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A FACTORING THEOREM FOR A WEIGHTED BERGMAN SPACE

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Dedicated to the memory of Allen Lowell Shields

0. Introduction

Let $\mathbf{D}$ be the open unit disc in the complex plane $\mathbf{C}$, $\mathbf{T}$ the unit circle. For $\alpha > -1$, put

$$\omega_\alpha(z) = \pi^{-1}(\alpha+1)(1-|z|^2)^\alpha, \qquad z \in \overline{\mathbf{D}},$$

and write $d\sigma_\alpha(z) = \omega_\alpha(z)\, d\sigma(z)$, where $d\sigma$ is ordinary area measure on $\mathbf{D}$. Moreover, let $L^2(\mathbf{D}, d\sigma_\alpha)$ denote the Hilbert space of (equivalence classes of) Borel measurable complex-valued functions on $\mathbf{D}$ satisfying

$$\|f\|^2_{L^2(\alpha)} = \int_{\mathbf{D}} |f(z)|^2\, d\sigma_\alpha(z) < \infty,$$

supplied with the above norm and the associated inner product

$$\langle f, g\rangle_{L^2(\alpha)} = \int_{\mathbf{D}} f(z)\overline{g}(z)\, d\sigma_\alpha(z).$$

Let $L^2_a(\mathbf{D}, d\sigma_\alpha)$ denote the subspace of $L^2(\mathbf{D}, d\sigma_\alpha)$ consisting of all functions that are holomorphic on $\mathbf{D}$; this subspace is easily seen to be closed. We will refer to the spaces $L^2_a(\mathbf{D}, d\sigma_\alpha)$, $\alpha > -1$, as weighted Bergman spaces. Note that the Hardy space $H^2(\mathbf{D})$ may be regarded as the limiting case of $L^2_a(\mathbf{D}, d\sigma_\alpha)$ as $\alpha \to -1$.

In [1], the author discovered an analog of the Hardy space factoring theorem valid for the standard (unweighted) Bergman space $L^2_a(\mathbf{D}, d\sigma_0)$. We formulate it for finite sequences, but it remains valid, mutatis mutandis, also for infinite zero sequences.

Factoring Theorem (Hedenmalm). *Suppose $A = \{a_j\}_0^N$ is a finite sequence of points in the open unit disk $\mathbf{D}$. Then there exists a function $G_A \in L^2_a(\mathbf{D}, d\sigma_0)$, unique up to multiplication by a unimodular constant, such that*

(a) *G_A vanishes precisely on A in the open unit disk $\mathbf{D}$,*

(b) *G_A has norm 1,*

(c) *Every $f \in L^2_a(\mathbf{D}, d\sigma_0)$ that vanishes on A has a factoring $f = G_A \cdot g$, where $g \in L^2_a(\mathbf{D}, d\sigma_0)$ has $\|g\|_{L^2(0)} \le \|f\|_{L^2(0)}$.*

This function G_A has a holomorphic extension across the circle $\mathbf{T}$, $|G_A| \ge 1$ holds on $\mathbf{T}$, and $|G_A|^2\, d\sigma_0$ is a representing measure for the origin, that is,

$$h(0) = \int_{\mathbf{D}} h(z)|G_A(z)|^2\, d\sigma_0(z)$$

holds for all bounded harmonic functions h on $\mathbf{D}$.

Remark. We may choose to call these functions G_A finite zero-based inner functions for the Bergman space, or why not finite Blaschke-type functions for the Bergman

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space. Another terminology, suggested by Peter Duren, Dmitry Khavinson, and Harold Shapiro, is to call G_A the *canonical divisor* associated with the set A.

In view of the author's joint work with Kehe Zhu [3], it is extremely likely that this theorem can be generalized in the following way.

Conjecture. *Fix a parameter value α, $-1 < \alpha \leq 1$. Suppose $A = \{a_j\}_0^N$ is a finite sequence of points in the open unit disk $\mathbf{D}$. Then there exists a function $G_A^\alpha \in L_a^2(\mathbf{D}, d\sigma_\alpha)$, unique up to multiplication by a unimodular constant, such that*

(a) *G_A^α vanishes precisely on A in the open unit disk $\mathbf{D}$,*

(b) *G_A^α has norm 1,*

(c) *every $f \in L_a^2(\mathbf{D}, d\sigma_\alpha)$ that vanishes on A has a factoring $f = G_A^\alpha \cdot g$, where $g \in L_a^2(\mathbf{D}, d\sigma_\alpha)$ has $\|g\|_{L^2(\alpha)} \leq \|f\|_{L^2(\alpha)}$.*

This function G_A^α has a holomorphic extension across the circle $\mathbf{T}$, $|G_A^\alpha| \geq 1$ holds on $\mathbf{T}$, and $|G_A^\alpha|^2\, d\sigma_\alpha$ is a representing measure for the origin, that is,

$$h(0) = \int_{\mathbf{D}} h(z)|G_A^\alpha(z)|^2\, d\sigma_\alpha(z)$$

holds for all bounded harmonic functions h on $\mathbf{D}$.

