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On the Failure of Optimal Factorization for Certain Weighted Bergman Spaces

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In this note we show that the factorization theorem in [2] fails to generalize to the weighted Bergman L^2-space on the disc with weight $(\alpha + 1)(1 - |z|^2)^\alpha$ for $\alpha > 1$; the limit 1 is sharp, because for $\alpha = 1$, it does generalize [3]. We also show that a similar factorization theorem fails to hold generally for the ordinary Bergman L^2-space in multiply connected planar domains.

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1. INTRODUCTION

Let Ω be a domain in the complex plane $\mathbf{C}$ and f be an analytic function on Ω. We denote by $Z(f)$ the zero set of f; multiple zeros are repeated according to multiplicity. Suppose X is a space of analytic functions on Ω. A set A (with possibly repeating elements) is called an X-zero set if there exists a function f in X such that $Z(f) = A$.

Let $\mathbf{D}$ be the open unit disc in $\mathbf{C}$ and $L_a^2(\mathbf{D})$ be the Bergman space of analytic functions in $L^2(\mathbf{D}, dA)$, where dA is the (normalized) area measure on $\mathbf{D}$. $L_a^2(\mathbf{D})$ is a Banach space (as a closed subspace of $L^2(\mathbf{D}, dA)$). The following result was proved recently by H. Hedenmalm.

THEOREM 1 [2] *For any $L_a^2(\mathbf{D})$-zero set A not containing* 0 *there exists a (unique) function G_A in $L_a^2(\mathbf{D})$ satisfying the following conditions:*

(1) $\|G_A\| = 1$, $G_A(0) > 0$;
(2) $Z(G_A) = A$;
(3) $\|f/G_A\| \le \|f\|$ *for all* $f \in L_a^2(\mathbf{D})$ *with* $A \subset Z(f)$.

The function G_A in the above theorem is simply the (unique) solution to the following extremal problem:

$$\sup\{\operatorname{Re} f(0) : \|f\| \le 1,\ A \subset Z(f)\}.$$

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Suppose X is a Banach space of analytic functions on $\mathbf{D}$ containing the constant function 1, and assume that the norm of the constant function 1 is 1. We say that X has the optimal factorization property if for each X-zero set A (not containing the origin) there exists a function G_A^X satistying the three properties in the above theorem (with $L_a^2(\mathbf{D})$ replaced by X in (3), and the norm of $L_a^2(\mathbf{D})$ replaced by that of X). In this case, we call the functions G_A^X *contractive zero-divisors*. Expressed in this terminology, Theorem 1 states that $L_a^2(\mathbf{D})$ has the optimal factorization property.

There are several directions in which one might try to extend Hedenmalm's result. We mention three of them:

(1) Does optimal factorization hold for Bergman L^p-spaces on $\mathbf{D}$?
(2) Does optimal factorization hold for weighted Bergman spaces on $\mathbf{D}$?
(3) Can we extend the notion of optimal factorization so that it also holds for other domains in $\mathbf{C}$?

The first question was answered affirmatively in [1] using an argument involving the bi-harmonic operator Δ^2 (the square of the Laplace operator). More specifically, the biharmonic Green's function, that is, the solution to the boundary-value problem

$$\begin{cases} \Delta^2 u = \delta_\zeta \\ u = 0, \qquad \dfrac{\partial}{\partial n} u = 0 \quad \text{on} \quad \partial\mathbf{D}, \end{cases}$$

is known to be positive on $\mathbf{D}$ for each $\zeta \in \mathbf{D}$, where δ_ζ is the Dirac mass at ζ, and in [1], this fact is the essential ingredient used to prove that optimal factorization holds for Bergman L^p-spaces on $\mathbf{D}$ $(1 \le p < \infty)$.

In this paper we consider questions (2) and (3) above. The results obtained are somewhat negative. The following is a sketch of the contents of our paper.

