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BOUNDARY VALUE PROBLEMS FOR WEIGHTED BIHARMONIC OPERATORS

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§1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 1, 2, 3, \dots$) with C^∞-boundary; this means that $\partial\Omega$ has finitely many connected components, each being a compact $(n-1)$-dimensional C^∞-surface. Let dV_n be the volume measure in $\mathbb{R}^n$, and dV_{n-1} the $(n-1)$-dimensional area measure on $\partial\Omega$.

For any integer $m = 0, 1, 2, \dots$, the Sobolev space $W^m(\Omega)$ consists of all complex-valued functions in $L^2(\Omega)$ whose partial derivatives of order not exceeding m also belong to $L^2(\Omega)$. In particular, $W^0(\Omega)$ coincides with $L^2(\Omega)$. The subspace $W_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$, the space of C^∞-smooth compactly supported complex-valued functions on Ω, in the Hilbert space $W^m(\Omega)$. It is useful to view $W_0^m(\Omega)$ as the subspace of $W^m(\Omega)$ consisting of functions whose normal derivatives of order not exceeding $m-1$ vanish on $\partial\Omega$. We shall need the (fractional exponent) Sobolev spaces $W^{m-1/2}(\partial\Omega)$ on the boundary $\partial\Omega$; these spaces are defined as follows. Locally, $\partial\Omega$ is C^∞-equivalent to $\mathbb{R}^{n-1}$, and Fourier analysis allows us to define $W^{m-1/2}(\mathbb{R}^{n-1})$ as a space of distributions (functions for $m > 0$); this gives us $W^{m-1/2}(\partial\Omega)$ locally, and hence globally. A standard technique based on partitions of unity offers a possibility to work locally with $W^{m-1/2}(\mathbb{R}^{n-1})$ whenever we want to demonstrate some regularity property of the functions in $W^{m-1/2}(\partial\Omega)$. For instance, one can show that, for $m = 1, 2, 3, \dots$, the functions in $W^m(\Omega)$ have well-defined boundary values in the sense of distribution theory, and that the restriction to $\partial\Omega$ of the space $W^m(\Omega)$ is precisely $W^{m-1/2}(\partial\Omega)$ (see [15]).

The *Green solver* for the Laplacian Δ is the operator $\Gamma\colon W^0(\Omega) \to W_0^1(\Omega) \cap W^2(\Omega)$ with $\Delta\Gamma\varphi = \varphi$ for $\varphi \in W^0(\Omega)$. In a more classical language, one would say that for any $f \in W^0(\Omega)$, Γf solves the Poisson equation

$$\begin{cases} \Delta\Gamma f = f & \text{on } \Omega, \\ \Gamma f = 0 & \text{on } \partial\Omega, \end{cases}$$

at least in a weak sense. The operator Γ is usually expressed in terms of a kernel function, $\Gamma(x, y)$, as follows:

$$\Gamma\varphi(x) = \int_\Omega \Gamma(x, y)\, \varphi(y)\, dV_n(y), \quad \varphi \in W^0(\Omega).$$

By the general ellipticity theory (see [1, 13, 15]), Γ maps $W^m(\Omega)$ to $W^{m+2}(\Omega)$, and the kernel $\Gamma(x, y)$ is C^∞ on $(\overline{\Omega} \times \overline{\Omega}) \setminus \delta(\Omega)$, where $\delta(\Omega)$ is the diagonal,

$$\delta(\Omega) = \left\{ (x, x) : x \in \overline{\Omega} \right\}.$$

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The Green solver Γ is related to the Poisson solver P; for $f \in W^{1/2}(\partial\Omega)$ the latter produces a function $Pf \in W^1(\Omega)$ satisfying

$$\begin{cases} \Delta Pf = 0 & \text{on } \Omega, \\ Pf = f & \text{on } \partial\Omega. \end{cases}$$

In a natural way, the operator P extends to a mapping $W^{-1/2}(\partial\Omega) \to HW^0(\Omega)$, where $HW^0(\Omega)$ is the closed subspace of $W^0(\Omega)$ consisting of harmonic functions. In general, P maps $W^{m-1/2}(\partial\Omega)$ to (and onto) $HW^m(\Omega)$, the harmonic subspace of $W^m(\Omega)$. For $f \in W^{1/2}(\Omega)$, Pf is given by the formula

$$Pf(x) = \int_{\partial\Omega} P(x,y) f(y)\, dV_{n-1}(y), \quad x \in \Omega,$$

where $P(x,y)$ is the Poisson kernel. The Poisson kernel is obtained from the Green function with the help of the identity

$$P(x,y) = -\frac{\partial}{\partial n(y)} \Gamma(x,y), \quad (x,y) \in \Omega \times \partial\Omega.$$

From the regularity properties of the Green kernel $\Gamma(x,y)$ it follows that $P(x,y)$ is C^∞ on $(\overline{\Omega} \times \partial\Omega) \setminus \delta(\partial\Omega)$, where $\delta(\partial\Omega)$ is the boundary diagonal,

$$\delta(\partial\Omega) = \{\, (x,x) : x \in \partial\Omega \,\}.$$

The Green kernel $\Gamma(x,y)$ is negative, and the Poisson kernel $P(x,y)$ is positive. In a sense, these facts are the two faces of a coin, the maximum principle.

For elliptic partial differential operators of order 4 or higher, there is no simple general maximum principle. Still, there is a good reason to believe that strong interplay exists between the Green functions and the analogs of the Poisson kernel for the operator in question. Let ω be a real-valued C^∞-function on the closure $\overline{\Omega}$ of Ω that satisfies $\omega(x) > 0$ throughout $\overline{\Omega}$; such functions will be referred to as *weights*. The Green solver for the weighted biharmonic operator $\Delta\omega^{-1}\Delta$ is the operator $U_\omega \colon W^0(\Omega) \to W_0^2(\Omega) \cap W^4(\Omega)$ for which $\Delta\omega^{-1}\Delta U_\omega$ is the identity operator $W^0(\Omega) \to W^0(\Omega)$. Again, in more classical terms, for $f \in W^0(\Omega)$ the function $U_\omega f$ solves the boundary value problem

$$\begin{cases} \Delta\omega^{-1}\Delta\, U_\omega f = f & \text{on } \Omega, \\ U_\omega f,\ (\partial/\partial n)\, U_\omega f = 0 & \text{on } \partial\Omega, \end{cases}$$

in the weak sense; here $\partial/\partial n$ stands for differentiation in the direction of the inward normal. From the general ellipticity theory [1, 13, 15] it follows that U_ω maps $W^m(\Omega)$ to $W_0^2(\Omega) \cap W^{m+4}(\Omega)$; moreover, the associated kernel $U_\omega(x,y)$ is C^∞ on $(\overline{\Omega} \times \overline{\Omega}) \setminus \delta(\Omega)$. The analog of the Poisson solver for the operator $\Delta\omega^{-1}\Delta$ is an operator D_ω, which we call the *Dirichlet solver* for $\Delta\omega^{-1}\Delta$; for any $f \in W^{1/2}(\partial\Omega)$ this operator produces a function $D_\omega f \in W^2(\Omega)$ such that (in the weak sense)

$$\begin{cases} \Delta\omega^{-1}\Delta\, D_\omega f = 0 & \text{on } \Omega, \\ D_\omega f = 0 & \text{on } \partial\Omega, \\ (\partial/\partial n)\, D_\omega f = f & \text{on } \partial\Omega. \end{cases}$$

