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Formal power series and nearly analytic functions

By

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0. Introduction. Consider a formal power series

$$\sum_{n=0}^{\infty} a_n z^n, \tag{0.1}$$

with complex coefficients a_n. If

$$|a_n| \leqq C^n \quad \text{for all } n \tag{0.2}$$

for some constant C, then (0.1) is convergent for $|z| < 1/C$, and it is analytic there. Conversely, given a function f, analytic on $\{z \in \mathbb{C} : |z| < r\}$, its power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

converges for $|z| < r$, and we have $|a_n| < R^n$ for all sufficiently large n, provided that $R > r$. It seems natural to ask whether we can associate to a given power series $\sum_0^{\infty} a_n z^n$ a function f having that power series expansion around $z = 0$, also when (0.2) is violated, in which case the power series diverges unless $z = 0$. Certainly, this f cannot be analytic near $z = 0$, because its power series would then satisfy (0.2). It turns out that we can expect f to be nearly analytic, in the sense that $\partial f / \partial \bar{z}$ be small near $z = 0$. This will be made more precise in the following section; in particular, the degree of smallness of $\partial f / \partial \bar{z}$ around $z = 0$ is related to the growth of the coefficients $\{a_n\}_0^{\infty}$. A theorem of E. Borel [1] states that to any given formal power series $\sum_0^{\infty} a_n x^n$, there is a C^{∞} function f on the real line such that $f^{(n)}(0)/n! = a_n$ for all n. The extensions obtained here need not be C^{∞}, but the technique easily modifies to provide C^{∞} extensions f to the unit disc, and allows us to control $|\bar{\partial} f|$ near the origin.

Extensions need not be unique. In Section 2 we study how far two functions representing the same divergent power series expansion can deviate from one another.

1. Nearly analytic functions. In the sequel, we shall use the notation $\partial = \partial / \partial z$ and $\bar{\partial} = \partial / \partial \bar{z}$. Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$. Suppose $\omega : [0, 1] \to [0, \infty[$ is a continuous increasing function. Extend ω to $\overline{\mathbb{D}}$ by defining

$\omega(z) = \omega(|z|)$, $z \in \overline{\mathbb{D}}$. Introduce the space $\mathcal{D}(\omega)$ consisting of all functions $f \in C(\overline{\mathbb{D}})$ with $\bar{\partial} f \in C(\overline{\mathbb{D}})$ and

$$|\bar{\partial} f(z)| \leqq C_f \, \omega(z), \qquad z \in \overline{\mathbb{D}},$$

where C_f is a constant depending on f. Endowed with the norm

$$\| f \|_{\mathcal{D}(\omega)} = \max \{ \| f \|_\infty, \| \bar{\partial} f / \omega \|_\infty \},$$

$\mathcal{D}(\omega)$ is a Banach space, and supplied with pointwise multiplication on $\overline{\mathbb{D}}$, it is a Banach algebra. Here, $\| \cdot \|_\infty$ denotes the supremum norm on the region $\mathbb{D} \setminus \{0\}$; we also used the convention $0/0 = 0$. In what follows, we shall assume that ω decreases faster than polynomially, that is,

$$\omega(r) = O(r^n) \quad \text{as} \quad r \to 0 \tag{1.1}$$

for every positive integer n. For $0 < r \leqq 1$, let

$$D_r = \{ z \in \mathbb{C} : |z| < r \},$$

and orient its boundary ∂D_r counterclockwise. Let $f \in \mathcal{D}(\omega)$. By the Cauchy integral formula, we have

$$f(z) = (2\pi i)^{-1} \left\{ \int_{\partial D_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta + \iint_{D_r} \frac{\bar{\partial} f(\zeta)}{\zeta - z} \, \delta\zeta \wedge d\bar{\zeta} \right\}, \quad z \in D_r. \tag{1.2}$$

For $n = 0, 1, 2, \ldots$, let

$$a_n(f) = (2\pi i)^{-1} \left\{ \int_{\partial D_r} \zeta^{-n-1} f(\zeta) \, d\zeta + \iint_{D_r} \zeta^{-n-1} \, \bar{\partial} f(\zeta) \, d\zeta \wedge d\bar{\zeta} \right\}; \tag{1.3}$$

the right hand side is well defined because of our assumption (1.1). Also, it is easy to check that a_n is independent of the parameter r. We shall see that $\sum_0^\infty a_n(f) z^n$ is the formal power series expansion associated with f, in the appropriate sense.

