BEURLING TYPE INVARIANT SUBSPACES OF THE BERGMAN SPACES

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1. *Introduction*

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For $0 < p < +\infty$, the Bergman space $AL^p(\mathbb{D})$ consists of analytic functions f in $\mathbb{D}$ with

$$\|f\|_{AL^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z)\right)^{1/p} < +\infty,$$

where dA is area measure on $\mathbb{D}$, normalized so that $\mathbb{D}$ has mass 1. Equipped with the above norm, $AL^p(\mathbb{D})$ is a Banach space for $1 \leqslant p < +\infty$; when $0 < p < 1$, we have that $AL^p(\mathbb{D})$ is a complete metric space if supplied with the distance $d(f,g) = \|f-g\|^p_{AL^p}$. The reader is referred to **[20]** for the basic theory of Bergman and Hardy spaces.

Let z be the coordinate function on $\mathbb{D}$: $z(\lambda) = \lambda$ with $\lambda \in \mathbb{D}$ (the reader is asked not to confuse this notational convention with the use of z as a complex number or a complex variable elsewhere in the paper). Suppose that X is a topological vector space of analytic functions on $\mathbb{D}$, with the property that $zf \in X$ whenever $f \in X$. Multiplication by z is thus an operator on X, and if X is a Banach space, then it is automatically a bounded operator. A subspace I of X is said to be *invariant* (or z-invariant) if it is closed and $f \in I$ implies that $zf \in I$. Beurling's successful characterization **[2]** (for $p = 2$) of the invariant subspaces of the Hardy spaces $H^p(\mathbb{D})$ suggests the problem of describing the invariant subspaces of the Bergman spaces $AL^p(\mathbb{D})$. The purpose of the present paper is to study a class of invariant subspaces of $AL^p(\mathbb{D})$ which may be said to be of Beurling type.

To every function f in the Hardy space $H^p(\mathbb{D})$, which does not vanish identically, there corresponds a zero set (Blaschke sequence) $Z(f)$ and a singular Borel measure μ_f on $\mathbb{T}$. Here $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. Beurling's characterization of invariant subspaces of $H^p(\mathbb{D})$ states that every invariant subspace I (with the exception of $\{0\}$) of $H^p(\mathbb{D})$ is uniquely determined by a zero set Z and a singular measure μ on $\mathbb{T}$ in the following fashion:

$$I = \mathscr{I}(Z,\mu;H^p) = \{f \in H^p(\mathbb{D}) : Z \subset Z(f),\ \mu \leqslant \mu_f\}.$$

Here the inclusion $Z \subset Z(f)$ is to be understood as counting multiplicities. Another way to state Beurling's theorem is to say that every invariant subspace of $H^p(\mathbb{D})$ (other than $\{0\}$) is generated by a classical inner function. We are inclined to say that

Received 23 November 1993; revised 25 February 1994.

1991 *Mathematics Subject Classification* 47A15.

Research of the first author supported by the Swedish Natural Science Research Council and the Wallenberg Prize from the Swedish Mathematical Society.

Research of the second and third authors supported by the National Science Foundation.

J. London Math. Soc. (2) 53 (1996) 601–614

an invariant subspace of $AL^p(\mathbb{D})$ is of *Beurling type* provided it is given by a zero sequence and a Borel measure μ as above, with $H^p(\mathbb{D})$ replaced by $AL^p(\mathbb{D})$. The problem with this definition is that it is not clear what we mean by the measure μ_f for $f \in AL^p(\mathbb{D})$, because $AL^p(\mathbb{D})$ is not contained in the Nevanlinna class of holomorphic quotients of bounded analytic functions.

To remedy this difficulty, we recall some of the $A^{-\infty}$ theory developed in **[13, 14]**. For positive real-valued n, let A^{-n} be the Banach space of all holomorphic functions f on the unit disk $\mathbb{D}$ subject to the condition

$$|f(z)| \leqslant C(f)(1-|z|)^{-n} \quad \text{for } z \in \mathbb{D}$$

for some positive constant $C(f)$. The union of all these spaces A^{-n} is denoted by $A^{-\infty}$, and it is not difficult to show that it coincides with the union of all the spaces $AL^p(\mathbb{D})$, with $0 < p < +\infty$. The space $A^{-\infty}$ is a topological algebra if thought of as the inductive limit of the spaces A^{-n} (with $0 < n < +\infty$), and the topology remains the same if we think of it as the inductive limit of the spaces $AL^p(\mathbb{D})$ (with $0 < p < +\infty$). In analogy (see **[13, 14]**) with the theory of Hardy spaces, we associate to every $f \in A^{-\infty}$ with $f \neq 0$, its zero set $Z(f)$ (counting multiplicities) and a so-called κ-singular measure (the definition of which is deferred to Section 2) σ_f on $\mathbb{T}$. It is proved in **[14]** that every invariant subspace I of $A^{-\infty}$ is uniquely determined by a zero set Z and a κ-singular measure σ on $\mathbb{T}$ in the following way:

$$I = \mathscr{I}(Z, \sigma; A^{-\infty}) = \{f \in A^{-\infty} : Z \subset Z(f),\ \sigma \leqslant \sigma_f\}.$$

It is also shown in **[14]** that every invariant subspace of $A^{-\infty}$ is singly generated.

Since every Bergman space $AL^p(\mathbb{D})$ is contained in $A^{-\infty}$, we can consider invariant subspaces of $AL^p(\mathbb{D})$ of the following form

$$\mathscr{I}(Z, \sigma; AL^p) = \{f \in AL^p(\mathbb{D}) : Z \subset Z(f),\ \sigma \leqslant \sigma_f\},$$

where Z is an $AL^p(\mathbb{D})$-zero set and σ is a κ-singular measure on $\mathbb{T}$. Such invariant subspaces of $AL^p(\mathbb{D})$ will be said to be of *κ-Beurling type*. In the Banach space setting, the permitted class of measures σ may be larger than for $A^{-\infty}$, provided that the zero set Z is thick, because the space $\mathscr{I}(Z, 0; AL^p)/G_Z$ (here, G_Z is the canonical divisor, as in **[10, 7]**) approaches the Hardy space $H^p(\mathbb{D})$ as Z gets saturated, and in $H^p(\mathbb{D})$, all singular positive Borel measures play a role in the description of invariant subspaces. One would then expect to have a class of Beurling type invariant subspaces $\mathscr{I}(Z, \mu; AL^p)$, where the measure μ belongs to a set $\mathfrak{P}(Z; AL^p)$ of positive singular Borel measures on $\mathbb{T}$, containing the κ-singular ones. A characterization of the set $\mathfrak{P}(Z : AL^p)$ as the zero set Z varies would be desirable. To illustrate the possibility of having a bigger relevant collection of singular Borel measures in the Banach space case than for the soft topology space $A^{-\infty}$, consider the space A^{-n} for a fixed $n > 0$, and the saturated zero sequence Z constructed by Kristian Seip in **[19]**. Seip **[19]** finds a function F_Z in A^{-n} which vanishes precisely on Z, with the property that if $f \in A^{-n}$ vanishes on Z, then f/F_Z belongs to $H^\infty(\mathbb{D})$. The sequence Z is thus maximal, in the sense that a bigger zero sequence can only differ from Z by a Blaschke sequence. If u_μ is the singular inner function associated with a positive singular Borel measure μ, then

$$\mathscr{I}(Z, \mu; A^{-n}) = F_Z u_\mu H^\infty(\mathbb{D})$$

is an invariant subspace of A^{-n} of Beurling type. Since A^{-n} is nonseparable, only weakly closed (with respect to an appropriate predual) invariant subspaces should be considered, so it should be pointed out that the above subspace qualifies. Thus $\mathfrak{P}(Z;A^{-n})$ (defined analogously) consists of all positive singular finite Borel measures on $\mathbb{T}$.