The associated kernel $D_\omega(x,y)$ is obtained from $U_\omega(x,y)$ via the formula

$$D_\omega(x,y) = \omega(y)^{-1} \frac{\partial^2}{\partial n(y)^2} U_\omega(x,y) = \omega(y)^{-1} \Delta_y\, U_\omega(x,y), \quad (x,y) \in \Omega \times \partial\Omega.$$

The regularizing properties of U_ω imply that D_ω maps $W^{m-1/2}(\partial\Omega)$ to $W^{m+1}(\Omega)$ for $m > 0$; moreover, D_ω has a natural extension to a map $W^{-1/2}(\partial\Omega) \to W^1(\Omega)$. Also, it follows that the associated kernel $D_\omega(x,y)$ is C^∞ on $(\bar{\Omega} \times \partial\Omega) \setminus \delta(\partial\Omega)$. With the help of the above identity, it can be shown (see [10]) that if the kernel $U_\omega(x,y)$ is positive on $\Omega\times\Omega$ (the word *positive* is used in the Bourbaki sense, requiring only that $U_\omega(x,y) \geq 0$), then $D_\omega(x,y)$ is also positive, on $\Omega \times \partial\Omega$. On the other hand, if Ω is starlike and ω is constant, then the reverse implication is also true [10]. In fact, no concrete example seems to be known in which $D_\omega(x,y)$ is positive but $U_\omega(x,y)$ fails to be positive, although it is very likely that this can occur. On the other hand, there are plenty of examples of regions Ω in $\mathbb{R}^2$ for which $U(x,y)$ is not of fixed sign (we drop the subscript ω when we talk about $\omega = 1$); for instance, Paul Garabedian [4] found that elongated ellipses (with the quotient of axes exceeding 1.5934) can serve as such examples. The biharmonic Green function $U(x,y)$ is related to plate bendings and creeping flows. The function $U(x,y)$ expresses the vertical deflection at $x \in \Omega$ of an infinitesimally thin planar plate having the shape of Ω and clamped at the boundary, under a point load at $y \in \Omega$. A standard reference is the Hadamard treatise [5, 6]. The relationship with creeping flows is as follows: if we have a thin layer of viscous fluid spread over Ω, and make the fluid rotate slowly about a point $y \in \Omega$, then the level sets of $U(x,y)$ are the flow lines, and if for fixed $y \in \Omega$ the function $U(x,y)$ has a local minimum in Ω (which is necessarily the case if $U(x,y)$ is negative somewhere), then the flow has an eddy (i.e., the flow rotates in the direction opposite to that of the main flow around y).

More general weighted biharmonic operators of the form $\Delta\omega^{-1}\Delta$ and their Green solvers were studied by Garabedian [4]. In $\mathbb{R}^2$, they arise naturally when one studies the biharmonic equation with the help of conformal mappings. It is of value to know general criteria on Ω and ω guaranteeing that $U_\omega(x,y)$ is positive throughout $\Omega \times \Omega$. The highly nonlinear dependence of U_ω on the weight ω makes explicit computation awkward. In this paper we propose a variational method, which can potentially provide a large family of weights ω for which $U_\omega(x,y)$ is positive.

Suppose we know that $U_\omega(x,y)$ is positive for some weight ω. In which directions in the space of weights may we deviate from ω so that to keep positivity, and how far away from ω may we go? More precisely, if μ is another weight and $\omega_t = \omega + t\mu$ (t is a positive real number), then when can we claim that $U_{\omega_t}(x,y)$ is positive? Our purpose in the present paper is to show that if $D_\mu(x,y) \leq D_\omega(x,y)$ on $\Omega\times\partial\Omega$, then $D_\mu(x,y) \leq D_{\omega_t}(x,y)$ on $\Omega \times \partial\Omega$, and $U_\omega(x,y) + tU_\mu(x,y) \leq U_{\omega_t}(x,y)$ on $\Omega \times \Omega$. Then, the result generalizes as follows: if $\omega_1, \omega_2, \dots, \omega_k$ are finitely many weights subject to the condition

$$D_{\omega_1}(x,y) \leq D_{\omega_2}(x,y) \leq \cdots \leq D_{\omega_k}(x,y), \quad (x,y) \in \Omega \times \partial\Omega,$$

and $\omega = t_1\omega_1 + \cdots + t_k\omega_k$, where the t_j are positive real numbers, then $D_{\omega_1}(x,y) \leq D_\omega(x,y)$ on $\Omega \times \partial\Omega$, and

$$t_1 U_{\omega_1}(x,y) + \cdots + t_k U_{\omega_k}(x,y) \leq U_\omega(x,y), \quad (x,y) \in \Omega \times \Omega.$$

In particular, if all the $U_{\omega_j}(x,y)$ are positive, then U_ω is positive as well. This result is applied in §7 to show that the factorization theory for the standard Bergman spaces on the unit disk developed in [2, 3, 7, 9, 10, 11] extends to a much more general context of weighted Bergman spaces with radial weights.

§2. Sobolev spaces and boundary values

We mention a few standard results about restrictions of Sobolev spaces to the boundary (traces); see, for instance, [15]. The simplest way to convince oneself of the validity of Lemmas 2.1 and 2.3 is to use a C^∞-deformation of Ω such that a given piece of the resulting domain look like a half-space in $\mathbb{R}^n$, and then do the necessary computations there. Lemma 2.2 can be obtained as a consequence of Green's theorem and Lemma 2.3. Lemma 2.4 is implied by the ellipticity of the Laplacian.

Lemma 2.1. *The space of restrictions to $\partial\Omega$ of the functions in $W^1(\Omega)$ coincides with $W^{1/2}(\partial\Omega)$. The restriction of a function to $\partial\Omega$ is 0 if and only if this function belongs to $W_0^1(\Omega)$.*

Let $HW^0(\Omega) = HL^2(\Omega)$ be the closed subspace of $W^0(\Omega)$ consisting of harmonic functions.

Lemma 2.2. *The space of restrictions to $\partial\Omega$ of the functions in $HW^0(\Omega)$ coincides with $W^{-1/2}(\partial\Omega)$.*

Lemma 2.3. *The image of both spaces $W^2(\Omega)$ and $W^2(\Omega)\cap W_0^1(\partial\Omega)$ under the operation of taking the (inward) normal derivative at the boundary coincides with $W^{1/2}(\partial\Omega)$. A function in $W^2(\Omega)\cap W_0^1(\Omega)$ has normal derivative 0 if and only if it belongs to $W_0^2(\Omega)$.*

Let $W^{-m}(\Omega)$ denote the dual space of $W_0^m(\Omega)$; this is a space of distributions on Ω. Consider the space $\Sigma(\Omega)$ of all functions $f \in W^0(\Omega)$ for which $\Delta f \in W^{-1}(\Omega)$.

