_1, iy_2) be the coordinates on $(i\mathbf{R})^2$. If G is a function in $L^2((i\mathbf{R})^2)$, then there is a function $g \in L^2(\mathbf{R}^2)$ such that $\hat{g} = G$, and

$$\|g\|_{L^2} = (2\pi)^{-1/2}\|G\|_{L^2}.$$

If, in addition, $\partial G/\partial y_1, \partial G/\partial y_2, \partial^2 G/\partial y_1 \partial y_2 \in L^2((i\mathbf{R})^2)$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+t_1^2)(1+t_2^2)|g(t_1,t_2)|^2\,dt_1\,dt_2 < \infty,$$

so by Hölder's inequality,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(t_1,t_2)|\,dt_1\,dt_2 < \infty.$$

Moreover, there is a norm inequality:

$$\|g\|_{L^1} \leqslant C\max\{\|G\|_{L^2}, \|\partial G/\partial y_1\|_{L^2}, \|\partial G/\partial y_2\|_{L^2}, \|\partial^2 G/\partial y_1\partial y_2\|_{L^2}\}. \tag{3.14}$$

From (3.13) it is easy to deduce that

$$\max_{|\alpha|+|\beta|\leqslant 2}\|D^{(\alpha,\beta)}u_\lambda\| \leqslant C\frac{|\lambda|^{12}}{(\operatorname{Re}\lambda)^4}\|1/\hat{f}\|^3_{\Omega(\lambda)}$$

for $|\lambda| \geqslant r$. A few more computations show that

$$\max_{|\alpha|+|\beta|\leqslant 2}\|(1+z_1)(1+z_2)D^{(\alpha,\beta)}K_\lambda\| \leqslant C\frac{|\lambda|^{12}}{(\operatorname{Re}\lambda)^4}\|1/\hat{f}\|^3_{\Omega(\lambda)}$$

for $|\lambda| \geqslant r$, and since $(z_1,z_2)\mapsto(1+z_1)^{-1}(1+z_2)^{-1}$, $(z_1,z_2)\in(i\mathbf{R})^2$, belongs to $L^2((i\mathbf{R})^2)$, we conclude by (3.14) that there is a function $k_\lambda \in L^1(\mathbf{R}^2)$ such that $\hat{k}_\lambda = K_\lambda$ on $(i\mathbf{R})^2$, and

$$\|k_\lambda\|_{L^1} \leqslant C\frac{|\lambda|^{12}}{(\operatorname{Re}\lambda)^4}\|1/\hat{f}\|^3_{\Omega(\lambda)},$$

again for $|\lambda| \geqslant r$. Let

$$Q_\lambda = \frac{1-\chi_\lambda}{1-(\lambda+4)\widehat{A^2}} - 1 + \hat{f}\cdot K_\lambda = -\chi_\lambda + (\lambda+4)\frac{1-\chi_\lambda}{w-\lambda} + \hat{f}\cdot K_\lambda;$$

we intend to show that Q_λ is the Fourier transform of a function $q_\lambda \in L^1(\mathbf{R}^2_+)$, and we want to control $\|q_\lambda\|_{L^1}$. By (3.9) and the fact that χ_λ is supported on the set

$$\{z \in \overline{\Pi}^2_+ : |w(z)-\lambda| \leqslant \tfrac{2}{3}\operatorname{Re}\lambda\},$$

we have

$$\|(z_1+1)(z_2+1)D^{(\alpha,\beta)}\chi_\lambda\| \leqslant C\frac{|\lambda|^4+1}{(\mathrm{Re}\,\lambda)^2}$$

for $|\alpha|+|\beta| \leqslant 2$. By (3.14), we obtain

$$\|\chi_\lambda\|_{\widehat{L^1}} \leqslant C\frac{|\lambda|^4+1}{(\mathrm{Re}\,\lambda)^2};$$

here $\|\hat{g}\|_{\widehat{L^1}} = \|g\|_{L^1}$ for $g \in L^1(\mathbf{R}^2)$. Observe that we should actually have restricted χ_λ to $(i\mathbf{R})^2$ in the above expression. The formula

$$\frac{1-\chi_\lambda}{w-\lambda} = \widehat{A^2}\cdot\left(1-\chi_\lambda+(\lambda+4)\frac{1-\chi_\lambda}{w-\lambda}\right)$$

shows that what remains for us to do is to estimate

$$\left\|\widehat{A^2}\cdot\frac{1-\chi_\lambda}{w-\lambda}\right\|_{\widehat{L^1}}.$$

A few computations show that

$$\left\|(z_1+1)(z_2+1)D^{(\alpha,\beta)}\left(\widehat{A^2}\cdot\frac{1-\chi_\lambda}{w-\lambda}\right)\right\| \leqslant C\frac{|\lambda|^3}{(\mathrm{Re}\,\lambda)^3},$$

from which we can conclude that

$$\left\|\widehat{A^2}\cdot\frac{1-\chi_\lambda}{w-\lambda}\right\|_{\widehat{L^1}} \leqslant C\frac{|\lambda|^3}{(\mathrm{Re}\,\lambda)^3},$$

and so

$$\left\|-\chi_\lambda+(\lambda+4)\frac{1-\chi_\lambda}{w-\lambda}\right\|_{\widehat{L^1}} \leqslant C\frac{|\lambda|^6}{(\mathrm{Re}\,\lambda)^3}.$$

Thus we obtain

$$\begin{aligned}\|q_\lambda\|_{L^1} = \|Q_\lambda\|_{\widehat{L^1}} &\leqslant \left\|-\chi_\lambda+(\lambda+4)\frac{1-\chi_\lambda}{w-\lambda}\right\|_{\widehat{L^1}} + \|f\|_{L^1}\cdot\|k_\lambda\|_{L^1} \\ &\leqslant C\left(\frac{|\lambda|^6}{(\mathrm{Re}\,\lambda)^3}+\frac{|\lambda|^{12}}{(\mathrm{Re}\,\lambda)^4}\|1/\hat{f}\|^3_{\Omega(\lambda)}\right) \leqslant C\frac{|\lambda|^{12}}{(\mathrm{Re}\,\lambda)^4}(1+\|1/\hat{f}\|^3_{\Omega(\lambda)})\end{aligned} \tag{3.15}$$

for $|\lambda| \geqslant r$. But we also wanted to show that $q_\lambda \in L^1(\mathbf{R}_+^2)$, or in other words, that $\operatorname{supp} q_\lambda \subset [0, \infty)^2$. The following observation will be helpful: If $g \in L^1(\mathbf{R}^2)$ has a Fourier transform that is the restriction to $(i\mathbf{R})^2$ of a function in $A_0(\Pi_+^2)$, then $\operatorname{supp} g \subset [0, \infty)^2$. Expressed differently, $L^1(\mathbf{R}_+^2)\hat{} = L^1(\mathbf{R}^2)\hat{} \cap A_0(\Pi_+^2)$. One way to show this is to take a sequence $\{\varepsilon_n\}_1^\infty \subset L^1(\mathbf{R}_+^2) \cap C^\infty(\mathbf{R}^2)$ such that $\|\varepsilon_n\|_{L^1} = 1$, $\varepsilon_n \geqslant 0$, and $\operatorname{supp} \varepsilon_n \subset [0, 1/n]^2$, and observe that

$$(\varepsilon_n * g)\hat{} = \hat{\varepsilon}_n \cdot \hat{g} \in H^2(\Pi_+^2),$$

which by standard Fourier analysis implies that $\operatorname{supp}(\varepsilon_n * g) \subset [0, \infty)^2$; here $H^2(\Pi_+^2)$ is the usual Hardy space on Π_+^2, which is the image of $L^2(\mathbf{R}_+^2)$ under the Fourier transform. The conclusion follows if we let $n \to \infty$.

By the definition of Q_λ,

$$\begin{aligned}\partial Q_\lambda/\partial \bar{z}_j &= -\frac{\partial \chi_\lambda/\partial \bar{z}_j}{1 - (\lambda + 4)\widehat{A^2}} + \hat{f} \cdot \partial K_\lambda/\partial \bar{z}_j \\ &= -\frac{\partial \chi_\lambda/\partial \bar{z}_j}{1 - (\lambda + 4)\widehat{A^2}} + \hat{f} \cdot (1 + z_1)^{-1}(1 + z_2)^{-1} \partial u_\lambda/\partial \bar{z}_j = 0\end{aligned}$$

for $j = 1, 2$, and hence Q_λ is analytic on Π_+^2. Since Q_λ is also continuous on $\overline{\Pi}_+^2$ and vanishes at infinity, we have $Q_\lambda \in A_0(\Pi_+^2)$, so by the previous observation, $q_\lambda \in L^1(\mathbf{R}_+^2)$. Our next step is to check that $\delta_0 + q_\lambda$, where δ_0 is the Dirac measure at $(0, 0)$, as in the proof of Theorem 2.3, is an element of the coset $(\delta_0 - (4 + \lambda)A^2 + I(f))^{*-1}$. Now

$$\begin{aligned}((\delta_0 + q_\lambda) * (\delta_0 - (4 + \lambda)A^2))\hat{} &= (1 + Q_\lambda)(1 - (4 + \lambda)\widehat{A^2}) \\ &= ((1 - \chi_\lambda)/(1 - (4 + \lambda)\widehat{A^2}) + \hat{f} \cdot K_\lambda)(1 - (4 + \lambda)\widehat{A^2}) \\ &= 1 + \hat{f} \cdot G_\lambda,\end{aligned}$$

where

$$G_\lambda = -\chi_\lambda/\hat{f} + (1 - (4 + \lambda)\widehat{A^2}) \cdot K_\lambda.$$