Proposition 1.1. *Suppose $f \in \mathcal{D}(\omega)$, and the $a_n(f)$ are given by* (1.3). *Then*

$$f(z) = a_0(f) + a_1(f) z + \cdots + a_{n-1}(f) z^{n-1} + z^n h_n(z), \tag{1.4}$$

where

$$h_n(z) = (2\pi i)^{-1} \left\{ \int_{\partial D_r} \frac{f(\zeta)}{\zeta^n(\zeta - z)} \, d\zeta + \iint_{D_r} \frac{\bar{\partial} f(\zeta)}{\zeta^n(\zeta - z)} \, \delta\zeta \wedge d\bar{\zeta} \right\}, \quad z \in D_r.$$

In particular,

$$f(z) = a_0(f) + a_1(f) z + \cdots + a_{n-1}(f) z^{n-1} + O(|z|^n), \quad \text{as } z \to 0.$$

Proof. We will prove the assertion by induction. When $n = 0$, (1.4) coincides with the true statement (1.2). Secondly, assuming (1.4) holds for $n = k$, we will show that it holds for $n = k + 1$ as well. By (1.3), the definition of h_n, and the equality

$$\zeta^{-k}(\zeta - z)^{-1} - \zeta^{-k-1} = z \zeta^{-k-1} (\zeta - z)^{-1},$$

we obtain

$$h_k(z) - a_k(f) = z(2\pi i)^{-1}\left\{\int_{\partial D_r} \frac{f(\zeta)}{\zeta^{k+1}(\zeta - z)}\,d\zeta + \iint_{D_r} \frac{\bar\partial f(\zeta)}{\zeta^{k+1}(\zeta - z)}\,\delta\zeta \wedge d\bar\zeta\right\},\quad z \in D_r,$$

that is,

$$h_k(z) - a_k(f) = z\,h_{k+1}(z), \quad z \in D_r.$$

It follows that

$$z^k h_k(z) = a_k(f)\,z^k + z^{k+1} h_{k+1}(z), \quad z \in D_r,$$

and the proof is complete.

Introduce the weight $W[\omega]\colon \mathbb{N} = \{0, 1, 2, \ldots\} \to [0, \infty[$ by the relation

$$W[\omega](n) = \sup\{\omega(t)/t^n : 0 < t \leqq 1\}.$$

Observe that $W[\omega]$ is an increasing function on $\mathbb{N}$, and that $W[\omega](n) > 0$ for all n unless $\omega \equiv 0$. From (1.3) we get the estimate

$$|a_n(f)| \leqq \|f\|_\infty + \|\bar\partial f/\omega\|_\infty \int_0^1 t^{-n}\omega(t)\,dt$$

by putting $r = 1$. It follows that

$$|a_n(f)| \leqq \|f\|_{\mathscr{D}(\omega)}(1 + W[\omega](n)).$$

Let's write this as a proposition.

Proposition 1.2. *Suppose $f \in \mathscr{D}(\omega)$ and that $a_n(f)$ is given by* (1.3). *Then*

$$|a_n(f)| \leqq \|f\|_{\mathscr{D}(\omega)}(1 + W[\omega](n)).$$

If we are given an increasing weight sequence $w\colon \mathbb{N} \to [0, \infty[$, we can construct an associated weight $\Omega[w]\colon [0, 1] \to [0, \infty[$ via the relation

$$\Omega[w](t) = \inf\{t^n w(n) : n \in \mathbb{N}\}.$$

The following result explains the duality between the operations $W[\cdot]$ and $\Omega[\cdot]$.

Proposition 1.3. *Suppose $\omega\colon [0, 1] \to [0, \infty[$ and $w\colon \mathbb{N} \to [0, \infty[$ are continuous increasing functions, and that $\omega(0) = 0$. Then $W[\Omega[W[\omega]]] = W[\omega]$, and $\Omega[W[\Omega[w]]] = \Omega[w]$. We have that $W[\Omega[w]] = w$ if and only if $\log w = \{\log w(n)\}_0^\infty$ is a convex sequence. In fact, in general, $\log W[\Omega[w]]$ is the largest increasing convex minorant to $\log w$. Moreover, $\Omega[W[\omega]] = \omega$ if and only if the function*

$$h(t) = \log \omega(e^t), \quad t \leqq 0,$$

is concave, and $h'(t) \in \mathbb{N}$ whenever $h'(t)$ is defined. In general, the function

$$H(t) = \log \Omega[W[\omega]](e^t), \quad t \leqq 0,$$

is the smallest concave majorant to h with the property that $H'(t) \in \mathbb{N}$ whenever $H'(t)$ is defined.