Unlike the cases of the Hardy spaces $H^p(\mathbb{D})$ and $A^{-\infty}$, there exist in $AL^p(\mathbb{D})$ invariant subspaces which are definitely not of κ-Beurling type, and not even of Beurling type. Moreover, not every invariant subspace of $AL^p(\mathbb{D})$ is singly generated. The existing examples of non-Beurling type and non-singly generated invariant subspaces (see, for instance, **[11]**) have the codimension n property, with $2 \leqslant n \leqslant +\infty$. An invariant subspace J is said to have the codimension n property if J/zJ has (complex) dimension n. It is not difficult to see that all Beurling type and singly generated invariant subspaces have the codimension 1 property. The question poses itself: do the classes of (a) singly generated invariant subspaces, (b) Beurling type invariant subspaces, and (c) invariant subspaces J having the codimension 1 property, all coincide? The term Beurling type has never been properly defined, because the relevant set of measures is not properly understood, so a part of the question is to clarify this point.

A partial answer to the above question is supplied below. The main results of the paper are Theorems A, B, C, and D below.

THEOREM A. *Suppose that f is in $AL^p(\mathbb{D})$. If f belongs to the Nevanlinna class, then the invariant subspace $I(f)$ of $AL^p(\mathbb{D})$ generated by f is of κ-Beurling type. Moreover,*

$$I(f) = \mathscr{I}(Z(f), \sigma_f; AL^p) = \{g \in AL^p(\mathbb{D}) : Z(f) \subset Z(g), \sigma_f \leqslant \sigma_g\}.$$

For an invariant subspace I of $AL^p(\mathbb{D})$ we consider the extremal problem

$$\sup\{\operatorname{Re} f^{(n)}(0) : \|f\|_{AL^p} \leqslant 1, f \in I\},$$

where n is the smallest non-negative integer such that there exists an $f \in I$ with $f^{(n)}(0) \neq 0$. It is shown in **[8]** that for $1 \leqslant p < +\infty$, this extremal problem has a unique solution. We denote the solution by G_I and call it the extremal function of I. It is not known whether the above extremal problem has a unique solution for $0 < p < 1$.

THEOREM B ($1 \leqslant p < +\infty$). *If I is a κ-Beurling type invariant subspace of $AL^p(\mathbb{D})$ induced by a Blaschke sequence and a finite κ-singular measure, then the extremal function G_I of I belongs to the Nevanlinna class and generates the whole invariant subspace I.*

Recall from **[8]** that if G_I is the extremal function of an invariant subspace I of $AL^p(\mathbb{D})$ (with $1 \leqslant p < +\infty$), then $\|f/G_I\|_{AL^p} \leqslant \|f\|_{AL^p}$ for all $f \in I(G_I)$, the invariant subspace generated by G_I. Note that $I(G_I)$ may be smaller than I. Our next theorem strengthens the result of **[8]** in the case of κ-Beurling type invariant subspaces.

THEOREM C ($1 \leqslant p < +\infty$). *Let I be a κ-Beurling type invariant subspace of $AL^p(\mathbb{D})$. If G_I is the extremal function of I, then $\|f/G_I\|_{AL^p} \leqslant \|f\|_{AL^p}$ for all $f \in I$.*

The last theorem tells us how to find the extremal function of an invariant subspace generated by a singular inner function whose measure is supported on a finite set.

THEOREM D ($1 \leqslant p < +\infty$). *Suppose that u is a singular inner function, whose singularity set is a finite subset E of $\mathbb{T}$. Then the extremal function G associated with the invariant subspace in $AL^p(\mathbb{D})$ generated by u has the form*

$$G(z) = (R(z))^{2/p} u(z) \quad \textit{for } z \in \mathbb{D},$$

where R is a rational function with simple poles on E and nowhere else in the extended complex plane, determined uniquely by the conditions that $G(0) > 0$, $G(0)\,G(\infty) = 1$ and

$$(G(z))^{p/2} = R(z)\,(u(z))^{p/2},$$

holomorphically extended to $\mathbb{C}\backslash E$, should have no residue at the points of E (id est, *its antiderivative should be holomorphic and single-valued throughout $\mathbb{C}\backslash E$*).

2. *Preliminaries on κ-singular measures*

The notion of κ-singular measures is essential in the function theory of Bergman spaces. The definition of a κ-singular measure is based on the notion of Carleson sets on $\mathbb{T}$.

DEFINITION 2.1. A set $F \subset \mathbb{T}$ is called a *Carleson set* if it is closed, has Lebesgue measure zero, and satisfies the condition

$$\hat{\kappa}(F) = \sum_n \frac{|J_n|}{2\pi}\left(\log\frac{2\pi}{|J_n|} + 1\right) < +\infty,$$

where $\{J_n\}$ is the collection of complementary arcs of F in $\mathbb{T}$ and $|J_n|$ is the Euclidean length of J_n.

We use $\mathscr{F}$ to denote the collection of all Carleson sets and $\mathscr{B}$ to denote the collection of all Borel sets $B \subset \mathbb{T}$ such that $\bar{B} \in \mathscr{F}$. The collection of all $G = \bigcup_{n=1}^{\infty} F_n$ with $F_n \in \mathscr{F}$, will be denoted by $\mathscr{F}_s$.

The concept of Carleson sets is connected with the study on boundary zero sets of certain function algebras. Recall that the disk algebra, denoted by $\mathscr{A}$, consists of analytic functions f in $\mathbb{D}$ which are continuous up to the boundary. It is well known [**1**, **5**, **12**] that Carleson sets are precisely the boundary zero sets of functions in $\mathscr{A}_n = \{f : f^{(n)} \in \mathscr{A}\}$, where n is any positive integer. Let $A^{+\infty}$ be the space of all analytic functions in $\mathbb{D}$ all of whose derivatives belong to $\mathscr{A}$. We shall need the following slightly stronger version of the above characterization of Carleson sets.

LEMMA 2.2 [**12**]. *For every Carleson set F there exists a function Φ in $A^{+\infty}$ such that $F = \{z \in \bar{\mathbb{D}} : \Phi(z) = 0\}$ and $\Phi^{(n)}(z) = 0$ for all $z \in F$ and $n \geqslant 1$.*

We can now introduce the notion of κ-singular measures.

DEFINITION 2.3. A function $\sigma : \mathscr{B} \to [0, +\infty)$ is called a (*non-negative*) *κ-singular measure* if σ is a finite (positive) Borel measure on every Carleson set and there exists a constant $C > 0$ such that $\sigma(F) \leqslant C\hat{\kappa}(F)$ for every Carleson set F. A κ-singular measure σ is finite if $\sup\{\sigma(F) : F \in \mathscr{F}\} < +\infty$.