We wish to show that $G_\lambda \in L^1(\mathbf{R}_+^2)\hat{}$. To see that $\chi_\lambda/\hat{f}|_{(i\mathbf{R})^2} \in L^1(\mathbf{R}^2)\hat{}$, find a function $\varphi_\lambda \in L^1(\mathbf{R}^2)$ such that $\hat{\varphi}_\lambda \cdot \hat{f} = 1$ on $\operatorname{supp} \chi_\lambda \cap (i\mathbf{R})^2$, which is possible since $\hat{f}$ does not vanish there and $L^1(\mathbf{R}^2)$ is a regular algebra, and observe that

$$\chi_\lambda/\hat{f}|_{(i\mathbf{R})^2} = \hat{\varphi}_\lambda \cdot \chi_\lambda|_{(i\mathbf{R})^2} \in L^1(\mathbf{R}^2)\hat{},$$

because $\chi_\lambda|_{(i\mathbf{R})^2} \in L^1(\mathbf{R}^2)\hat{}$. It is now obvious that $G_\lambda|_{(i\mathbf{R})^2} \in L^1(\mathbf{R}^2)\hat{}$, and since $G_\lambda \in A_0(\Pi_+^2)$, which one checks just as with Q_λ, we conclude $G_\lambda \in L^1(\mathbf{R}_+^2)\hat{}$. We have now verified that $\delta_0 + q_\lambda$ is an element of the coset $(\delta_0 - (4+\lambda)A^2 + I(f))^{*-1}$, so by the proof of Theorem 2.3,

$$\mathscr{C}[\phi](\lambda) = \langle A^1 + A^1 * q_\lambda, \phi\rangle, \qquad \operatorname{Re}\lambda > 0,$$

and hence by (3.15),

$$|\mathscr{C}[\phi](\lambda)| \leqslant C \cdot \frac{|\lambda|^{12}}{(\operatorname{Re}\lambda)^4}(1 + \|1/\hat{f}\|^3_{\Omega(\lambda)}) \tag{3.16}$$

for λ such that $\operatorname{Re}\lambda > 0$ and $|\lambda| \geqslant r$. From here on, λ is no longer confined to the right half plane. By (3.8), we already know that

$$|\mathscr{C}[\phi](\lambda)| \leqslant C(|\lambda| + 1)(1 + |\operatorname{Re}\lambda|^{-2}), \qquad \operatorname{Re}\lambda < 0. \tag{3.17}$$

Our next and final step will be to show that (3.6) and (3.7), together with the above estimates (3.16) and (3.17), will force the entire function $d^2/d\lambda^2\, \mathscr{C}[\phi]$ to vanish identically, from which the assertion follows, by Theorem 2.3. Some of the function-theoretic arguments used below are similar to those used in [Hed1].

If we modify the function M slightly, we obtain from (3.7), (3.16), and the definition of $\Omega(\lambda)$ the estimate

$$|\mathscr{C}[\phi](\lambda)| \leqslant \exp\{C_\varepsilon(\exp(\varepsilon|\lambda|^{1/2}) + M(\operatorname{Re}\lambda))\} \tag{3.18}$$

for $\lambda \in \Pi_+$, and all $\varepsilon > 0$. Introduce the functions

$$\Phi_{t,\varepsilon}(\lambda) = \exp\{-C_\varepsilon \exp(\varepsilon|t|^{1/2})\}\mathscr{C}[\phi](\lambda + it), \qquad \lambda \in \mathbf{C}$$

for $t \in \mathbf{R}$ and $\varepsilon > 0$. We will restrict λ to the region

$$\mathscr{D}_2 = \{\lambda \in \mathbf{C}: |\operatorname{Re}\lambda| < 2, |\operatorname{Im}\lambda| < 2\}.$$

If the C_ε used to define $\Phi_{t,\varepsilon}$ is chosen adroitly, we have by (3.18)

$$|\Phi_{t,\varepsilon}(\lambda)| \leqslant \exp\{C_\varepsilon \cdot M(\operatorname{Re}\lambda)\}, \qquad \lambda \in \mathscr{D}_2 \cap \Pi_+,$$

for all $\varepsilon > 0$. By (3.17), we have

$$|\Phi_{t,\varepsilon}(\lambda)| \leqslant C_\varepsilon \cdot |\operatorname{Re}\lambda|^{-2}, \qquad \lambda \in \mathscr{D}_2 \cap \Pi_-,$$

for all $\varepsilon > 0$. The functions $\Phi_{t,\varepsilon}$ are holomorphic on $\mathscr{D}_2$ for all $t \in \mathbf{R}$ and $\varepsilon > 0$. Now, because

$$\int_0^1 \log M(x)\, dx < \infty,$$

we can apply a classical theorem in function theory known as the Levinson-Sjöberg log-log-theorem [NLe, p. 127], [Gur, p. 40], which states that

$$|\Phi_{t,\varepsilon}(\lambda)| \leqslant C_\varepsilon, \qquad \lambda \in \mathscr{D}_1,$$

independently of $t \in \mathbf{R}$, where

$$\mathscr{D}_1 = \{\lambda \in \mathbf{C}: |\mathrm{Re}\,\lambda| < 1, |\mathrm{Im}\,\lambda| < 1\}.$$

Together with (3.18), this implies that

$$|\mathscr{C}[\phi](\lambda)| \leqslant \exp\{C_\varepsilon \exp(\varepsilon|\lambda|^{1/2})\} \tag{3.19}$$

for $\mathrm{Re}\,\lambda \geqslant -1$, and all $\varepsilon > 0$. From (3.6), (3.16), and the definition of $\Omega(\lambda)$, we get for all $\delta > 0$ the estimate

$$|\mathscr{C}[\phi](\lambda)| \leqslant C\cdot\frac{|\lambda|^{12}}{(\mathrm{Re}\,\lambda)^4}\exp(\delta|\lambda|^2/\mathrm{Re}\,\lambda) \tag{3.20}$$

for $\lambda \in \Pi_+$ such that $A(\delta)(1 + |\lambda|^{1/2}) \leqslant \mathrm{Re}\,\lambda$ and $|\lambda| \geqslant \mathscr{R}(\delta)$, where we may have to modify our constants $A(\delta)$ and $\mathscr{R}(\delta)$. It will be more convenient to state (3.20) in the form

$$|\mathscr{C}[\phi](\lambda) \leqslant C_\delta \exp(\delta|\lambda|^2/\mathrm{Re}\,\lambda) \tag{3.21}$$

for $\lambda \in \Pi_+$ such that $A(\delta)(1 + |\lambda|^{1/2}) \leqslant \mathrm{Re}\,\lambda$, where again a modification of $A(\delta)$ may be necessary. Consider the region

$$\mathscr{V}(\delta) = \{\lambda \in \mathbf{C}: -1 < \mathrm{Re}\,\lambda < A(\delta)(1 + |\lambda|^{1/2})\}.$$

We will assume that $A(\delta)$ is chosen $\geqslant 1$. Then, on the right boundary component of $\mathscr{V}(\delta)$, given by the equation $\mathrm{Re}\,\lambda = A(\delta)(1 + |\lambda|^{1/2})$, we have, by (3.21),

$$|\mathscr{C}[\phi](\lambda)| \leqslant C_\delta \exp(\delta|\lambda|^{3/2}). \tag{3.22}$$

On the left boundary component $\mathrm{Re}\,\lambda = -1$, we have, by (3.17),

$$|\mathscr{C}[\phi](\lambda) \leqslant C(1 + |\lambda|). \tag{3.23}$$

We will use a Phragmén-Lindelöf type argument to show that (3.22) holds inside $\mathscr{V}(\delta)$ as well. For $t \in \mathbf{R}$, let

$$\Theta_t = \{\lambda \in \mathbf{C}: \mathrm{Im}\,\lambda = t \text{ and } -1 \leqslant \mathrm{Re}\,\lambda \leqslant A(\delta)(1 + |\lambda|^{1/2})\},$$

and let $\theta(t)$ denote the length of this interval. It is not hard to see that

$$\theta(t) \leqslant C_\delta \cdot (1 + |t|)^{1/2}. \tag{3.24}$$

Introduce the regions $\mathscr{V}_+(\delta) = \{\lambda \in \mathscr{V}(\delta) \colon \operatorname{Im} \lambda > 0\}$ and $\mathscr{V}_-(\delta) = \{\lambda \in \mathscr{V}(\delta) \colon \operatorname{Im} < 0\}$. By (3.22) and (3.23), the holomorphic function

$$\Psi_\delta(\lambda) \equiv \exp(2\delta(\lambda + 2)^{3/2}) \cdot \mathscr{C}[\phi](\lambda), \qquad \operatorname{Re} \lambda > -2,$$

has the estimate

$$|\Psi_\delta(\lambda)| \leqslant C_\delta, \qquad \lambda \in \partial\mathscr{V}_+(\delta) \cup \partial\mathscr{V}_-(\delta).$$

Then the function

$$v_\delta(\lambda) \equiv \log|\Psi_\delta(\lambda)/C_\delta|, \qquad \operatorname{Re} \lambda > -2,$$

is subharmonic, and $v_\delta \leqslant 0$ on $\partial\mathscr{V}_+(\delta) \cup \partial\mathscr{V}_-(\delta)$. For $\lambda_0 \in \mathscr{V}(\delta)$ and $\xi > \operatorname{Im} \lambda_0$, let $\omega_\delta(\lambda_0, \xi)$ be the harmonic measure at λ_0 of the interval Θ_ξ in the domain $\mathscr{V}_+(\delta, \xi) = \{\lambda \in \mathscr{V}_+(\delta) \colon \operatorname{Im} \lambda < \xi\}$. If $M_\delta(\xi)$ denotes the maximum of v_δ on Θ_ξ, we have the estimate

$$v_\delta(\lambda) \leqslant M_\delta(\xi)\omega_\delta(\lambda, \xi), \qquad \lambda \in \mathscr{V}_+(\delta, \xi).$$

By [Hal, p. 3], the Ahlfors distortion theorem can be used to show that

$$\omega_\delta(\lambda, \xi) \leqslant \frac{4}{\pi} \exp\left(4\pi - \pi \int_{\operatorname{Im} \lambda}^{\xi} \frac{dt}{\theta(t)}\right), \qquad \lambda \in \mathscr{V}_+(\delta, \xi),$$

when $\int_{\operatorname{Im} \lambda}^{\xi} dt/\theta(t) > 2$, so that by (3.24),

$$\begin{aligned} v_\delta(\lambda) &\leqslant \frac{4}{\pi} M_\delta(\xi) \exp\left(4\pi - \frac{\pi}{C_\delta} \int_{\operatorname{Im} \lambda}^{\xi} \frac{dt}{(1 + t)^{1/2}}\right) \\ &= \frac{4}{\pi} M_\delta(\xi) \exp\left\{4\pi - \frac{2\pi}{C_\delta}((\xi + 1)^{1/2} - (\operatorname{Im} \lambda + 1)^{1/2})\right\}, \qquad \lambda \in \mathscr{V}_+(\delta, \xi), \end{aligned} \tag{3.25}$$

again when $\int_{\operatorname{Im} \lambda}^{\xi} dt/\theta(t) > 2$, which holds if ξ is sufficiently large. By (3.19), we have for every $\varepsilon > 0$

$$M_\delta(\xi) \leqslant C_{\delta,\varepsilon} \cdot \exp(\varepsilon \xi^{1/2}),$$

so if we pick an $\varepsilon < 2\pi/C_\delta$ and let $\xi \to \infty$ in (3.25), it follows that $v_\delta \leqslant 0$ on $\mathscr{V}_+(\delta)$.