Proof. This result is probably well-known in convexity theory. But for the sake of completeness, we include a proof. Let us first show that $\log W[\Omega[w]]$ is the largest convex minorant to $\log w$. Observe that by a change of variables,

$$\log W[\Omega[w]](n) = \sup\{\inf\{n - m)s + \log w(m) : m \in \mathbb{N}\} : s \geqq 0\}.$$

In other words, $\log W[\Omega[w]]$ is the upper envelope of all the lines

$$(1.5) \qquad n \mapsto \inf\{(n - m)s + \log w(m) : m \in \mathbb{N}\} \quad s \geqq 0.$$

The line (1.5) is precisely the line with slope s, lying below the graph of $\log w$, which is closet to the graph of $\log w$. It follows that the upper envelope of the lines (1.5) coincides with the largest increasing convex minorant to $\log w$. We proceed to show that H is the smallest concave majorant to h with the property that $H'(t) \in \mathbb{N}$ whenever $H'(t)$ is defined. Observe that

$$H(t) = \inf\{\sup\{n(t - s) + h(s) : s \leqq 0\} : n \in \mathbb{N}\};$$

in other words, H is the lower envelope of all the lines

$$(1.6) \qquad t \mapsto \sup\{n(t - s) + h(s) : s \leqq 0\}, \quad n \in \mathbb{N}.$$

The line (1.6) is precisely the line with slope n, lying above the graph of h, which is closest to the graph of h. It follows that H, being the lower envelope of the lines (1.6), coincides with the smallest concave majorant to h with piecewise constant $\mathbb{N}$-valued derivative.

To show that $W[\Omega[W[\omega]]] = W[\omega]$ and $\Omega[W[\Omega[w]]] = \Omega[w]$, one need only verify that the pertinent conditions are met by the functions $W[\omega]$ and $\Omega[w]$. This is left to the interested reader.

The following theorem is a partial converse to Proposition 1.2.

Theorem 1.4. *Let $\omega : [0, 1] \to [0, \infty[$ be an increasing continuous function, which has $\omega(1) > 0$ and satisfies* (1.1). *Suppose $\{a_n\}_0^\infty$ is a complex-valued sequence such that*

$$(1.7) \qquad |a_n| \leqq C W[\omega](n - 1)/(n + 1)^\alpha, \quad n = 1, 2, 3, \ldots,$$

for some constants C and α, $\alpha > 2$. Then there exists a function $f \in \mathscr{D}(\omega)$ such that

$$a_n(f) = a_n, \quad n \in \mathbb{N}.$$

Proof. Let g be a continuous function on the unit circle $\mathbb{T} = \partial\mathbb{D}$, to be specified later, and set

$$f(z) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{\omega(r) g(e^{i\theta})}{r e^{i\theta} - z} r\, dr\, d\theta, \quad z \in \overline{\mathbb{D}}.$$

Then, by computation, $f \in C(\overline{\mathbb{D}})$, and

$$\bar{\partial} f(r e^{i\theta}) = \omega(r) g(e^{i\theta}),$$

so that $f \in \mathscr{D}(\omega)$. From (1.2) and the definition of f, we obtain

$$\int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0, \quad z \in \mathbb{D},$$

and by differentiating this equality, we obtain

$$\int_{\mathbb{T}} \zeta^{-n-1} f(\zeta)\, d\zeta = 0, \quad n \in \mathbb{N}.$$

We conclude that

$$\begin{aligned} a_n(f) &= (2\pi i)^{-1} \iint_{\mathbb{D}} \zeta^{-n-1} \bar{\partial} f(\zeta)\, d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{\pi} \int_0^1 r^{-n} \omega(r)\, dr \cdot \int_0^{2\pi} e^{-i(n+1)\theta} g(e^{i\theta})\, d\theta \\ &= 2\hat{g}(n+1) \int_0^1 r^{-n} \omega(r)\, dr, \end{aligned}$$

where

$$\hat{g}(k) = \int_0^{2\pi} e^{-ik\theta} g(e^{i\theta})\, d\theta / 2\pi.$$

It follows that

$$\hat{g}(n+1) = a_n(f) \left\{ \int_0^1 t^{-n} \omega(t)\, dt \right\}^{-1}, \quad n \in \mathbb{N}.$$