It is clear that a κ-singular measure is completely determined by its values of $\mathscr{F}$. We shall need the following approximation theorem concerning κ-singular measures.

THEOREM 2.4 **[14]**. *For every κ-singular measure σ there exists an increasing sequence $\{F_n\}$ of Carleson sets such that*

$$\sigma(B) = \lim_n \sigma(B \cap F_n)$$

for all $B \in \mathscr{B}$.

As a result of the above theorem, every κ-singular measure can be uniquely extended to a non-negative Borel measure $\tilde{\sigma}$ on $\mathbb{T}$ supported on a set $G \in \mathscr{F}_s$ so that $\sigma(B) = \tilde{\sigma}(B)$ for all $B \in \mathscr{B}$ and

$$\tilde{\sigma}(M) = \tilde{\sigma}(M \cap G) = \sup\{\sigma(B) : B \in \mathscr{B}, B \subset M\}$$

for all Borel sets $M \subset \mathbb{T}$. Note that $\tilde{\sigma}(\mathbb{T}) = \tilde{\sigma}(G) = +\infty$ is not excluded. In what follows we shall make no distinction between a κ-singular measure σ and its extension to $\tilde{\sigma}$.

As another consequence of the above approximation theorem, we can conclude that every finite positive Borel measure μ on $\mathbb{T}$ admits a unique decomposition $\mu = \sigma + \mu_0$, where σ is a κ-singular measure and μ_0 is a positive Borel measure on $\mathbb{T}$ with $\mu_0(B) = 0$ for all $B \in \mathscr{B}$. We shall call σ the *κ-singular part* of μ and μ_0 the *κ-smooth part* of μ. A finite positive Borel measure μ on $\mathbb{T}$ is called a *κ-smooth measure* if $\mu(B) = 0$ for all $B \in \mathscr{B}$.

Given a (nonzero) function $f \in A^{-\infty}$ and a Carleson set F let Φ be a function in $A^{+\infty}$ such that $F = \{z \in \bar{\mathbb{D}} : \Phi(z) = 0\}$ and $\Phi^{(n)}(z) = 0$ for all $z \in F$ and $n \geqslant 0$. For a nonnegative Borel measure μ on F set

$$f_{F,\mu}(z) = f(z)\,\Phi(z) \exp\left[\int_F \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right] \quad \text{for } z \in \mathbb{D}.$$

It is shown in **[14]** that the set

$$\mathscr{M}_{F,f} = \{\mu : f_{F,\mu} \in A^{-\infty}\}$$

is independent of the choice of Φ and has a maximal element $\sigma_{F,f}$. It is further shown in **[14]** that there exists a unique κ-singular measure σ_f on $\mathbb{T}$ such that $\sigma_{F,f}$ is the restriction of σ_f to F from every $F \in \mathscr{F}$. This is how a κ-singular measure is associated to a function in $A^{-\infty}$. We note in passing that the function $f_{F,\mu}$ above belongs to $AL^p(\mathbb{D})$ provided that $f \in AL^p(\mathbb{D})$ and $\mu \leqslant \sigma_{F,f}$.

It follows from **[13, 14]** that the κ-singular measures of functions in $A^{-\infty}$ have the property that $\sigma_{fg} = \sigma_f + \sigma_g$ whenever f and g are in $A^{-\infty}$.

LEMMA 2.5 ($0 < p < +\infty$). *Suppose that Z is a zero set for $AL^p(\mathbb{D})$ and σ is a κ-singular measure on $\mathbb{T}$. Then*

$$\mathscr{I}(Z, \sigma; AL^p) = \{f \in AL^p(\mathbb{D}) : Z \subset Z(f), \sigma \leqslant \sigma_f\}$$

is an invariant subspace of $AL^p(\mathbb{D})$. Here $Z(f)$ is the zero set of f and the inclusion $Z \subset Z(f)$ accounts for multiplicities.

Proof. Let

$$J = \{f \in A^{-\infty} : Z \subset Z(f), \sigma \leqslant \sigma_f\}.$$

It is shown in **[14]** that J is a (closed) invariant subspace of $A^{-\infty}$. Since $\mathscr{I}(Z, \sigma; AL^p) = J \cap AL^p(\mathbb{D})$ and $A^{-\infty}$ is topologized as the inductive limit of the Bergman spaces, we conclude that $\mathscr{I}(Z, \sigma; AL^p)$ is a (closed) invariant subspace of $AL^p(\mathbb{D})$.

Note that it may well happen that $\mathscr{I}(Z,\sigma;AL^p)$ is trivial in $AL^p(\mathbb{D})$; Z is called an $AL^p(\mathbb{D})$-zero set if $\mathscr{I}(Z,0;AL^p)$ is nontrivial. Similarly, σ is called an $AL^p(\mathbb{D})$-κ-singular measure if $\mathscr{I}(\emptyset,\sigma;AL^p)$ is nontrivial, where $\emptyset$ is the empty set.

DEFINITION 2.6 ($0<p<+\infty$). An invariant subspace I of $AL^p(\mathbb{D})$ is said to be of *κ-Beurling type* if $I=\mathscr{I}(Z,\sigma;AL^p)$ for some $AL^p(\mathbb{D})$-zero set Z and some κ-singular measure σ on $\mathbb{T}$.

3. *The invariant subspace generated by a Nevanlinna function*

In this section, we investigate the invariant subspace generated by a Nevanlinna function in $AL^p(\mathbb{D})$. First recall that if $f\in AL^p(\mathbb{D})$, then the invariant subspace $I(f)$ generated by f in $AL^p(\mathbb{D})$ is the closure in $AL^p(\mathbb{D})$ of the set $f\mathscr{P}=\{f\phi:\phi\in\mathscr{P}\}$, where $\mathscr{P}$ is the set of all complex polynomials. If $I(f)=AL^p(\mathbb{D})$, then we say that f is cyclic in $AL^p(\mathbb{D})$.

Suppose f is in the Nevanlinna class with canonical factorization $f=B(S_{\mu_1}/S_{\mu_2})F$. Thus B is a Blaschke product, S_{μ_1} and S_{μ_2} are singular inner functions with (relatively prime) singular measures μ_1 and μ_2, and F is an outer function. It follows from **[13]** that if f also belongs to $A^{-\infty}$ then μ_2 is κ-smooth and σ_f is the κ-singular part of μ_1. Recall from **[15]** that S_{μ_2} is cyclic in every $AL^p(\mathbb{D})$. It follows easily that f and $S_{\mu_2}f$ generate the same invariant subspace in $AL^p(\mathbb{D})$, provided that f is in $AL^p(\mathbb{D})$.

LEMMA 3.1 ($0<p<+\infty$). *Suppose that $f=BS$, where B is a Blaschke product with zero set z and S is a classical singular inner function with singular measure μ. If μ is κ-singular, then $I(f)=\mathscr{I}(Z,\mu;AL^p)$; in particular, $I(f)$ is of κ-Beurling type.*

Proof. Since $Z(f)=Z$ and $\sigma_f=\mu$, we have $f\in\mathscr{I}(Z,\mu;AL^p)$. But $\mathscr{I}(Z,\mu;AL^p)$ is an invariant subspace by Lemma 2.5, we conclude that $I(f)\subset\mathscr{I}(Z,\mu;AL^p)$. To prove the other inclusion, we first assume that Z is a finite sequence and μ is carried on a single Carleson set.