A similar argument shows that $v_\delta \leqslant 0$ on $\mathscr{V}_-(\delta)$, and hence

$$|\mathscr{C}[\phi](\lambda)| \leqslant C_\delta \exp(\delta|\lambda|^{3/2}), \qquad \lambda \in \mathscr{V}(\delta), \tag{3.26}$$

for all $\delta > 0$. Similar Phragmén-Lindelöf–type arguments for the regions $\{\lambda \in \mathbf{C}: \operatorname{Re}\lambda > -1 \text{ and } \operatorname{Im}\lambda > 0\}$ and $\{\lambda \in \mathbf{C}: \operatorname{Re}\lambda > -1 \text{ and } \operatorname{Im}\lambda < 0\}$ show that (3.21), (3.23), and (3.26) imply

$$|\mathscr{C}[\phi](\lambda)| \leqslant C_\delta \exp(\delta|\lambda|), \qquad \operatorname{Re}\lambda > -1, \tag{3.27}$$

for every $\delta > 0$. Yet another application of the Phragmén-Lindelöf principle for the region $\{\lambda \in \mathbf{C}: \operatorname{Re}\lambda > -1\}$ shows that (3.23) and (3.27) together imply that

$$|\mathscr{C}[\phi](\lambda)| \leqslant C(1 + |\lambda|), \qquad \operatorname{Re}\lambda > -1,$$

so by (3.17) and the fact that $\mathscr{C}[\phi]$ is entire, $\mathscr{C}[\phi]$ must in fact be a polynomial of order $\leqslant 1$. Hence $d^2/d\lambda^2\, \mathscr{C}[\phi](\lambda) \equiv 0$, so by Theorem 2.3, $I(f) = L^1(\mathbf{R}^2_+)$. The proof is complete.

Remarks. (a) If we study the proof of Theorem 3.1 carefully, we realize that the theorem will remain valid if we replace condition (d) by the weaker condition

$$\log 1/|\hat{f}(z)| = o(|z|^2/\operatorname{Re} w(z)) \text{ as } \overline{\Pi^2_+} \ni z \to (\infty, \infty) \text{ with } 1 + |z|^{1/2} \leqslant \delta \operatorname{Re} w(z),$$

for some fixed $\delta > 0$, and we may also replace the quantity $|z|^{1/2}$ in (e) by $|\operatorname{Im} w(z)|^{1/2} + |\operatorname{Re} w(z)|$, which grows faster as $|z| \to \infty$.

(b) One may wonder why we need to assume that $\hat{f}$ has bounded derivatives of order $\leqslant 2$ off a compact set. One reason is that $\bar\partial$ problems have not been studied with spaces like $\widehat{L^1}$ in mind. But even in situations where this is not a problem, there is a need to impose regularity conditions on $\hat{f}$ due to a property of the L^1 norm, which is related to the fact that the Wiener algebra $l^1(\mathrm{N})$ does not possess the so-called uniformly bounded inverse property, which was discovered by Harold S. Shapiro [Sha]; see [Hed2, section 5] for more details.

Theorem 3.1 has the following analog for the uniform algebra $A_0(\Pi^2_+)$, which is a lot easier to prove, in part because we only need to control the supremum norm while solving the relevant $\bar\partial$ problem, which can be done using [HeC, p. 676]. We omit the proof; if the reader needs guidance beyond the proof of Theorem 3.1, he is referred to [Hed3]. An interesting feature is the fact that we do not need to impose any additional regularity on the function in question.

THEOREM 3.4. *Let f be a function in $A_0(\Pi^2_+)$, set $w(z) = (\sqrt{z_1+1} + \sqrt{z_2+1})^2 - 4$, and, moreover, let $M\colon (0,\infty) \to [1,\infty)$ be a continuous decreasing function such that*

$$\int_0^1 \log M(x)\, dx < \infty.$$

Then f generates a dense ideal in $A_0(\Pi_+^2)$ if

(a) $f(z) \neq 0$ *for all* $z \in \overline{\Pi}_+^2$,
(b) $f(z_1, \cdot)$ *is outer for all* $z_1 \in \overline{\Pi}_+$,
(c) $f(\cdot, z_2)$ *is outer for all* $z_2 \in \overline{\Pi}_+$,
(d) $\log 1/|f(z)| = o(|z|^2/\mathrm{Re}\, w(z))$ *for* $\mathrm{Re}\, w(z) \geqslant 1$, *as* $\overline{\Pi}_+^2 \ni z \to (\infty, \infty)$, *and*
(e) $\log 1/|f(z)| = O(\exp(\varepsilon|z|^{1/2}) + M(\mathrm{Re}\, w(z)))$ *as* $|z| \to \infty$ *with* $z \in \overline{\Pi}_+^2$ *for all* $\varepsilon > 0$.

Remarks. (a) By Corollary 1.7 and the remark thereafter, conditions (a)–(c) are necessary; (d) is somewhat stronger than condition (iii) of Corollary 1.7.

(b) It appears that Theorem 3.4 remains true when we change the function w to $w(z) = (\sqrt{z_1} + \sqrt{z_2})^2$, although I haven't checked all the details. The main difficulty is that this w does not have bounded first-order derivatives, so one has to be more careful when defining the function χ_λ. However, the author has no idea whether this is possible for the algebra $L^1(\mathbf{R}_+^2)$ as well, that is, whether we may replace w by the above expression in Theorem 3.1. It should be mentioned that a simple argument makes it possible to replace w in both Theorems 3.1 and 3.4 by the expression $w(z) = \sqrt{z_1 + \varepsilon} + \sqrt{z_2 + \varepsilon})^2 - 4\varepsilon$, if $\varepsilon > 0$.

Just like Theorem 3.1, Theorem 3.4 has a number of corollaries.

COROLLARY 3.5. *Let f be a function in $A_0(\Pi_+^2)$, and set $w(z) = (\sqrt{z_1 + 1} + \sqrt{z_2 + 1})^2 - 4$. Then f generates a dense ideal in $A_0(\Pi_+^2)$ if*

(a) $f(z) \neq 0$ *for all* $z \in \overline{\Pi}_+^2$, *and*
(b) $\log 1/|f(z)| = o(|z|^2/\mathrm{Re}\, w(z))$ *as* $|z| \to \infty$ *with* $z \in \overline{\Pi}_+^2$.

COROLLARY 3.6. *A function $f \in A_0(\Pi_+^2)$ generates an ideal that is dense in $A_0(\Pi_+^2)$ if*

(a) $f(z) \neq 0$ *for all* $z \in \overline{\Pi}_+^2$, *and*
(b) $\log 1/|f(z)| = o(|z|)$ *as* $|z| \to \infty$ *with* $z \in \overline{\Pi}_+^2$.

The following corollary, together with Lemma 1.5 and Theorem 1.6, provides a partial answer to Problem 1.1.

COROLLARY 3.7. *If $f \in A(\mathbf{D}^2)$ has $Z(f) = (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$, then f is BR-outer, that is, $I(f) = \mathscr{I}(Z(f))$ holds, if*

$$\log 1/|f(z)| = o(1/d(z)) \quad as \quad \mathbf{D}^2 \ni z \to Z(f),$$

where $d(z)$ is the Euclidean distance between z and $Z(f)$.

4. The two-dimensional Volterra algebra. Let $[0, 1]^2 = [0, 1] \times [0, 1]$ be the unit square in $\mathbf{R}^2$. The space $L^1([0, 1]^2)$, endowed with restricted convolution multiplication, is called the Volterra algebra. More precisely, the algebraic (and topological) properties of $L^1([0, 1]^2)$ come from identifying it with the quotient algebra $L^1(\mathbf{R}_+^2)/\mathbf{J}$, where $\mathbf{J}$ is the closed ideal

$$\mathbf{J} = \{f \in L^1(\mathbf{R}_+^2): f = 0 \text{ almost everywhere on } [0, 1]^2\}.$$

The Volterra algebra can also be thought of as a subspace of $L^1(\mathbf{R}_+^2)$ by extending the functions to vanish outside $[0, 1]^2$; it should be observed, however, that the extension mapping $L^1([0, 1]^2) \to L^1(\mathbf{R}_+^2)$ is not an algebra homomorphism. We will use the symbol $\tilde{*}$ for the restricted convolution multiplication in $L^1([0, 1]^2)$. As was mentioned in the introduction, Elizabeth Strouse suggested the following problem in [Str].

Strouse's problem. Which functions $f \in L^1([0, 1]^2)$ are cyclic, that is, generate an ideal $f \tilde{*} L^1([0, 1]^2)$ that is dense in $L^1([0, 1]^2)$?

The spaces $L^p([0, 1]^2)$, with $1 < p < \infty$, are also Banach algebras with restricted convolution multiplication, and each of them is dense in $L^1([0, 1]^2)$. The following result states that the structure of closed ideals in $L^p([0, 1]^2)$, $1 \leqslant p < \infty$, is independent of p. The proof is very similar to that of Theorem 4.1 [Str].

PROPOSITION 4.1. *The mapping $I \mapsto I \cap L^p([0, 1]^2)$ is a bijection from the closed ideals of $L^1([0, 1]^2)$ onto the closed ideals of $L^p([0, 1]^2)$, $1 \leqslant p < \infty$.*

Proof. We cannot apply Theorem 4.1 [Str] directly, because $L^p([0, 1]^2)$ does not possess a bounded approximate identity. Let $\{e_n\}_1^\infty \subset C^\infty(\mathbf{R}^2)$ be a sequence such that $e_n \geqslant 0$, $\operatorname{supp} e_n \subset [0, 1/n]^2$, and $\|e_n\|_{L^1} = 1$. Then $\{e_n\}_1^\infty$ is a bounded approximate identity in $L^1([0, 1]^2)$. Let I be a closed ideal in $L^p([0, 1]^2)$, and let $\bar{I}$ be the closure of I in $L^1([0, 1]^2)$, which is an ideal in the Volterra algebra because $L^p([0, 1]^2)$ is dense in $L^1([0, 1]^2)$. We want to show that $\bar{I} \cap L^p([0, 1]^2) = I$. So, let $f_n \to f$ in $L^1([0, 1]^2)$ for some sequence $\{f_n\}_1^\infty \subset I$, and assume $f \in L^p([0, 1]^2)$. Now since $e_k \in C^\infty([0, 1]^2)$, $f_n \tilde{*} e_k \in L^p([0, 1]^2)$, and by Minkowski's inequality for integrals,

$$\|f_n \tilde{*} e_k - f \tilde{*} e_k\|_{L^p} = \|(f_n - f) \tilde{*} e_k\|_{L^p} \leqslant \|f_n - f\|_{L^1} \cdot \|e_k\|_{L^p} \to 0 \quad \text{as} \quad n \to \infty,$$

so that we may conclude that $f \tilde{*} e_k \in I$. Letting $k \to \infty$, $f \tilde{*} e_k \to f$ in $L^p([0, 1]^2)$, and the assertion follows.