If we can find a $g \in C(\mathbb{T})$ having

$$\hat{g}(n+1) = a_n \left\{ \int_0^1 t^{-n} \omega(t)\, dt \right\}^{-1}, \quad n \in \mathbb{N}, \tag{1.8}$$

the assertion $a_n(f) = a_n$ will follow. We will need to estimate $\int_0^1 t^{-n} \omega(t)\, dt$ from below. Let $t_n \in \,]0, 1]$ be a point where $\sup \{t^{-n+1} \omega(t) : 0 < t \leqq 1\}$ is attained, that is,

$$t_n^{-n+1} \omega(t_n) = W[\omega](n-1).$$

Observe that $t_n \to \beta$ as $n \to \infty$, where

$$\beta = \inf \operatorname{supp} \omega = \inf \{t \in [0, 1] : \omega(t) > 0\},$$

which has $0 \leqq \beta < 1$, since we assumed that $\omega(1) > 0$. Now since ω was increasing,

$$\int_0^1 t^{-n} \omega(t)\, dt \geqq \int_{t_n}^1 t^{-n} \omega(t)\, dt \geqq \omega(t_n) \int_{t_n}^1 t^{-n}\, dt = \omega(t_n) \cdot (t_n^{-n+1} - 1)/n.$$

Since $t_n \to \beta \in [0, 1[$, we have for sufficiently large n,

$$(t_n^{-n+1} - 1)/n \geqq t_n^{-n+1}/2n,$$

so that

$$\int_0^1 t^{-n} \omega(t)\, dt \geqq t_n^{-n+1} \omega(t_n)/2n = W[\omega](n-1)/2n. \tag{1.9}$$

By (1.7) and (1.9), we have that for big n,

$$|a_n| \left\{ \int_0^1 t^{-n} \omega(t)\, dt \right\}^{-1} \leqq 2C/n^{\alpha-1},$$

so that

$$\sum_{n=0}^{\infty} |a_n| \left\{ \int_0^1 t^{-n} \omega(t)\, dt \right\}^{-1} < \infty .$$

It is now possible to solve the problem (1.8); in fact, we can solve it with g being a disc algebra function with summable Taylor series. The proof is complete.

Remark. The technique used in the proof of Theorem 1.4 resembles that used by Dynkin in [2], where he studied classes of almost analytic functions on the unit circle $\mathbb{T}$.

2. Uniqueness of extensions. Suppose two functions $F, G \in \mathscr{D}(\omega)$ have the same formal power series expansions at 0, that is, $a_n(F) = a_n(G)$ for all $n = 0, 1, 2, \ldots$. By Proposition 1.1, this means that the function $f = F - G$ satisfies

(2.1) $$f(z) = O(|z|^n) \quad \text{as} \quad z \to 0$$

for all $n = 1, 2, 3, \ldots$. If $\omega(t) = 0$ on $[0, \varepsilon]$ for some $\varepsilon > 0$, then $f \in \mathscr{D}(\omega)$ implies that f is analytic on $D(0, \varepsilon)$, the open disc with radius ε, centered at 0, so by (2.1),

(2.2) $$f \equiv 0 \quad \text{on} \quad D(0, \varepsilon).$$

Assertion (2.2) is considerably stronger than (2.1). This leads one to wonder if perhaps in general, provided $f \in \mathscr{D}(\omega)$, (2.1) can be strengthened to something like

(2.3) $$|f(z)| \leqq C_f\, \omega(z), \quad z \in \overline{\mathbb{D}},$$

for some constant C_f depending on f. Unless ω vanishes on some interval $[0, \varepsilon]$ with $\varepsilon > 0$, we certainly cannot hope to get anything as strong as (2.2). To see this, let $\{\varepsilon_n\}_0^\infty \subset]0, 1[$ be a strictly decreasing sequence, converging to 0, and let $\{f_n\}_1^\infty$ a corresponding sequence of smooth nonidentically vanishing functions on $\overline{\mathbb{D}}$, satisfying $f_n(z) = 0$ if $|z| \leqq \varepsilon_n$ or $|z| \geqq \varepsilon_{n-1}$. Given an ω which is positive on $]0, 1]$, it is not difficult to choose positive real numbers δ_n so as to make the series

$$f(z) = \sum_{n=1}^{\infty} \delta_n f_n(z), \quad z \in \overline{\mathbb{D}},$$

converge in $\mathscr{D}(\omega)$. The function f is an element of $\mathscr{D}(\omega)$, satisfies (2.1), and yet it does not meet (2.2) for any $\varepsilon > 0$.