For weighted Bergman spaces on $\mathbf{D}$, we will show that if there exists a contractive zero-divisor, then it must be a solution to a corresponding extremal problem. We are particularly interested in the weights

$$\omega_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha, \qquad z \in \mathbf{D},$$

for $\alpha > -1$. Extremal functions for one prescribed zero can be calculated explicitly, and for the weights ω_α, with $\alpha > 4$, they have additional zeros in $\mathbf{D}$ if the original zero is close enough to the boundary; moreover, for $\alpha > 1$, if the prescribed zero is close enough to the boundary, the extremal function will have some boundary values of modulus less than 1. From these facts it is not difficult to derive that optimal factorization fails for weighted Bergman spaces on $\mathbf{D}$ with weights ω_α, for $\alpha > 1$. This result is sharp, because Hedenmalm [3] has shown that optimal factorization does hold for the weighted Bergman space with weight function $\omega_1(z) = 2(1 - |z|^2)$.

For ordinary Bergman spaces on more general planar domains, the extremal function for one prescribed zero will be calculated for any points where the kernel function vanishes, and shown to be just a constant multiple of the kernel function. But for multiply connected domains there always exist parameter values for which the kernel function has more than one zero [4]. Consequently, optimal factorization fails for the Bergman space on such domains.

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2. WEIGHTED BERGMAN SPACES

For $\alpha > -1$, let $L_a^2(dA_\alpha)$ be the weighted Bergman space of analytic functions in $L^2(\mathbf{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z).$$

Being a closed subspace of $L^2(\mathbf{D}, dA_\alpha)$, $L_a^2(dA_\alpha)$ is a Hilbert space. We will use $\| \ \|_\alpha$ and $\langle \ , \ \rangle_\alpha$ to denote the norm and inner product on $L_a^2(dA_\alpha)$, respectively.

PROPOSITION 2 *Suppose $a \in \mathbf{D}$ and $a \neq 0$; then the function*

$$G_a^\alpha(z) = \left[1 - \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\alpha+2}\right] \Big/ \sqrt{1-(1-|a|^2)^{\alpha+2}}$$

(the principal branch) is the solution to the following extremal problem:

$$\sup\{\operatorname{Re} f(0) : \|f\|_\alpha \leq 1, \ f(a) = 0\}.$$

Proof Let H_a be the closed subspace of $L_a^2(dA_\alpha)$ consisting of functions which vanish at a. It is clear that H_a is a Hilbert space and $f \mapsto f(0)$ is a bounded linear functional on H_a. By Riesz representation theorem, there exists a unique function g_a in H_a such that

$$f(0) = \langle f, g_a \rangle_\alpha, \qquad f \in H_a.$$

It is easy to check (see 4.2.1 of [5]) that the function (the principal branch)

$$1 - \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{2+\alpha}$$

is in H_a and has the property stated above. Thus by uniqueness we have

$$g_a(z) = 1 - \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\alpha+2}.$$

Also

$$\|g_a\|_\alpha^2 = \langle g_a, g_a \rangle_\alpha = g_a(0) = 1 - (1-|a|^2)^{\alpha+2}.$$

Let

$$G_a^\alpha(z) = \frac{g_a(z)}{\|g_a\|_\alpha} = \frac{1 - \left(\dfrac{1-|a|^2}{1-\bar{a}z}\right)^{\alpha+2}}{\sqrt{1-(1-|a|^2)^{\alpha+2}}}.$$

Then $G_a^\alpha(0) > 0$ and the Cauchy-Schwarz inequality shows that G_a^α is the solution to the extremal problem

$$\sup\{\operatorname{Re} f(0) : \|f\|_\alpha \leq 1, \ f(a) = 0\}.$$

PROPOSITION 3 *The optimal function G_a^α has a unique zero in* $\mathbf{D}$ *($z = a$) for each $a \in \mathbf{D}$ if and only if $-1 < \alpha \leq 4$.*