Next, let J be a closed ideal in $L^1([0, 1]^2)$. We want to show that the $L^1([0, 1]^2)$-closure of $J \cap L^p([0, 1]^2)$ equals J. So, let $f \in J$ be arbitrary. Then $e_k \tilde{*} f \in J \cap L^p([0, 1]^2)$, and $e_k \tilde{*} f \to f$ in $L^1([0, 1]^2)$, and the assertion follows. The proof is complete.

Proposition 4.1 states in particular that a function $f \in L^2([0, 1]^2)$ generates a dense ideal in the Hilbert space $L^2([0, 1]^2)$ if and only if it generates a dense ideal in $L^1([0, 1]^2)$. This should make the problem interesting to operator theorists.

The Fourier transform defines a continuous monomorphism $L^1(\mathbf{R}_+^2) \to A_0(\Pi_+^2)$. Let $\mathbf{K}$ denote the ideal

$$\mathbf{K} = e^{-z_1} A_0(\Pi_+^2) + e^{-z_2} A_0(\Pi_+^2),$$

and $\bar{\mathbf{K}}$ its closure. If $\mathbf{J}$ denotes the closed $L^1(\mathbf{R}_+^2)$-ideal

$$\mathbf{J} = \{f \in L^1(\mathbf{R}_+^2)\colon f = 0 \text{ almost everywhere on } [0, 1]^2\},$$

its image under the Fourier transform is contained within $\mathbf{K}$, and (see [Str])

$$\mathbf{J} = \{f \in L^1(\mathbf{R}_+^2)\colon \hat{f} \in \bar{\mathbf{K}}\}.$$

It follows that the Fourier transform induces a continuous monomorphism

$$\mathscr{F}\colon L^1([0, 1]^2) = L^1(\mathbf{R}_+^2)/\mathbf{J} \to A_0(\Pi_+^2)/\bar{\mathbf{K}}.$$

For a function $f \in L^1([0, 1]^2)$, let

$$\hat{f}(z) = \int_0^1 \int_0^1 e^{-t_1 z_1 - t_2 z_2} f(t_1, t_2)\, dt_1\, dt_2, \qquad z \in \mathbf{C}^2.$$

The next proposition follows from Theorem 4.6 [Str].

PROPOSITION 4.2. *The map $I \mapsto \mathscr{F}^{-1}(I) = \{f \in L^1([0, 1]^2)\colon \mathscr{F} f \in I\}$ is a bijection from the set of all closed ideals in $A_0(\Pi_+^2)/\bar{\mathbf{K}}$ onto the set of all closed ideals in $L^1([0, 1]^2)$. In other words, the map $I \mapsto \{f \in L^1(\mathbf{R}_+^2)\colon \hat{f} \in I\}$ is a bijection from the set of all closed ideals of $A_0(\Pi_+^2)$ containing $\bar{\mathbf{K}}$ onto the set of all closed ideals in $L^1(\mathbf{R}_+^2)$ containing $\mathbf{J}$. In particular, a function $f \in L^1([0, 1]^2)$ generates a dense ideal in $L^1([0, 1]^2)$ if and only if $\hat{f}$ and $\bar{\mathbf{K}}$ together generate a dense ideal in $A_0(\Pi_+^2)$.*

The following proposition is proved the same way as Lemma 1.5. The class $\mathscr{U}_*$ was defined back in section 1.

PROPOSITION 4.3. *If I is a dense ideal in $A_0(\Pi_+^2)$, $I \circ L$ generates a dense ideal in $A_0(\Pi_+)$ for all $L \in \mathscr{U}_*$.*

COROLLARY 4.4. *Let $f \in A_0(\Pi_+^2)$, and let $L = (L_1, L_2) \in \mathscr{U}_*$ be such that*

$$\liminf_{t \to +\infty} \operatorname{Re} L_j(t)/t > 0, \qquad j = 1, 2.$$

Then if f and $\bar{\mathbf{K}}$ generate a dense ideal in $A_0(\Pi_+^2)$,

$$\log 1/|f \circ L(re^{i\theta})| = o(r) \quad \text{as} \quad r \to +\infty,$$

for almost all $\theta \in (-\pi/2, \pi/2)$.

Proof. The closed ideal $\bar{\mathbf{K}}$ is generated (after closure) by the two functions $k_1(z) = (z_1 + 1)^{-1}(z_2 + 1)^{-1}e^{-z_1}$ and $k_2(z) = (z_1 + 1)^{-1}(z_2 + 1)^{-1}e^{-z_2}$. By Proposition 4.3, $f \circ L$, $k_1 \circ L$, and $k_2 \circ L$ must generate a dense ideal in $A_0(\Pi_+)$. For this to be possible, either

$$\limsup_{t \to +\infty} t^{-1} \log|k_1 \circ L(t)| = 0,$$

$$\limsup_{t \to +\infty} t^{-1} \log|k_2 \circ L(t)| = 0, \qquad \text{or}$$

$$\limsup_{t \to +\infty} t^{-1} \log|f \circ L(t)| = 0$$

must hold. The first two possibilities are excluded by the condition on L, and now the assertion follows from an application of the Ahlfors-Heins theorem [Boa, p. 116].

COROLLARY 4.5. *Let $f \in L^1([0, 1]^2)$, and let $L = (L_1, L_2) \in \mathcal{U}_*$ be such that*

$$\liminf_{t \to +\infty} \operatorname{Re} L_j(t)/t > 0, \qquad j = 1, 2.$$

Then if f generates a dense ideal in $L^1([0, 1]^2)$,

$$\log 1/|\hat{f} \circ L(re^{i\theta})| = o(r) \quad as \quad r \to +\infty,$$

for almost all $\theta \in (-\pi/2, \pi/2)$.

Our main result is the following.

THEOREM 4.6. *For $\varepsilon > 0$, set $S_\varepsilon = \{z \in \mathbf{C}: |z| \leqslant 1, \operatorname{Re} z \geqslant \varepsilon\}$, and let K_ε be the cone $K_\varepsilon = \bigcup_{t \geqslant 0}(tS_\varepsilon \times tS_\varepsilon)$. Let $f \in L^1([0, 1]^2)$. Then f generates a dense ideal in $L^1([0, 1]^2)$ if for all $\varepsilon > 0$,*

$$\log 1/|\hat{f}(z)| = o(|z|) \quad as \quad K_\varepsilon \ni z \to (\infty, \infty). \tag{4.1}$$

Proof. The assumption (4.1) implies that

$$\min\{\operatorname{Re} z_1, \operatorname{Re} z_2, \log 1/|\hat{f}(z)|\} = o(|z|) \tag{4.2}$$

as $|z| \to \infty$ with $z \in \overline{\Pi}_+^2$. To see this, observe that it is sufficient to show that $z \in K_\varepsilon$ if $\min\{\operatorname{Re} z_1, \operatorname{Re} z_2\} \geqslant \varepsilon|z|$. Let $z = t\zeta$, with $t \geqslant 0$ and $|\zeta| = 1$. Then if $\min\{\operatorname{Re} z_1, \operatorname{Re} z_2\} \geqslant \varepsilon|z|$, $\min\{\operatorname{Re} \zeta_1, \operatorname{Re} \zeta_2\} \geqslant \varepsilon$, and since $\zeta_1, \zeta_2 \in \overline{\mathbf{D}}$, $\zeta \in S_\varepsilon \times S_\varepsilon$, so that $z = t\zeta \in K_\varepsilon$, as desired.

To a functional $\phi \in L^\infty(\mathbf{R}_+^2) = L^1(\mathbf{R}_+^2)^*$, we can associate a functional $\tilde{\phi}$ acting on the Fourier transforms of $L^1(\mathbf{R}_+^2)$ functions via the relation

$$\langle \hat{f}, \tilde{\phi} \rangle = \langle f, \phi \rangle.$$

Some functionals ϕ will have the property that $\tilde{\phi}$ extends to a continuous linear functional on the uniform closure of $L^1(\mathbf{R}^2_+)^\wedge$, which equals $A_0(\Pi^2_+)$. Such an extension is necessarily unique whenever it exists, so we will keep the symbol $\tilde{\phi}$ for it as well.

Let $I(f, \mathbf{J})$ denote the closed ideal in $L^1(\mathbf{R}^2_+)$ generated by f and $\mathbf{J}$, and let $I(\hat{f}, \mathbf{K})$ be the closed ideal in $A_0(\Pi^2_+)$ generated by $\hat{f}$ and $\mathbf{K}$. By Proposition 4.2,

$$I(f, \mathbf{J}) = \{g \in L^1(\mathbf{R}^2_+) : \hat{g} \in I(\hat{f}, \mathbf{K})\}. \tag{4.3}$$

We want to show that $I(f, \mathbf{J}) = L^1(\mathbf{R}^2_+)$, or, equivalently, that $I(\hat{f}, \mathbf{K}) = A_0(\Pi^2_+)$. Let $\phi \in L^\infty(\mathbf{R}^2_+)$ be an arbitrary functional that annihilates $I(f, \mathbf{J})$ and satisfies $\tilde{\phi} \in A_0(\Pi^2_+)^*$. By Theorem 2.3, the function $\mathscr{C}[\phi]$ is entire, because $Z(I(f, \mathbf{J})) = \varnothing$. If we run through the proof of Theorem 2.3, we see that

$$\mathscr{C}[\phi](\lambda) = \langle (\widehat{A^1} + I(\hat{f}, \mathbf{K}))\cdot(1 - (4 + \lambda)\widehat{A^2} + I(\hat{f}, \mathbf{K}))^{-1}, \tilde{\phi}\rangle, \qquad \lambda \in \mathbf{C}, \tag{4.4}$$

where the inverse is taken modulo $I(\hat{f}, \mathbf{K})$ in the unitization of $A_0(\Pi^2_+)$. If we can show that $d^2/d\lambda^2\, \mathscr{C}[\phi](\lambda) \equiv 0$ for all such ϕ's, then $\widehat{A^5} \in I(\hat{f}, \mathbf{K})$, so by (4.3), $A^5 \in I(f, \mathbf{J})$, and $I(f, \mathbf{J}) = L^1(\mathbf{R}^2_+)$ follows just as in the proof of Theorem 2.3.