Now suppose we have an arbitrary $f \in \mathscr{D}(\omega)$ with $a_n(f) = 0$ for $n = 0, 1, 2, \ldots$. We may assume without loss of generality that $f = 0$ on $\mathbb{T}$; otherwise we simply replace f by $f_1 = f \cdot \chi$, where $\chi \in C^\infty$, $0 \leqq \chi \leqq 1$ on $\mathbb{D}$, $\chi = 1$ near 0, and $\chi = 0$ near $\mathbb{T}$, and observe that $a_n(f_1) = a_n(f)$, since these coefficients are purely local in nature, in view of Proposition 1.1. Since $a_n(f) = 0$ for all $n = 0, 1, 2, \ldots$, we have, again by Proposition 1.1,

(2.4) $$f(z) = z^n (2\pi i)^{-1} \iint_{\mathbb{D}} \frac{\bar\partial f(\zeta)}{\zeta^n(\zeta - z)}\, d\zeta \wedge d\bar\zeta, \quad z \in \mathbb{D}.$$

Since

$$|\bar\partial f(\zeta)/\zeta^n| \leqq C_f\, |\zeta|^{-n}\, \omega(|\zeta|)$$

for some constant C_f depending on f, and

$$\iint_{\mathbb{D}} \frac{dA(\zeta)}{|\zeta - z|} \leqq 2\pi, \quad z \in \mathbb{D},$$

where dA is planar area measure, we have the estimate

$$\left|(2\pi i)^{-1} \iint_{\mathbb{D}} \frac{\bar{\partial} f(\zeta)}{\zeta^n(\zeta - z)} d\zeta \wedge d\bar{\zeta}\right| \leqq 2 C_f \sup\{r^{-n}\omega(r): 0 < r \leqq 1\}$$
$$= 2 C_f \cdot W[\omega](n).$$

By (2.4), it now follows that

$$|f(z)| \leqq 2 C_f \cdot W[\omega](n)\,|z|^n$$

for all $n = 0, 1, 2, \ldots$, so that

$$|f(z)| \leqq 2 C_f \inf\{W[\omega](n)\,|z|^n : n = 0, 1, 2, \ldots\} = 2 C_f \Omega[W[\omega]](|z|). \tag{2.5}$$

Proposition 1.3 descripes in detail when $\Omega[W[\omega]] = \omega$. At any rate, the previously mentioned observation (2.2) follows from (2.5) in the case when $\omega = 0$ on $D(0, \varepsilon)$.

It is possible to refine estimate (2.5).

Let $q \in \mathbb{N}\backslash\{0\}$, and put

$$f_q(z) = f(z^q), \quad z \in \overline{\mathbb{D}}.$$

Now

$$\bar{\partial} f_q(z) = q\bar{z}^{q-1} \cdot \bar{\partial} f(z^q)$$

from which we deduce that $f \in \mathscr{D}(\omega_q)$, where

$$\omega_q(r) = r^{q-1}\,\omega(r^q), \quad 0 \leqq r \leqq 1.$$

One easily checks that ω_q is a continuous increasing weight which satisfies (1.1) because ω does. Since $|f(z)| = O(|z|^n)$ as $z \to 0$ for all $n = 1, 2, 3, \ldots$, we have $|f_q(z)| = O(|z|^n)$ as $z \to 0$ for all $n = 1, 2, 3, \ldots$ as well, and hence $a_n(f_q) = 0$ for all $n \in \mathbb{N}$. Let $k \in \mathbb{N}$. By Proposition 1.1,

$$f_q(z) = z^k \cdot (2\pi i)^{-1} \cdot \iint_{\mathbb{D}} q\bar{\zeta}^{q-1}\zeta^{-k}(\zeta - z)^{-1}\bar{\partial} f(\zeta^q)\,d\zeta \wedge d\bar{\zeta},$$