No such statement holds for parameter values $\alpha > 1$ [3]. In this paper we shall verify this conjecture in the special case $\alpha = 1$.

1. A LEMMA

The following lemma, without being stated explicitly, was used several times in [1]. This time we shall make even more extensive use of it. The assertion is proved by checking that both sides of the equality have the same Laplacians and the same boundary values. Given an $L^1(\mathbf{T})$ function u, we denote by $P[u]$ its harmonic extension to the interior:

$$P[u](z) = \int_{-\pi}^{\pi} P_z(e^{i\theta})u(e^{i\theta})\, d\theta/2\pi, \qquad z \in \mathbf{D}.$$

Here, $P_z(\zeta)$ is the Poisson kernel

$$P_z(\zeta) = \frac{1 - |\zeta z|^2}{|1 - \bar{\zeta} z|^2}, \qquad z \in \mathbf{D}, \quad \zeta \in \mathbf{T}.$$

The Hardy space $H^2(\mathbf{D})$ consists of those functions $f \in L^2(\mathbf{T})$ that have Poisson extensions to the interior $\mathbf{D}$ that are analytic. One then frequently identifies the space $H^2(\mathbf{D})$ with the space of Poisson extensions.

Lemma 1.1. *If $f \in H^2(\mathbf{D})$ has the power series expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad z \in \mathbf{D},$$

we have

$$P[|f|^2](z) = |f(z)|^2 + (1 - |z|^2) \sum_{k=0}^{\infty} \left| \sum_{n=0}^{\infty} c_{n+k+1} z^n \right|^2, \qquad z \in \mathbf{D}.$$

If we use the notation $T\varphi(z) = (\varphi(z) - \varphi(0))/z$, this expression takes the form

$$P[|f|^2](z) = |f(z)|^2 + (1 - |z|^2) \sum_{k=1}^{\infty} |T^k f(z)|^2, \qquad z \in \mathbf{D}.$$

Note also that if

$$\Gamma(z, \zeta) = \frac{1}{\pi} \log \left| \frac{z-\zeta}{1-\bar{\zeta} z} \right|^2, \qquad (z, \zeta) \in \mathbf{D}^2,$$

is the Green function associated with the Laplace operator

$$\Delta = \tfrac{1}{4}(\partial^2/\partial x^2 + \partial^2/\partial y^2),$$

and $f \in H^2(\mathbf{D})$, then

$$P[|f|^2](z) - |f(z)|^2 = -\int_{\mathbf{D}} \Gamma(z, \zeta) |f'(\zeta)|^2 \, d\sigma(\zeta), \qquad z \in \mathbf{D}.$$

2. Definition of the canonical divisors G_A^α

Definition 2.1. Fix the real parameter α, $-1 < \alpha \le 1$. Suppose A is a finite or infinite zero sequence for the weighted Bergman space $L_a^2(\mathbf{D}, d\sigma_\alpha)$. Then if 0 is not in the sequence A, G_A^α is defined as the unique extremal function for the problem

$$\sup\{\Re g(0) : g \in L_a^2(\mathbf{D}, d\sigma_\alpha),\, g = 0 \text{ on } A,\, \|g\|_{L^2(\alpha)} \le 1\}.$$

More generally, if 0 belongs to the sequence A with multiplicity n, we define G_A^α as the unique extremal function solving

$$\sup\{\Re g^{(n)}(0) : g \in L_a^2(\mathbf{D}, d\sigma_\alpha),\, g = 0 \text{ on } A,\, \|g\|_{L^2(\alpha)} \le 1\}.$$

The following result is a more or less immediate consequence of this definition. The proof is analogous to the one for the case $\alpha = 0$ presented in [1], and we see no reason to duplicate the argument here.

Proposition 2.2. *Fix the parameter α, $-1 < \alpha \le 1$. Suppose $A = \{a_j\}_1^N$ is a finite or infinite zero sequence for the space $L_a^2(\mathbf{D}, d\sigma_\alpha)$. Then G_A^α is perpendicular to all functions vanishing on the sequence $A \cup \{0\}$ (counting multiplicities); in particular, it is perpendicular to $z^n G_A^\alpha$, for all $n = 1, 2, 3, \ldots$. Consequently, $|G_A^\alpha|^2 d\sigma_\alpha$ is a representing measure for 0, meaning that*

$$h(0) = \int_{\mathbf{D}} h(z) |G_A^\alpha(z)|^2 \, d\sigma_\alpha(z)$$

holds for all bounded harmonic functions h on $\mathbf{D}$.