Take a function g from $\mathscr{I}(Z,\mu;AL^p)$, where Z is a finite sequence in $\mathbb{D}$ and μ is a κ-singular measure carried on a Carleson set F. Choose a function Φ from $A^{+\infty}$ such that $F=\{z\in\bar{\mathbb{D}}:\Phi(z)=0\}$ and $\Phi^{(n)}(z)=0$ for all $z\in F$ and $n\geqslant 0$. By **[16]**, the function $g_\delta=(g\Phi^\delta)/(BS)$ belongs to $AL^p(\mathbb{D})$ for every $\delta>0$, where B is the Blaschke product with zero set Z and S is the classical singular inner function with singular measure μ. Since $\|g-g\Phi^\delta\|_{AL^p}\to 0$ as $\delta\to 0^+$ and $g\Phi^\delta=BSg_\delta$ belongs to $I(BS)$, we conclude that g is in $I(BS)=I(f)$.

To prove that $\mathscr{I}(Z,\mu;AL^p)\subset I(f)$ in the general case, we assume that $Z=\{z_n\}$ with $|z_1|\leqslant|z_2|\leqslant\ldots\leqslant|z_n|\leqslant\ldots$ and μ is supported on the union of an increasing sequence $\{F_n\}$ of Carleson sets. For every integer $N\geqslant 1$ let $Z_N=\{z_1,\ldots,z_N\}$ and let μ_N be the restriction of μ to F_N. Now if f is a function in $\mathscr{I}(Z,\mu;AL^p)$, then f belongs to each $I(B_N S_N)$ by the special case proved in the preceding paragraph; here B_N is the Blaschke product with zero set ZS_N and S_N is the singular inner function with singular (as well as κ-singular) measure μ_N. This easily implies that each $f\tilde{B}_N\tilde{S}_N$ belongs to $I(BS)$, where $\tilde{B}_N$ is the Blaschke product with zero set $Z-Z_N$ and $\tilde{S}_N$ is the singular inner function with singular measure $\mu-\mu_N$. Letting $N\to+\infty$ and applying the dominated convergence theorem, we conclude that $\|f\tilde{B}_N\tilde{S}_N-f\|_{AL^p}\to 0$. This shows that $f\in I(BS)=I(f)$ and hence the proof of Lemma 3.1 is completed.

We can now prove the main result of this section.

THEOREM 3.2 ($0 < p < +\infty$). *If f is in $AL^p(\mathbb{D})$, and if f is in the Nevanlinna class, then $I(f) = \mathscr{I}(Z, \sigma; AL^p)$, where Z is the zero set of f and $\sigma = \sigma_f$ is the κ-singular part of the singular measure in the canonical Nevanlinna factorization of f.*

Proof. Since $\mathscr{I}(Z, \sigma; AL^p)$ is an invariant subspace of AL^p ($\mathbb{D}$) and f is in $\mathscr{I}(Z, \sigma; AL^p)$, we have $I(f) \subset \mathscr{I}(Z, \sigma; AL^p)$.

To prove the other inclusion we may as well assume that f is in the Smirnov class, that is, f admits the factorization $f = BSH$, where B is the Blaschke product with zero set Z, S is a singular inner function with singular measure μ (whose κ-singular part is σ), and H is an outer function; see the paragraph preceding Lemma 3.1. For every $\varepsilon > 0$, let H_ε be the outer function whose modulus on $\mathbb{T}$ satisfies

$$|H_\varepsilon(\zeta)| = \begin{cases} |H(\zeta)| & \text{if } |H(\zeta)| \geqslant \varepsilon, \\ 1 & \text{if } |H(\zeta)| < \varepsilon. \end{cases}$$

Since $1/H_\varepsilon$ is bounded, we see that the function

$$f_\varepsilon = \frac{f}{H_\varepsilon} = \frac{BSH}{H_\varepsilon}$$

belongs to $I(f)$. The construction of H_ε ensures that $|H(z)| \leqslant |H_\varepsilon(z)|$ on $\mathbb{D}$, so that $\|f_\varepsilon - BS\|_{AL^p} \to 0$ (as $\varepsilon \to 0^+$) by dominated convergence. It follows that BS is in $I(f)$. Write $\mu = \sigma + \mu_0$ with σ being κ-singular and μ_0 being κ-smooth. Accordingly, $S = S_\sigma S_{\mu_0}$, where S_σ is the singular inner function corresponding to σ and S_{μ_0} is the singular inner function corresponding to μ_0. Since S_{μ_0} is cyclic in $AL^p(\mathbb{D})$ by **[15]**, the relation $BS = BS_\sigma S_{\mu_0} \in I(f)$ easily implies that $BS_\sigma \in I(f)$, and hence $I(BS_\sigma) \subset I(f)$. Using Lemma 3.1 we conclude that $\mathscr{I}(Z, \sigma; AL^p) = I(BS_\sigma) \subset I(f)$. This completes the proof of Theorem 3.2.

Recall that for $1 \leqslant p < +\infty$ the extremal function of an invariant subspace I in $AL^p(\mathbb{D})$ is the solution to the extremal problem

$$\sup\{\operatorname{Re} f^{(n)}(0) : \|f\|_{AL^p} \leqslant 1, f \in I\},$$

where n is the smallest nonnegative integer such that there exists a function f in I with $f^{(n)}(0) \neq 0$. We now obtain some properties for the extremal function of an invariant subspace generated by a Nevanlinna function in $AL^p(\mathbb{D})$.

THEOREM 3.3 ($1 \leqslant p < +\infty$). *Suppose that $I = \mathscr{I}(Z, \sigma; AL^p)$ is a κ-Beurling type invariant subspace of $AL^p(\mathbb{D})$ with Z being a Blaschke sequence and $\sigma(\mathbb{T}) < +\infty$. Then the extremal function G of I belongs to the Nevanlinna class with $Z_G = Z$ and $\sigma_G = \sigma$.*

Proof. Let B be the Blaschke product with zero set Z and S be the singular inner function with singular measure σ. If Z is finite and $\sigma = 0$, then by [**7**, **10**] G is the product of the finite Blaschke product B and a function $H(z)$ which is analytic in a neighbourhood of $\bar{\mathbb{D}}$ and satisfies $|H(z)| \geqslant 1$ on $\mathbb{D}$. Theorem 3.3 is obviously true in this case.

Next consider the case where Z is finite and σ is carried on a single Carleson set F. By [**16**, Theorem 3] (slightly modified to include a finite zero set Z) we can construct a sequence $\{Z_n\}$ of finite sets in $\mathbb{D}$ such that $Z \subset Z_n$ (counting multiplicities) for all n and $\|G_n - G\|_{AL^p} \to 0$ as $n \to +\infty$, where G_n is the extremal function of the κ-

Beurling type invariant subspace $\mathcal{I}(Z_n, 0; AL^p)$ (or the invariant subspace generated by the finite Blaschke product B_n whose zero set is Z_n). Since the construction of Z_n in [**16**] ensures that the Blaschke product B_n with zero set Z_n tends to BS as $n \to +\infty$, we obtain the factorization $G = BSH$, where $|H(z)| \geqslant 1$ on $\mathbb{D}$. This clearly implies that G is in the Nevanlinna class with $Z_G = Z$. By Theorem 3.2 and the fact that $1/H$ is bounded in $\mathbb{D}$, we have

$$I = I(BS) = I(G/H) \subset I(G) \subset I.$$

It follows that $I(G) = I$ and hence $\sigma_G = \sigma$ by Theorem 3.2 again.