Since $\phi \perp \mathbf{J}$, ϕ is supported on $[0, 1]^2$, and therefore

$$\mathscr{C}[\phi](\lambda) = \int_0^1 \int_0^1 H_\lambda(t_1, t_2)\phi(t_1, t_2)\, dt_1\, dt_2, \qquad \lambda \in \mathbf{C}.$$

The estimates of H_λ obtained in the proof of Proposition 2.2 now show that

$$|\mathscr{C}[\phi](\lambda)| \leqslant C\cdot(1 + |\lambda|), \qquad \operatorname{Re}\lambda \leqslant 0,$$

and

$$|\mathscr{C}[\phi](\lambda)| \leqslant C\cdot(1 + |\lambda|)\exp(\operatorname{Re}\lambda/2), \qquad \operatorname{Re}\lambda \geqslant 0.$$

It follows that the entire function $\mathscr{C}[\phi]$ has finite exponential type (see [Boa, p. 66]), and that the type is $\leqslant 1/2$. If we can show that

$$\mathscr{C}[\phi](\lambda) = O(\exp(\varepsilon\lambda)) \quad \text{as} \quad \mathbf{R} \ni \lambda \to +\infty, \tag{4.5}$$

for all $\varepsilon > 0$, a Phragmén-Lindelöf–type argument will force $\mathscr{C}[\phi]$ to collapse, in the sense that it has to be a polynomial of degree $\leqslant 1$, and so $d^2/d\lambda^2\, \mathscr{C}[\phi](\lambda) \equiv 0$. By the previous discussion, the assertion $I(f, \mathbf{J}) = L^1(\mathbf{R}^2_+)$ would then follow. To obtain the estimate (4.5), we need to construct elements of the coset

$$(1 - (4 + \lambda)\widehat{A^2} + I(\hat{f}, \mathbf{K}))^{-1}$$

for $\mathbf{R} \ni \lambda \geqslant 1$, just as in the proof of Theorem 3.1. For the time being, let λ be a complex parameter with $\operatorname{Re} \lambda > 0$. Introduce the regions

$$U(\lambda) = \{z \in \overline{\Pi}_+^2 : |w(z) - \lambda| \leqslant \tfrac{2}{3} \operatorname{Re} \lambda\} \quad \text{and}$$

$$V(\lambda) = \{z \in \overline{\Pi}_+^2 : |w(z) - \lambda| < \tfrac{1}{3} \operatorname{Re} \lambda\},$$

and put

$$f_1(z) = \hat{f}(z),$$

$$f_2(z) = (z_1 + 1)^{-1}(z_2 + 1)^{-1} e^{-z_1},$$

$$f_3(z) = (z_1 + 1)^{-1}(z_2 + 1)^{-1} e^{-z_2}.$$

As was mentioned in the proof of Corollary 4.4, f_2 and f_3 generate (after closure) the ideal $\overline{\mathbf{K}}$, so that f_1, f_2, and f_3 together generate $I(\hat{f}, \mathbf{K})$. We need to construct functions $g_1, g_2, g_3 \in A_0(\Pi_+^2)$, depending on the parameter λ, such that

$$q_\lambda = \frac{1 - \sum_{j=1}^{3} f_j g_j}{1 - (4 + \lambda)\widehat{A^2}} - 1 \in A_0(\Pi_+^2),$$

and in doing so, we want to control the norm of q_λ. In particular,

$$\sum_{j=1}^{3} f_j g_j = 1 \quad \text{on} \quad \mathscr{V}(\lambda) = \{z \in \overline{\Pi}_+^2 : w(z) = \lambda\},$$

so that the g_j are solutions to a corona-type problem on $\mathscr{V}(\lambda)$. The function $1 + q_\lambda$ will then be an element of the coset

$$(1 - (4 + \lambda)\widehat{A^2} + I(\hat{f}, \mathbf{K}))^{-1},$$

so that

$$\mathscr{C}[\phi](\lambda) = \langle \widehat{A^1} \cdot (1 + q_\lambda), \tilde{\phi} \rangle, \qquad \operatorname{Re} \lambda > 0.$$

Fortunately, the functions f_j have certain regularity properties, which makes our task easier.

For $j = 1, 2, 3$, put

$$\varphi_j = \bar{f}_j \Big/ \left(\sum_{k=1}^{3} |f_k|^2 \right) \in C^\infty(\overline{\Pi}_+^2),$$

and observe that

$$\sum_{j=1}^{3} f_j(z)\varphi_j(z) = 1, \qquad z \in \overline{\Pi}_+^2 .$$

The φ_j's need not be analytic. To rectify this, we will use the Koszul complex (see [Hör] and [Gar, pp. 364–366]), a general algebraic mechanism which converts smooth corona solutions into analytic ones. First we need a few estimates. Let $\|\cdot\|$ be the supremum norm on $\overline{\Pi}_+^2$, and let $\|\cdot\|_{U(\lambda)}$ be the supremum norm on $U(\lambda)$. Put

$$A(\lambda) = \sup_{z \in U(\lambda)} \min_{j=1,2,3} \left\{ \frac{1}{|f_j(z)|} \right\},$$

and observe that

$$\left\| \left(\sum_{k=1}^{3} |f_k|^2 \right)^{-1} \right\|_{U(\lambda)} \leqslant A(\lambda)^2 . \tag{4.6}$$

By (4.2), the definition of $U(\lambda)$, and the fact that $|w(z)|$ is proportional to $|z|$ as $|z| \to \infty$ with $z \in \overline{\Pi}_+^2$,

$$\log A(\lambda) = o(|\lambda|) \quad \text{as} \quad |\lambda| \to \infty \tag{4.7}$$

with $\lambda \in \Pi_+$. It is easy to check that

$$D^{(\alpha,0)} f_j \in A_0(\Pi_+^2), \qquad j = 1, 2, 3, \tag{4.8}$$

for all multi-indices $\alpha \in \mathbf{N}^2$; for $j = 1$ this follows from the fact that f is supported on $[0, 1]^2$, and for $j = 2$, 3, it follows by direct computation. The differential operator $D^{(\alpha,\beta)}$ was introduced in the proof of Theorem 3.1. After a few computations, (4.6) and (4.8) imply that

$$\begin{cases} \|\varphi_j\|_{U(\lambda)} \leqslant C \cdot A(\lambda)^2, \\ \|D^{(0,\beta)}\varphi_j\|_{U(\lambda)} \leqslant C \cdot A(\lambda)^4 \quad \text{for} \quad |\beta| = 1, \text{and} \\ \|D^{(0,\beta)}\varphi_j\|_{U(\lambda)} \leqslant C \cdot A(\lambda)^6 \quad \text{for} \quad |\beta| = 2. \end{cases} \tag{4.9}$$

Here and in the rest of the proof, C stands for a positive constant, not necessarily the same at different occurrences.

If $\mathscr{R}$ is a subalgebra of $C(\Pi_+^2)$, let $\mathscr{R}_{(p,q)}$ denote the $\mathscr{R}$-module of (p, q)-forms with coefficients in $\mathscr{R}$. Clearly, $\mathscr{R}_{(p,q)} = \{0\}$ if $p > 2$ or $q > 2$. For $\mathscr{R} = C^k(\Pi_+^2)$, we write

$C^k_{(p,q)}(\Pi^2_+)$ instead of $(C^k(\Pi^2_+))_{(p,q)}$. Define $\bar\partial$: $C^1_{(0,q)}(\Pi^2_+) \to C_{(0,q+1)}(\Pi^2_+)$ by

$$\bar\partial h = \frac{\partial h}{\partial \bar z_1} \wedge d\bar z_1 + \frac{\partial h}{\partial \bar z_2} \wedge d\bar z_2,$$

$$\begin{aligned}\bar\partial(h_1 \wedge d\bar z_1 + h_2 \wedge d\bar z_2) &= \frac{\partial h_1}{\partial \bar z_2} \wedge d\bar z_2 \wedge d\bar z_1 + \frac{\partial h_2}{\partial \bar z_1} \wedge d\bar z_1 \wedge d\bar z_2 \\ &= \left(\frac{\partial h_2}{\partial \bar z_1} - \frac{\partial h_1}{\partial \bar z_2}\right) \wedge d\bar z_1 \wedge d\bar z_2, \qquad \text{and}\end{aligned}$$

$$\bar\partial(h \wedge d\bar z_1 \wedge d\bar z_2) = 0,$$

for $h, h_1, h_2 \in C^1(\Pi^2_+)$. Let $\Lambda^0(\mathscr{R}_{(p,q)}) = \mathscr{R}_{(p,q)}$, and let $\Lambda^1(\mathscr{R}_{(p,q)})$ be the vector space of all expressions

$$\sum_{j=1}^{3} \omega_j \wedge \mathbf{e}_j, \qquad \omega_j \in \mathscr{R}_{(p,q)},$$

where the $\mathbf{e}_j$ are place markers.

The space $\Lambda^2(\mathscr{R}_{(p,q)})$ consists of all vectors

$$\sum_{1 \leqslant j,k \leqslant 3} \omega_{j,k} \wedge \mathbf{e}_j \wedge \mathbf{e}_k, \qquad \omega_{j,k} \in \mathscr{R}_{(p,q)},$$

and $\Lambda^3(\mathscr{R}_{(p,q)})$ consists of all vectors

$$\sum_{1 \leqslant j,k,l \leqslant 3} \omega_{j,k,l} \wedge \mathbf{e}_j \wedge \mathbf{e}_k \wedge \mathbf{e}_l, \qquad \omega_{j,k,l} \in \mathscr{R}_{(p,q)},$$

where we require that

$$\mathbf{e}_j \wedge \mathbf{e}_k = -\mathbf{e}_k \wedge \mathbf{e}_j.$$

The vector spaces $\Lambda^s(\mathscr{R}_{(p,q)})$ are defined similarly for all $s \in \mathbf{N}$. However, $\Lambda^s(\mathscr{R}_{(p,q)}) = \{0\}$ for $s > 3$, because $\mathbf{e}_j \wedge \mathbf{e}_j = 0$. Let $\tau \in \Lambda^s(\mathscr{R}_{(0,q_1)})$ and $\omega \in \Lambda^t(\mathscr{R}_{(0,q_2)})$. The wedge product $\tau \wedge \omega \in \Lambda^{s+t}(\mathscr{R}_{(0,q_1+q_2)})$ is defined by declaring that

$$f \wedge g = fg, \quad f \wedge \mathbf{e}_k = \mathbf{e}_k \wedge f, \quad f \wedge d\bar z_j = d\bar z_j \wedge f, \quad d\bar z_j \wedge \mathbf{e}_k = \mathbf{e}_k \wedge d\bar z_j,$$

for all $f, g \in \mathscr{R}$, $j = 1, 2$, and $k = 1, 2, 3$, and by requiring that

$$(\tau_1 + \tau_2) \wedge (\omega_1 + \omega_2) = \tau_1 \wedge \omega_1 + \tau_1 \wedge \omega_2 + \tau_2 \wedge \omega_1 + \tau_2 \wedge \omega_2$$

for all $\tau_1, \tau_2 \in \Lambda^s(\mathscr{R}_{(0,q_1)})$ and $\omega_1, \omega_2 \in \Lambda^t(\mathscr{R}_{(0,q_2)})$. We define