3. Statement of results

To simplify our notation, let us write ∂ and $\bar{\partial}$ instead of $\partial/\partial z$ and $\partial/\partial \bar{z}$, respectively. Writing $z = x + iy$, with $i^2 = -1$, we shall use as the Laplacian Δ the operator

$$\Delta_z = \Delta = \partial \bar{\partial} = \tfrac{1}{4}(\partial^2/\partial x^2 + \partial^2/\partial y^2).$$

Introduce the Green function for the Laplacian

$$\Gamma(z, \zeta) = \frac{1}{\pi} \log \left| \frac{z-\zeta}{1-\bar{\zeta} z} \right|^2, \qquad (z, \zeta) \in \mathbf{D}^2,$$

and put, for parameter values α, $-1 < \alpha \le 1$,

$$\Phi_A^\alpha(z) = \pi \int_{\mathbf{D}} \Gamma(z, \zeta)(|G_A^\alpha(\zeta)|^2 - 1)\, d\sigma_\alpha(\zeta), \qquad z \in \mathbf{D}.$$

Here, A denotes an arbitrary zero set for an $L_a^2(\mathbf{D}, d\sigma_\alpha)$ function. This function Φ_A^α has the property

$$\Delta_z \Phi_A^\alpha(z) = \pi\omega_\alpha(z)(|G_A^\alpha(\zeta)|^2 - 1), \qquad z \in \mathbf{D}.$$

Theorem 3.1. *Suppose $A = \{a_j\}_1^N$ is a finite sequence of points in $\mathbf{D}$. Then the function Φ_A^1 extends to an infinitely differentiable function on the closed unit disk $\overline{\mathbf{D}}$, and enjoys the properties*

(a) *$\partial\Phi_A^1/\partial n = 0$ and $\partial^2\Phi_A^1/\partial n^2 = 0$ on $\mathbf{T}$, where $\partial/\partial n$ is differentiation in the outward normal direction, and*

(b) *$0 \le \Phi_A^1(z) \le 2(1-|z|^2)(3/4 - |z|^2/4)$ for all $z \in \mathbf{D}$.*

If we apply the same methods as in [1], with Green's formula and an extremality argument as the chief tools, we obtain from the above technical result the desired factoring theorem, which we state here for completeness.

Theorem 3.2. *Suppose $A = \{a_j\}_0^N$ is a finite sequence of points in the open unit disk $\mathbf{D}$. Then the function G_A^1 has the properties*

(a) *G_A^1 vanishes precisely on A in the open unit disk $\mathbf{D}$,*

(b) *G_A^1 has norm 1,*

(c) *every $f \in L_a^2(\mathbf{D}, d\sigma_1)$ that vanishes on A has a factoring $f = G_A^1 \cdot g$, where $g \in L_a^2(\mathbf{D}, d\sigma_1)$ has*

$$\|g\|_{L^2(1)} \le \|g\|_{L^2(1)} + \int_{\mathbf{D}} \Phi_A^1(z)|f'(z)|^2\, d\sigma_0(z) = \|G_A^1 g\|_{L^2(1)} = \|f\|_{L^2(1)}.$$

This function G_A^1 has a holomorphic extension across the circle $\mathbf{T}$, and $|G_A^1| \ge 1$ holds on $\mathbf{T}$.

Remark. When we allow A to be a general zero set for the space $L_a^2(\mathbf{D}, d\sigma_1)$, some of the statements in the above theorem must be modified. Parts (a) and (b) remain the same, but in (c) an equality is replaced by an inequality:

$$\|g\|_{L^2(1)} \le \|g\|_{L^2(1)} + \int_{\mathbf{D}} \Phi_A^1(z)|f'(z)|^2\, d\sigma_0(z) \le \|G_A^1 g\|_{L^2(1)}.$$

The inequality appears when we apply Fatou's lemma in an approximation argument; it is not clear whether it is ever strict. Also, we clearly no longer can expect the function G_A^1 to extend holomorphically across a portion of $\mathbf{T}$ containing an accumulation point of A. One can show, however, that it does extend holomorphically across any arc of $\mathbf{T}$ containing no accumulation point for the sequence A.

As in [1], the function G_A^1 is a contractive multiplier $H^2(\mathbf{D}) \to L_a^2(\mathbf{D}, d\sigma_1)$. This is a consequence of the estimate from above of the function Φ_A^1 in Theorem 3.1(b).

Corollary 3.3. *For all $L_a^2(\mathbf{D}, d\sigma_1)$-zero sequences A in $\mathbf{D}$, the function G_A^1 is a contractive multiplier $H^2(\mathbf{D}) \to L_a^2(\mathbf{D}, d\sigma_1)$. From this it follows that we have the estimate*

$$|G_A^1(z)| \le (1-|z|^2)^{-1}, \qquad z \in \mathbf{D}.$$

The size estimate follows from [6], p. 232.

Proof of Theorem 3.1. Let us for convenience of notation write G_A instead of G_A^1, and Φ_A instead of Φ_A^1. Note that by the regularity up to the boundary $\mathbf{T}$ of the function

$$\Delta_z \Phi_A^1(z) = \pi\omega_1(z)(|G_A^1(z)|^2 - 1), \qquad z \in \mathbf{D},$$

the function Φ_A^1 itself must be C^∞ (even real analytic) on the closed unit disk, with value 0 on the unit circle. Let Ψ be the real analytic function on $\overline{\mathbf{D}}$ which has value 0 on $\mathbf{T}$, and solves the partial differential equation $\Delta\Psi(z) = -\pi\omega_1(z)$ inside $\mathbf{D}$. One checks that Ψ is given by the formula

$$\Psi(z) = 2(1-|z|^2)(3/4-|z|^2/4), \qquad z \in \overline{\mathbf{D}}.$$

Clearly, $\Phi_A - \Psi \leq 0$ on $\mathbf{D}$, since the left-hand side is subharmonic with boundary values 0. This proves the upper estimate of Φ_A in (b).