Lemma 2.4. $\Sigma(\Omega) = HW^0(\Omega) + W_0^1(\Omega)$.

By Lemmas 2.1, 2.2, and 2.4, the mapping of restriction to the boundary $\partial\Omega$, which we refer to as R, is well defined and maps $\Sigma(\Omega)$ onto $W^{-1/2}(\partial\Omega)$.

§3. Preliminaries

We equip $L^2(\Omega)$ with the standard dual pairing

$$\langle f, g\rangle_\Omega = \int_\Omega f(x)\, g(x)\, dV_n(x).$$

Then the *adjoint* T^* of an operator T is defined by the formula

$$\langle T^* f, g\rangle_\Omega = \langle f, Tg\rangle_\Omega.$$

If the operator T is given by an integral kernel $T(x,y)$,

$$Tf(x) = \int_\Omega T(x,y)\, f(y)\, dV_n(y),$$

then so is T^*, and in fact $T^*(x,y) = T(y,x)$.

Let ω be a weight on Ω, as before. Since $\Delta\omega^{-1}\Delta U_\omega \colon W^0(\Omega) \to W^0(\Omega)$ is the identity operator, it follows that $\omega^{-1}\Delta U_\omega = \Gamma + H_\omega$, where $\Delta H_\omega = 0$. Since Γ acts from $W^m(\Omega)$ to $W^{m+2}(\Omega)$ and U_ω from $W^m(\Omega)$ to $W^{m+4}(\Omega)$, we see that H_ω acts from $W^m(\Omega)$ to $HW^{m+2}(\Omega)$ ($m = 0, 1, 2, \dots$). The kernel $H_\omega(x,y)$ associated with H_ω is real-valued. The operator H_ω equals $-Q_\omega\omega\Gamma$, where the operator

$$Q_\omega\omega f(x) = \int_\Omega Q_\omega(x,y)\, f(y)\, \omega(y)\, dV_n(y), \quad x \in \Omega,$$

is the orthogonal harmonic projection in the weighted Hilbert space $L^2(\Omega,\omega)$ with the norm

$$\|f\|_{L^2(\Omega,\omega)} = \left(\int_\Omega |f(x)|^2 \omega(x)\, dV_n(x)\right)^{1/2}.$$

One way to see this is to use the Green theorem; this yields

$$\int_\Omega \Delta U_\omega f(x)\, g(x)\, dV_n(x) = \int_\Omega U_\omega f(x)\, \Delta g(x)\, dV_n(x), \qquad f, g \in C^\infty(\overline{\Omega}),$$

whence it follows that ΔU_ω maps $W^0(\Omega)$ to the orthocomplement of the harmonic functions in $W^0(\Omega)$. Since $\Delta U_\omega = \omega\,(\Gamma + H_\omega)$, momentary reflection leads to the identity $H_\omega = -Q_\omega \omega \Gamma$, as stated. The associated kernel $Q_\omega(x,y)$ is real-valued and selfadjoint: $Q_\omega(x,y) = Q_\omega(y,x)$. Let $H_\omega^*\colon W^0(\Omega) \to W^0(\Omega)$ be the adjoint to $H_\omega\colon W^0(\Omega) \to W^2(\Omega) \subset W^0(\Omega)$; its kernel is $H_\omega^*(x,y) = H_\omega(y,x)$. Since $H_\omega = -Q_\omega \leq \Gamma$, we have $H_{\leq}^* = -\Gamma \omega Q_\omega$.

Lemma 3.1 (regularity of kernels). *The kernel $U_\omega(x,y)$ is C^∞ on $(\overline{\Omega} \times \overline{\Omega}) \setminus \delta(\Omega)$, and the kernels $H_\omega(x,y)$ and $Q_\omega(x,y)$ are both C^∞ on $(\overline{\Omega} \times \overline{\Omega}) \setminus \delta(\partial\Omega)$.*

Proof. It has already been mentioned that $U_\omega(x,y)$ has the above degree of regularity. The formula

$$Q_\omega(x,y) = -\omega(x)^{-1}\omega(y)^{-1}\Delta_x\Delta_y U_\omega(x,y), \quad (x,y) \in (\Omega \times \Omega) \setminus \delta(\Omega),$$

shows that $Q_\omega(x,y)$ is also C^∞ on $(\overline{\Omega} \times \overline{\Omega}) \setminus \delta(\Omega)$. The kernel $Q_\omega(x,y)$ is harmonic in each variable separately on Ω; hence, it must be C^∞ on $(\overline{\Omega} \times \overline{\Omega}) \setminus \delta(\partial\Omega)$. The regularity of $Q_\omega(x,y)$ implies the regularity of $H_\omega(x,y)$ claimed above, because $Q_\omega(x,y) = -\omega(y)^{-1}\Delta_y H_\omega(x,y)$. □

The following statement is a consequence of the fact that the operator $\Delta\omega^{-1}\Delta$ with the zero Dirichlet boundary data is selfadjoint.

Lemma 3.2. *The kernel $U_\omega(x,y)$ is real-valued and selfadjoint, $U_\omega(x,y) = U_\omega(y,x)$. The corresponding fact pertaining to the operator U_ω is that it maps the real-valued functions to real-valued functions, and $\langle U_\omega f, g\rangle_\Omega = \langle f, U_\omega g\rangle_\Omega$ for $f, g \in W^0(\Omega)$.*

We turn to the properties of the mapping H_ω^*.

Lemma 3.3. *The identity $U_\omega \Delta \omega^{-1} = \Gamma + H_\omega^*$ is valid on $W^2(\Omega)$. It follows that, for $m = 0, 2, 3, 4, \ldots$, the operator $\Gamma + H_\omega^*$ acts from $W^m(\Omega)$ to $W^{m+2}(\Omega) \cap W_0^2(\Omega)$, so that $H_\omega^*\colon W^m(\Omega) \to W^{m+2}(\Omega) \cap W_0^1(\Omega)$.*

Proof. Let $f \in W^2(\Omega)$ and $g \in W^0(\Omega)$. By Lemma 3.2 and the Green formula, we have

$$\langle U_\omega \Delta f, g\rangle_\Omega = \langle \Delta f, U_\omega g\rangle_\Omega = \langle f, \Delta U_\omega g\rangle_\Omega,$$

and the definition of the operation of taking the adjoint implies that

$$\langle (\Gamma + H_\omega^*)\omega f, g\rangle_\Omega = \langle \omega f, (\Gamma + H_\omega) g\rangle_\Omega = \langle f, \omega(\Gamma + H_\omega) g\rangle_\Omega.$$