$$\bar{\partial}: \Lambda^s(C^1_{(0,q)}(\Pi^2_+)) \to \Lambda^s(C_{(0,q+1)}(\Pi^2_+))$$

by differentiating the coefficients of the place markers $\mathbf{e}_j$, $\mathbf{e}_j \wedge \mathbf{e}_k$, or $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. That is,

$$\bar{\partial}\left(\sum_j u_j \wedge \mathbf{e}_j\right) = \sum_j (\bar{\partial} u_j) \wedge \mathbf{e}_j,$$

$$\bar{\partial}\left(\sum_{j,k} u_{j,k} \wedge \mathbf{e}_j \wedge \mathbf{e}_k\right) = \sum_{j,k} (\bar{\partial} u_{j,k}) \wedge \mathbf{e}_j \wedge \mathbf{e}_k,$$

$$\bar{\partial}(u \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = \bar{\partial} u \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

for all u_j, $u_{j,k}$, $u \in C^1_{(0,q)}(\Pi^2_+)$. Let

$$\mathbf{f} = f_1 \wedge \mathbf{e}_1 + f_2 \wedge \mathbf{e}_2 + f_3 \wedge \mathbf{e}_3 \in \Lambda^1(A_0(\Pi^2_+))$$

and

$$\mathbf{\Phi} = \varphi_1 \wedge \mathbf{e}_1 + \varphi_2 \wedge \mathbf{e}_2 + \varphi_3 \wedge \mathbf{e}_3 \in \Lambda^1(C^\infty(\overline{\Pi}^2_+)).$$

Define

$$P_{\mathbf{f}}: \Lambda^s(C_{(0,q)}(\Pi^2_+)) \to \Lambda^{s-1}(C_{(0,q)}(\Pi^2_+))$$

by

$$P_{\mathbf{f}}\left(\sum_j u_j \wedge \mathbf{e}_j\right) = \sum_j f_j \wedge u_j,$$

$$P_{\mathbf{f}}\left(\sum_{j,k} u_{j,k} \wedge \mathbf{e}_j \wedge \mathbf{e}_k\right) = \sum_{j,k} (f_j \wedge u_{j,k} \wedge \mathbf{e}_k - f_k \wedge u_{j,k} \wedge \mathbf{e}_j),$$

$$P_{\mathbf{f}}(u \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = f_1 \wedge u \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 - f_2 \wedge u \wedge \mathbf{e}_1 \wedge \mathbf{e}_3 + f_3 \wedge u \wedge \mathbf{e}_1 \wedge \mathbf{e}_2,$$

for all u_j, $u_{j,k}$, $u \in C_{(0,q)}(\Pi^2_+)$. Also, $P_{\mathbf{f}}\omega = 0$ for $\omega \in \Lambda^s(C_{(0,q)}(\Pi^2_+))$ if $s > 3$. In particular,

$$P_{\mathbf{f}}\mathbf{\Phi} = \sum_{j=1}^{3} f_j \varphi_j = 1.$$

Important properties of $\bar{\partial}$ and $P_{\mathbf{f}}$ are

$$\bar{\partial}^2 = 0, \qquad P_{\mathbf{f}}^2 = 0, \quad \text{and} \quad P_{\mathbf{f}}\bar{\partial} = \bar{\partial} P_{\mathbf{f}}.$$

Extend the norms $\|\cdot\|$ and $\|\cdot\|_{U(\lambda)}$ to $\Lambda^s(\mathscr{R}_{(0,q)})$ ($s \leqslant 3$, $q \leqslant 2$) by defining the respective norms of a form $\omega = \sum_{I,J} u_{I,J} \wedge d\bar{z}^I \wedge e^J$ ($|I| = q$, $|J| = s$) to be

$$\|\omega\| = \max_{I,J} \|u_{I,J}\| \quad \text{and}$$

$$\|\omega\|_{U(\lambda)} = \max_{I,J} \|u_{I,J}\|_{U(\lambda)},$$

where $I = (I_1, I_2) \in \{0, 1\}^2$ and $J = (J_1, J_2, J_3) \in \{0, 1\}^3$ are multi-indices. Here, $d\bar{z}^I = d\bar{z}_1^{I_1} \wedge d\bar{z}_2^{I_2}$ and $\mathbf{e}^J = \mathbf{e}_1^{J_1} \wedge \mathbf{e}_2^{J_2} \wedge \mathbf{e}_3^{J_3}$, where $d\bar{z}_j^0 = 1$, $\mathbf{e}_j^0 = 1$, $d\bar{z}_j^1 = d\bar{z}_j$, and $\mathbf{e}_j^1 = \mathbf{e}_j$. Let χ_λ be as in the proof of Theorem 3.1, and set

$$\mathbf{h} = \chi_\lambda \wedge \mathbf{\Phi} \wedge \bar{\partial}\mathbf{\Phi} \in \Lambda^2(C_{(0,1)}(\overline{\Pi}_+^2)),$$

which is supported on $U(\lambda)$, and has the estimate

$$\|\mathbf{h}\| = \|\mathbf{h}\|_{U(\lambda)} \leqslant \|\mathbf{\Phi}\|_{U(\lambda)} \cdot \|\bar{\partial}\mathbf{\Phi}\|_{U(\lambda)} \leqslant C \cdot A(\lambda)^6, \tag{4.10}$$

by (4.9). Observe that

$$P_{\mathfrak{f}}\mathbf{h} = \chi_\lambda \wedge \bar{\partial}\mathbf{\Phi} \quad \text{and}$$

$$\bar{\partial}\mathbf{h} = \bar{\partial}\chi_\lambda \wedge \mathbf{\Phi} \wedge \bar{\partial}\mathbf{\Phi}.$$

Let

$$\mathbf{u} = (1 - (4 + \lambda)\widehat{A^2})^{-1} \wedge \bar{\partial}\mathbf{h} = \frac{w(z) + 4}{w(z) - \lambda} \wedge \bar{\partial}\chi_\lambda \wedge \mathbf{\Phi} \wedge \bar{\partial}\mathbf{\Phi};$$

the zeros in the denominator are absorbed by the factor $\bar{\partial}\chi_\lambda$, because $\bar{\partial}\chi_\lambda = 0$ on $V(\lambda)$. Now because $\mathbf{u}$ is supported on $\Omega(\lambda) = U(\lambda) \setminus V(\lambda)$, we get by (3.9) and (4.9),

$$\|\mathbf{u}\| \leqslant C \cdot (\operatorname{Re} \lambda)^{-2}(1 + |\lambda|)^2 \cdot A(\lambda)^6. \tag{4.11}$$

Since $\mathbf{u} \in \Lambda^2(C_{(0,2)}(\overline{\Pi}_+^2))$, it has the form

$$\mathbf{u} = \sum_{j,k: j<k} u_{j,k} \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge \mathbf{e}_j \wedge \mathbf{e}_k,$$

with $u_{j,k} \in C(\overline{\Pi}_+^2)$ supported on $\Omega(\lambda)$. Consider the $\bar{\partial}$ problem

$$\frac{\partial v_{j,k}}{\partial \bar{z}_1} = u_{j,k};$$

in terms of the (bidisc) coordinates

$$\zeta_j = \frac{1 - z_j}{1 + z_j}, \qquad z_j \in \Pi_+, \qquad j = 1, 2,$$

it becomes

$$\frac{\partial v_{j,k}}{\partial \bar{\zeta}_1} = -\frac{1}{2}(z_1 + 1)^2 u_{j,k}. \tag{4.12}$$

The supremum norm of the right-hand side of (4.12) is bounded by

$$C \cdot (\operatorname{Re} \lambda)^{-2} \cdot (1 + |\lambda|)^4 \cdot A(\lambda)^6,$$

because of (4.11) and the estimate

$$|z_1 + 1| \leqslant C \cdot (1 + |\lambda|) \quad \text{on} \quad U(\lambda).$$

By a solution formula for the $\bar{\partial}$ problem on the disc $\mathbf{D}$ (see [Gar, p. 319]), we can find a $v_{j,k} \in C((\overline{\Pi}_+ \cup \{\infty\})^2)$ solving (4.12), such that

$$\|v_{j,k}\| \leqslant C \cdot (\operatorname{Re} \lambda)^{-2}(1 + |\lambda|)^4 \cdot A(\lambda)^6.$$

Then

$$\mathbf{v} = \sum_{j,k:j<k} v_{j,k} \wedge d\bar{z}_2 \wedge \mathbf{e}_j \wedge \mathbf{e}_k \in \Lambda^2(C_{(0,1)}((\overline{\Pi}_+ \cup \{\infty\})^2))$$

solves the $\bar{\partial}$ problem $\bar{\partial}\mathbf{v} = \mathbf{u}$, and satisfies

$$\|\mathbf{v}\| \leqslant C(\operatorname{Re} \lambda)^{-2}(1 + |\lambda|)^4 \cdot A(\lambda)^6. \tag{4.13}$$

Let

$$\mathbf{k} = \mathbf{h} - (1 - (4 + \lambda)\widehat{A^2}) \wedge \mathbf{v} \in \Lambda^2(C_{(0,1)}((\overline{\Pi}_+ \cup \{\infty\})^2)).$$

Then $\bar{\partial}\mathbf{k} = 0$,

$$P_{\mathbf{f}}\mathbf{k} = P_{\mathbf{f}}\mathbf{h} - (1 - (4 + \lambda)\widehat{A^2}) \wedge P_{\mathbf{f}}\mathbf{v} = \chi_\lambda \wedge \bar{\partial}\Phi - (1 - (4 + \lambda)\widehat{A^2}) \wedge P_{\mathbf{f}}\mathbf{v},$$

and

$$\|\mathbf{k}\| \leqslant C \cdot (\operatorname{Re} \lambda)^{-2}(1 + |\lambda|)^4 \cdot A(\lambda)^6, \tag{4.14}$$

by (4.10) and (4.13). Consider the $\bar\partial$ problem

$$\bar\partial \mathbf{w} = (z_1+1)^{-2}(z_2+1)^{-2} \wedge \mathbf{k},$$

which is soluble because $\bar\partial \mathbf{k} = 0$. If we rewrite this differential equation in terms of the coordinates (ζ_1, ζ_2) and use the solution to the $\bar\partial$ problem on the bidisc $\mathbf{D}^2$ obtained in [HeC, p. 676], we get a solution $\mathbf{w} \in \Lambda^2(C((\overline{\Pi}_+ \cup \{\infty\})^2))$ with

$$\|\mathbf{w}\| \leqslant C\cdot(\operatorname{Re}\lambda)^{-2}(1+|\lambda|)^4\cdot A(\lambda)^6, \tag{4.15}$$

by (4.14). Let

$$\mathbf{P} = \chi_\lambda^2 \wedge \mathbf{\Phi} - (z_1+1)^2(z_2+1)^2 \wedge \chi_\lambda \wedge P_{\mathbf{f}}\mathbf{w} \in \Lambda^1(C(\overline{\Pi}_+^2)).$$