For technical reasons, we are going to prove the positivity of the function Φ_A under the additional assumption that the points of the sequence A are distinct, and all different from 0. An approximation argument similar to the one used in [1] shows that we may do so without loss of generality. By Lemma 2.2, G_A is perpendicular to all functions in $L^2_a(\mathbf{D}, d\sigma_1)$ vanishing on $A \cup \{0\}$, and so it must be a linear combination of kernel functions (for a more detailed discussion, see [2]), that is,

$$G_A(z) = \lambda_0 + \sum_{j=1}^{N} \lambda_j (1-\overline{a}_j z)^{-3}, \qquad z \in \mathbf{D},$$

for some scalars $\lambda_0, \dots, \lambda_N \in \mathbf{C}$, which we can compress to

$$G_A(z) = \sum_{j=0}^{N} \lambda_j (1-\overline{a}_j z)^{-3}, \qquad z \in \mathbf{D},$$

by putting $a_0 = 0$. We want to solve the Laplace equation

$$\Delta\Phi_A(z) = 2(1-|z|^2)(|G_A(z)|^2-1) = 2(|G_A|^2 - |zG_A|^2 + |z|^2 - 1)$$

with $\Phi_A = 0$ on $\mathbf{T}$. If, as in Lemma 1.1, $P[\phi]$ denotes the Poisson integral of the function ϕ, and for analytic functions f on $\mathbf{D}$, $\int f$ denotes the analytic function F with $F(0) = 0$ and $F'(z) = f(z)$, we have

$$\Phi_A(z)/2 = \left|\int G_A\right|^2 - P\left[\left|\int G_A\right|^2\right] + P\left[\left|\int zG_A\right|^2\right] - \left|\int zG_A\right|^2 + \frac{|z|^4}{4} - |z|^2 + \frac{3}{4}.$$

Here $\int G_A$ and $\int zG_A$ are given by the formulas

$$\int G_A = \sum_{j=0}^{N} \lambda_j z \frac{1-\overline{a}_j z/2}{(1-\overline{a}_j z)^2}$$

and

$$\int zG_A = \sum_{j=0}^{N} \lambda_j \frac{z^2/2}{(1-\overline{a}_j z)^2},$$

where we use the convention $a_0 = 0$, as always. On the other hand, we know that if $f \in H^2(\mathbf{D})$ and T denotes the backward shift operator

$$T\varphi(z) = (\varphi(z) - \varphi(0))/z, \qquad z \in \mathbf{D},$$

acting on functions φ analytic on $\mathbf{D}$, then Lemma 1.1 offers us the formula

$$P[|f|^2](z) - |f(z)|^2 = (1-|z|^2)\sum_{n=1}^{\infty} |T^n f(z)|^2, \qquad z \in \mathbf{D}.$$

It now follows that

$$\begin{aligned}\Phi_A(z)/2 = & -(1-|z|^2)\sum_{n=1}^{\infty}\left|T^n \int G_A\right|^2 \\ & + (1-|z|^2)\sum_{n=1}^{\infty}\left|T^n \int zG_A\right|^2 + (1-|z|^2)(3/4-|z|^2/4),\end{aligned}$$

that is,

$$\Phi_A(z)/2 = (1-|z|^2) \times \left\{ -\sum_{n=1}^{\infty} \left| T^n \int G_A \right|^2 + \sum_{n=1}^{\infty} \left| T^n \int zG_A \right|^2 + \frac{3}{4} - \frac{|z|^2}{4} \right\}, \quad z \in \mathbf{D}. \tag{3.1}$$

For $n \geq 1$ we have the formulas

$$T^n \int G_A = \frac{1}{2} \sum_{j=0}^{N} \lambda_j \overline{a}_j^{n-1} \frac{n+1-n\overline{a}_j z}{(1-\overline{a}_j z)^2}$$

and

$$T^{n+1} \int zG_A = \frac{1}{2} \sum_{j=0}^{N} \lambda_j \overline{a}_j^{n-1} \frac{n-(n-1)\overline{a}_j z}{(1-\overline{a}_j z)^2};$$

in addition, we have the formula

$$T \int zG_A = \frac{1}{2} \sum_{j=0}^{N} \frac{\lambda_j z}{(1-\overline{a}_j z)^2}.$$

