BOUNDED ANALYTIC FUNCTIONS AND CLOSED IDEALS

By

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0. Introduction

The first complete description of the closed ideals in a Banach algebra of analytic functions was obtained for the disc algebra $A(\mathbf{D})$ independently by Arne Beurling (unpublished) and Walter Rudin [RUD 1]; $\mathbf{D}$ is the open unit disc. Later, Boris Korenblum [KOR] described the closed ideals of the algebra

$$A_n(\mathbf{D}) = \{f \in A(\mathbf{D}): f^{(n)} \in A(\mathbf{D})\},$$

using a classical duality technique going back to ideas of Beurling and Carleman, which is usually referred to as the Carleman transform. The Carleman transform method has been applied successfully to other function algebras in [FEL], [GUR], and [BEG]. However, it has never been applied to a nonseparable algebra like $H^\infty = H^\infty(\mathbf{D})$. In Section 2, we use it in the spirit of [DOM] to show that every closed ideal in H^∞ whose hull is contained in the fiber $\mathcal{M}_1(H^\infty)$ has the form $k \cdot J$, where k is the singular inner function

$$k(z) = \exp(\alpha(z+1)/(z-1)), \qquad z \in \mathbf{D}, \quad \text{for some } \alpha \geqq 0,$$

and J is a closed H^∞-ideal containing all functions whose Gelfand transforms vanish on $\mathcal{M}_1(H^\infty)$. We also show that for analytic points in $\mathcal{M}(H^\infty)$, that is, points with nontrivial Gleason parts, the primary ideals considered by Kenneth Hoffman [HOF, pp. 100–101] are the only ones there are. A primary ideal is a closed ideal that is contained in only one maximal ideal.

The Carleman transform technique can be used to reveal interesting properties of the structure of closed ideals in other nonseparable algebras, too. We outline what can be shown for the algebra $H_1^\infty = \{f \in H^\infty: f' \in H^\infty\}$. The maximal ideal space of H_1^∞ can be identified with $\bar{\mathbf{D}}$ in the obvious way. Every primary ideal H_1^∞-ideal I at 1 with $\alpha(I) = 0$ (see Section 2) has the form

$$I = \{f \in H_1^\infty: f(1) = 0 \text{ and } f' \in K\},$$

where k is the singular inner function $k(z) = \exp(\alpha(z+1)/(z-1))$, $z \in \mathbf{D}$, for some $\alpha \geqq 0$, and K is a closed subspace of H^∞ that contains all functions whose Gelfand transforms vanish on $\mathcal{M}_1(H^\infty)$.

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Section 3 is devoted to the problem of comparing the structure of closed ideals in a Banach algebra B of analytic functions on some planar domain (= nonempty open connected set) W with that of a subalgebra A which is characterized by the condition that its elements are holomorphically extendable onto some of the components of $\mathbf{C}\setminus W$. Our main tool will be the fact that we can use the holomorphic functional calculus to produce continuous epimorphisms $B\to A/I$ that are canonical on A for a large class of closed A-ideals I. The technique has some slight resemblance to Stout's [STO 1] method of proving the corona theorem for finitely connected planar domains.

Charles Stanton [STA] and others have generalized the Beurling–Rudin theorem to finitely connected domains. The point here is that we can relate the structures of closed ideals for very large classes of algebras of analytic functions.

Michael Behrens [BEH] was the first to discover a class of infinitely connected planar domains for which the corona theorem is true. He also obtained a description of the Gleason parts. For a fairly restricted class of infinitely connected domains, we will try to investigate to what extent our epimorphism idea can be developed along Behrens' lines. Section 4 is devoted to this topic.

1. Notation and basic concepts

In our terminology, all algebras are complex and commutative. Recall that a uniform algebra is a Banach algebra with a norm that is equivalent to the spectral norm, or, which is the same, the supremum norm of the Gelfand transform. The bilinear form linking any Banach space A with its dual Banach space A^* will always be denoted by $\langle\cdot,\cdot\rangle$.

For any Banach algebra B, we write $\mathcal{M}(B)$ for the Gelfand (or carrier) space of B, equipped with the Gelfand topology. If B has a unit, this is its maximal ideal space. For any element $x\in B$, $\sigma(x,B)$ is its spectrum. The hull of a B-ideal I is the set

$$h(I,B)=\{m\in\mathcal{M}(B)\colon \hat{x}(m)=0 \text{ for all } x\in I\},$$

which is a closed subset of $\mathcal{M}(B)$. It is well known that if I is closed, one can identify $h(I,B)$ and $\mathcal{M}(B/I)$ (see [STO 2], p. 27). It is easy to see that if I is closed and B has a unit, $\hat{x}(h(I,B))=\sigma(x+I,B/I)$ for every $x\in B$. A subalgebra A of B is said to be a *Banach subalgebra* if it is equipped with a norm stronger than that of B and which makes A a Banach algebra. By the closed graph theorem, a subalgebra can have (within equivalence) at most one Banach subalgebra norm.

For an open set $\Omega\subset\mathbf{C}$, $\mathcal{O}(\Omega)$ denotes the Fréchet algebra of all holomorphic functions on Ω, and if K is a compact subset of $\mathbf{C}$, $\mathcal{O}(K)$ denotes the algebra of germs of functions analytic in neighborhoods of K, endowed with the natural inductive limit topology. $H^\infty(\Omega)$ is the Banach algebra of analytic functions on Ω, and if Ω is bounded, $A(\Omega)$ is the Banach algebra of analytic functions on Ω that extend continuously to $\bar{\Omega}$. If Ω is unbounded, $A(\Omega)$ consists of those analytic functions that extend continuously to $\bar{\Omega} \cup \{\infty\}$.

We denote by z the coordinate function $z(\zeta) = \zeta$, $\zeta \in \mathbf{C}$. It will be obvious from the context when we use z as a function or as a complex variable. We will now describe more closely the class of Banach algebras that we will deal with. Until Remark 1.2, W will be an arbitrary bounded domain in the complex plane $\mathbf{C}$.

Definition 1.1. B is an admissible algebra on W if:

(a) B is a Banach subalgebra of $H^\infty(W)$ containing $\mathcal{O}(\bar{W})$, and

(b) for each $\zeta \in W$, the point evaluation at ζ is the only element of the set $\mathcal{M}_\zeta(B) = \{m \in \mathcal{M}(B)\colon \hat{z}(m) = \zeta\}$.