Proof The linear transformation

$$w = \frac{1-\bar{a}z}{1-|a|^2}$$

takes $\mathbf{D}$ onto the disc with center $1/(1-|a|^2)$ and radius $|a|/(1-|a|^2)$. Note that this disc is symmetric with respect to the real axis and its boundary intersects this axis at the points $1/(1+|a|)$ and $1/(1-|a|)$. Thus as $|a|$ increases to 1 these discs increase to the half plane $\operatorname{Re} w > \frac{1}{2}$. Now $G_a^\alpha(z) = 0$ has a solution in $\mathbf{D}$ other than $z = a$ if and only if thereis a point $w \neq 1$ in this half plane such that $w^{\alpha+2} = 1$. This is clearly equivalent to the statement that

$$\operatorname{Re}\left[\exp\frac{2\pi i}{\alpha+2}\right] > \frac{1}{2}.$$

Since we are dealing with the principal branch of the logarithm, that is, "arg" is chosen between $-\pi$ and π, the above statement is equivalent to

$$\frac{2\pi}{\alpha+2} < \frac{\pi}{3} \quad \text{or} \quad \alpha > 4,$$

completing the proof of Proposition 3.

PROPOSITION 4 *If $\alpha > 4$, then for any a in* $\mathbf{D}$ *with*

$$|a| = 2\sin\frac{\pi}{\alpha+2}$$

the function

$$G_a^\alpha(z) = \left[1-\left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\alpha+2}\right] \Big/ \sqrt{1-(1-|a|^2)^{\alpha+2}}$$

(the principal branch) has at least one zero on $\partial\mathbf{D}$.

Proof This follows from direct calculation. In fact, if

$$|a| = 2\sin\frac{\pi}{\alpha+2},$$

then the point

$$z = \frac{1-(1-|a|^2)\exp\left(\dfrac{2\pi i}{\alpha+2}\right)}{\bar{a}}$$

is a zero of G_a^α on $\partial\mathbf{D}$.

PROPOSITION 5 *If $\alpha > 1$, then there exists $a \in \mathbf{D}$ and $z \in \partial\mathbf{D}$ such that $|G_a^\alpha(z)| < 1$.*

Proof If $\alpha > 4$, the desired result follows from Proposition 4. So we may as well assume that $1 < \alpha < 5$. In this case we shall prove that

$$\min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| < 1$$

for $a \in \mathbf{D}$ close enough to $\partial\mathbf{D}$.

Note that

$$w = \frac{1-|a|^2}{1-\bar{a}z}$$

maps the unit circle onto the circle $|w-1| = |a|$. In fact, if $|z| = 1$, then

$$|w-1| = \left|\frac{1-|a|^2-1+\bar{a}z}{1-\bar{a}z}\right| = |a|\left|\frac{z-a}{1-\bar{a}z}\right| = |a|.$$

It follows that for all $|a| < 1$

$$\min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| = \frac{1}{\sqrt{1-(1-|a|^2)^{\alpha+2}}} \min_{|t|\le\pi} |1-(1+|a|e^{it})^{\alpha+2}|.$$

Letting $a \to \partial\mathbf{D}$, we get

$$\lim_{|a|\to 1^-} \min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| = \inf_{|t|<\pi} |1-(1+e^{it})^{\alpha+2}|.$$

Since

$$1+e^{it} = 2\cos\frac{t}{2}e^{it/2},$$

we conclude that

$$\lim_{|a|\to 1^-} \min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| = \inf_{|t|<\pi/2} |1-(2\cos t e^{it})^{\alpha+2}| = \inf_{|t|\le\pi/2} |1-(2\cos t)^{\alpha+2}e^{(\alpha+2)ti}|.$$

By symmetry, we have

$$\lim_{|a|\to 1^-} \min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| = \min_{0\le t\le\pi/2} |1-(2\cos t)^{\alpha+2}e^{(\alpha+2)ti}|.$$

Observe that

$$|1-e^{5\pi i/3}| = |1-e^{7\pi i/3}| = 1.$$

This implies that for $3 < \alpha < 5$ (simply let $t = \pi/3$)

$$\lim_{|a|\to 1^-} \min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| \le |1-e^{(\alpha+2)\pi i/3}| < 1.$$

Thus we have

$$\min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| < 1$$

for $3 < \alpha < 5$ and $a \in \mathbf{D}$ close enough to $\partial\mathbf{D}$.