We now get

$$\begin{aligned}
4\sum_{n=1}^{\infty} \left| T^n \int G_A \right|^2 &= \sum_{n=1}^{\infty} \sum_{j,k=0}^{N} \lambda_j \overline{\lambda}_k \overline{a}_j^{n-1} a_k^{n-1} \frac{n+1-n\overline{a}_j z}{(1-\overline{a}_j z)^2} \cdot \frac{n+1-na_k\overline{z}}{(1-a_k\overline{z})^2} \\
&= \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1-\overline{a}_j z)^2 (1-a_k\overline{z})^2} \\
&\qquad \times \sum_{n=1}^{\infty} (\overline{a}_j a_k)^{n-1} (n+1-n\overline{a}_j z)(n+1-na_k\overline{z}) \\
&= \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1-\overline{a}_j z)^2 (1-a_k\overline{z})^2} \\
&\qquad \times \sum_{n=1}^{\infty} (\overline{a}_j a_k)^{n-1} (n(n+1)(1-\overline{a}_j z)(1-a_k\overline{z}) + n+1-n\overline{a}_j a_k |z|^2) \\
&= \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1-\overline{a}_j z)^2 (1-a_k\overline{z})^2} \\
&\qquad \times \left\{ \frac{2(1-\overline{a}_j z)(1-a_k\overline{z})}{(1-\overline{a}_j a_k)^3} + \frac{2-\overline{a}_j a_k - \overline{a}_j a_k |z|^2}{(1-\overline{a}_j a_k)^2} \right\}.
\end{aligned}$$

On the other hand we also have

$$4\sum_{n=1}^{\infty}\left|T^{n+1}\int zG_A\right|^2 = \sum_{n=1}^{\infty}\sum_{j,k=0}^{N}\lambda_j\bar{\lambda}_k\bar{a}_j^{n-1}a_k^{n-1}\frac{n-(n-1)\bar{a}_jz}{(1-\bar{a}_jz)^2}\cdot\frac{n-(n-1)a_k\bar{z}}{(1-a_k\bar{z})^2}$$
$$= \sum_{j,k}\frac{\lambda_j\bar{\lambda}_k}{(1-\bar{a}_jz)^2(1-a_k\bar{z})^2} \times\sum_{n=0}^{\infty}(\bar{a}_ja_k)^n(n+1-n\bar{a}_jz)(n+1-na_k\bar{z})$$
$$= \sum_{j,k}\frac{\lambda_j\bar{\lambda}_k}{(1-\bar{a}_jz)^2(1-a_k\bar{z})^2} \times\sum_{n=0}^{\infty}(\bar{a}_ja_k)^n(n(n+1)(1-\bar{a}_jz)(1-a_k\bar{z})+n+1-n\bar{a}_ja_k|z|^2)$$
$$= \sum_{j,k}\frac{\lambda_j\bar{\lambda}_k}{(1-\bar{a}_jz)^2(1-a_k\bar{z})^2} \times\left\{\frac{2\bar{a}_ja_k(1-\bar{a}_jz)(1-a_k\bar{z})}{(1-\bar{a}_ja_k)^3}+\frac{1-\bar{a}_j^2a_k^2|z|^2}{(1-\bar{a}_ja_k)^2}\right\},$$

and hence by (3.1),

$$2\Phi_A(z) = (1-|z|^2) \times\left\{\sum_{j,k}\frac{\lambda_j\bar{\lambda}_k|z|^2}{(1-\bar{a}_jz)^2(1-a_k\bar{z})^2}-\sum_{j,k}\frac{\lambda_j\bar{\lambda}_k}{(1-\bar{a}_jz)^2(1-a_k\bar{z})^2} \times\left(\frac{2(1-\bar{a}_jz)(1-a_k\bar{z})}{(1-\bar{a}_ja_k)^2}+\frac{1-\bar{a}_ja_k|z|^2}{1-\bar{a}_ja_k}\right)+3-|z|^2\right\}$$
$$= (1-|z|^2) \times\left\{-\sum_{j,k}\frac{\lambda_j\bar{\lambda}_k}{(1-\bar{a}_jz)^2(1-a_k\bar{z})^2}\left(\frac{2(1-\bar{a}_jz)(1-a_k\bar{z})}{(1-\bar{a}_ja_k)^2}+\frac{1-|z|^2}{1-\bar{a}_ja_k}\right)+3-|z|^2\right\},$$

that is,

$$2\frac{\Phi_A(z)}{1-|z|^2} = -2\sum_{j,k}\frac{\lambda_j\bar{\lambda}_k}{(1-\bar{a}_ja_k)^2(1-\bar{a}_jz)(1-a_k\bar{z})} - (1-|z|^2)\sum_{j,k}\frac{\lambda_j\bar{\lambda}_k}{(1-\bar{a}_ja_k)(1-\bar{a}_jz)^2(1-a_k\bar{z})^2}+3-|z|^3. \tag{3.2}$$

Let us now check that the right-hand side of (3.2) has boundary value 0 on $\mathbf{T}$, which implies that $\partial\Phi_A/\partial n = 0$ on $\mathbf{T}$, and hence allows us to apply Green's theorem. Now let ϕ be the function

$$\phi(z) = \sum_{j,k}\frac{\lambda_j\bar{\lambda}_k}{(1-\bar{a}_ja_k)^2(1-\bar{a}_jz)(1-a_k\bar{z})}.$$