For each $f \in \mathcal{O}(W)$ and $\alpha \in W$, let

$$T_\alpha f(\zeta) = (f(\zeta) - f(\alpha))/(\zeta - \alpha), \qquad \zeta \in W \setminus \{\alpha\}.$$

Thomas Wolff [WOL] calls B *stable* if $T_\alpha f \in B$ for all $\alpha \in W$ and $f \in B$. The relation

$$f = f(\alpha) + (z - \alpha) T_\alpha f, \qquad f \in B, \quad \alpha \in W,$$

shows that if B is a stable Banach subalgebra of $H^\infty(W)$ containing $\mathcal{O}(\bar{W})$, B is admissible on W.

Let B be an admissible algebra on W. Then $\hat{z}^{-1}(W)$ is an open subset of $\mathcal{M}(B)$, and $\hat{z}$ is a bijective continuous mapping $\hat{z}^{-1}(W) \to W$. It follows from the definition of the Gelfand topology that $\hat{z}$ is a homeomorphism $\hat{z}^{-1}(W) \to W$, making it justified to identify $\hat{z}^{-1}(W)$ with W. For every $\alpha \in \bar{W}$, let

$$\mathcal{M}_\alpha(B) = \{m \in \mathcal{M}(B)\colon \hat{z}(m) = \alpha\};$$

this gives a natural fibering of $\mathcal{M}(B)$ over $\bar{W}$. $A_\alpha(B)$ is the restriction of $\hat{B}$ to $\mathcal{M}_\alpha(B)$.

For ease of notation, we will write $Z(I, B)$ instead of $\hat{z}(h(I, B))$ for every B-ideal I; of course this can be done for nonadmissible algebras, too. One easily realizes that

$$Z(I, B) \cap W = \{z \in W\colon f(z) = 0 \text{ for all } f \in I\};$$

here condition (b) of Definition 1.1 is important.

Remark 1.2. The concept of admissibility can easily be extended to include all planar domains W such that $\bar{W} \neq \mathbf{C}$. We shall see how this is done. Suppose W is unbounded. Let $a \in \mathbf{C} \setminus \bar{W}$, and denote by φ the function $\zeta \mapsto (\zeta - a)^{-1}$, $\zeta \in \bar{W} \cup \{\infty\}$, extended to have the value 0 at ∞. Define B to be an admissible algebra on W if $B \circ \varphi^{-1}$ is an admissible algebra on the bounded domain $\varphi(W)$.

For an unbounded domain W, let $\mathcal{O}_0(W)$ denote the subalgebra of $\mathcal{O}(W)$ consisting of those functions $f(z)$ which tend to 0 as $|z| \to \infty$ within W. $H_0^\infty(W)$ is the algebra $H^\infty(W)\cap \mathcal{O}_0(W)$, which is an ideal in $H^\infty(W)$, and $A_0(W)$ is the algebra $A(W)\cap \mathcal{O}_0(W)$.

2. Closed ideals in $H^\infty(D)$

For any family $\mathcal{F}$ of functions in $H^\infty = H^\infty(\mathbf{D})$, put

$$\alpha(\mathcal{F}) = \inf_{f\in\mathcal{F}} \left(-\limsup_{t\to 1-} (1-t)\log|f(t)| \right),$$

which is a nonnegative real number. Also, recall the notation

$$T_\zeta f(z) = (f(z)-f(\zeta))/(z-\zeta), \qquad z,\zeta\in\mathbf{D}, \quad z\neq\zeta.$$

Proposition 2.1. *Let I be a closed ideal in H^∞ such that $h(I,H^\infty)\subset \mathcal{M}_1(H^\infty)$ and $\alpha(I)=0$. Then I contains the ideal*

$$I_0 = \{f\in H^\infty\colon \hat{f}(m)=0 \text{ for all } m\in\mathcal{M}_1(H^\infty)\}.$$

Proof. For $\zeta\in\mathbf{C}\backslash\bar{\mathbf{D}}$, $(z-\zeta+I)^{-1} = (z-\zeta)^{-1}+I$, and consequently

$$\|(z-\zeta+I)^{-1}\| \leqq \|(z-\zeta)^{-1}\| = \sup_{z\in\mathbf{D}} |z-\zeta|^{-1} = (|\zeta|-1)^{-1}.$$

Pick an arbitrary functional $\phi\in I^\perp = (H^\infty/I)^*$. Then the function

$$\Phi(\zeta) = \langle (z-\zeta+I)^{-1},\phi\rangle, \qquad \zeta\in\mathbf{C}\backslash\hat{z}(h(I,H^\infty))\supset \mathbf{C}\backslash\{1\},$$

is holomorphic on its domain of definition, and

$$(2.1) \qquad |\Phi(\zeta)| \leqq \|\phi\|\cdot(|\zeta|-1)^{-1}, \qquad \zeta\in\mathbf{C}\backslash\bar{\mathbf{D}}.$$

Select an arbitrary function $f\in I$ which does not vanish identically. The function $T_\zeta f(z) = (f(z)-f(\zeta))/(z-\zeta)$ belongs to H^∞ for $\zeta\in\mathbf{D}$, and one easily checks that

$$-(z-\zeta)T_\zeta f/f(\zeta) - 1\in I$$

for $\zeta\in\mathbf{D}\backslash Z(f)$, where $Z(f)$ is the set $\{z\in\mathbf{D}\colon f(z)=0\}$. Hence

$$\Phi(\zeta) = -\langle T_\zeta f,\phi\rangle/f(\zeta), \qquad \zeta\in\mathbf{D}\backslash Z(f).$$

By the maximum modulus principle,

$$(2.2) \qquad \|T_\zeta f\| \leqq 2\|f\|(1-|\zeta|)^{-1}, \qquad \zeta\in\mathbf{D}.$$

By (2.1–2) and our assumption $\alpha(I)=0$, Theorem 3.2 [BEG] shows that there is an integer N such that

$$\Phi(\zeta) = O(|\zeta-1|^{-N}) \qquad \text{as } \zeta\to 1.$$

Hence Φ has a pole at 1, and by the estimate (2.1), it must be a simple pole. Observing that $\Phi(t)\to 0$ as $|\zeta|\to\infty$, we conclude that

$$\Phi(\zeta)=\lambda(1-\zeta)^{-1}, \qquad \zeta\in\mathbf{C}\setminus\{1\},$$

for some complex constant λ, $|\lambda|\leqq\|\phi\|$. This has the interpretation that ϕ acts as λ times the point evaluation at 1 on the functions $(z-\zeta)^{-1}$ for $\zeta\in\mathbf{C}\setminus\mathbf{D}$. Since finite linear combinations of these functions form a dense subspace of $A(\mathbf{D})$ and ϕ is continuous, ϕ acts as λ times the point evaluation on all of $A(\mathbf{D})$. In particular, $\langle z-1,\phi\rangle=0$, and hence $z-1\in I$, because $\phi\in I^{\perp}$ was arbitrary. It is well known and not too hard to see that the closure of the H^{∞}-ideal generated by $(z-1)$ equals I_0, and therefore $I\supset I_0$. The proof is complete.