Next we prove the above result for $1 < \alpha < 3$. For $0 \le t \le \pi/2$ write

$$\begin{aligned} g(t) &= |1-(2\cos t)^{\alpha+2}e^{(\alpha+2)ti}|^2 \\ &= 1+(2\cos t)^{2(\alpha+2)} - 2(2\cos t)^{\alpha+2}\cos(\alpha+2)t. \end{aligned}$$

Elementary calculations show that

$$g'(t) = 4(\alpha+2)(2\cos t)^{\alpha+1}[\sin(\alpha+3)t - (2\cos t)^{\alpha+2}\sin t].$$

If $1 < \alpha < 3$, then

$$2\pi < \frac{\alpha+3}{2}\pi < 3\pi \qquad \text{and so} \qquad \sin\frac{\alpha+3}{2}\pi > 0.$$

It follows that $g'(t) > 0$ if $t \in (0, \pi/2)$ is close enough to $\pi/2$. This implies that $g(t)$ increases near $\pi/2$. Since $g(\pi/2) = 1$, we see that $g(t)$ is strictly less than 1 for t close enough to $\pi/2$. Therefore,

$$\min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| < 1$$

for $1 < \alpha < 3$ and a close enough to $\partial\mathbf{D}$.

The case $\alpha = 3$ can be handled as follows. Recall that in this case

$$\lim_{|a|\to 1^-} \min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| = \min_{0\le t\le\pi/2} |1 - (2\cos t)^5 e^{5ti}|.$$

Letting $t = 2\pi/5$, we conclude that

$$\lim_{|a|\to 1^-} \min_{|\theta|\le\pi} |G_a^\alpha(e^{i\theta})| \le \left|1 - \left(2\cos\frac{2\pi}{5}\right)^5\right| < 1.$$

This completes the proof of Proposition 5.

THEOREM 6 *If $\alpha > 1$, then $L_a^2(dA_\alpha)$ does not have the optimal factorization property. In other words, it is not possible to find a function G_A^α for each $L_a^2(dA_\alpha)$-zero set A $(0 \notin A)$ satisfying the following conditions:*

(1) $\|G_A^\alpha\|_\alpha = 1$, $G_A^\alpha(0) > 0$;
(2) $Z(G_A^\alpha) = A$;
(3) $\|f/G_A^\alpha\|_\alpha \le \|f\|_\alpha$ *for all* $f \in L_a^2(dA_\alpha)$ *with* $A \subset Z(f)$.

Proof Assume that $L_a^2(dA_\alpha)$ has the factorization property; then G_A^α must be the solution to the following extremal problem:

$$\sup\{\operatorname{Re} f(0) : \|f\|_\alpha \le 1;\ A \subset Z(f)\}.$$

In fact, if $\|f\|_\alpha \le 1$, $A \subset Z(f)$, and $f(0) \ge 0$, then condition (3) implies that

$$0 \le \frac{f(0)}{G_A^\alpha(0)} \le \left\|\frac{f}{G_A^\alpha}\right\|_\alpha \le \|f\|_\alpha \le 1.$$

Thus $f(0) \le G_A^\alpha(0)$ and G_A^α is the unique solution to the above extremal problem.