We want to show that $\phi(z) = 1$ on $\mathbf{T}$, because then it follows that the right-hand side of (3.2) vanishes on $\mathbf{T}$, as asserted. According to [1], p. 55, we have

$$\int_{\mathbf{T}} z^n (1 - \overline{a}_j)^{-1}(1 - a_k \overline{z})^{-1}\, ds(z)/2\pi = a_k^n (1 - \overline{a}_j a_k)^{-1}$$

for $n \geq 0$, where ds denotes arc length measure on $\mathbf{T}$, so that we get

$$\hat{\phi}(-n) = \int z^n \phi(z)\, ds(z)/2\pi = \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k a_k^n}{(1 - \overline{a}_j a_k)^3} = \langle z^n G_A, G_A \rangle_{L^2(1)}$$

for $n \geq 0$, and since ϕ is real-valued, we also have $\hat{\phi}(n) = \overline{\hat{\phi}(-n)}$. The fact that G_A is extremal implies that $\langle z^n G_A, G_A \rangle_{L^2(1)} = 0$ for $n > 0$, and since its norm is 1, $\langle G_A, G_A \rangle_{L^2(1)} = 1$. This shows that $\phi(z) = 1$ on $\mathbf{T}$, as desired. Let us now look more closely at the function $1 - \phi$. Since

$$\phi(z) = \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1 - \overline{a}_j z)(1 - a_k \overline{z})} \sum_{n=0}^{\infty} (n+1)(\overline{a}_j a_k)^n = \sum_{n=0}^{\infty} (n+1) \left| \sum_j \lambda_j \overline{a}_j^n / (1 - \overline{a}_j z) \right|^2 ,$$

we have, by Lemma 1.1,

$$\begin{aligned}
\frac{1 - \phi(z)}{1 - |z|^2} = \frac{P[\phi](z) - \phi(z)}{1 - |z|^2} &= \sum_{n=0}^{\infty} (n+1) \sum_{m=1}^{\infty} \left| \sum_j \lambda_j \overline{a}_j^n T^m (1/(1 - \overline{a}_j z)) \right|^2 \\
&= \sum_{n=0}^{\infty} (n+1) \sum_{m=1}^{\infty} \left| \sum_j \lambda_j \overline{a}_j^{n+m} / (1 - \overline{a}_j z) \right|^2 \\
&= \sum_{n=0}^{\infty} (n+1) \sum_{m=1}^{\infty} \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k \overline{a}_j^{n+m} a_k^{n+m}}{(1 - \overline{a}_j z)(1 - a_k \overline{z})} \\
&= \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1 - \overline{a}_j z)(1 - a_k \overline{z})} \sum_{n=0}^{\infty} (n+1)(\overline{a}_j a_k)^n \sum_{m=1}^{\infty} (\overline{a}_j a_k)^m \\
&= \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1 - \overline{a}_j z)(1 - a_k \overline{z})} \frac{\overline{a}_j a_k}{(1 - \overline{a}_j a_k)^3}.
\end{aligned}$$

By (3.2) we now have

$$2\Phi_A(z) = (1 - |z|^2)^2 \left\{ 1 + 2 \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1 - \overline{a}_j z)(1 - a_k \overline{z})} \frac{\overline{a}_j a_k}{(1 - \overline{a}_j a_k)^3} - \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1 - \overline{a}_j a_k)(1 - \overline{a}_j z)^2 (1 - a_k \overline{z})^2} \right\}.$$

We shall now see that

$$\Phi_A(z)/(1 - |z|^2)^2 = 0 \quad \text{on } \mathbf{T}.$$

Introduce the function

$$\psi(z) = 2\Phi_A(z)/(1 - |z|^2)^2 = 1 + 2 \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1 - \overline{a}_j z)(1 - a_k \overline{z})} \frac{\overline{a}_j a_k}{(1 - \overline{a}_j a_k)^3} - \sum_{j,k} \frac{\lambda_j \overline{\lambda}_k}{(1 - \overline{a}_j a_k)(1 - \overline{a}_j z)^2 (1 - a_k \overline{z})^2};$$

since we have

$$\begin{aligned}
\int_{\mathbf{T}} \frac{z^n}{(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}\, ds(z)/2\pi
&= \int_{\mathbf{T}} \frac{z^{n+2}}{(1-\overline{a}_j z)^2(z-a_k)^2}\, ds(z)/2\pi \\
&= \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{z^{n+1}}{(1-\overline{a}_j z)^2(z-a_k)^2}\, dz \\
&= \operatorname*{Res}_{z=a_k}\left\{\frac{z^{n+1}}{(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}\right\} \\
&= (z^{n+1}/(1-\overline{a}_j z)^2)'|_{z=a_k} \\
&= \frac{a_k^n}{(1-\overline{a}_j a_k)^3}(n+1-(n-1)\overline{a}_j a_k),
\end{aligned}$$