We identify the right-hand sides, and hence the left-hand sides. It follows that $U_\omega \Delta \omega^{-1} = \Gamma + H_\omega^*$ on $W^2(\Omega)$, so that $\Gamma + H_\omega^*$ maps $W^m(\Omega)$ to $W^{m+2}(\Omega) \cap W_0^2(\Omega)$ for $m = 2, 3, 4, \ldots$. The operator Q_ω acts on $W^0(\Omega)$, whence $H_\omega^* = -\Gamma\omega Q_\omega\colon W^0(\Omega) \to W^2(\Omega)$. We conclude that $\Gamma + H_\omega^*\colon W^0(\Omega) \to W^2(\Omega)$. Approximating functions in $W^0(\Omega)$ by elements of $W^2(\Omega)$, we see that, actually, $\Gamma + H_\omega^*\colon W^0(\Omega) \to W_0^2(\Omega)$. The lemma follows. □

Lemma 3.4. *For $m = 0, 2, 3, 4, \ldots$, the operator Q_ω maps $W^m(\Omega)$ to $HW^m(\Omega)$. Moreover, the kernel $Q_\omega(x,y)$ is real-valued and selfadjoint: $Q_\omega(x,y) = Q_\omega(y,x)$.*

Proof. We have already seen that $Q_\omega(x,y)$ is real-valued and selfadjoint. By Lemma 3.3, the operator H_ω^* acts from $W^m(\Omega)$ to $W^{m+2}(\Omega)$, so that $Q_\omega = -\omega^{-1}\Delta H_\omega^*\colon W^m(\Omega) \to W^m(\Omega)$. By definition, the image under Q_ω consists of harmonic functions. □

In view of Lemma 3.3, the following assertion is immediate.

Lemma 3.5. *The operator* $U_\omega \Delta \omega^{-1} \Delta$ *is the identity on* $W^4(\Omega) \cap W_0^2(\Omega)$; *since it equals* $\Gamma\Delta + H_\omega^* \Delta$, *and* $\Gamma\Delta$ *is the identity on* $W_0^1(\Omega)$, *it follows that* $H_\omega^* \Delta = 0$ *on* $W^4(\Omega) \cap W_0^2(\Omega)$. *In particular, if* ν *is another weight on* Ω, *then* $H_\omega^* \Delta U_\nu = 0$ *on* $W^0(\Omega)$.

Lemma 3.6. *If* ω *and* ν *are weights, then*

$$U_\omega - U_\nu = (\Gamma + H_\nu^*)(\omega - \nu)(\Gamma + H_\omega) = (\Gamma + H_\omega^*)(\omega - \nu)(\Gamma + H_\nu).$$

Proof. By Lemma 3.5, we have

$$U_\omega = \Gamma \Delta U_\omega = (\Gamma + H_\nu^*) \Delta U_\omega = (\Gamma + H_\nu^*) \leq (\Gamma + H_\omega).$$

On the other hand,

$$U_\nu = U_\nu \Delta (\Gamma + H_\omega) = (U_\nu \Delta \nu^{-1}) \nu (\Gamma + H_\omega) = (\Gamma + H_\nu^*) \nu (\Gamma + H_\omega).$$

Now, the first identity in the formulation of the lemma follows; the second can be obtained by interchanging the roles of ω and ν. □

Remark. Lars Hörmander has pointed out to me that Lemma 3.6 expresses the usual resolvent identity

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$$

for the operators $A, B \colon W^4(\Omega) \cap W_0^2(\Omega) \to W^0(\Omega)$ given by $A = \Delta \omega^{-1} \Delta$ and $B = \Delta \nu^{-1} \Delta$ (in the calculation, the identity $\omega^{-1}(\omega - \nu)\nu^{-1} = \nu^{-1} - \omega^{-1}$ must be used).

§4. The Dirichlet problem

We introduce duality on the boundary by the formula

$$\langle f, g \rangle_{\partial\Omega} = \int_{\partial\Omega} f(x)\, g(x)\, dV_{n-1}(x);$$

this formula makes sense at least for $f, g \in L^2(\partial\Omega)$. In terms of this dual action, the spaces $W^{1/2}(\partial\Omega)$ and $W^{-1/2}(\partial\Omega)$ are dual to each other. Let $R \colon HL^2(\Omega) \to W^{-1/2}(\partial\Omega)$ be the restriction operator; by Lemma 2.2, R is onto. We recall that $Q_\omega \colon L^2(\Omega) \to HL^2(\Omega)$ is defined in such a way that $Q_\omega \colon L^2(\Omega, \omega) \to HL^2(\Omega, \omega)$ is the orthogonal projection. The operator $RQ_\omega \colon L^2(\Omega) \to W^{-1/2}(\partial\Omega)$ is associated with a kernel $Q_\omega(x, y)$, $(x, y) \in \partial\Omega \times \Omega$,

$$RQ_\omega f(x) = \int_\Omega Q_\omega(x, y)\, f(y)\, dV_n(y), \quad x \in \partial\Omega,$$

and the adjoint operator $(RQ_\omega)^* \colon W^{1/2}(\partial\Omega) \to HL^2(\Omega) \subset L^2(\Omega)$ is associated with a kernel $Q_\omega(x, y)$, $(x, y) \in \Omega \times \partial\Omega$,

$$(RQ_\omega)^* f(x) = \int_{\partial\Omega} Q_\omega(x, y)\, f(y)\, dV_{n-1}(y), \quad x \in \Omega.$$

Theorem 4.1. *The mapping* $D_\omega = (RH_\omega)^* = -\Gamma\omega(RQ_\omega)^*\colon W^{1/2}(\partial\Omega) \to W^2(\Omega) \cap W_0^1(\Omega)$ *can be written as follows:*

$$D_\omega f(x) = \int_{\partial\Omega} H_\omega(y,x)\, f(y)\, dV_{n-1}(y), \quad f \in W^{1/2}(\partial\Omega).$$

It has the property that for $f \in W^{1/2}(\partial\Omega)$ *the function* $u = D_\omega f$ *solves the Dirichlet problem*

$$\begin{cases} \Delta\omega^{-1}\Delta u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial u/\partial n = f & \text{on } \partial\Omega \end{cases}$$

(*n is the inward normal*).