Let $A = \{a\}$ with $a \in \mathbf{D} - \{0\}$. In this case $G_A^\alpha = G_a^\alpha$. It is easy to see that condition (3) implies that

$$\|G_a^\alpha f\|_\alpha \ge \|f\|_\alpha$$

for all $f \in L^2_a(dA_\alpha)$. We show that this implies the boundary values of G^α_a have modulus greater than or equal to 1. For $\lambda \in \mathbf{D}$, let

$$f(z) = \frac{\sqrt{(1-|\lambda|^2)^{\alpha+2}}}{(1-\bar{\lambda}z)^{\alpha+2}}$$

in the above inequality (Note that the above function is the normalized reproducing kernel of $L^2_a(dA_\alpha)$ at λ) and make a change of variable on the left hand side; then

$$\int_{\mathbf{D}} |G^\alpha_a \circ \varphi_\lambda(z)|^2 \, dA_\alpha(z) \geq 1,$$

where

$$\varphi_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}.$$

Suppose $z_0 \in \partial\mathbf{D}$ and let $\{\lambda_n\}$ be a sequence in $\mathbf{D}$ which tends to z_0. Then $\varphi_{\lambda_n}(z)$ tends to z_0 (as $n \to +\infty$) for each $z \in \mathbf{D}$. Since G^α_a is continuous on the closed disk $\overline{\mathbf{D}}$ and dA_α is a probability measure, it follows from the dominated convergence theorem that $|G^\alpha_a(z_0)| \geq 1$. The proof of Theorem 6 is now complete in view of Proposition 5.

Remark Consider the following boundary value problem

$$\begin{cases} \Delta((1-|z|^2)^{-\alpha}\Delta u) = \delta_z \\ u, \quad \dfrac{\partial}{\partial n}u = 0 \quad \text{on} \quad \partial\mathbf{D}. \end{cases}$$

The method in [1] shows that if the solutions to the above equations are all positive on $\mathbf{D}$, then $\|fG^\alpha_A\|_\alpha \geq \|f\|_\alpha$ for all f in $L^2_a(dA_\alpha)$. Since the inequality is not always true, we conclude that the solutions to the above boundary value problem are not always positive on $\mathbf{D}$.

Note that in the proof of Proposition 5 we actually showed that if $\alpha > 1$ then

$$\lim_{|a|\to 1^-} \min_{|\theta|\leq\pi} |G^\alpha_a(e^{i\theta})| < 1.$$

Next we show that this inequality holds only if $\alpha > 1$.

PROPOSITION 7 *If* $-1 < \alpha \leq 1$, *then*

$$\lim_{|a|\to 1^-} \min_{|\theta|\leq\pi} |G^\alpha_a(e^{i\theta})| \geq 1.$$

Proof Recall from the proof of Proposition 5 that

$$\lim_{|a|\to 1^-} \min_{|\theta|\leq\pi} |G^\alpha_a(e^{i\theta})| = \min_{0\leq t\leq\pi/2} |1-(2\cos t)^{\alpha+2}e^{(\alpha+2)ti}|.$$

Since

$$|1-(2\cos t)^{\alpha+2}e^{(\alpha+2)ti}|^2 = 1+(2\cos t)^{2(\alpha+2)}-2(2\cos t)^{\alpha+2}\cos(\alpha+2)t$$

$$= 1+(2\cos t)^{\alpha+2}[(2\cos t)^{\alpha+2}-2\cos(\alpha+2)t],$$

it suffices to show that

$$(2\cos t)^{\lambda} \geq 2\cos\lambda t$$

for all $1 \leq \lambda \leq 3$ and $0 \leq t \leq \pi/2$.

First observe that for $1 \leq \lambda \leq 3$ we have

$$\cos\lambda t \leq 0, \qquad \frac{\pi}{2\lambda} \leq t \leq \frac{\pi}{2}.$$

Thus it is enough to show that

$$(2\cos t)^{\lambda} \geq 2\cos\lambda t, \qquad 0 \leq t \leq \frac{\pi}{2\lambda}.$$

Since the function $\cos t$ is decreasing on $[0,\pi/2]$, we clearly have $2\cos t \geq 2\cos\lambda t$ for all $0 \leq t \leq \pi/(2\lambda)$ and $1 \leq \lambda \leq 3$.