Keep the notation

$$I_0=\{f\in H^{\infty}\colon \hat{f}(m)=0 \text{ for all } m\in\mathcal{M}_1(H^{\infty})\}.$$

Since I_0 is an intersection of maximal ideals, the quotient algebra H^{∞}/I_0 is canonically isomorphic to its image under the Gelfand transform, which is the fiber algebra $A_1(H^{\infty})=H^{\infty}|_{\mathcal{M}_1(H^{\infty})}$. According to Gamelin [GAM 2], $A_1(H^{\infty})$ is a uniform algebra with maximal ideal space $\mathcal{M}_1(H^{\infty})$. There is an obvious bijective correspondence between the $A_1(H^{\infty})$-ideals and those H^{∞}-ideals that contain I_0; certainly, closed ideals correspond to closed ideals. Let I be a closed ideal in H^{∞} such that $h(I,H^{\infty})\subset\mathcal{M}_1(H^{\infty})$. If k is the singular inner function

$$k(z)=\exp\left(-\frac{\alpha(I)}{2}\frac{1+z}{1-z}\right), \qquad z\in\mathbf{D},$$

the closed ideal

$$J=\{f\in H^{\infty}\colon kf\in I\}$$

satisfies $h(J,H^{\infty})\subset\mathcal{M}_1(H^{\infty})$ and $\alpha(J)=0$. Hence J contains I_0 by Proposition 2.1, and by the previous discussion, J corresponds uniquely to a closed $A_1(H^{\infty})$-ideal, namely the one formed by taking the restriction of J to $\mathcal{M}_1(H^{\infty})$. Since $I=k\cdot J$, we find that k takes care of all "outside influence", and that J only depends on its behavior on the fiber $\mathcal{M}_1(H^{\infty})$. We collect our observations in a theorem.

Theorem 2.2. *Let I be a closed H^{∞}-ideal such that $h(I,H^{\infty})\subset\mathcal{M}_1(H^{\infty})$. Then $I=k\cdot J$, where k is the singular inner function*

$$k(z)=\exp\left(-\frac{\alpha(I)}{2}\frac{1+z}{1-z}\right), \qquad z\in\mathbf{D},$$

and J is a closed H^{∞}-ideal which contains

$$\{f\in H^{\infty}\colon \hat{f}(m)=0 \text{ and all } m\in\mathcal{M}_1(H^{\infty})\}.$$

As a consequence, we have the following

Proposition 2.3. *Let I be a closed H^{∞}-ideal such that $h(I,H^{\infty})\subset\mathcal{M}_1(H^{\infty})$.*

Then

$$\|(z-\zeta+I)^{-1}\| \leqq 4(1-|\zeta|)^{-1} e^{\alpha(I)|1-\zeta|^{-1}}, \qquad \zeta \in \mathbf{D}.$$

Proof. By Theorem 2.2, $(z^n - 1)\cdot k \in I$ for every $n \in \mathbf{N}$. Once easily checks that the function

$$-T_\zeta((z^n-1)k)/((\zeta^n-1)k(\zeta))$$

belongs to the coset $(z-\zeta+I)^{-1}$ for $\zeta \in \mathbf{D}$. Observe that $\|z^n - 1\| = 2$. By the maximum modulus principle,

$$\begin{aligned}\|T_\zeta((z^n-1)k)\| &\leqq 2\|(z^n-1)k\|\cdot(1-|\zeta|)^{-1}\\ &\leqq 4(1-|\zeta|)^{-1}, \qquad \zeta \in \mathbf{D},\end{aligned}$$

and consequently

$$\begin{aligned}\|(z-\zeta+I)^{-1}\| &\leqq \|T_\zeta((z^n-1)k)/((\zeta^n-1)k(\zeta))\|\\ &\leqq 4(1-|\zeta|)^{-1}\cdot|1-\zeta^n|^{-1}\cdot|k(\zeta)|^{-1}\\ &\leqq 4(1-|\zeta|)^{-1}\cdot|1-\zeta^n|^{-1}e^{\alpha(I)|1-\zeta|^{-1}}\\ &\to 4(1-|\zeta|)^{-1}e^{\alpha(I)|1-\zeta|^{-1}}\end{aligned}$$

as $n \to \infty$ for $\zeta \in \mathbf{D}$, which is the desired estimate.

We wish to state an open problem. An ideal I in H^∞ is called *primary at* $m \in \mathcal{M}(H^\infty)$ if it is closed and $h(I, H^\infty) = \{m\}$. If m is not specified, we simply say primary.

Problem 2.4. Determine the primary ideals of H^∞.

John Garnett told me that this problem has been raised by Kenneth Hoffman. For $m \in \mathbf{D}$, the situation is well known. It m belongs to a nontrivial Gleason part, that is, by Hoffman's theorem [HOF], [GAR, pp. 410–415], an analytic disc, the primary ideals are easily computed, as we shall see. Let $L: \mathbf{D} \to \mathcal{M}(H^\infty)$ be an analytic disc mapping, and suppose $m = L(z_0)$, $z_0 \in \mathbf{D}$. Without loss of generality, we can take z_0 to be 0. Hoffman [HOF, pp. 100–101] considered the primary ideals

$$I_n(m) = \{f \in H^\infty : f \circ L \in z^n H^\infty\}, \qquad n = 1, 2, \ldots,$$

and showed that in an appropriate sense, $I_n(m)$ is the nth power of the maximal ideal, $\ker m$. In Theorem 2.5 below, we will prove that each primary ideal at m coincides with one of $\{I_n(m)\}_{n=1}^\infty$. So the remaining case is when m belongs to a one-point Gleason part. It is sufficiently challenging to attack this problem when m lies in the Šilov boundary of H^∞, which one usually identifies with the maximal ideal space of $L^\infty(\partial\mathbf{D})$.

Recall the definition of the primary ideals $I_n(m)$.

Theorem 2.5. *If $m \in \mathcal{M}(H^\infty)$ belongs to a nontrivial Gleason part, every primary ideal at m has the form $I_n(m)$ for some positive integer n.*

Proof. Let I be an arbitrary primary ideal at m, and denote by n the largest positive integer such that $I \subset I_n(m)$. We intend to show that $I = I_n(m)$.