Now consider the general case. Let $\{Z_n\} = \{z \in Z : |z| \leqslant n/(n+1)\}$ (counting multiplicities) and let $\{\sigma_n\}$ be an increasing sequence of κ-singular measures carried on single Carleson sets such that $\sigma_n \to \sigma$ as $n \to +\infty$. Let $I_n = \mathcal{I}(Z_n, \sigma_n; AL^p)$. It is clear that $I = \bigcap_n I_n$. By the special case discussed in the preceding paragraph, the extremal function G_n of I_n admits the factorization $G_n = B_n S_n H_n$, where B_n is the Blaschke product with zero set Z_n, S_n is the singular inner function with singular measure σ_n, and H_n satisfies $|H_n(z)| \geqslant 1$ on $\mathbb{D}$. Since the G_n are unit vectors, we can use a normal family argument to find a subsequence $\{n_k\}$ such that $G_{n_k} \to BSH$ uniformly on compact sets, where H is an analytic function with modulus greater than or equal to 1 on $\mathbb{D}$. By Fatou's lemma, the function $f = BSH$ belongs to $AL^p(\mathbb{D})$ with $\|f\|_{AL^p} \leqslant 1$. If we can show that $f = G$, then G is in the Nevanlinna class with $Z_G = Z$, and the argument at the end of the preceding paragraph shows that $\sigma_G = \sigma$. Thus it remains to show that $f = G$.

It is easy to see that $G_{n_k} \to f$ in the topology of $A^{-\infty}$. Let

$$J_n = \{g \in A^{-\infty} : Z_n \subset Z(g), \sigma_n \leqslant \sigma_g\}$$

and

$$J = \{g \in A^{-\infty} : Z \subset Z(g), \sigma \leqslant \sigma_g\}.$$

Since $G_{n_k} \in I_{n_k} \subset J_{n_k} \subset J_{n_i}$ for all $k \geqslant i$, letting $k \to +\infty$ leads to $f \in J_{n_i}$ for all i. It follows that $f \in \bigcap_i J_i = J$. But f is in $AL^p(\mathbb{D})$, so that $f \in J \cap AL^p(\mathbb{D}) = I$. On the other hand, if m is the number of times 0 appears in Z, then the inequalities

$$G_1^{(m)}(0) \geqslant G_2^{(m)}(0) \geqslant \ldots \geqslant G^{(m)}(0)$$

imply that $f^{(m)}(0) \geqslant G^{(m)}(0)$. Recall that $\|f\|_{AL^p} \leqslant 1$. It follows from the extremal property of G that $f = G$. This completes the proof of Theorem 3.3.

Corollary 3.4 ($1 \leqslant p < +\infty$). *Suppose that $I = \mathcal{I}(Z, \sigma; AL^p)$ is a κ-Beurling type invariant subspace of $AL^p(\mathbb{D})$ with Z being Blaschke and $\sigma(\mathbb{T}) < +\infty$ (or, equivalently, I is the invariant subspace generated by a Nevanlinna function in $AL^p(\mathbb{D})$). Then I is generated by its extremal function.*

Proof. This follows from Theorems 8 and 9. It is also a consequence of the proof of Theorem 3.3.

Corollary 3.5 ($1 \leqslant p < +\infty$). *Suppose that $I = \mathcal{I}(Z, \sigma; AL^p)$ is a κ-Beurling type invariant subspace of $AL^p(\mathbb{D})$. Then the following conditions are equivalent:*

(1) *Z is Blaschke and $\sigma(\mathbb{T}) < +\infty$;*
(2) *the extremal function G of I belongs to the Nevanlinna class;*
(3) *I contains nonzero elements of the Nevanlinna class.*

Proof. It is trivial that (2) implies (3). That (1) implies (2) follows from Theorem 3.3; (1) follows from (3) by the definition of Beurling type spaces.

Explicit formulas for extremal functions G_I are hard to come by, except in very simple situations, such as for the invariant subspace associated with a multiple zero at a point in $\mathbb{D}$, or a point mass at the boundary $\mathbb{T}$. Recently, however, Jonas Hansbo **[9]** obtained a beautiful formula for the case with two points, inside the disk, or on the boundary.

THEOREM 3.6 (Hansbo) ($1 \leqslant p < +\infty$). *Suppose that the positive Borel measure μ on $\mathbb{T}$ is supported on the two-point set $\{\alpha, \beta\}$, and write $\mu_\alpha = \mu(\{\alpha\})$, $\mu_\beta = \mu(\{\beta\})$. Let J be the Bessel-type function*

$$J(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2} \quad \textit{with } z \in \mathbb{C},$$

and write $Q(z) = J'(z)/J(z)$. Then the extremal function for the κ-Beurling type invariant subspace $\mathscr{I}(\varnothing, \mu; AL^p)$ is given by the formula

$$G_\mu(z) = H_\mu(0)^{-1/2} H_\mu(z) \quad \textit{with } z \in \mathbb{D},$$

where

$$H_\mu(z) = \exp\left(-\mu_\alpha \frac{\alpha+z}{\alpha-z} - \mu_\beta \frac{\beta+z}{\beta-z}\right)$$
$$\times \left\{1 + \frac{p\mu_\alpha}{1-\bar{\alpha}z} + \frac{p\mu_\beta}{1-\bar{\beta}z} + \frac{p^2\mu_\alpha\mu_\beta}{(1-\bar{\alpha}z)(1-\bar{\beta}z)} Q(p^2\mu_\alpha\mu_\beta|\alpha-\beta|^{-2})\right\}^{2/p} \quad \textit{with } z \in \mathbb{D}.$$

Hansbo's result is an explicit special case of the following statement on the structure of extremal functions of invariant subspaces generated by a singular inner function.

THEOREM 3.7 ($1 \leqslant p < +\infty$). *Suppose that u is a singular inner function, whose singularity set is a finite subset E of $\mathbb{T}$. Then the extremal function G associated with the invariant subspace in $AL^p(\mathbb{D})$ generated by u has the form*

$$G(z) = (R(z))^{2/p} u(z) \quad \textit{with } z \in \mathbb{D},$$

where R is a rational function with simple poles on E and nowhere else in the extended complex plane, determined uniquely by the conditions that $G(0) > 0$, $G(0)\,G(\infty) = 1$, and

$$(G(z))^{p/2} = R(z)\,(u(z))^{p/2},$$

holomorphically extended to $\mathbb{C} \backslash E$, should have no residue at the points of E (id est, *its antiderivative should be holomorphic and single-valued throughout $\mathbb{C} \backslash E$*).

Proof. We first show that the extremal function G has the above-mentioned properties. Assume that p is rational; a continuity argument can then be used to cover the general case. Approximate u with the nth power of a Blaschke product $B_{A(n)}$ with a single zero along each ray emanating from the origin with endpoint on E, and no other zeros. Since p is rational, we may assume the integer n is such that $\frac{1}{2}np$ is also an integer.