Then $\mathbf{p}$ is supported on $U(\lambda)$,

$$P_{\mathbf{f}}\mathbf{p} = \chi_\lambda^2 \wedge P_{\mathbf{f}}\mathbf{\Phi} - (z_1+1)^2(z_2+1)^2 \wedge \chi_\lambda \wedge P_{\mathbf{f}}^2\mathbf{w} = \chi_\lambda^2,$$

and

$$\begin{aligned}\bar\partial\mathbf{p} &= 2\chi_\lambda \wedge \bar\partial\chi_\lambda \wedge \mathbf{\Phi} + \chi_\lambda^2 \wedge \bar\partial\mathbf{\Phi} - (z_1+1)^2(z_2+1)^2 \wedge (\bar\partial\chi_\lambda \wedge P_{\mathbf{f}}\mathbf{w} + \chi_\lambda \wedge P_{\mathbf{f}}\bar\partial\mathbf{w}) \\ &= 2\chi_\lambda \wedge \bar\partial\chi_\lambda \wedge \mathbf{\Phi} + (1-(4+\lambda)\widehat{A^2})\chi_\lambda \wedge P_{\mathbf{f}}\mathbf{v} - (z_1+1)^2(z_2+1)^2 \wedge \bar\partial\chi_\lambda \wedge P_{\mathbf{f}}\mathbf{w}.\end{aligned}$$

Let $\mathbf{b}$ be a solution to

$$\begin{aligned}\bar\partial\mathbf{b} &= (z_1+1)(z_2+1)(1-(4+\lambda)\widehat{A^2})^{-1} \wedge \bar\partial\mathbf{p} = (z_1+1)(z_2+1)(1-(4+\lambda)\widehat{A^2})^{-1} \\ &\quad \wedge \bar\partial\chi_\lambda \wedge (2\chi_\lambda \wedge \mathbf{\Phi} - (z_1+1)^2(z_2+1)^2 \wedge P_{\mathbf{f}}\mathbf{w}) + (z_1+1)(z_2+1)\chi_\lambda \wedge P_{\mathbf{f}}\mathbf{v};\end{aligned}$$

it is not hard to check that the right-hand side is $\bar\partial$-closed. If we write this $\bar\partial$ problem in the coordinates (ζ_1, ζ_2) and use [HeC, p. 676], (3.9), (4.9), (4.13), and (4.15), we find a $\mathbf{b} \in \Lambda^1(C((\overline{\Pi}_+ \cup \{\infty\})^2))$ such that

$$\|\mathbf{b}\| \leqslant C\cdot(1+|\lambda|)^{12}\cdot(\operatorname{Re}\lambda)^{-4}\cdot A(\lambda)^6. \tag{4.16}$$

Write

$$\mathbf{b} = \sum_{j=1}^{3} b_j \wedge \mathbf{e}_j,$$

and set

$$q_\lambda = \frac{1-\chi_\lambda^2}{1-(4+\lambda)\widehat{A^2}} - 1 + (z_1+1)^{-1}(z_2+1)^{-1}\sum_{j=1}^{3} f_j b_j.$$

Then $q_\lambda \in C((\overline{\Pi}_+ \cup \{\infty\})^2)$, q_λ vanishes at infinity, and $\bar\partial q_\lambda = 0$, so we must have $q_\lambda \in A_0(\Pi_+^2)$. Moreover,

$$(1 - (4 + \lambda)\widehat{A^2})(1 + q_\lambda) = 1 - \chi_\lambda^2 + (z_1 + 1)^{-1}(z_2 + 1)^{-1}(1 - (4 + \lambda)\widehat{A^2}) \sum_{j=1}^{3} f_j b_j$$

$$= 1 - \sum_{j=1}^{3} f_j g_j ,$$

where

$$g_j = p_j - (z_1 + 1)^{-1}(z_2 + 1)^{-1}(1 - (4 + \lambda)\widehat{A^2})b_j ,$$

and $\mathbf{p} = \Sigma_j p_j \wedge \mathbf{e}_j$. Now $g_j \in C(\overline{\Pi}_+ \cup \{\infty\})^2)$ vanishes at infinity and satisfies $\bar\partial g_j = 0$, so that $g_j \in A_0(\Pi_+^2)$. Thus $1 + q_\lambda$ is an element of the coset

$$(1 - (4 + \lambda)\widehat{A^2} + I(\hat{f}, \mathbf{K}))^{-1} ,$$

and so

$$\mathscr{C}[\phi](\lambda) = \langle \widehat{A^1} \cdot (1 + q_\lambda), \tilde{\phi} \rangle, \qquad \operatorname{Re} \lambda > 0,$$

by (4.4). By (4.16) and a trivial estimate of the first term in the definition of q_λ, we obtain

$$(4.17) \qquad |\mathscr{C}[\phi](\lambda)| \leqslant C \cdot (\operatorname{Re} \lambda)^{-4}(1 + |\lambda|)^{12} \cdot A(\lambda)^6, \qquad \operatorname{Re} \lambda > 0 .$$

In particular, (4.5) holds by (4.7), so the assertion $I(f, \mathbf{J}) = L^1(\mathbf{R}_+^2)$ follows. The proof is complete.

The sufficient condition (4.1) of Theorem 4.6 is not necessary. Part of the reason for this is that a function f can be cyclic in $L^1([0, 1]^2)$ and yet have a Fourier transform that vanishes along some sequence $\{z_n\}_0^\infty \subset K_\varepsilon$ converging to (∞, ∞), for some fixed $\varepsilon > 0$. It is, however, possible to weaken (4.1).

In the proof of Theorem 4.6, we only used the fact that we could control $A(\lambda)$ for real λ, although the assumption (4.1) implied control of $A(\lambda)$ for all $\lambda \in \Pi_+$. If we use the Ahlfors-Heins theorem in combination with the Phragmén-Lindelöf principle, we only need to control $A(\lambda)$ for a rather scattered set of λ's. Also, we can shrink the support set of χ_λ somewhat without changing the proof of Theorem 4.6. The result obtained is the following.

THEOREM 4.7. *Let $\{\lambda_n\}_0^\infty$ be a sequence of positive real numbers tending to infinity, and set*

$$G(\delta) = \{z \in \overline{\Pi}_+^2 \colon |w(z) - \lambda_n| \leqslant \delta \lambda_n \quad \textit{for some } n\} ,$$

where $w(z) = (\sqrt{z_1+1} + \sqrt{z_2+1})^2 - 4$. *If* $f \in L^1([0,1]^2)$, *and, for some* δ, $0 < \delta < 1$, *and all* $\varepsilon > 0$,

$$\log 1/|\hat{f}(z)| = o(|z|) \quad \text{as} \quad K_\varepsilon \cap G(\delta) \ni z \to (\infty, \infty), \tag{4.18}$$

where K_ε *is as in Theorem* 4.6, *then* f *generates a dense ideal in* $L^1([0,1]^2)$.

Proof. It is possible to find a function χ_λ which is supported on

$$U_\delta(\lambda) = \left\{ z \in \overline{\Pi}_+^2 : |w(z) - \lambda| \leqslant \frac{\delta}{2} \operatorname{Re} \lambda \right\},$$

equals 1 on

$$V_\delta(\lambda) = \left\{ z \in \overline{\Pi}_+^2 : |w(z) - \lambda| < \frac{\delta}{4} \operatorname{Re} \lambda \right\},$$

and has the estimate (3.9), possibly with different constants. If we put

$$A_\delta(\lambda) = \sup_{z \in U_\delta(\lambda)} \min_{j=1,2,3} \left\{ \frac{1}{|f_j(z)|} \right\},$$

we obtain from (4.18) just as in the proof of Theorem 4.6 that

$$\log A_\delta(\lambda) = o(|\lambda|) \quad \text{as} \quad |\lambda| \to \infty$$

with $\lambda \in H(\delta) = \{\zeta \in \Pi_+ : |\zeta - \lambda_n| \leqslant \delta\lambda_n/3 \text{ for some } n\}$. From the estimate (4.17), with $A(\lambda)$ replaced by $A_\delta(\lambda)$, we get

$$\mathscr{C}[\phi](\lambda) = O(\exp(\varepsilon|\lambda|)) \quad \text{as} \quad H(\delta) \ni \lambda \to \infty$$

for all $\varepsilon > 0$, so that

$$\liminf_{r \to +\infty} r^{-1} \log|\mathscr{C}[\phi](re^{i\theta})| \leqslant 0$$

for all θ in the interval $(-\alpha, \alpha)$, where $\alpha = \arctan(\delta/3)$. By the Ahlfors-Heins theorem (see [Boa, p. 116]), the limit

$$\lim_{r \to +\infty} r^{-1} \log|\mathscr{C}[\phi](re^{i\theta})|$$

exists for almost all θ, $-\pi/2 < \theta < \pi/2$, and equals $\beta \cos\theta$, where

$$\beta = \limsup_{t \to +\infty} t^{-1} \log|\mathscr{C}[\phi](t)|.$$

We conclude that $\beta \leqslant 0$, so that (4.5) holds. The assertion $I(f, \mathbf{J}) = L^1(\mathbf{R}_+^2)$ now follows just as in the proof of Theorem 4.6.

Remarks. (a) The proofs of Theorems 4.6 and 4.7 can be modified so as to show that if $f_1, \ldots, f_n \in L^1([0, 1]^2)$ and (4.18) holds with $|f(z)|$ replaced by $\max_j |f_j(z)|$, then the functions $f_1, \ldots, f_n$ together generate a dense ideal in $L^1([0, 1]^2)$.