First assume that $\frac{3}{2} \leq \lambda \leq 3$. Then $\pi/2\lambda \leq \pi/3$ and hence $2\cos t \geq 1$ for all $0 \leq t \leq \pi/(2\lambda)$. It follows that

$$(2\cos t)^{\lambda} \geq 2\cos t \geq 2\cos\lambda t, \qquad 0 \leq t \leq \frac{\pi}{2\lambda}.$$

Next assume that $1 \leq \lambda < \frac{3}{2}$. In this case we have $\pi/2\lambda > \pi/3$ and

$$(2\cos t)^{\lambda} \geq 2\cos t \geq 2\cos\lambda t, \qquad 0 \leq t \leq \frac{\pi}{3}.$$

It remains to show that

$$f(t) = (2\cos t)^{\lambda} - 2\cos\lambda t \geq 0, \qquad \frac{\pi}{3} \leq t \leq \frac{\pi}{2\lambda}.$$

Taking the derivative of f, we find that

$$f'(t) = 2\lambda[\sin\lambda t - \sin t(2\cos t)^{\lambda-1}].$$

Since the function $\sin t$ is increasing on $[0,\pi/2]$, and $0 \leq 2\cos t \leq 1$ for $\pi/3 \leq t \leq \pi/2\lambda$, we see that

$$\sin\lambda t \geq \sin t \geq \sin t(2\cos t)^{\lambda-1}$$

for all $\pi/3 \leq t \leq \pi/2\lambda$. Thus the function f is increasing on $[\pi/3,\pi/2\lambda]$. The desired result now follows from the fact that

$$f\left(\frac{\pi}{3}\right) = 1 - 2\cos\frac{\lambda\pi}{3} > 0.$$

3. MULTIPLY CONNECTED DOMAINS

In this section we show that Hedenmalm's theorem fails for many planar domains. We need the following theorem of N. Suita and A. Yamada:

THEOREM 8 [4] *Let Ω be a finite open Riemann surface which is not simply connected. Then the Bergman kernel $K(z,w)$ of Ω has exactly $n+2p-1$ zeros for suitable w's, where p is the genus of Ω and n is number of boundary contours of Ω.*

The above theorem shows that there exist a lot of planar domains satisfying the hypothesis of the following proposition.

PROPOSITION 9 *Let Ω be a bounded planar domain with the following property: there exists a point w_0 in Ω such that $K(z,w_0)$ (as a function of z) has at least two zeros in Ω. Then there exists a point z_0 in Ω (not equal to w_0), such that the solution to the following extremal problem has at least two zeros in Ω:*

$$\sup\{\operatorname{Re} f(w_0) : \|f\| \le 1,\ f(z_0) = 0\}.$$

Here $\|\cdot\|$ denotes the norm in the Bergman space $L^2_a(\Omega)$ of analytic functions in $L^2(\Omega, dA)$ (dA is the normalized area measure on Ω).

Proof Fix z_0 in Ω so that $K(z_0,w_0)=0$. It suffices to show that

$$f_0(z) = \frac{K(z,w_0)}{\sqrt{K(w_0,w_0)}}$$

is the solution to the extremal problem

$$\sup\{\operatorname{Re} f(w_0) : \|f\| \le 1,\ f(z_0) = 0\}.$$

Let H_0 be the Hilbert space of analytic functions f in $L^2(\Omega, dA)$ with $f(z_0)=0$. Since $f \mapsto f(w_0)$ is a bounded linear functional on H_0, the Riesz representation theorem implies that there exists a unique function g_0 in H_0 such that

$$f(w_0) = \langle f, g_0\rangle, \qquad f \in H_0.$$

Obviously, $K(z,w_0)$ satisfies the above condition. Thus $g_0(z) = K(z,w_0)$ by uniqueness. Let

$$f_0(z) = \frac{g_0(z)}{\|g_0\|} = \frac{K(z,w_0)}{\sqrt{K(w_0,w_0)}}.$$

Then the Cauchy-Schwarz inequality shows that f_0 is the optimal function for the extremal problem

$$\sup\{\operatorname{Re} f(w_0) : \|f\| \le 1,\ f(z_0) = 0\}.$$

This completes the proof of Proposition 9.