Since $I \not\subset I_{n+1}(m)$, we can find an $f \in I$ such that $(f \circ L)^{(n)}(0) \neq 0$. The H^p factorization theorem shows $f = bg$, where b is a Blaschke product and $g \in H^\infty$ has no zeros in $\mathbf{D}$. By Lemma 2.2 [HOF], we cannot have $\hat{g}(m) = 0$, because then g would have to vanish identically on the Gleason part containing m. Hence $\hat{b}(m) = 0$, and $b \in I$, because g is invertible modulo I. A theorem of Hoffman (Theorem 5.3 [HOF]) now provides us with interpolating Blaschke products $b_1, \ldots, b_n$ with $\hat{b}_1(m) = \cdots = \hat{b}_n(m) = 0$ and a Blaschke product b_0 with $\hat{b}_0(m) \neq 0$, such that $b = b_0 \cdot b_1 \cdot \cdots \cdot b_n$. We conclude $b_1 \cdot \cdots \cdot b_n \in I$.

Now is a good time to make the following remark. Applying our considerations this far to the special case $I = I_n(m)$, we find that $I_n(m) = (\ker m)^n$, where $(\ker m)^n$ denotes the set of all products $f_1 \cdot \cdots \cdot f_n$, $f_j \in \ker m$, which is the smallest conceivable candidate for the nth power of the maximal ideal $\ker m$.

Let us carry on with the proof, and choose a b_k, $1 \leqq k \leqq n$. Denote by $\{z_{j,k}\}_{j=0}^\infty$ the (interpolating) sequence of zeros of b_k. For an arbitrary open neighborhood U of m, let $b_{k,U}$ be the (interpolating) Blaschke product corresponding to the subsequence $\{z_{j,k}\}_{j=0}^\infty \cap U$. Then $b_{1,U} \cdot \cdots \cdot b_{n,U} \in I$ for every open neighborhood U of m, since the closure of an interpolating sequence equals the zero set of the Gelfand transform of the corresponding Blaschke product (Lemma IX.3.3 in [GAR]), and two disjoint subsequences of an interpolating sequence always have disjoint closures in $\mathcal{M}(H^\infty)$ (Theorem X.4.1 [GAR]). Since b_k is interpolating,

$$H^\infty / b_k H^\infty \cong l^\infty(\mathbf{N}) \cong C(\beta \mathbf{N}),$$

where $\beta\mathbf{N}$ denotes the Stone–Čech compactification of the nonnegative integers $\mathbf{N}$. By a theorem of Šilov (the corollary is §36 of [GRS]) which we apply to the algebra $C(\beta\mathbf{N})$, the H^∞-ideal generated by the functions $b_{k,U}$ with U varing over all open neighborhoods of m is dense in the maximal ideal $\ker m$. That is, given an arbitrary $h \in H^\infty$ with $\hat{h}(m) = 0$, there exist for every $\varepsilon > 0$ an open neighborhood $U(\varepsilon)$ of m and a function $g_\varepsilon \in H^\infty$ such that

$$\| b_{k,U(\varepsilon)} \cdot g_\varepsilon - h \| < \varepsilon.$$

It is now fairly obvious that every function in $I_n(m) = (\ker m)^n$ can be approximated by expressions of the form

$$b_{1,U} \cdot \cdots \cdot b_{n,U} \cdot g, \qquad g \in H^\infty,$$

which belong to I, of course. Since I is closed, the assertion now follows.

Remark 2.6. Employing some of the arguments used in the proof of Theorem 2.5, one can easily show that every closed ideal in H^∞ that contains an interpolating Blaschke product equals the intersection of those maximal ideals which contain it.

An antisymmetric set for the Douglas algebra $H^\infty + C$ is a set $S \subset \mathcal{M}(H^\infty + C) = \mathcal{M}(H^\infty)\backslash\mathbf{D}$ such that whenever $f \in H^\infty + C$ and $\hat{f}|_S$ is real valued, then $\hat{f}|_S$ is constant. A set that is maximal with respect to this property is called a maximal antisymmetric set. The maximal antisymmetric sets for $H^\infty + C$ are much smaller than the fibers $\mathcal{M}_\alpha(H^\infty)$, $|\alpha| = 1$. For a general primary ideal, we can say the following.

Proposition 2.7. *If I is a primary H^∞-ideal at a point in $\mathcal{M}(H^\infty)\backslash\mathbf{D}$, there is a unique maximal antisymmetric set S for $H^\infty + C$ such that I contains all functions in H^∞ that vanish on S.*

Proof. After a simple rotation, we may without loss of generality assume that $h(I, H^\infty) \subset \mathcal{M}_1(H^\infty)$. Our first step is to show that I contains all functions that vanish on $\mathcal{M}_1(H^\infty)$. By Theorem 2.2, it will be sufficient to show that the Gelfand transform of the singular inner function k has more than one zero in $\mathcal{M}(H^\infty)$ if $\alpha(I) > 0$. Of course, all zeros of $\hat{k}$ lie in $\mathcal{M}_1(H^\infty)$. Choose an interpolating sequence $\{z_n\}_0^\infty$ on the interval $[0, 1)$. Then $k(z_n) \to 0$ as $n \to \infty$, so $\hat{k}$ vanishes on the cluster set in $\mathcal{M}(H^\infty)$ of $\{z_n\}_0^\infty$, which is homeomorphic to $\beta\mathbf{N}\backslash\mathbf{N}$ [GAR, p. 190], where $\beta\mathbf{N}$ is the Stone–Čech compactification of the nonnegative integers $\mathbf{N}$. Since $\beta\mathbf{N}$ is very huge, $\hat{k}$ clearly vanishes at more than one point.

Now we know that $I \supset \{f \in H^\infty : f|_{\mathcal{M}_1(H^\infty)} \equiv 0\}$. Consider the closed $(H^\infty + C)$-ideal

$$J = \{f \in H^\infty + C : \hat{f}|_{\mathcal{M}_1(H^\infty)} \in \overline{I|_{\mathcal{M}_1(H^\infty)}}\},$$

which has the property that $J \cap H^\infty = I$. Bishop's decomposition theorem for ideals [GAM 1, p. 61], which is due to Glicksberg, states that a function $f \in H^\infty + C$ belongs to J if (and only if) $\hat{f}|_S \in J|_S$ for every maximal antisymmetric set S for $H^\infty + C$. It is not hard to see that the fact that I is a primary ideal in H^∞ entails that J must be primary in $H^\infty + C$. Now $J|_S = H^\infty + C|_S = H^\infty|_S$ for all maximal antisymmetric sets S not containing the singleton $h(J, H^\infty + C) = h(I, H^\infty)$, because $J|_S$ would not be contained in any maximal ideal of $H^\infty|_S$. This is so because $\mathcal{M}(H^\infty|_S) = S$ [STO 2, pp. 118–119]. Hence in order to check that a function $f \in H^\infty + C$ is in J it suffices to check that $\hat{f}|_S \in J|_S$ for the unique maximal antisymmetric set S that contains $h(J, H^\infty + C)$. If we restrict our attention to H^∞ functions and recall that $J \cap H^\infty = I$, the assertion follows.