from which we derive the estimate

$$\begin{aligned}|f_q(z)| &\leqq q\,|z|^k \cdot \|\zeta \mapsto (\zeta - z)^{-1}\|_{L^1(\mathbb{D})} \cdot \|\zeta \mapsto \bar{\zeta}^{q-1}\zeta^k\bar{\partial} f(\zeta^q)\|_{L^\infty(\mathbb{D})}\\ &\leqq 2q\,C_f\,|z|^k \sup\{r^{q-k-1}\,\omega(r^q): 0 < r \leqq 1\}\\ &= 2q\,C_f\,|z|^k \sup\{t^{1-(k+1)/q}\,\omega(t): 0 < t \leqq 1\}\\ &= 2q\,C_f\,|z|^k\,W[\omega]((k+1)/q - 1),\end{aligned}$$

where C_f is the same constant as last time. If we write $p = k + 1 - q$, we get

$$|f(z)| \leqq 2q\,C_f\,|z|^{(p-1)/q+1}\,W[\omega](p/q).$$

The condition $k \in \mathbb{N}$ translates to $p + q \in \mathbb{Z}_+ = \mathbb{N}\setminus\{0\}$. If we substitute np for p and nq for q, where $n \in \mathbb{Z}_+$, we get

$$(2.6) \qquad |f(z)| \leqq 2q\,C_f\,|z|^{p/q+1}\,W[\omega]\,(p/q)\inf\{n\,|z|^{-1/(nq)}: n \in \mathbb{Z}_+\};$$

this is permissible because $np + nq \in \mathbb{Z}_+$ if $p + q \in \mathbb{Z}_+$. We should study the quantity

$$\inf\{nq\,|z|^{-1/(nq)}: n \in \mathbb{Z}_+\}.$$

Calculus tells us to choose n near $q^{-1}\log 1/|z|$. So, let n_0 be the smallest integer $\geqq q^{-1}\log 1/|z|$; then

$$(2.7) \qquad \inf\{nq\,|z|^{-1/(nq)}: n \in \mathbb{Z}_+\} \leqq n_0\,q\,|z|^{-1/(n_0 q)} \leqq n_0\,q \cdot e \leqq e(q + \log 1/|z|).$$

From (2.6) and (2.7) we obtain the estimate

$$(2.8) \qquad |f(z)| \leqq 2e\,C_f\,|z|\inf\{|z|^{p/q}(q + \log 1/|z|)\,W[\omega]\,(p/q): q, p + q \in \mathbb{Z}_+\}.$$

If we put $q = 1$, (2.8) tells us that

$$(2.9) \qquad |f(z)| \leqq 2e\,C_f\,|z|\,(1 + \log 1/|z|)\,\Omega[W[\omega]]\,(|z|),$$

which is a substantial improvement of (2.5). Introduce the function

$$\Omega_q[W[\omega]]\,(r) = \inf\{r^{p/q}\,W[\omega]\,(p/q)\}.$$

Just as in Proposition 1.3, one easily verifies that the function

$$H_q(t) = \log \Omega_q[W[\omega]]\,(e^t), \quad t \leqq 0,$$

is the smallest concave majorant to $t \mapsto \log \omega(e^t)$ with the property that $q\,H_q'(t) \in \mathbb{N}$ whenever $H_q'(t)$ is defined. With this notation, it follows from (2.8) that

$$(2.10) \qquad |f(z)| \leqq 2e\,C_f\,|z|\inf\{(q + \log 1/|z|)\,\Omega_q[W[\omega]]\,(|z|): q \in \mathbb{Z}_+\}.$$

Let us assume ω is such that the function

$$g(t) = -\log \omega(e^{-t}), \quad t \geqq 0,$$

is smooth, convex, and increasing. Let us decide that we want to know when (2.10) implies that

$$|f(z)| \leqq K_{f,\omega} \cdot |z|\,(1 + \log 1/|z|) \cdot \omega(z), \quad z \in \mathbb{D}\setminus\{0\},$$

for some constant $K_{f,\omega}$ depending on f and ω. Put $G_q(t) = -H_q(-t)$, where H_q is as above. Then G_q can be interpreted as the biggest convex minorant to g with the property that $q\,G_q'(t) \in \mathbb{N}$ whenever $G_q'(t)$ is defined. If we substitute $z = e^{-t+i\theta}$, where $t \geqq 0$ and $\theta \in \mathbb{R}$, (2.10) becomes

$$(2.11) \qquad |f(e^{-t+i\theta})| \leqq 2e\,C_f\,e^{-t}\inf\{(q + t)\exp(-G_q(t)): q \in \mathbb{Z}_+\}.$$