Proof. We check that $u = (RH_\omega)^* f$ solves the above Dirichlet problem. To this end, we start with the observation that, by the Green formula, $U_\omega\Delta = (\Delta U_\omega)^*$ on $W^2(\Omega)$, that is, for $\varphi \in W^2(\Omega)$ and $\psi \in W^0(\Omega)$,

$$\langle U_\omega\Delta\varphi, \psi\rangle_\Omega = \langle \Delta\varphi, U_\omega\psi\rangle_\Omega = \langle \varphi, \Delta U_\omega\psi\rangle_\Omega.$$

It follows that $(\Delta U_\omega)^* = \Gamma\omega + H_\omega^*\omega$ maps $W^0(\Omega)$ to $W_0^2(\Omega)$, so that H_ω^* maps $W^0(\Omega)$ to $W^2(\Omega) \cap W_0^1(\Omega)$. By Lemma 3.3 and the properties of Q_ω, the image of $W^0(\Omega)$ under H_ω^* is contained in the kernel of $\Delta\omega^{-1}\Delta$, i.e., in the space of functions h satisfying $\Delta\omega^{-1}\Delta h = 0$. Writing $H_\omega^* = (\Delta U_\omega)^*\omega^{-1} - \Gamma$, we see that for any $\varphi \in W^0(\Omega)$ the function $H_\omega^*\varphi$ satisfies

$$\begin{cases} \Delta\omega^{-1}\Delta H_\omega^*\varphi = 0 & \text{on } \Omega, \\ H_\omega^*\varphi = 0 & \text{on } \partial\Omega, \\ (\partial/\partial n)\, H_\omega^*\varphi = P^*\varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $P^*\colon W^0(\Omega) \to W^{1/2}(\partial\Omega)$ is the adjoint to the Poisson solver $P\colon W^{-1/2}(\partial\Omega) \to HW^0(\Omega) \subset W^0(\Omega)$. This follows from the identity

$$P^*\varphi = -(\partial/\partial n)\,\Gamma\varphi \quad \text{on } \partial\Omega.$$

In order to convert (4.1) into information useful for us, we need to further analyze the restriction operator R, and in particular, to introduce an adjoint to it. Let $R^*\colon W^{1/2}(\partial\Omega) \to W^0(\Omega)$ denote any of the operators that satisfy

$$\langle Rg, h\rangle_{\partial\Omega} = \langle g, R^*h\rangle_\Omega, \quad g \in HW^0(\Omega),\ h \in W^{1/2}(\partial\Omega);$$

we observe that $(RH_\omega)^* = H_\omega^* R^*$ and that P^*R^* is the identity operator. If we put $\varphi = R^*f$ in (4.1), the assertion follows. □

§5. A variational formula

By Lemma 3.6, for any weights ω and ν we have

$$U_\omega - U_\nu = (\Gamma + H_\nu^*)(\omega - \nu)(\Gamma + H_\omega). \tag{5.1}$$

Putting $\omega_t = \omega + t\mu$, where $0 \le t < +\infty$ and μ is a weight, we observe that the operator $\Delta\omega_t^{-1}\Delta\colon W^4(\Omega) \cap W_0^2(\Omega) \to W^0(\Omega)$ is a real-analytic function of t. Since this operator is invertible, the inverse $U_{\omega_t}\colon W^0(\Omega) \to W^4(\Omega) \cap W_0^2(\Omega)$ possesses the same property.

It follows that $H_{\omega_t}: W^0(\Omega) \to HW^2(\Omega)$, and the adjoint operator $H^*_{\omega_t}$, which we regard both as a mapping $W^0(\Omega) \to W^2(\Omega) \cap W^1_0(\Omega)$ and $W^2(\Omega) \to W^4(\Omega) \cap W^1_0(\Omega)$, are real-analytic functions of t. Hence, from (5.1) we conclude that

$$\begin{aligned}\frac{d}{dt} U_{\omega_t} &= (\Gamma + H^*_{\omega_t})\mu(\Gamma + H_{\omega_t}) \\ &= (\Gamma + H^*_{\omega_t})\mu(\Gamma + H_\mu) + (\Gamma + H^*_{\omega_t})\mu(H_{\omega_t} - H_\mu) \\ &= U_\mu + (\Gamma + H^*_\mu)\mu(H_{\omega_t} - H_\mu) + (H^*_{\omega_t} - H^*_\mu)\mu(H_{\omega_t} - H_\mu) \\ &= U_\mu + (H^*_{\omega_t} - H^*_\mu)\mu(H_{\omega_t} - H_\mu),\end{aligned}$$

that is,

$$\frac{d}{dt} U_{\omega_t} = U_\mu + (H^*_{\omega_t} - H^*_\mu)\mu(H_{\omega_t} - H_\mu). \tag{5.2}$$

This is the basic identity of the paper. From it we shall be able to derive the main result by appealing to the Picard method of solving first-order differential equations. To see why this is at all possible, we first resort to a heuristic argument.

Heuristic argument. Suppose we know that $H_\mu \le H_{\omega_t}$ for a particular value of t, (here the inequality between the operators is to be interpreted as the corresponding relationship between the associated kernels). Then $H^*_\mu \le H^*_{\omega_t}$, so that $U_\mu \le (d/dt)\, U_{\omega_t}$, by (5.2). When applied to a suitably smooth positive function f on $\overline{\Omega}$, both sides of the inequality vanish along with their normal derivatives on $\partial\Omega$. Consequently, the inequality remains valid if we apply the Laplacian to both sides and restrict to the boundary, i.e., $\mu H_\mu f \le (d/dt)\,(\omega_t H_{\omega_t})f$ on $\partial\Omega$. This can be rewritten as follows:

$$\frac{d}{dt}\big(\omega_t(H_{\omega_t} - H_\mu)\big) f \ge 0 \quad \text{on } \partial\Omega.$$

Thus, as long as $H_\mu \le H_{\omega_t}$, the operator $R\omega_t(H_{\omega_t} - H_\mu)$ increases with t (R is the operator of restriction to $\partial\Omega$). Combined with the maximum principle for harmonic functions, this reinforces the first inequality, so that the inequality $H_\mu \le H_{\omega_t}$ remains valid if t is replaced by $t + dt$, for an infinitesimally small positive dt. So, starting with $H_\mu \le H_\omega$, we successively get $H_\mu \le H_{\omega_t}$, for all positive t. Combined with (5.2), this leads to $U_\omega + tU_\mu \le U_{\omega_t}$, which is an inequality of required type.

We apply the Laplacian Δ to both sides of (5.2) to get

$$\begin{aligned}\frac{d}{dt}\big(\omega_t(H_{\omega_t} - H_\mu)\big) &= \Delta(H^*_{\omega_t} - H^*_\mu)\mu(H_{\omega_t} - H_\mu) \\ &= (\mu Q_\mu - \omega_t Q_{\omega_t})\mu(H_{\omega_t} - H_\mu).\end{aligned} \tag{5.3}$$

The right-hand side of (5.3) is an operator $W^0(\Omega) \to W^2(\Omega)$. Using the general identity $H_\omega = -Q_\omega \le \Gamma$, we rewrite (5.3) as follows:

$$\frac{d}{dt}\big(\omega_t(Q_\mu\mu - Q_{\omega_t}\omega_t)\Gamma\big) = (\mu Q_\mu - \omega_t Q_{\omega_t})\mu(H_{\omega_t} - H_\mu).$$

If we apply the restriction operator R to both sides, we get

$$\frac{d}{dt}\big(\omega_t R(Q_\mu\mu - Q_{\omega_t}\omega_t)\Gamma\big) = (\mu RQ_\mu - \omega_t RQ_{\omega_t})\mu(H_{\omega_t} - H_\mu),$$

and if we then pass to the adjoint operators, the result is

$$\frac{d}{dt}\left(\Gamma\big(\mu(RQ_\mu)^* - \omega_t(RQ_{\omega_t})^*\big)\omega_t\right) = (H^*_{\omega_t} - H^*_\mu)\mu((RQ_\mu)^*\mu - (RQ_{\omega_t})^*\omega_t).$$