Remark 2.8. I recently learned that Pamela Gorkin, in collaboration with Raymond Mortini, has used Proposition 2.7 and a factorization theorem of Sheldon

Axler to prove that every primary ideal at a point in the Šilov boundary is maximal, so the only case of Problem 2.4 that still evades us is when the ideal is primary at a point outside the Šilov boundary whose Gleason part is trivial. This result will appear in a paper by Raymond Mortini, which also includes a similar result about closed prime ideals in H^∞.

3. An epimorphism

The following is the program I wish to carry through. Throughout this section, B is an admissible algebra on some bounded domain W. Let A be a closed subalgebra of B which contains the unit and the coordinate function z, and let I be a proper closed A-ideal. Lucien Waelbroeck [WAE 1] provides us with a morphism (= a continuous homomorphism mapping unit on unit) $\mathcal{O}(Z(I, A)) \to A/I$ taking z onto $z + I$, which we will call the HFC (holomorphic functional calculus) morphism. If $B = [A, B \cap \mathcal{O}(Z(I, A))]$, the algebra generated by A and $B \cap \mathcal{O}(Z(I, A))$, and the canonical epimorphism (= surjective homomorphism) $A \to A/I$ and the HFC morphism are compatible in such a way that they define a continuous epimorphism $L_I\colon B \to A/I$, the quotient Banach algebras A/I and $B/\ker L_I$ will be canonically isomorphic. It is natural to link I and $\ker L_I$. If this can be done for a large class of closed A-ideals, we would have a very satisfactory situation.

In the sequel, W will be a bounded planar domain, and two planar domains W_1 and W_2 will be given, of which W_1 is bounded, such that $W_1 \cap W_2 = W$ and $W_1 \cup W_2 = \mathbf{C}$. We will assume that W_2 is not dense in $\mathbf{C}$. It is not hard to construct two unique continuous projections $P_1\colon \mathcal{O}(W) \to \mathcal{O}(W_1)$ and $P_2\colon \mathcal{O}(W) \to \mathcal{O}_0(W_2)$ that add up to identity; the uniqueness follows from Liouville's theorem. Indeed, let $\{V_n\}_0^\infty$ be a growing sequence of finitely connected smoothly bordered subdomains of W_1 such that V_1 contains $\mathbf{C} \setminus W_2$ and $\bigcup_{n=1}^\infty V_n = W_1$, and define

$$P_1 f(z) = \frac{1}{2\pi i} \int_{\partial V_n} (\zeta - z)^{-1} f(\zeta)\, d\zeta, \qquad z \in V_n.$$

This expression is independent of the chosen V_n and defines a continuous projection $\mathcal{O}(W) \to \mathcal{O}(W_1)$. A similar formula shows that there exists a P_2, and that P_1 and P_2 add up to identity. Clearly, P_1 takes $H^\infty(W)$ onto $H^\infty(W_1)$, and P_2 takes $H^\infty(W)$ onto $H_0^\infty(W_2)$.

Recall that B was an admissible algebra on W. Put $B_1 = B \cap H^\infty(W_1)$, $B_2 = B \cap H^\infty(W_2)$, and $B_2^0 = B \cap H_0^\infty(W_2)$, which we equip with the obvious norms; then B_1, B_2, and B_2^0 are Banach subalgebras of B.

We have arrived at a point where it is convenient to specify our assumptions on B. *We assume that B_1 and B_2 are admissible algebras on W_1 and W_2, respectively, and that $B = B_1 \oplus B_2^0$*, meaning $B = B_1 + B_2^0$ and $B_1 \cap B_2^0 = \{0\}$. An equivalent formulation of the condition $B = B_1 \oplus B_2^0$ is that $P_1(B) \subset B$. By the closed graph theorem, B_1 and B_2^0, and consequently B_2, are *closed* subalgebras of B.

Now assume I is a proper closed ideal in B_1. B_1 will play the role of A in the previously described general setting. Waelbroeck's HFC morphism $\mathcal{O}(Z(I, B_1)) \to B_1/I$ which takes z onto $z + I$, gives us a continuous homomorphism $B_2^0 \to B_1/I$ if $Z(I, B_1) \subset W_2$. And the image of an arbitrary $f \in B_2^0$ is by his definition [WAE 1, p. 517]

$$\frac{1}{2\pi i} \int_{\partial U} (\zeta - z + I)^{-1} f(\zeta) d\zeta,$$

where $U \subset\subset W_2$ is some finitely connected smoothly bordered domain which contains $Z(I, B_1)$. Denote by L_I the linear mapping $B = B_1 \oplus B_2^0 \to B_1/I$ defined to be the canonical epimorphism on B_1 and the HFC morphism on B_2^0. L_I is continuous by the closed graph theorem.

Lemma 3.1. *Let I be a closed B_1-ideal such that $Z(I, B_1) \subset W_2$. If $\mathcal{O}(\bar{W}_2 \cup \{\infty\})$ is dense in B_2, L_I is a continuous epimorphism $B \to B_1/I$.*

Proof. By the holomorphic functional calculus, the assumption that $\mathcal{O}(\bar{W}_2 \cup \{\infty\})$ is dense in B_2 entails that finite linear combinations of $(z - \zeta)^{-1}$, $\zeta \in \mathbf{C} \setminus \bar{W}_2$, form a dense subspace of B_2^0. In order to show that L_I is a homomorphism, it will be sufficient to show that

$$L_I((z - \zeta)^{-1} \cdot f) = L_I((z - \zeta)^{-1}) \cdot L_I(f)$$

for all $f \in B_1$, $\zeta \in \mathbf{C} \setminus \bar{W}_2$. So, let $f \in B_1$ and $\zeta \in \mathbf{D} \setminus \bar{W}_2$ be arbitrary. Since the HFC morphism takes z onto $z + I$,

$$L_I((z - \zeta)^{-1}) = (z - \zeta + I)^{-1}.$$

Since $\mathbf{C} \setminus \bar{W}_2 \subset W_1$, $T_\zeta f$, which was defined in Section 1, will belong to B and $H^\infty(W_1)$, and consequently to B_1. By the definition of $T_\zeta f$, $(z - \zeta) T_\zeta f = f - f(\zeta)$, and hence

$$L_I(T_\zeta f) = (f - f(\zeta) + I)(z - \zeta + I)^{-1}.$$

Now the formula

$$L_I((z - \zeta)^{-1} \cdot f) = L_I(T_\zeta f) + f(\zeta) L_I((z - \zeta)^{-1})$$

does the rest of the job.

Unfortunately, the assumption of Lemma 3.1 is pretty restrictive; for example, it is not satisfied for $B = H^\infty(W)$. However, there is a cure for this difficulty, as we shall see.