If we take the infimum in (2.11) only over those $q \in \mathbb{Z}_+$ such that for some fixed constant $K \geqq 1$, $q \leqq q_K(t)$, where $q_K(t)$ is the biggest integer $\leqq K + (K - 1)\,t$, then (2.11) becomes

$$(2.12) \qquad |f(e^{-t+i\theta})| \leqq 2e\,K\,C_f(1 + t)\,e^{-t}\exp(-\sup\{G_q(t): \mathbb{Z}_+ \ni q \leqq q_K(t)\}).$$

Our problem becomes to determine for which smooth, convex, and increasing functions g it is true that

$$\sup\{G_q(t): \mathbb{Z}_+ \ni q \leqq q_K(t)\} \geqq g(t) - M,$$

for some constant $M > 0$. After a moment's thought we realize that this certainly holds if

(i) $$g'(\tau) - g'(t) < 1/q_K(t)$$

implies that

(ii) $$g(\tau) - g(t) - (\tau - t)g'(t) \leqq M,$$

for all $\tau \geqq t \geqq 0$. Let $\phi: [0, \infty[\to]0, \infty[$ be a continuous function, and let Φ denote an antiderivative to ϕ, that is, a solution to $\Phi'(t) = \phi(t)$. Let us assume that $g''(t) \geqq \phi(t)$ for all $t \geqq 0$. We hope that for some ϕ, this condition will imply that (i) $\Rightarrow$ (ii). So, assume (i) holds, but not (ii); we hope to arrive at a contradiction. Observe that

$$g'(\tau) - g'(t) = \int_t^\tau g''(s)\,ds \geqq \int_t^\tau \phi(s)\,ds = \Phi(\tau) - \Phi(t),$$

so that

(2.13) $$\Phi(\tau) - \Phi(t) < 1/q_K(t).$$

Also, since g is convex,

$$g(\tau) - g(t) - (\tau - t)g'(t) = \int_t^\tau (g'(s) - g'(t))\,ds \leqq \int_t^\tau (g'(\tau) - g'(t))\,ds$$
$$= (\tau - t)(g'(\tau) - g'(t)),$$

so that since (i) holds and (ii) doesn't,

(2.14) $$\tau - t \geqq M \cdot q_K(t).$$

We want to find a Φ such that (2.13) and (2.14) are incompatible for all $\tau \geqq t \geqq 0$, that is, we should have

(2.15) $$\Phi(t + M \cdot q_K(t)) - \Phi(t) \geqq 1/q_K(t)$$

for all $t \geqq 0$. It is not difficult to check that the function

$$\Phi(t) = -\varepsilon(1 + t)^{-1}, \quad t \geqq 0,$$

satisfies (2.15), provided that

$$\varepsilon \geqq \frac{1 + MK}{M(K - 1)^2}.$$

If $\varepsilon \leqq 1$, this inequality holds if we choose $M = 1$ and $K = 4/\varepsilon$. We have obtained the following theorem.

Theorem 2.1. *Suppose ω is such that the function*

$$g(t) = -\log \omega(e^{-t}), \quad t \geqq 0,$$

is smooth, convex, increasing, and satisfies

$$g''(t) \geqq \varepsilon(1+t)^{-2}, \quad t \geqq 0,$$

for some ε, $0 < \varepsilon \leqq 1$. *Let* $f \in \mathscr{D}(\omega)$ *vanish on* $\mathbb{T}$ *and satisfy* $a_n(f) = 0$ *for all* $n = 0, 1, 2, \ldots$; *then*

$$|f(z)| \leqq 8\varepsilon^{-1} e^2 C_f |z| (1 + \log 1/|z|) \omega(z), \quad z \in \overline{\mathbb{D}} \backslash \{0\},$$

where $C_f = \|\bar{\partial} f/\omega\|_\infty$.

Remark. If we want a statement for the case when f might not vanish on $\mathbb{T}$, we have to carry though the argument with the cut-off function χ quantitatively. After a few computations, one then obtains the estimate

$$|f(z)| \leqq 8(1 + 10/\omega(1/2))\varepsilon^{-1} e^2 \|\bar{\partial} f/\omega\|_\infty |z| (1 + \log 1/|z|) \omega(z),$$

$$\text{for} \quad 0 < |z| \leqq 1/2.$$

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