(b) It should be observed that what determines whether a function $f \in L^1([0, 1]^2)$ generates a dense ideal in $L^1([0, 1]^2)$ is the behavior of f near the origin (0, 0). More precisely, f generates a dense ideal in $L^1([0, 1]^2)$ if and only if its restriction to $[0, \varepsilon]^2$ generates a dense ideal in the corresponding Volterra algebra $L^1([0, \varepsilon]^2)$ $(0 < \varepsilon \leqslant 1)$. In fact, something even stronger is true: a collection $\mathscr{K}$ of functions in $L^1([0, 1]^2)$ generates a dense ideal in $L^1([0, 1]^2)$ if and only if the restriction $\mathscr{K}|_{[0,\varepsilon]^2}$ generates a dense ideal in $L^1([0, \varepsilon]^2)$. The "only if" part of the assertion is trivial. For the "if" part, observe that it is sufficient to prove the assertion for $\varepsilon = 1/2$, because we can then iteratively obtain the result for $\varepsilon = 2^{-n}$, $n = 1, 2, 3, \ldots$. If $\mathscr{K}$, restricted to $[0, 1/2]^2$, generates a dense ideal in $L^1([0, 1/2]^2)$, then by Proposition 4.2,

$$\hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1/2} A_0(\Pi_+^2) + e^{-z_2/2} A_0(\Pi_+^2)$$

is dense in $A_0(\Pi_+^2)$, where $\hat{\mathscr{K}}$ denotes the collection of Fourier transforms of functions in $\mathscr{K}$. We need to show that

$$\hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1} A_0(\Pi_+^2) + e^{-z_2} A_0(\Pi_+^2)$$

is also dense in $A_0(\Pi_+^2)$, because then $\mathscr{K}$ generates a dense ideal in $L^1([0, 1]^2)$, again by Proposition 4.2. Because

$$\hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1/2} A_0(\Pi_+^2) + e^{-z_2/2} A_0(\Pi_+^2)$$

is assumed dense in $A_0(\Pi_+^2)$, we must have that

$$\begin{aligned}
&\hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1/2}(\hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1/2} A_0(\Pi_+^2) + e^{-z_2/2} A_0(\Pi_+^2)) \\
&\quad + e^{-z_2/2}(\hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1/2} A_0(\Pi_+^2) + e^{-z_2/2} A_0(\Pi_+^2)) \\
&= \hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1} A_0(\Pi_+^2) + e^{-z_2} A_0(\Pi_+^2) + e^{-z_1/2 - z_2/2} A_0(\Pi_+^2)
\end{aligned}$$

is dense in $A_0(\Pi_+^2)$ as well. But we also see that

$$e^{-z_1/2 - z_2/2}(\hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1/2} A_0(\Pi_+^2) + e^{-z_2/2} A_0(\Pi_+^2)),$$

which is a subset of

$$\hat{\mathscr{K}} \cdot A_0(\Pi_+^2) + e^{-z_1} A_0(\Pi_+^2) + e^{-z_2} A_0(\Pi_+^2),$$

is dense in $e^{-z_1/2 - z_2/2} A_0(\Pi_+^2)$. The assertion now follows.

5. Comments and further results. The reader has probably wondered silently what happens in higher dimensions than 2. Basically, one should be able to do everything in higher dimensions as well, but some work needs to be done. One has to obtain a concrete formula for A^α, $\operatorname{Re} \alpha > 0$, where

$$\widehat{A^\alpha}(z) = (\sqrt{z_1 + 1} + \cdots + \sqrt{z_n + 1})^{-\alpha}, \qquad z \in \overline{\Pi}_+^n.$$

Also, it appears that no one has published the kind of solution we need to $\bar{\partial}$ problems on the polydisc $\mathbf{D}^n$ when $n > 2$.

Consider the space $L^1(\mathbf{R} \times \mathbf{R}_+)$, which is a convolution subalgebra of $L^1(\mathbf{R}^2)$ if we extend the functions to vanish outside $\mathbf{R} \times \mathbf{R}_+$. The following problem is the $L^1(\mathbf{R} \times \mathbf{R}_+)$ analog to Levin's problem.

Problem 5.1. Characterize those functions $f \in L^1(\mathbf{R} \times \mathbf{R}_+)$ for which $f * L^1(\mathbf{R} \times \mathbf{R}_+)$ is dense in $L^1(\mathbf{R} \times \mathbf{R}_+)$.

This problem should be a lot easier than Levin's problem because the solution to the uniform algebra version of it is well known, as we shall see. Let $A_0(i\mathbf{R} \times \Pi_+)$ be the closed subalgebra of $C((i\mathbf{R} \times \overline{\Pi}_+) \cup \{\infty\})$ consisting of those functions which are analytic in the second variable and vanish at infinity. The Fourier transform defines a monomorphism $L^1(\mathbf{R} \times \mathbf{R}_+) \to A_0(i\mathbf{R} \times \Pi_+)$ with dense range. By [Gam, p. 61], all the closed ideals in $A_0(i\mathbf{R} \times \Pi_+)$ can be described in terms of the Beurling-Rudin theorem. In particular, a function $f \in A_0(i\mathbf{R} \times \Pi_+)$ generates a dense ideal if and only if

(a) $f(z) \neq 0$ for all $z \in i\mathbf{R} \times \overline{\Pi}_+$, and
(b) $f(z_1, \cdot)$ is outer for all $z_1 \in i\mathbf{R}$.

For $f \in L^1(\mathbf{R} \times \mathbf{R}_+)$, let $\mathscr{F}_1 f$ be the partial Fourier transform with respect to z_1:

$$\mathscr{F}_1 f(z_1, t_2) = \int_{-\infty}^{\infty} e^{-t_1 z_1} f(t_1, t_2)\, dt_1, \qquad t_2 > 0, z_1 \in i\mathbf{R}.$$

Let $f \in L^1(\mathbf{R} \times \mathbf{R}_+)$. If f generates a dense ideal in $L^1(\mathbf{R} \times \mathbf{R}_+)$, then $\hat{f}$ must satisfy the conditions (a)–(b) above, or equivalently,

(i) $\hat{f}(z) \neq 0$ for all $z \in i\mathbf{R} \times \overline{\Pi}_+$, and
(ii) $0 \in \operatorname{supp} \mathscr{F}_1 f(z_1, \cdot)$ for all $z_1 \in i\mathbf{R}$.

Question 5.2. Do (i)–(ii) imply that f generates a dense ideal?

One can probably show that (i)–(ii) imply that f generates a dense ideal, provided we assume that $\hat{f}$ has some regularity property, like having bounded derivatives of order ≤ 2 off some compact subset of $i\mathbf{R} \times \overline{\Pi}_+$. This is a test question for whether the regularity condition imposed on $\hat{f}$ in Theorem 3.1 is superfluous.

Let $\mathbf{B}$ be the open unit ball of $\mathbf{C}^2$, and let $A(\mathbf{B}) = C(\overline{\mathbf{B}}) \cap H^\infty(\mathbf{B})$ be the ball algebra. Let $f \in A(\mathbf{B})$ be a function which vanishes at the point $(1, 0)$ only. Using techniques similar to those found in section 1, it can be shown that the condition

$$\log 1/|f(z)| = o(1/(1 - |z|)) \quad \text{as} \quad z \to (1, 0), \tag{5.1}$$

is necessary for f to generate (after closure) the maximal ideal $\{g \in A(\mathbf{B})\colon g(1, 0) = 0\}$. In [Hed3], the author showed that the slightly stronger condition

$$\log 1/|f(z)| = o(1/(1 - |z_1|)) \quad \text{as} \quad z \to (1, 0)$$

is sufficient.

Question 5.3. Is (5.1) a sufficient condition for f to generate the maximal ideal at $(1, 0)$?

This question should be easier to answer than Question 1.9. The author believes that both questions have the same answer, and suspects that it is negative.

Part of the reason why we cannot expect the method used in section 3 to prove that the necessary conditions obtained in section 2 are sufficient is that we need to impose restrictions on the decrease of $\hat{f}$ (or f, in the case of Theorem 3.4) along the distinguished boundary $(i\mathbf{R})^2$ of $\overline{\Pi}_+^2$. This is due to the fact that all nonempty level set $\widehat{A^2}(z) = \lambda$ of the function

$$\widehat{A^2}(z) = (\sqrt{z_1 + 1} + \sqrt{z_2 + 1})^{-2}, \qquad z \in \overline{\Pi}_+^2,$$

intersect the distinguished boundary. As we shall see, this is a phenomenon that cannot be avoided.

PROPOSITION 5.4. *Let $a \in A(\mathbf{D}^2)$ have $Z(a) \supset (\{1\} \times \overline{\mathbf{D}}) \cup (\overline{\mathbf{D}} \times \{1\})$. Then $a(\mathbf{T}^2) = a(\overline{\mathbf{D}}^2)$.*

Proof. As a first step, let us show that $a(\mathbf{T} \times \overline{\mathbf{D}}) = a(\overline{\mathbf{D}}^2)$. Fix a $(z_0, w_0) \in \mathbf{D} \times \overline{\mathbf{D}}$, and let $\lambda = a(z_0, w_0)$. We plan to show that there is a point $(\alpha, \beta) \in \mathbf{T} \times \overline{\mathbf{D}}$ such that $a(\alpha, \beta) = \lambda$. Without loss of generality, we may assume that $\lambda \notin a(\mathbf{T}, w_0)$, so that in particular, $\lambda \neq 0$. Let

$$a_w(z) = a(z, w), \qquad z, w \in \overline{\mathbf{D}}.$$

Consider the set $U(\lambda)$ of all $w \in \overline{\mathbf{D}}$ for which $\lambda \in a_w(\mathbf{D})$, which is nonempty because $w_0 \in U(\lambda)$. We wish to show that this set is relatively open in $\overline{\mathbf{D}}$. To this end, let $(z_1, w_1) \in \mathbf{D} \times \overline{\mathbf{D}}$ be such that $\lambda = a_{w_1}(z_1)$, and introduce the function $\varphi_w(z) = a_w(z) - \lambda$. Let γ be a circle in $\mathbf{D}$ surrounding z_1 such that φ_{w_1} has no other zeros than z_1 inside γ, and $\varphi_{w_1} \neq 0$ on γ. By Rouché's theorem, if w is sufficiently close to

w_1 (so that $\|\varphi_{w_1} - \varphi_w\|$ is small), then φ_w must have the same number of zeros inside γ as does φ_{w_1}, and hence $0 \in \varphi_w(\mathbf{D})$, that is, $\lambda \in a_w(\mathbf{D})$. This shows that $U(\lambda)$ is relatively open. Clearly, there is a neighborhood of 1 which is disjoint from $U(\lambda)$, because $\lambda \neq 0$, and so the boundary of $U(\lambda)$ relative to $\overline{\mathbf{D}}$ must be nonempty. Pick a $\beta \in \partial U(\lambda) \backslash \mathbf{T}$. Let $w_n \in U(\lambda)$ form a sequence converging to β. Then there is an associated sequence $\{z_n\} \subset \mathbf{D}$ such that $a(z_n, w_n) = \lambda$. By replacing $\{(z_n, w_n)\}$ with a subsequence, we can make it converge to some point (α, β), and since $\beta \notin U(\lambda)$ and $a(\alpha, \beta) = \lambda$, we must have $|\alpha| = 1$, so the assertion follows.

The second step is to show that $a(\mathbf{T}^2) = a(\mathbf{T} \times \overline{\mathbf{D}})$. This is done by iterating the argument used for the first step.

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