For the time being, let V_1 and V_2 be two finitely connected smoothly bordered domains such that

$$\mathbf{C}\setminus W_2 \subset V_1 \subset\subset W_1,$$

$$\mathbf{C}\setminus W_1 \subset V_2 \subset \bar{V}_2 \subset W_2, \qquad \text{and}$$

$$\bar{V}_1 \cap \bar{V}_2 = \varnothing.$$

Definition 3.2. The algebra B satisfies the SD condition with respect to (V_1, V_2) if there exist an admissible algebra $B^{(1)}$ on $W_1 \cap V_2$ and an admissible algebra $B^{(2)}$ on $V_1 \cap W_2$, such that $B^{(1)} \cap H^\infty(W_1) = B_1$, and $B^{(2)} \cap H^\infty(W_2) = B_2$.

Remarks 3.3. (a) SD stands for shrinking domain.
(b) By the holomorphic functional calculus, $B^{(1)}$ contains B_2 and $B^{(2)}$ contains B_1, and hence both contain B as a subalgebra.

Proposition 3.4. *If B satisfies the SD condition with respect to (V_1, V_2), L_I is a continuous epimorphism $B \to B_1/I$ that is canonical on B_1 for every closed B_1-ideal I with $Z(I, B_1) \subset W_2$.*

Proof. Let U be some finitely connected planar domain with smooth boundary such that $\bar{V}_2 \cup Z(I, B_1) \subset U \subset \bar{U} \subset W_2$. We intend to show that $B_1 \oplus A_0(U)$ is an admissible algebra on $W_1 \cap U$. Clearly, $H^\infty(W_1) \oplus A_0(U)$ is a uniform admissible algebra on $W_1 \cap U$ containing $B_1 \oplus A_0(U)$. Our first step is to show that

$$B_1 \oplus A_0(U) = B^{(1)} \cap (H^\infty(W_1) \oplus A_0(U)).$$

Let P_1 and P_2 denote the natural projections of $H^\infty(W_1) \oplus A_0(U)$ onto $H^\infty(W_1)$ and $A_0(U)$, respectively. If $f \in B^{(1)} \cap (H^\infty(W_1) \oplus A_0(U))$, the holomorphic functional calculus tells us that $P_2 f \in A_0(U)$ belongs to $B^{(1)}$, and hence $P_1 f \in B^{(1)} \cap H^\infty(W_1) = B_1$. The other inclusion is evident since $A_0(U) \subset B^{(1)}$, and the conclusion follows. So far we have shown that $B_1 \oplus A_0(U)$ is a Banach subalgebra of $H^\infty(W_1 \cap U)$. To show that $B_1 \oplus A_0(U)$ is an admissible algebra on $W_1 \cap U$, the only condition that needs verification is (b) of Definition 1.1. So, assume m is a complex homomorphism on $B_1 \oplus A_0(U)$ such that $m(z) = \alpha$ for some $\alpha \in W_1 \cap U$. Since B_1 is admissible, $m(f) = f(\alpha)$ for each $f \in B_1$. The same holds for $f \in A_0(U)$, and we're done.

Since $Z(I, B_1) \subset U$, we can define L_I on the larger algebra $B_1 \oplus A_0(U)$. It follows from Lemma 3.1 and Mergelyan's theorem that L_I is a homomorphism, and since we know L_I to be continuous and surjective, the assertion follows.

We will now present a simple factorization technique that will prove useful. Introduce the notation

$$Z_\Omega(f) = \{z \in \Omega : f(z) = 0\}, \qquad f \in \mathcal{O}(\Omega),$$

for arbitrary planar domains Ω.

Lemma 3.5. *Assume B satisfies the SD condition with respect to (V_1, V_2), and let f be a non-identically vanishing function in B. Then, for any set E, $\bar{V}_2 \subset E \subset \mathbf{C} \setminus \bar{V}_1$, there exist $f_1 \in B_1$ and $f_2 \in B_2$ such that $f = f_1 \cdot f_2$, $Z_{W_1}(f_1) = Z_W(f) \cap E$, and $Z_{W_2}(f_2) = Z_W(f) \setminus E$.*

Proof. Observe that $Z_W(f)$ is a countable set of isolated points in W, since f does not vanish identically. According to a theorem of Weierstrass (see [BUR], p. 237), there exist two functions, $\varphi_1 \in \mathcal{O}(W_1)$ and $\varphi_2 \in \mathcal{O}(W_2 \cup \{\infty\})$, such that

$$Z_{W_1}(\varphi_1) = Z_W(f) \cap E \quad \text{and} \quad Z_{W_2}(\varphi_2) = Z_W(f) \setminus E,$$

and the zeros of $\varphi_1 \cdot \varphi_2$ have the same multiplicity as those of f; then

$$f/(\varphi_1 \cdot \varphi_2) \in (\mathcal{O}(W))^{-1}.$$

We would like to take the logarithm of $f/(\varphi_1 \cdot \varphi_2)$, but unfortunately, this may turn out not to be a well-defined holomorphic function, since W is not simply connected. There is a way to get around that difficulty. Choose two finitely connected smoothly bordered domains U_1, $\bar{V}_1 \subset U_1 \subset\subset W_1$, and U_2, $\bar{V}_2 \subset U_2 \subset \bar{U}_2 \subset W_2$, such that $U_1 \cup U_2 = \mathbf{C}$. Then $U \equiv U_1 \cap U_2$ is easily seen to be a finitely connected subdomain of W with boundary $\partial U = \partial U_1 \cup \partial U_2$. Clearly, P_1 and P_2 extend to be continuous projections $\mathcal{O}(U) \to \mathcal{O}(U_1)$ and $\mathcal{O}(U) \to \mathcal{O}_0(U_2)$, respectively, adding up to identity. Since U is finitely connected and smoothly bordered, it is not hard to find rational functions $h_1 \in (\mathcal{O}(U_1))^{-1}$ and $h_2 \in (\mathcal{O}(U_2 \cup \{\infty\}))^{-1}$ such that the winding number of

$$f/(\varphi_1 \cdot \varphi_2 \cdot h_1 \cdot h_2) \in (\mathcal{O}(U))^{-1}$$

is zero around each component of $\mathbf{C} \setminus U$. This enables us to form a logarithm of it in $\mathcal{O}(U)$, which we denote by g. We have obtained a factorization $f = f_1 \cdot f_2$ on U, where

$$f_1 \equiv \varphi_1 h_2 \exp(P_1 g) \in \mathcal{O}(U_1) \qquad \text{and}$$

$$f_2 \equiv \varphi_2 h_2 \exp(P_2 g) \in \mathcal{O}(U_2 \cup \{\infty\}).$$

As soon as we have shown that $f_1 \in B_1$ and $f_2 \in B_2$, the assertion will follow, since $f_1 = f/f_2 \in \mathcal{O}(W_1)$ and $f_2 = f/f_1 \in \mathcal{O}(W_2)$ have the correct zeros. By the choice of E, $f_1 \in (\mathcal{O}(\bar{V}_1))^{-1}$ and $f_2 \in (\mathcal{O}(\bar{V}_2))^{-1}$, and the holomorphic functional calculus tells us that $f_1 \in (B^{(2)})^{-1}$ and $f_2 \in (B^{(1)})^{-1}$. Hence

$$f_1 = f/f_2 \in B^{(1)} \cap \mathcal{O}(U_1 \cup (W \cap U_2)) = B^{(1)} \cap \mathcal{O}(W_1) = B^{(1)} \cap H^\infty(W_1) = B_1.$$

A similar argument for f_2 does the trick.