In order to generalize Theorem 1 to other planar domains, we need to formulate Theorem 1 in a conformally invariant way. Let Ω be a bounded planar domain and let $K(z,w)$ be the Bergman kernel of Ω. For each $w \in \Omega$, let k_w be the unit vector in $L^2_a(\Omega)$ defined by

$$k_w(z) = \frac{K(z,w)}{\sqrt{K(w,w)}}, \qquad z \in \Omega.$$

We call the k_w's normalized reproducing kernels on Ω. It is easy to see that for each $w \in \Omega$ the probability measure $|k_w(z)|^2\,dA(z)$ is a representing measure with respect to the point w, that is,

$$\int_\Omega f(z)|k_w(z)|^2\,dA(z) = f(w)$$

for all area integrable analytic functions f in Ω. In the rest of this section all unspecified norms will denote the norm on $L^2_a(\Omega)$.

THEOREM 10 *Suppose Ω is a bounded simply connected planar domain and w_0 is a point in Ω. Then for each $L^2_a(\Omega)$-zero set A not containing w_0 there exists a unique function G_A in $L^2_a(\Omega)$ satisfying the following conditions:*

(1) $\|G_A\| = 1$, $G_A(w_0) > 0$;
(2) $Z(G_A) = A$;
(3) $\|f k_{w_0}/G_A\| \le \|f\|$ *for all f in $L^2_a(\Omega)$ with $A \subset Z(f)$.*

Proof Suppose A is an $L^2_a(\Omega)$-zero set not containing w_0 and suppose G_A is a function in $L^2_a(\Omega)$ satisfying the above three conditions. Then G_A has to be the optimal function of the following extremal problem (this will prove uniqueness):

$$\sup\{\operatorname{Re} f(w_0) : \|f\| \le 1,\ A \subset Z(f)\}.$$

In fact, if f is in $L^2_a(\Omega)$ with $\|f\| \le 1$, $f(w_0) > 0$, and $A \subset Z(f)$, then

$$0 < \frac{f(w_0)}{G_A(w_0)} = \int_\Omega \frac{f(z)}{G_A(z)} |k_{w_0}(z)|^2\,dA(z)$$

$$\le \left[\int_\Omega \left|\frac{f(z)}{G_A(z)}\right|^2 |k_{w_0}(z)|^2\,dA(z)\right]^{1/2}$$

$$= \left\|\frac{f k_{w_0}}{G_A}\right\| \le \|f\| \le 1,$$

and hence $G_A(w_0) \ge f(w_0)$. This shows that G_A is the optimal function.

It remains to show that the optimal function G_A of the extremal problem

$$\sup\{\operatorname{Re} f(w_0) : \|f\| \le 1,\ A \subset Z(f)\}$$

satisfies the conditions (1), (2), and (3). Condition (1) is obvious. Next, we prove (2) and (3).