it follows that for $n \geq 0$

$$\begin{aligned}
\hat{\psi}(-n) &= \int_{\mathbf{T}} z^n \psi(z)\, ds(z)/2\pi \\
&= \delta_0(n) + 2\sum_{j,k}\frac{\lambda_j\overline{\lambda}_k\overline{a}_j a_k^{n+1}}{(1-\overline{a}_j a_k)^4} - \sum_{j,k}\frac{\lambda_j\overline{\lambda}_k a_k^n}{(1-\overline{a}_j a_k)^4}(n+1-(n-1)\overline{a}_j a_k) \\
&= \delta_0(n) + \sum_{j,k}\frac{\lambda_j\overline{\lambda}_k a_k^n}{(1-\overline{a}_j a_k)^4}(2\overline{a}_j a_k - n - 1 + (n-1)\overline{a}_j a_k) \\
&= \delta_0(n) - (n+1)\sum_{j,k}\frac{\lambda_j\overline{\lambda}_k a_k^n}{(1-\overline{a}_j a_k)^3} = \delta_0(n) - (n+1)\delta_0(n) = 0.
\end{aligned}$$

It follows, since ψ is real-valued, that $\psi(z) = 0$ on $\mathbf{T}$. Let us now rewrite ψ in order to be able to apply Lemma 1.1:

$$\begin{aligned}
\psi(z) &= 1 + \sum_{j,k}\frac{\lambda_j\overline{\lambda}_k\overline{a}_j a_k}{(1-\overline{a}_j z)(1-a_k\overline{z})}\sum_{n=0}^{\infty}(n+1)(n+2)(\overline{a}_j a_k)^n \\
&\quad - \sum_{j,k}\frac{\lambda_j\overline{\lambda}_k}{(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}\sum_{n=0}^{\infty}(\overline{a}_j a_k)^n \\
&= 1 + \sum_{n=0}^{\infty}(n+1)(n+2)\left|\sum_j \frac{\lambda_j\overline{a}_j^{n+1}}{1-\overline{a}_j z}\right|^2 - \sum_{n=0}^{\infty}\left|\sum_j \frac{\lambda_j\overline{a}_j^n}{(1-\overline{a}_j z)^2}\right|^3.
\end{aligned}$$

Since

$$T^k((1-\overline{a}_j z)^{-1}) = \overline{a}_j^k/(1-\overline{a}_j z),$$

$$T^k((1-\overline{a}_j z)^{-2}) = \overline{a}_j^k(k+1-k\overline{a}_j z)/(1-\overline{a}_j z)^2,$$

we obtain that

$$
\begin{aligned}
-\psi(z)/(1-|z|^2) &= (P[\psi](z)-\psi(z))/(1-|z|^2)\\
&= \sum_{n=0}^{\infty}(n+1)(n+2)\sum_{m=1}^{\infty}\left|\sum_j \lambda_j \overline{a}_j^{n+1} T^m(1/(1-\overline{a}_j z))\right|^2\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left|\sum_j \lambda_j \overline{a}_j^{n} T^m(1/(1-\overline{a}_j z)^2)\right|^2\\
&= \sum_{n=0}^{\infty}(n+1)(n+2)\sum_{m=1}^{\infty}\left|\sum_j \lambda_j \overline{a}_j^{n+m+1} T^m((1-\overline{a}_j z)^{-1})\right|^2\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left|\sum_j \lambda_j \overline{a}_j^{n+m} \frac{m+1-m\overline{a}_j z}{(1-\overline{a}_j z)^2}\right|^2\\
&= \sum_{n=0}^{\infty}(n+1)(n+2)\sum_{m=1}^{\infty}\sum_{j,k}\lambda_j\overline{\lambda}_k \frac{\overline{a}_j^{n+m+1}a_k^{n+m+1}}{(1-\overline{a}_j z)(1-a_k\overline{z})}\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{j,k}\lambda_j\overline{\lambda}_k \overline{a}_j^{n+m}a_k^{n+m}\frac{(m+1-m\overline{a}_j z)(m+1-ma_k\overline{z})}{(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}\\
&= \sum_{j,k}\frac{\lambda_j\overline{\lambda}_k}{(1-\overline{a}_j z)(1-a_k\overline{z})}\sum_{n=0}^{\infty}(n+1)(n+2)(\overline{a}_j a_k)^n\sum_{m=1}^{\infty}(\overline{a}_j a_k)^{m+1}\\
&\quad -\sum_{j,k}\frac{\lambda_j\overline{\lambda}_k}{(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}\\
&\qquad \times\sum_{n=0}^{\infty}(\overline{a}_j a_k)^n\sum_{m=1}^{\infty}(m+1-m\overline{a}_j z)(m+1-ma_k\overline{z})(\overline{a}_j a_k)^m\\
&= 2\sum_{j,k}\frac{\lambda_j\overline{\lambda}_k\overline{a}_j^2a_k^2}{(1-\overline{a}_j a_k)^4(1-\overline{a}_j z)(1-a_k\overline{z})}\\
&\quad -\sum_{j,k}\frac{\lambda_j\overline{\lambda}_k}{(1-\overline{a}_j a_k)(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}\\
&\quad \times\left\{2\overline{a}_j a_k\frac{(1-\overline{a}_j z)(1-a_k\overline{z})}{(1-\overline{a}_j a_k)^3}+\frac{\overline{a}_j a_k}{1-\overline{a}_j a_k}+\frac{\overline{a}_j a_k}{(1-\overline{a}_j a_k)^2}-\frac{\overline{a}_j^2a_k^2|z|^2}{(1-\overline{a}_j a_k)^2}\right\}\\
&= \sum_{j,k}\frac{\lambda_j\overline{\lambda}_k\overline{a}_j a_k\{-2(1-\overline{a}_j z)(1-a_k\overline{z})-1+\overline{a}_j a_k-1+\overline{a}_j a_k|z|^2\}}{(1-\overline{a}_j a_k)^3(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}\\
&= -2\sum_{j,k}\frac{\lambda_j\overline{\lambda}_k\overline{a}_j a_k}{(1-\overline{a}_j a_k)^3(1-\overline{a}_j z)(1-a_k\overline{z})}\\
&\quad -2\sum_{j,k}\frac{\lambda_j\overline{\lambda}_k\overline{a}_j a_k}{(1-\overline{a}_j a_k)^2(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}\\
&\quad -(1-|z|^2)\sum_{j,k}\frac{\lambda_j\overline{\lambda}_k\overline{a}_j^2a_k^2}{(1-\overline{a}_j a_k)^3(1-\overline{a}_j z)^2(1-a_k\overline{z})^2}.
\end{aligned}
$$