Now let J be any nonzero ideal in B. Then, according to Waelbroeck [WAE 2], there exist two unique B-ideals J_1 and J_2 such that

(i) $Z(J_1, B) = Z(J, B) \cap E$,
(ii) $Z(J_2, B) = Z(J, B) \setminus E$, and

(iii) $J = J_1 \cap J_2$,

where E is as in the formulation of Lemma 3.5. If J is closed, J_1 and J_2 are closed, too.

Proposition 3.6. *Assume B satisfies the SD condition with respect to (V_1, V_2), and let J be a nonzero ideal in B; J_1 and J_2 are as in* (i)–(iii). *Then each $f \in J$ has a factorization $f = f_1 \cdot f_2$, where $f_1 \in J_1 \cap B_1$ and $f_2 \in J_2 \cap B_2$.*

Remark 3.7. The functions f_1 and f_2 can be chosen as in Lemma 3.5.

Proof of Proposition 3.6. By Lemma 3.5, $f_1 \in B_1$ and $f_2 \in B_2$, so it remains to show that $f_1 \in J_1$ and $f_2 \in J_2$. In order to obtain this conclusion, we will first show that $f_1 \cdot B + J_2 = B$ and $f_2 \cdot B + J_1 = B$. We will only attack the first relation, since the second one is verified in exactly the same manner. Since the left hand side is an ideal, it suffices to show that it is not contained in any maximal ideal. So, let $m \in \mathcal{M}(B)$ be arbitrary. In order to annihilate J_2, m must belong to $h(J_2, B)$. Since B_1 is an admissible algebra on W_1, $f_1 \in B_1$, and $m(z) \in \hat{z}(h(J_2, B)) = Z(J_2, B) \subset W_1$, we have that $m(f_1) = f_1(m(z))$. The observation that f_1 has no zeros on $Z(J_2, B)$ rules out the possibility that m can annihilate $f_1 \cdot B + J_2$, and the assertion follows. Hence there exist $g_1, g_2 \in B$ such that $g_1 f_1 - 1 \in J_2$ and $g_2 f_2 - 1 \in J_1$. Since $f \in J$, $f \cdot g_2 = f_1 \cdot (f_2 \cdot g_2) \in J$, and applying the second of the last-mentioned relations, we obtain $f_1 \in J_1$; similarly, $f_2 \in J_2$. The proof is complete.

Remark 3.8. Proposition 3.6 tells us that J equals $(J \cap B_1) \cdot (J \cap B_2)$ in the strongest conceivable way, since trivially,

$$(J_1 \cap B_1) \cdot (J_2 \cap B_2) \subset J_1 \cdot J_2 \subset J_1 \cap J_2 = J.$$

Definition 3.9. The algebra B is said to satisfy the *full* SD condition if to each open neighborhood Ω of $\mathbf{C} \setminus W_2$ there exists an admissible pair (V_1, V_2) with respect to which B satisfies the SD condition, such that $V_1 \subset \Omega$.

For a proper closed B-ideal J with $Z(J, B) \subset W_2$, let $\Lambda_J \colon B \to B/J$ be the linear mapping defined to be the canonical quotient mapping $B_1 \to (B_1 + J)/J$ on B_1, and the HFC morphism

$$B_2^0 \to \mathcal{O}(Z(J, B)) \to B/J$$

on B_2^0. Λ_J is continuous by the closed graph theorem. In case J contains some closed B_1-ideal I with $Z(I, B_1) \subset W_2$, Λ_J is the composition of L_I and the canonical homomorphism $B_1/I \to B/J$.

Proposition 3.10. *If B satisfies the full SD condition, Λ_J coincides with the canonical epimorphism $B \to B/J$ for all closed B-ideals J with $Z(J, B) \subset W_2$.*

Proof. Recall that by definition, Λ_J is canonical on B_1, and

$$\Lambda_J f = \frac{1}{2\pi i}\int_{\partial U} (\zeta - z + J)^{-1} f(\zeta)\,d\zeta$$

for $f \in B_2^0$, where $U \subset\subset W_2$ is some finitely connected smoothly bordered domain which contains $Z(J, B)$. Since

$$\|(\zeta - z + J)^{-1}\| = O(|\zeta|^{-1}) \qquad \text{as } |\zeta| \to \infty,$$

and the functions in B_2^0 tend to 0 as $|\zeta| \to \infty$, we have the same formula if U is exchanged by its union V with the unbounded component of the complement $\mathbf{C} \setminus U$. It will be sufficient to prove that

$$\Lambda_J f = f + J, \qquad f \in B_2^0.$$

By assumption, we can choose an admissible pair (V_1, V_2) with respect to which B satisfies the SD condition, such that $V_1 \subset\subset W_1 \setminus Z(J, B)$. By Proposition 3.6 and Remark 3.7, there exists a function $\varphi \in J \cap B_1$ which has no zeros on $\bar{V}_1$. Then

$$T_\zeta \varphi(z) = (\varphi(z) - \varphi(\zeta))/(z - \zeta), \qquad z \in W_1,$$

belongs to B_1 for $\zeta \in \mathbf{C} \setminus \bar{V}_2 \supset V_1$ and varies continuously with ζ, because B satisfies the SD condition with respect to (V_1, V_2) and because the norm on B_1 is equivalent to the natural one on $B^{(1)} \cap H^\infty(W_1)$. One easily checks that $T_\zeta\varphi/\varphi(\zeta)$ is a function in the coset $(\zeta - z + J)^{-1}$ for each $\zeta \in V_1$. Readjusting our choice of U, we may assume $\mathbf{C} \setminus V \subset V_1$. By a simple calculation,