For a finite sequence Z of points in $\mathbb{D}$, let $B_Z(z)$ denote the Blaschke product with zeros Z, and $G_Z^p(z)$ the canonical zero divisor associated with Z relative to the space $AL^p(\mathbb{D})$. Let us agree to write kZ for the sequence where each point of Z is repeated k times. By [7], and the assumption that $\frac{1}{2}np$ is an integer,

$$G^p_{nA(n)}(z) = (G^{np}_{A(n)}(z))^n = (G^2_{(np/2)A(n)}(z))^{2/p} \quad \text{with } z \in \mathbb{D},$$

and moreover, the representation

$$G_Z^p(z) = (J_Z^p(0,0))^{-1/p} B_Z(z)\,(J_Z^p(z,0))^{2/p} \quad \text{with } z \in \mathbb{D}$$

is valid, where $J_Z^p(z,\zeta)$ is the reproducing kernel function in the L^2 Bergman space with weight $|B_Z|^p$:

$$f(z) = \int_{\mathbb{D}} f(\zeta)\, J_Z^p(z,\zeta)\, |B_Z(\zeta)|^p\, dA(\zeta) \quad \text{with } z \in \mathbb{D},$$

holds for $f \in AL^p(\mathbb{D})$ (Z is finite). By [10], $R_Z(z) = G_Z^2(z)/B_Z(z)$ is a rational function with simple poles at the conjugate points of Z,

$$Z_* = \{1/\bar{z} : z \in Z\},$$

and it has no other poles. This remains equally true if the zeros of Z have multiplicities. Moreover, the function G_Z^2 is the derivative of a rational function H_Z, whose poles are contained in the set Z_*. As n tends to $+\infty$, we have that $B_{nA(n)}(z)$ tends to $u(z)$, and $G^p_{nA(n)}(z)$ tends to the extremal function $G(z)$ [16]. The rational function with simple poles, $R_{(np/2)A(n)}(z)$, must then converge (as $n \to +\infty$) to a rational function $R(z)$, with simple poles on E and nowhere else. The function $G^2_{(np/2)A(n)}(z)$, which possesses a single-valued antiderivative throughout $\mathbb{C}\backslash A(n)_*$, tends to

$$(G(z))^{p/2} = R(z)\,(u(z))^{p/2}$$

as $n \to +\infty$, which thus also has a single-valued antiderivative, this time on $\mathbb{C}\backslash E$. The conditions $G(0) > 0$ and $G(0)\,G(\infty) = 1$ are more or less obvious consequences of similar statements about the zero divisors $G_Z^2(z)$ [10].

It remains to check that the above conditions determine $G(z)$ uniquely. Clearly, it suffices to show that if

$$f(z) = \rho(z)\,u(z) \quad \text{for } z \in \mathbb{C}\backslash E$$

is the derivative of a holomorphic single-valued function F on $\mathbb{C}\backslash E$, and ρ is a rational function with simple poles on E (and nowhere else), subject to $\rho(\infty) = 0$, then $f(z) \equiv 0$. Fix F by requesting that $F(0) = 0$. One checks that

$$f(z) = O((1-|z|^2)^{-1/2}) \quad \text{as } 1 > |z| \longrightarrow 1,$$

so that F is in Lipschitz $\frac{1}{2}$ on the unit disk $\mathbb{D}$, and in particular, F is bounded there. Make an elementary estimate of F in the region $\mathbb{C}\backslash\bar{\mathbb{D}}$, near $\mathbb{T}$ by integrating from a fixed starting point, and near infinity by noticing that $f(\infty) = 0$ forces F to be bounded there. An appeal to the Phragmén–Lindelöf principle then yields

$$|F(z)| \leqslant C(f)|u(z)| \quad \text{for } z \in \mathbb{C}\backslash\bar{\mathbb{D}}. \tag{3.1}$$

We now need the identity ($\mathbb{T}$ is assumed to be positively oriented, and $n = 0, 1, 2, \ldots$)

$$\int_{\mathbb{D}} f(z)\, \bar{z}^n\, \overline{u(z)}\, dA(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{F(z)}{z^{n+2} u(z)}\, dz, \tag{3.2}$$

where we make use of the fact that $\overline{u(z)} = 1/u(z)$ on $\mathbb{T}$, valid because u is an inner function. By (3.1), it is permitted to change the contour of integration in (3.2) to a larger circle $R\mathbb{T}$, and as the radius R tends to $+\infty$, the integrand approaches 0 so rapidly that the right hand side of (3.2) must be 0. It follows that f is perpendicular to $z^n u$ for all $n = 0, 1, 2, \ldots$, and hence to the invariant subspace $I(u)$ generated by u in $AL^2(\mathbb{D})$. But f itself is in $I(u)$, so f vanishes identically as asserted.

REMARK 3.8. For $p = 2$, Theorem 3.7 was known to Hedenmalm in 1991. Since then, related results have appeared, *exempli gratia*, see **[17]**.

4. *Contractive divisibility in κ-Beurling type subspaces*

The recently revived interest in the function theory of Bergman spaces stems mainly from the discovery of the contractive property of the extremal functions for invariant subspaces. Specifically, if G is the extremal function of an invariant subspace I in $AL^p(\mathbb{D})$, then $\|f/G\|_{AL^p} \leqslant \|f\|_{AL^p}$ for all $f \in I(G)$. This result is due to Hedenmalm **[10]** when $p = 2$ and to Duren, Khavinson, Shapiro and Sundberg **[7, 8]** when $1 \leqslant p < +\infty$. Since $I(G)$ is generally smaller than I itself, a natural question arises: does $\|f/G\|_{AL^p} \leqslant \|f\|_{AL^p}$ hold for all $f \in I$? The answer is yes if $I = \mathscr{I}(Z, 0; AL^p)$; see **[10]** for the case $p = 2$ and **[7]** for the more general case $1 \leqslant p < +\infty$. The answer to the question above is also yes if $I = \mathscr{I}(Z, \sigma; AL^p)$ with Z being Blaschke and $\sigma(\mathbb{T}) < +\infty$, because in this case we actually have $I = I(G)$ by Theorem 3.3. The purpose of this section is to give an affirmative answer to the question mentioned above in the case of a general κ-Beurling invariant subspace.

THEOREM 4.1 ($1 \leqslant p < +\infty$). *Suppose that $I = \mathscr{I}(Z, \sigma; AL^p)$ is a κ-Beurling type invariant subspace in $AL^p(\mathbb{D})$. If G_I is the extremal function of I, then $\|f/G_I\|_{AL^p} \leqslant \|f\|_{AL^p}$ for all $f \in I$.*

Proof. Let $Z_n = \{z \in Z : |z| < n/(n+1)\}$ (counting multiplicities). Choose a sequence $\{F_n\}$ of Carleson sets according to Theorem 2.4. Thus if σ_n is the restriction of σ to F_n, then $\sigma_n \to \sigma$. Let $I_n = \mathscr{I}(Z_n, \sigma_n; AL^p)$ and let G_n be the extremal function of I_n. Just as in the proof of Theorem 3.3, we can show that there exists a subsequence $\{G_{n_k}\}$ which converges to G uniformly on compact sets (and hence in norm, because we also have $\|G_{n_k}\|_{AL^p} \to \|G\|_{AL^p}$). Now if f is a function in I, then f belongs to every I_n, so that $\|f/G_n\|_{AL^p} \leqslant \|f\|_{AL^p}$ for all n. Let $n \to +\infty$ and apply Fatou's lemma again, we conclude that $\|f/G\|_{AL^p} \leqslant \|f\|_{AL^p}$, completing the proof of Theorem 4.1.