We recall that, in general, $(RQ_\omega)^*\colon W^{1/2}(\partial\Omega) \to HW^0(\Omega)$, so that the right-hand side of the above identity should be viewed as an operator acting from $W^{1/2}(\partial\Omega)$ to $W^2(\Omega)\cap W^1_0(\Omega)$. Now we apply the Laplacian to both sides to obtain

$$\frac{d}{dt}\left(\big(\mu(RQ_\mu)^* - \omega_t(RQ_{\omega_t})^*\big)\omega_t\right) = \big(\mu Q_\mu - \omega_t Q_{\omega_t}\big)\mu((RQ_\mu)^*\mu - (RQ_{\omega_t})^*\omega_t),$$

and rewrite the left-hand side as follows:

$$\frac{d}{dt}\left(\omega_t\big((RQ_\mu)^*\mu - (RQ_{\omega_t})^*\omega_t\big)\right) = (\mu Q_\mu - \omega_t Q_{\omega_t})\mu\big((RQ_\mu)^*\mu - (RQ_{\omega_t})^*\omega_t\big). \tag{5.4}$$

The expression written after the d/dt operation on the left and the expression on the right are operators $W^{1/2}(\partial\Omega) \to \Sigma(\Omega)$, where $\Sigma(\Omega) = HW^0(\Omega) + W^1_0(\Omega)$ is the space introduced in §2. Indeed, if $f \in HW^0(\Omega)$ and $g \in C^\infty(\overline{\Omega})$, then $fg \in \Sigma(\Omega)$, which can be shown by taking the Laplacian: $\Delta(fg) = f\Delta g + 2\nabla f\cdot\nabla g \in W^{-1}(\Omega)$. Since the restriction operator R is well defined on $\Sigma(\Omega)$, applying it to both sides of (5.4) we obtain

$$\frac{d}{dt}\left(R\omega_t\big((RQ_\mu)^*\mu - (RQ_{\omega_t})^*\omega_t\big)\right) = (\mu RQ_\mu - \omega_t RQ_{\omega_t})\mu\big((RQ_\mu)^*\mu - (RQ_{\omega_t})^*\omega_t\big). \tag{5.5}$$

In terms of the operators

$$\begin{aligned} T_t &= (RQ_\mu)^*\mu - (RQ_{\omega_t})^*\omega_t \colon W^{1/2}(\partial\Omega) \to HW^0(\Omega) \subset W^0(\Omega),\\ T^*_t &= \mu RQ_\mu - \omega_t RQ_{\omega_t}\colon W^0(\Omega) \to W^{-1/2}(\partial\Omega), \end{aligned}$$

equation (5.5) turns into

$$\frac{d}{dt}\left(R\omega_t T_t\right) = T^*_t\mu T_t. \tag{5.6}$$

The above identity involves only the operator T_t; this is an improvement of (5.3), where the Laplace operator occurs in addition to the difference $H_{\omega_t} - H_\mu$. We shall start with applying the Picard process to (5.6), and then work our way up to (5.3) and (5.2).

In terms of $G_t = H_{\omega_t} - H_\mu\colon W^0(\Omega) \to HW^2(\Omega)$, identity (5.3) simplifies as follows (after application of R to both sides):

$$\frac{d}{dt}(R\omega_t G_t) = T^*_t\mu G_t, \tag{5.7}$$

and (5.2) becomes

$$\frac{d}{dt}U_{\omega_t} = U_\mu + G^*_t\mu G_t. \tag{5.8}$$

In (5.7), we regard R and T^*_t as mappings $W^2(\Omega) \to W^{3/2}(\partial\Omega)$.

§6. The main result

We say that an operator T acting from a vector space of distributions on a manifold to a vector space of distributions on another manifold is *positive* (in symbols: $T \geq 0$) if it maps the positive distributions to positive distributions. We recall that the positive distributions may be identified with locally finite positive Borel measures. If T is associated with a kernel $T(x, y)$, then T is positive if and only if the kernel is positive.

Remark. The above concept of positivity differs from the standard one for operators on Hilbert space, where T is declared to be positive whenever $\langle Tu, u\rangle \geq 0$ for all vectors u.

Lemma 6.1. *If $T_0 \geq 0$, then $T_t \geq 0$ for all $0 \leq t < +\infty$.*

Proof. As an operator $W^{1/2}(\partial\Omega) \to HL^2(\Omega) \subset L^2(\Omega)$, T_t is uniquely determined as the solution of equation (5.6) with initial value T_0. We shall find approximate solutions by modifying the standard Picard process for ordinary differential equations (see [14]; the method dates back to Cauchy and Liouville). For continuous operator-valued functions $Y(t)\colon W^{1/2}(\partial\Omega) \to HW^0(\Omega)$, with adjoints $Y(t)^*\colon W^0(\Omega) \to W^{-1/2}(\partial\Omega)$, we put

$$F[Y](t) = P\omega_t^{-1}\Big(\omega R T_0 + \int_0^t Y(s)^*\mu Y(s)\,ds\Big),$$

where $P\colon W^{-1/2}(\partial\Omega) \to HW^0(\Omega)$ is the Poisson solver. For each continuous $Y(t)$, $F[Y](t)$ is continuously differentiable as a function with values in the operator space $W^{1/2}(\partial\Omega) \to HW^0(\Omega)$. Moreover, if $Y(t) \geq 0$ for all positive t, then also $F[Y](t) \geq 0$ for all positive t. In other words, F preserves the cone of positive-valued functions. The function $Y(t) = T_t$ is a fixed point of the mapping $F[Y]$, i.e., it satisfies the equation

$$Y(t) = F[Y](t).$$

By the standard fixed point theory associated with the Picard theorem, a fixed point is unique. Let us see how the argument runs when we restrict our attention to a short interval $[0, \delta]$. We have two competing fixed points $X(t)$ and $Y(t)$, and wish to show that they must coincide. The two operator-valued functions are assumed to be continuous; in particular, their norms are uniformly bounded on $[0, \delta]$. Thus, choosing a suitable positive constant M, we can make sure that

$$\|P\omega_t^{-1}\|\,\|\mu\|\,\big(\|X(t)\| + \|Y(t)\|\big) \leq M, \quad t \in [0, \delta], \tag{6.1}$$

where the norms are the operator norms on the appropriate spaces. Since $X(t)$ and $Y(t)$ are fixed points of F, we can write

$$\begin{aligned} X(t) - Y(t) &= F[X](t) - F[Y](t) \\ &= P\omega_t^{-1}\Big(\int_0^t (X(s)^* - Y(s)^*)\mu X(s)\,ds + \int_0^t Y(s)^*\mu(X(s) - Y(s))\,ds\Big), \end{aligned}$$