Since Ω is simply connected, there exists, by the Riemann mapping theorem, a conformal mapping φ from $\mathbf{D}$ onto Ω such that $\varphi(0) = w_0$ and $\varphi'(0) > 0$. Fix an $L^2_a(\Omega)$-zero set A with $w_0 \notin A$ and let $\tilde{G}$ be the solution to the extremal problem

$$\sup\{\operatorname{Re} g(0) : \|g\|_{L^2_a(\mathbf{D})} \le 1,\ \varphi^{-1}(A) \subset Z(g)\}.$$

We claim that

$$\tilde{G}(z) = G_A(\varphi(z))\varphi'(z), \qquad z \in \mathbf{D}.$$

In fact, if g is in the closed unit ball of $L_a^2(\mathbf{D})$ with $\varphi^{-1}(A) \subset Z(g)$ and $g(0) > 0$, then the function

$$G(w) = g \circ \varphi^{-1}(w)(\varphi^{-1})'(w), \qquad w \in \Omega,$$

is in $L_a^2(\Omega)$ and satisfies $\|G\| \le 1$, $A \subset Z(G)$. Thus

$$G(w_0) \le G_A(w_0), \quad \text{or} \quad g(0) \le G_A(\varphi(0))\varphi'(0).$$

This shows that $\tilde{G}(z) = G_A(\varphi(z))\varphi'(z)$ on $\mathbf{D}$. By Theorem 1 we have $Z(G_A) = A$ (hence condition 2)) and

$$\left\| \frac{g}{\tilde{G}} \right\|_{L_a^2(\mathbf{D})} \le \|g\|_{L_a^2(\mathbf{D})}$$

for all $g \in L_a^2(\mathbf{D})$ with $\varphi^{-1}(A) \subset Z(g)$. Replacing g by $(f \circ \varphi)\varphi'$ with $f \in L_a^2(\Omega)$ having $A \subset Z(f)$, and changing variables, we obtain

$$\int_\Omega \left| \frac{f(w)}{G_A(w)} \right|^2 |(\varphi^{-1})'(w)|^2 \, dA(w) \le \int_\Omega |f(w)|^2 \, dA(w).$$

Recall from the invariance of the Bergman kernel that

$$\varphi'(z) K(\varphi(z), \varphi(w)) \overline{\varphi'(w)} = \frac{1}{(1 - z\bar{w})^2}, \qquad z, w \in \mathbf{D}.$$

Let $w = 0$ and replace z by $\varphi^{-1}(w)$, $w \in \Omega$. Then

$$K(w, w_0)\varphi'(0) = (\varphi^{-1})'(w), \qquad w \in \Omega.$$

Setting $w = w_0$ we get

$$K(w_0, w_0)(\varphi'(0))^2 = 1,$$

and hence

$$(\varphi^{-1})'(w) = \frac{K(w, w_0)}{\sqrt{K(w_0, w_0)}} = k_{w_0}(w).$$

This completes the proof of Theorem 10.

Remark The above theorem is simply a conformally invariant version of Theorem 1. A similar reformulation of the L^p version of Theorem 1 is also possible; see [1].

DEFINITION 11 *Suppose Ω is a bounded planar domain and w_0 is a point in Ω. We say that $L_a^2(\Omega)$ has the optimal factorization property with respect to w_0, if for each $L_a^2(\Omega)$-zero set A not containing w_0, there exists a function G_A in $L_a^2(\Omega)$, such that*

(1) $\|G_A\| = 1$, $G_A(w_0) > 0$;
(2) $Z(G_A) = A$;
(3) $\|f k_{w_0}/G_A\| \le \|f\|$ *for all* $f \in L_a^2(\Omega)$ *with* $A \subset Z(f)$.

THEOREM 12 *If Ω is a bounded planar domain satisfying the hypothesis of Proposition 9, then there are points w in Ω such that $L_a^2(\Omega)$ does not have the optimal factorization property with respect to w.*

Proof Assume that $L_a^2(\Omega)$ has the optimal factorization property with respect to w_0. Then for an $L_a^2(\Omega)$-zero set A not containing w_0 the function G_A must be the solution to the following extremal problem (see the first part of the proof of Theorem 10):

$$\sup\{\mathrm{Re} f(w_0) : \|f\| \leq 1,\ A \subset Z(f)\}.$$

But by Proposition 9, the zero set of G_A is not equal to A in general. Thus condition (2) does not hold in general. This proves the theorem.

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