5. *Some applications*

In this section we apply our main results to obtain an operator theoretic result and a canonical factorization for Nevanlinna functions in $AL^p(\mathbb{D})$.

Let I_1 and I_2 be two invariant subspaces of $AL^p(\mathbb{D})$. We say that I_1 and I_2 are similar if there exists an invertible bounded linear operator $T: I_1 \to I_2$ such that $M_z T = TM_z$ on I_1, where M_z is the operator of multiplication by z. We say that I_1 and I_2 are quasi-similar if there exist bounded linear operators $T: I_1 \to I_2$ and $S: I_2 \to I_1$ such that T and S are one-to-one, have dense range, and satisfy $TM_z = M_z T$ on I_1 and $SM_z = M_z S$ on I_2. It is clear that I_1 and I_2 are similar (or quasi-similar) if and only if the operators $M_z: I_1 \to I_1$ and $M_z: I_2 \to I_2$ are similar (or quasi-similar). See **[6]** for the notions of similarity and quasi-similarity of linear operators on Banach spaces. It is shown in **[3]** that an invariant subspace I of $AL^p(\mathbb{D})$ is similar to the whole space

$AL^p(\mathbb{D})$ if and only if I is generated by a Blaschke product whose zero set is the union of finitely many interpolating sequences. Equivalently, I is similar to $AL^p(\mathbb{D})$ if and only if $I = \mathscr{I}(Z, 0; AL^p)$, where Z is the union of finitely many interpolating sequences in $\mathbb{D}$. We now characterize invariant subspaces of $AL^p(\mathbb{D})$ which are quasi-similar to the whole space $AL^p(\mathbb{D})$.

THEOREM 5.1. *Let I be an invariant subspace of $AL^p(\mathbb{D})$. Then the following conditions are equivalent:*

(1) *I is quasi-similar to $AL^p(\mathbb{D})$;*
(2) *I is generated by a bounded function;*
(3) *I is generated by a classical inner function;*
(4) *I is generated by a Nevanlinna function in $AL^p(\mathbb{D})$;*
(5) *$I = \mathscr{I}(Z, \sigma; AL^p)$ with Z being a Blaschke sequence and $\sigma(\mathbb{T}) < +\infty$.*

Proof. The equivalences of (2) through (5) follow from Theorems 8 and 9 and their proofs. We show that (1) and (2) are equivalent.

First assume that I is quasi-similar to $AL^p(\mathbb{D})$, so that there exist bounded linear operators $T: I \to AL^p(\mathbb{D})$ and $S: AL^p(\mathbb{D}) \to I$ such that S and T are one-to-one, have dense range, and satisfy $M_z S = SM_z$ on $AL^p(\mathbb{D})$ and $M_z T = TM_z$ on I. Let $f = S1$, where 1 is the constant function with value 1. Since $M_z S = SM_z$, it follows that $pf = Sp$ for every polynomial p. The boundedness of S easily implies that f is bounded, which in turn implies that $Sg = gf$ for all $g \in AL^p(\mathbb{D})$. Since S has dense range, we conclude that I is generated by f, and hence (1) implies (2).

Next assume that I is generated by a bounded function f. Let G be the extremal function of I. Then by Theorem 3.3 we have $I = I(G)$. Define $T: I \to AL^p(\mathbb{D})$ and $S: AL^p(\mathbb{D}) \to I$ as follows: $Tg = g/G$ for $g \in I$, and $Sg = fg$ for $g \in AL^p(\mathbb{D})$. Then S and T are both bounded, one-to-one, have dense range, and commute with M_z, so that I is quasi-similar to $AL^p(\mathbb{D})$, completing the proof of Theorem 5.1.

Next we present a factorization theorem for Nevanlinna functions in $AL^p(\mathbb{D})$. The factorization is similar to the classical inner-outer factorization of functions in Hardy spaces. Recall that a function f in $AL^p(\mathbb{D})$ is called a cyclic vector if $I(f) = AL^p(\mathbb{D})$. A unit vector G in $AL^p(\mathbb{D})$ will be called an *$AL^p(\mathbb{D})$-inner function* if it satisfies $\|Gf\|_{AL^p} \geqslant \|f\|_{AL^p}$ for all bounded analytic functions f. It is easy to show that a unit vector in $AL^p(\mathbb{D})$ is $AL^p(\mathbb{D})$-inner if and only if it is the extremal function of an invariant subspace. A unit vector in $AL^p(\mathbb{D})$ will be called *$AL^p(\mathbb{D})$-singular-inner* if it is nonvanishing on $\mathbb{D}$ and if it is $AL^p(\mathbb{D})$-inner. A unit vector in $AL^p(\mathbb{D})$ will be called a *contractive zero divisor* if it is the extremal function of some $\mathscr{I}(Z, 0; AL^p)$, where Z is a zero set for $AL^p(\mathbb{D})$.

THEOREM 5.2. *Suppose that $1 \leqslant p < +\infty$ and $f \in AL^p(\mathbb{D})$. If f belongs to the Nevanlinna class, then f admits a unique factorization: $f = GSH$, where G is a contractive zero divisor, S is an $AL^p(\mathbb{D})$-singular-inner function, and H is a cyclic vector in $AL^p(\mathbb{D})$ belonging to the Nevanlinna class (not all factors have to be present).*

Proof. Let Z be the zero set of f and let G be the contractive zero divisor corresponding to Z. Let $g = f/G$. Then g is nonvanishing on $\mathbb{D}$ and $\|g\|_{AL^p} \leqslant \|f\|_{AL^p}$; see [7]. By Theorem 3.3, the function G is in the Nevanlinna class, so that g is also in the Nevanlinna class. Let S be the extremal function of the invariant subspace generated by g. Then S is $AL^p(\mathbb{D})$-singular-inner and by Theorem 4.1 $H = g/S$

belongs to $AL^p(\mathbb{D})$ with $\|H\|_{AL^p} \leqslant \|g\|_{AL^p} \leqslant \|f\|_{AL^p}$. Since H is Nevanlinna, $Z_H = \varnothing$, and $\sigma_H = 0$ (because $\sigma_g = \sigma_S$), Theorem 3.2 implies that H is cyclic in $AL^p(\mathbb{D})$. Thus we have established the factorization $f = GSH$.

To prove the factorization $f = GSH$ is unique, assume that f also admits a factorization $f = G_1 S_1 H_1$, where G_1 is a contractive zero divisor, S_1 is $AL^p(\mathbb{D})$-singular-inner, and H_1 is cyclic in $AL^p(\mathbb{D})$ belonging to the Nevanlinna class. It is obvious that $G = G_1$, since they are both determined by the zero set of f. Let $g = SH = S_1 H_1$. Then g is in $AL^p(\mathbb{D})$ and g is in the Nevanlinna class. Since H_1 is in the Nevanlinna class, so must be S_1. It follows that $I(g) = I(S) = I(S_1)$, because the cyclicity of H_1 implies that its κ-singular measure is zero. This easily implies that $S = S_1$ and hence $H = H_1$.