whence, taking the norms and applying (6.1), we obtain

$$\|X(t) - Y(t)\| \leq M\,t\,L(t), \quad t \in [0, \delta], \tag{6.2}$$

where

$$L(t) = \sup\big\{\,\|X(s) - Y(s)\| : 0 \leq s \leq t\,\big\}.$$

Passing to the supremum over $t \in [0,\delta]$ in (6.2), we arrive at the inequality $L(\delta) \le M\delta\, L(\delta)$. By taking δ small, we can ensure that $M\delta < 1$, which implies that $L(\delta) = 0$, whence $X(t) = Y(t)$ on $[0,\delta]$. To get uniqueness on the entire interval $[0,+\infty[$, we argue as follows. If uniqueness does not occur, then there are two solutions $X(t)$ and $Y(t)$ and a point $t_0 \in]0,+\infty[$ such that the functions $X(t)$ and $Y(t)$ coincide on $[0,t_0]$, but fail to be equal at some points located arbitrarily close to t_0 and lying to the right of t_0. Then we make the coordinate shift $t \mapsto t - t_0$, which pushes t_0 to the origin. Then the above uniqueness argument applies again, showing that $X(t)$ and $Y(t)$ coincide in a small neighborhood of t_0, and uniqueness follows.

We shall show that the fixed point $Y(t)$ is a positive operator for each $t \in [0,+\infty[$ by approximating it with positive operator-valued functions. We define inductively $Y_0(t) = 0$ and $Y_{k+1}(t) = F[Y_k](t)$. It is clear that each $Y_k(t)$ is positive, for $t \in [0,+\infty[$. Finally, for small t (say, $t \in [0,\delta]$ with $\delta > 0$) the $Y_k(t)$ converge in norm to the fixed point $Y(t) = T_t$, which shows that $T_t \ge 0$.

Now we prove that $T_t \ge 0$ for all $t \in [0,+\infty[$, not only for t close to 0. Arguing by contradiction, we suppose that there exists a real number $\eta > 0$ for which the inequality $T_\eta \ge 0$ fails. Let η_0 be the infimum of all such η. From what we have done so far it follows that $\eta_0 > 0$. Since $T_t \ge 0$ for all t with $0 \le t < \eta_0$, we have $T_{\eta_0} \ge 0$ by continuity. Now, putting $S_t = T_{t-\eta_0}$ for $0 \le t < +\infty$, we obtain a family of operators of the same type as T_t, so that the above argument applies to S_t, and we get $S_t \ge 0$ for all $t \in [0,\varepsilon[$ for some small positive ε. This contradicts the definition of η_0. The proof is complete. □

Remark. In the above construction, with $Y(t) = T_t$, it can be checked that if $0 \le Y_k(t) \le Y(t)$ on $[0,+\infty[$, then

$$0 \le F[Y_k](t) = Y_{k+1}(t) \le F[Y](t) = Y(t) \quad \text{on } [0,+\infty[.$$

Since $0 \le Y_k \le Y$ for $k = 0$, this inequality is true for all $k = 0,1,2,\dots$. Consequently, the approximate solutions $Y_k(t)$ do not blow up even for large values of t, and we see that they actually converge on the entire interval $[0,+\infty[$.

Lemma 6.2. *If $G_0 \ge 0$, then $G_t \ge 0$ for all $0 \le t < +\infty$.*

Proof. First, we establish that $T_0 \ge 0$. We recall that, in general, $H_\nu^* = -\Gamma\nu Q_\nu$, so that

$$G_t^* = H_{\omega_t}^* - H_\mu^* = \Gamma\mu Q_\mu - \Gamma\omega_t Q_{\omega_t}. \tag{6.3}$$

By Lemma 3.3, G_t^* maps $W^m(\Omega)$ to $W^{m+2}(\Omega) \cap W_0^2(\Omega)$, for $m = 0,2,3,4,\dots$. The assumption $G_0 \ge 0$ implies that $G_0^* \ge 0$. If $f \in C^\infty(\bar\Omega)$ is an arbitrary positive function, then the function $G_0 f$ also belongs to $C^\infty(\bar\Omega)$, is positive on Ω, and vanishes along with its normal derivative on $\partial\Omega$. Its second normal derivative, which coincides with its Laplacian, must then be positive on $\partial\Omega$, that is, $R\Delta G_0^* f \ge 0$. By (6.3), we have

$$R\Delta G_t^* = \mu R Q_\mu - \omega_t R Q_{\omega_t} = T_t^*,$$

so that $T_0^* f \ge 0$. Since smooth functions are dense in $W^0(\Omega)$, this shows that $T_0^* \ge 0$, whence $T_0 \ge 0$. By Lemma 6.1, $T_t \ge 0$ for all $t \in [0,+\infty[$.

The remaining part of the proof is similar to that of Lemma 6.1. We observe that, as a function of t with values in the space of operators $W^0(\Omega) \to HW^2(\Omega)$, G_t is uniquely determined as the solution of equation (5.7) with the initial value G_0.

For continuous operator-valued functions $Y(t)\colon W^0(\partial\Omega) \to HW^2(\Omega)$, we put

$$F[Y](t) = P\omega_t^{-1}\Big(\omega R G_0 + \int_0^t T_s^* \mu Y(s)\, ds\Big),$$

where $P\colon W^{3/2}(\partial\Omega) \to HW^2(\Omega)$ is the Poisson solver. For each continuous $Y(t)$, the function $F[Y](t)$ is continuously differentiable. Moreover, if $Y(t) \geq 0$ for all positive t, then $F[Y](t) \geq 0$ for all positive t, because $T_s^* \geq 0$. In other words, F preserves the cone of positive-valued functions. The function $Y(t) = G_t$ is a fixed point of the mapping $F[Y]$, i.e., it satisfies the equation

$$Y(t) = F[Y](t).$$

By the standard fixed point theory associated with the Picard process, a fixed point is unique. We show that the fixed point $Y(t)$ is positive-valued on $[0, +\infty[$ by approximating it with positive operator-valued functions. We define inductively $Y_0(t) = 0$, $Y_{k+1}(t) = F[Y_k](t)$. It is clear that each $Y_k(t)$ is positive for $t \in [0, +\infty[$. Finally, for small t (say, $t \in [0, \delta]$ with $\delta > 0$) the $Y_k(t)$ converge in norm to the fixed point $Y(t) = G_t$, which implies that $G_t \geq 0$.

The argument showing that G_t is positive for all $t \in [0, +\infty[$, not only for t close to 0, is precisely the same as in the proof of Lemma 6.1 and, therefore, is omitted. □

Theorem 6.3. *If $D_\mu(x, y) \leq D_\omega(x, y)$ on $\Omega \times \partial\Omega$, and $\omega_t = \omega + t\mu$, then the inequalities*

$$D_\mu(x, y) \leq D_{\omega_t}(x, y), \quad (x, y) \in \Omega \times \partial\Omega,$$
$$U_\omega(x, y) + tU_\mu(x, y) \leq U_{\omega_t}(x, y), \quad (x, y) \in \Omega \times \Omega,$$

are valid for all $0 \leq t < +\infty$.