We now conclude that

$$\Phi_A(z) = \frac{1}{2}(1-|z|^2)^3 \left\{ 2\sum_{j,k} \frac{\lambda_j \bar{\lambda}_k \bar{a}_j a_k}{(1-\bar{a}_j a_k)^3 (1-\bar{a}_j z)(1-a_k \bar{z})} + 2\sum_{j,k} \frac{\lambda_j \bar{\lambda}_k \bar{a}_j a_k}{(1-\bar{a}_j a_k)^2 (1-\bar{a}_j z)^2 (1-a_k \bar{z})^2} + (1-|z|^2)\sum_{j,k} \frac{\lambda_j \bar{\lambda}_k \bar{a}_j^2 a_k^2}{(1-\bar{a}_j a_k)^3 (1-\bar{a}_j z)^2 (1-a_k \bar{z})^2} \right\},$$

and if we expand this in power series form we get

$$\Phi_A(z) = \frac{1}{2}(1-|z|^2)^3 \times \left\{ \sum_{n=0}^{\infty} (n+1)(n+2) \left| \sum_j \frac{\lambda_j \bar{a}_j^{n+1}}{1-\bar{a}_j z} \right|^2 + 2\sum_{n=0}^{\infty} (n+1) \left| \sum_j \frac{\lambda_j \bar{a}_j^{n+1}}{(1-\bar{a}_j z)^2} \right|^2 + \frac{1}{2}(1-|z|^2) \sum_{n=0}^{\infty} n(n+1) \left| \sum_j \frac{\lambda_j \bar{a}_j^{n+1}}{(1-\bar{a}_j z)^2} \right|^2 \right\},$$

from which it is obvious that $\Phi_A(z) \geq 0$ on $\mathbf{D}$.

The formula for Φ_A^1 obtained at the end of the proof of Theorem 3.2 suggests that the following should be true. If it could be verified, it would be possible to shorten considerably the proof of Theorem 3.3.

Conjecture 3.5. *Suppose F is holomorphic on $\mathbf{D}$ and satisfies $F'' \in L_a^2(\mathbf{D}, d\sigma_1)$. Moreover, let T denote the backwards shift: $T\varphi(z) = (\varphi(z) - \varphi(0))/z$. Then the function*

$$\Psi_F(z) = (1-|z|^2)^3 \left\{ \sum_{n=0}^{\infty} (n+1)(n+2)|T^{n+3}F|^2 + 2\sum_{n=0}^{\infty} (n+1)|(T^{n+2}F)'|^2 + \frac{1}{2}(1-|z|^2)\sum_{n=0}^{\infty} n(n+1)|(T^{n+2}F)'|^2 \right\}, \qquad z \in \bar{\mathbf{D}},$$

solves the sixth order elliptic partial differential equation

$$\Delta^3 \Psi_F(z) = \Delta^2((1-|z|^2)|F''(z)|^2), \qquad z \in \mathbf{D},$$

with 0 boundary data:

$$\Psi_F = (\partial/\partial n)\Psi_F = (\partial^2/\partial n^2)\Psi_F = 0 \quad \text{on } \mathbf{T}.$$

The boundary condition may have no meaning in the classical sense unless F has additional regularity; it is, however, possible to make sense of a weak formulation of it of Sobolev type in the general situation.

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