6. *Remarks and questions*

Recall that an invariant subspace I of $AL^p(\mathbb{D})$ is said to have the codimension 1 property if $\dim(I/zI) = 1$. In the case of Hardy spaces, every invariant subspace (except $\{0\}$) has the codimension 1 property. On the other hand, each $AL^p(\mathbb{D})$ has invariant subspaces which do not. Every κ-Beurling type invariant subspace I of $AL^p(\mathbb{D})$ has the codimension 1 property (and so does every Beurling type invariant subspace, if defined properly). This is so because multiplying or dividing a function in I by z does not change its κ-singular measure **[14]**, and the zero set is distorted in a predictable way.

Every singly generated invariant subspace of $AL^p(\mathbb{D})$ has the codimension 1 property (**[18]**, p. 596). The argument runs as follows, in case the generator $g \in AL^p(\mathbb{D})$ has $g(0) \neq 0$ (this restriction is really inessential). It suffices to show that if $f \in I(g)$ has $f(0) = 0$, then $f/z \in I(g)$. Choose a sequence $\{p_n\}$ of polynomials such that $\|p_n g - f\|_{AL^p} \to 0$ as $n \to +\infty$. Since $f(0) = 0$ and $g(0) \neq 0$, we can adjust p_n so that $p_n(0) = 0$. Write $f = zh$ and $p_n = zq_n$. It follows that $\|q_n g - h\|_{AL^p} \to 0$, so that $f/z = h \in I(g)$.

Our first question here was raised in the introduction; namely, is there a relationship between the class of singly generated invariant subspaces and that of Beurling type invariant subspaces of $AL^p(\mathbb{D})$? We suspect that the two classes coincide. We do not, however, believe that all singly generated invariant subspaces are of κ-Beurling type, because for saturated zero sets Z larger set $\mathfrak{P}(Z; AL^p)$ of singular measures can be expected to play a role.

Even the following special case remains an open problem: let Z be an $AL^p(\mathbb{D})$ zero set, so large that it is not Blaschke. Is the invariant subspace $\mathcal{I}(Z, 0; AL^p)$ then singly generated? Or, more boldly, is $\mathcal{I}(Z, 0; AL^p)$ generated by its extremal function G_Z?

Our second question involves cyclic vectors of $AL^p(\mathbb{D})$. It is easy to show that if f is cyclic in $AL^p(\mathbb{D})$, then $Z(f) = \varnothing$ and $\sigma_f = 0$. The converse is also known to be true in several special cases. By Theorem A of this paper, if f is in the Nevanlinna class, and if $Z(f) = \varnothing$ and $\sigma_f = 0$, then f is cyclic in $AL^p(\mathbb{D})$. It was proved in **[4]** that if $f \in AL^{p+\varepsilon}(\mathbb{D})$ for some $\varepsilon > 0$, and if $Z(f) = \varnothing$ and $\sigma_f = 0$, then f is cyclic in $AL^p(\mathbb{D})$. The following conjecture seems natural: f is cyclic in $AL^p(\mathbb{D})$ if and only if $Z(f) = \varnothing$ and $\sigma_f = 0$. Still, there may be a phenomenon of the following type, which could prevent the conjecture from being true: if a function f in $AL^p(\mathbb{D})$ with $Z(f) = \varnothing$ has critical growth near a large set $E \subset \mathbb{T}$, then cyclicity of f might require more than just $\sigma_f = 0$; on E, one may need to consider a larger class of measures than the κ-singular ones. At the time of writing this manuscript, Borichev and Hedenmalm have a construction which seems to produce a counterexample to the above conjecture.

Finally, we ask a question about contractive divisibility. Recall that if G_I is the extremal function of a κ-Beurling type invariant subspace I in $AL^p(\mathbb{D})$, then $\|f/G_I\|_{AL^p} \leqslant \|f\|_{AL^p}$ for all $f \in I$. We wonder if the same result holds for singly generated invariant subspaces. An affirmative answer will imply the following canonical factorization for functions in the Bergman spaces: every function f in $AL^p(\mathbb{D})$ admits a factorization $f = GSH$, where G is a contractive zero divisor, S is an $AL^p(\mathbb{D})$-singular-inner function, and H is a cyclic vector in $AL^p(\mathbb{D})$. Since this question is directly related to the first question, the main point here is that even though there (probably) are singly generated invariant subspaces which are not of κ-Beurling type, the contractive divisibility property may still hold for such spaces.

References

1. A. Beurling, 'Ensembles exceptionnels', *Acta Math.* 72 (1940) 1–13.
2. A. Beurling, 'On two problems concerning linear transformations in Hilbert space', *Acta Math.* 81 (1949) 239–255.
3. P. Bourdon, 'Similarity of parts to the whole for certain multiplication operators', *Proc. Amer. Math. Soc.* 99 (1987) 563–567.
4. L. Brown and B. Korenblum, 'Cyclic vectors in $A^{-\infty}$', *Proc. Amer. Math. Soc.* 102 (1988) 137–138.
5. L. Carleson, 'Sets of uniqueness for functions regular in the unit circle', *Acta Math.* 87 (1952) 325–345.
6. J. Conway, *Subnormal operators*, Pitman Research Notes in Mathematics Series 51 (Longman Scientific and Technical, Harlow, 1981).
7. P. Duren, D. Khavinson, H. Shapiro and C. Sundberg, 'Contractive zero divisors in Bergman spaces', *Pacific J. Math.* 157 (1993) 37–56.
8. P. Duren, D. Khavinson, H. Shapiro and C. Sundberg, 'Invariant subspaces in Bergman spaces and the biharmonic equation', *Michigan Math. J.* 41 (1994) 247–257.
9. J. Hansbo, 'Reproducing kernels and contractive divisors in Bergman spaces', preprint, Uppsala, 1993.
10. H. Hedenmalm, 'A factorization theorem for square area-integrable analytic functions', *J. Reine Angew. Math.* 422 (1991) 45–68.
11. H. Hedenmalm, 'An invariant subspace of the Bergman space having the codimension two property', *J. Reine Angew. Math.* 443 (1993), 1–9.
12. B. Korenblum, 'On functions holomorphic in the disk and smooth up to its boundary', *Dokl. Akad. Nauk SSSR* 200 (1971) 24–27 (in Russian).
13. B. Korenblum, 'An extension of the Nevanlinna theory', *Acta Math.* 135 (1975) 187–219.
14. B. Korenblum, 'A Beurling-type theorem', *Acta Math.* 138 (1977) 265–293.
15. B. Korenblum, 'Cyclic elements in some spaces of analytic functions', *Bull. Amer. Math. Soc.* 5 (1981) 317–318.
16. B. Korenblum, 'Outer functions and cyclic elements in Bergman spaces', *J. Funct. Anal.* 115 (1993) 104–118.
17. T. H. McGregor and M. I. Stessin, 'Weighted reproducing kernels in Bergman spaces', preprint, Albany, 1993.
18. S. Richter, 'Invariant subspaces in Banach spaces of analytic functions,' *Trans. Amer. Math. Soc.* 304 (1987) 585–616.
19. K. Seip, 'Regular sets of sampling and interpolation for weighted Bergman spaces', *Proc. Amer. Math. Soc.* 117 (1993) 213–220.
20. K. Zhu, *Operator theory in function spaces* (Marcell Dekker, New York, 1990).

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