Proof. For an arbitrary weight ν, we have $H_\nu(x, y) = D_\nu(y, x)$ for $(x, y) \in \partial\Omega \times \Omega$, and on $\Omega \times \Omega$ the function $H_\nu(x, y)$ is harmonic with respect to x. Hence, the assumption is equivalent to $H_\mu(x, y) \leq H_\omega(x, y)$ on $\Omega \times \Omega$. On the other hand, this is precisely the statement that $G_0 \geq 0$. By Lemma 6.2, $G_t \geq 0$ and $G_t^* \geq 0$ for $t \in [0, +\infty[$, so that

$$U_{\omega_t} = U_\omega + tU_\mu + \int_0^t G_s^* \mu G_s ds \geq U_\omega + tU_\mu, \quad t \in [0, +\infty[,$$

by (5.8), and the theorem follows. □

For any $t > 0$, if ν is a weight, then so is $t\nu$, and $U_{t\nu} = tU_\nu$, $D_{t\nu} = D_\nu$ (that is, U_ν is homogeneous of degree 1 in the weight space, and D_ν of degree 0). This observation allows us to prove the following generalization by iteration.

Corollary 6.4. *Let $\omega_1, \ldots, \omega_k$ be weights satisfying*

$$D_{\omega_1}(x, y) \leq \cdots \leq D_{\omega_k}(x, y), \quad (x, y) \in \Omega \times \partial\Omega.$$

Let ω be a weight of the form $\omega = t_1\omega_1 + \cdots + t_k\omega_k$, where the t_j are positive real numbers. Then $D_{\omega_1}(x, y) \leq D_\omega(x, y)$, and

$$t_1 U_{\omega_1}(x, y) + \cdots + t_k U_{\omega_k}(x, y) \leq U_\omega(x, y), \quad (x, y) \in \Omega \times \Omega.$$

Corollary 6.5. *Suppose $\omega_1, \ldots, \omega_k$ are weights satisfying the assumption of Corollary 6.4. Suppose, moreover, that $U_\omega(x, y) \geq 0$ on $\Omega \times \Omega$ for $\omega = \omega_1, \ldots, \omega_k$. Then $U_\omega \geq 0$ for all weights ω in the cone spanned by $\omega_1, \ldots, \omega_k$.*

Remark. The class of weight functions for which the above results can be obtained may be extended. For instance, one may consider weights of lower degree of regularity, or weights that vanish or are singular on some parts of $\partial\Omega$. Another (in my opinion, more interesting) possibility is to consider weights of variable sign. More precisely, we may require that ω be C^∞ up to the boundary of Ω, strictly positive on $\partial\Omega$, and such that, for harmonic functions $f \in HW^0(\Omega)$,

$$\int_\Omega |f(x)|^2\omega(x)\,dV_n(x) \asymp \int_\Omega |f(x)|^2\,dV_n(x),$$

where the symbol $\asymp$ means that two quantities are comparable in size (the left-hand side is bounded, from above and from below, by positive constants times the right-hand side). Then the harmonic kernel $Q_\omega(x,y)$ is well defined, being a sum over an orthonormal basis with respect to the weighted norm on $HW^0(\Omega)$. Going backwards, we see that the function $H_\omega(x,y)$ is well defined, as well as the Green function $U_\omega(x,y)$. This works in spite of the difficulties that arise from the surfaces where ω vanishes. It then appears that the main result, Theorem 6.3, remains valid if ω and μ are taken from this wider class of weights, provided that μ satisfies the following additional condition:

$$\int_\Omega f(x)\,g(x)\,\mu(x)\,dV_n(x) \geq 0$$

for all positive harmonic functions f and g.

§7. Applications to factorization theory in Bergman spaces

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$; the domain of interest is the open unit disk $\mathbb{D}$. For $\alpha > -1$, the weights

$$\nu_\alpha(z) = \pi^{-1}(\alpha+1)|z|^{2\alpha}, \quad z \in \mathbb{D},$$

were studied in [10], where the properties of the associated operators U_{ν_α} and H_{ν_α} were found. For instance, $U_{\nu_\alpha} \geq 0$ and $H_{\nu_\alpha} \leq H_{\nu_\beta}$ for all α, β such that $-1 < \beta \leq \alpha < +\infty$. It follows that $D_{\nu_\alpha} \leq D_{\nu_\beta}$ for all α, β, $-1 < \beta \leq \alpha < +\infty$. In general, the weights ν_α are not C^∞ near the origin, and they may have a zero there, or a singularity. Thus, they are not weights in the restrictive sense used earlier in this paper. However, off a fixed neighborhood of the origin, they are C^∞ up to the boundary, and are bounded away from 0. Modulo a few technical points based on this observation (which are omitted), the results of the previous section apply to such weights as well.

Corollary 7.1. *Let ρ be a Borel probability measure on $]-1,+\infty[$, and let*

$$\omega(z) = \pi^{-1}\int_{-1}^{+\infty}(\alpha+1)|z|^{2\alpha}d\rho(\alpha), \quad z \in \mathbb{D}.$$

Then

$$U_\omega \geq \int_{-1}^{+\infty} U_{\nu_\alpha}\,d\rho(\alpha) \geq 0.$$

Proof. The assertion follows immediately from Corollary 6.4 and the above remark if ρ is supported on a finite set. The general case is obtained with the help of an approximation argument. □

As in [3, 9, 10], this yields the following result on factorization in the weighted Bergman space $AL^2(\mathbb{D},\omega)$; we say that $f \in AL^2(\mathbb{D},\omega)$ provided f is holomorphic on $\mathbb{D}$, and

$$\|f\|_\omega = \left(\int_\mathbb{D} |f(z)|^2\omega(z)\,dS(z)\right)^{1/2} < +\infty.$$

Corollary 7.2. *If ω is as in Corollary 7.1, then for the zero set Z of an arbitrary function in $AL^2(\mathbb{D},\omega)$ there is a function G_Z in $AL^2(\mathbb{D},\omega)$ having norm 1, vanishing precisely on Z, and such that every function $f \in AL^2(\mathbb{D},\omega)$ vanishing on Z has a factorization $f = G_Z g$, where $g \in AL^2(\mathbb{D},\omega)$ and $\|g\|_\omega \leq \|f\|_\omega$.*

We shall refer to the assertion of Corollary 7.2 as stating that *factorization occurs for the weight* ω. In [16], Sergeĭ Shimorin showed that factorization occurs for the weights $\mu_\beta(z) = (\beta+1)(1-|z|^2)^\beta$, with $-1 < \beta < 0$. Expanding the weight $\mu_\beta(z)$ in a Taylor series in the variable $|z|^2$, we see that this is a special case of the weights covered by Corollary 7.2. In [8] it was shown that factorization occurs also for μ_β with $\beta = 1$; later, Shimorin [17] obtained factorization for $0 < \beta < 1$; in [12], it was shown that factorization fails for $\beta > 1$.

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