$$\frac{1}{2\pi i}\int_{\partial V} (T_\zeta\varphi(z)/\varphi(\zeta)) f(\zeta)\,d\zeta = f(z) - \varphi(z)\cdot\frac{1}{2\pi i}\int_{\partial V} \frac{f(\zeta)}{\varphi(\zeta)(\zeta - z)}\,d\zeta, \quad z \in W_1 \cap V,$$

for $f \in B_2^0$, and if we can show that the function

$$z \mapsto \frac{1}{2\pi i}\int_{\partial V} \frac{f(\zeta)}{\varphi(\zeta)(\zeta - z)}\,d\zeta, \qquad z \in W_1 \cap V,$$

is (the restriction to $W_1 \cap V$ of a function) in B, the proposition will be accomplished. This latter function equals $P_2(f/\varphi)$, where P_2 is the natural projection $\mathcal{O}(W_1 \cap V) \to \mathcal{O}_0(V)$. Since $\varphi \in (\mathcal{O}(\bar{V}_1 \cap W_2))^{-1}$, the holomorphic functional calculus tells us that $\varphi \in (B^{(2)})^{-1}$; hence $f/\varphi \in B^{(2)}$, and

$$P_2(f/\varphi) \in B^{(2)} \cap \mathcal{O}_0(W_2) = B^{(2)} \cap H_0^\infty(W_2) = B_2^0.$$

It is time to state the main result of this section. For a B_1-ideal I, $I \cdot B$ denotes the B-ideal generated by I.

Theorem 3.11. *Assume B satisfies the full SD condition. Then*
(a) *The mapping $I \mapsto I \cdot B$ is a bijection from the set of all closed B_1-ideals I with*

$Z(I, B_1) \subset W_2$ onto the set of all closed B-ideals J with $Z(J, B) \subset W_2$. Also $Z(I \cdot B, B) = Z(I, B_1)$, and the inverse mapping is given by $J \mapsto J \cap B_1$.

(b) *L_I is a continuous epimorphism $B \mapsto B_1/I$ that is canonical on B_1 with kernel $I \cdot B$ for all closed B_1-ideals I such that $Z(I, B_1) \subset W_2$.*

(c) *The quotient algebras B_1/I and $B/I \cdot B$ are canonically isomorphic for all closed B_1-ideals I with $Z(I, B_1) \subset W_2$.*

Proof. First we check (b). That L_I is a continuous epimorphism that is canonical on B_1 follows from Proposition 3.4. It remains to check that its kernel is $I \cdot B$. Obviously, $\ker L_I \supset \overline{I \cdot B}$, since L_I is canonical on B_1 and continuous, so part of the assertion is that $I \cdot B$ is closed. To simplify the notation, write $J = \overline{I \cdot B}$. Since obviously

$$Z(J, B) = \sigma(z + J, B/J) \subset \sigma(z + I, B_1/I) = Z(I, B_1),$$

Λ_J is well defined. More or less by the definitions of L_I and Λ_J, the composition of L_I and the canonical homomorphism $B_1/I \to B/J$ coincides with Λ_J. By Proposition 3.10, Λ_J coincides with the canonical epimorphism $B \to B/J$, and it is now immediate that $\ker L_I = J$. Since L_I is canonical on B_1, $J \cap B_1 = I$. If we select an admissible pair (V_1, V_2), with respect to which B satisfies the SD condition, such that

$$V_1 \subset\subset W_1 \setminus Z(J, B),$$

Proposition 3.6 tells us that $J = (J \cap B_1) \cdot B = I \cdot B$, and (b) follows.

Let us turn our attention to (c). For a closed B_1-ideal with $Z(I, B_1) \subset W_2$, the epimorphism L_I induces a Banach algebra isomorphism

$$\tilde{L}_I \colon B/\ker L_I = B/I \cdot B \to B_1/I.$$

Since L_I is canonical on B_1, $(\tilde{L}_I)^{-1}$ must coincide with the canonical homomorphism $B_1/I \to B/I \cdot B$.

Finally, we attack (a). While proving (b), we discovered that

$$\overline{I \cdot B} \cap B_1 = I \cdot B \cap B_1 = I$$

for closed B_1-ideals I with $Z(I, B_1) \subset W_2$, which makes the mapping $I \mapsto I \cdot B$ injective. We claim that it is surjective, too. So, let J be an arbitrary closed B-ideal with $Z(J, B) \subset W_2$. Keeping the previous choice of (V_1, V_2), Proposition 3.6 tells us that $J = (J \cap B_1) \cdot B$, and if we can show that $J \cap B_1$ satisfies $Z(J \cap B_1, B_1) \subset W_2$, the claim will follow. But this is obvious from the fact that $J \cap B_1$ contains a function which is nonzero on $\bar{V}_1$, which is true because of Proposition 3.6 and Remark 3.7. The only thing that remains to be verified is that $Z(I \cdot B_1, B) = Z(I, B_1)$ for closed B_1-ideals I with $Z(I, B_1) \subset W_2$. We have already obtained the inclusion $Z(I \cdot B, B) \subset Z(I, B_1)$. To obtain the reverse inclusion, let $m_1 \in h(I, B_1)$ be arbitrary. Then $m \equiv L_I^*(m_1) \in (I \cdot B)^\perp$ is a complex homomorphism in

$h(I \cdot B, B) = (I \cdot B)^{\perp} \cap \mathcal{M}(B)$, the restriction to B_1 of which is m_1, since L_I is canonical on B_1. Here, L_I^*: $I^{\perp} \to B^*$ is the adjoint mapping of L_I. Since $\hat{z}(m) \equiv m(z) = m_1(z)$, we conclude that $Z(I \cdot B, B) = Z(I, B_1)$. The proof of the theorem is complete.

Remarks 3.12. (a) Assume B satisfies the full SD condition. For a closed B-ideal J, let J_1 and J_2 be as in (i)–(iii) preceding Proposition 3.6. By Theorem 3.11, $Z(J_1 \cap B_1, B_1) = Z(J_1, B) \subset W_2$, $J_1 = (J_1 \cap B_1) \cdot B$, and the quotient algebras, $B_1/J_1 \cap B_1$ and B/J, are canonically isomorphic. We can change the rôles of B_1 and B_2 by a simple Möbius transformation $\mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$, at least if we slightly strengthen our assumption on B, so a similar statement can be made for J_2. In a sense, the problem of determining the closed ideals of B is solved once we can solve it for B_1 and B_2.

(b) Under the assumption of Theorem 3.11, L_I is the only epimorphism $B \to B_1/I$ that is canonical on B_1, for every closed B_1-ideal I with $Z(I, B_1) \subset W_2$. This follows from the fact that $\ker L_I = I \cdot B$ by a simple argument. If Φ is such an epimorphism, $\ker \Phi$ must contain $I \cdot B$, and since $B = B_1 + I \cdot B$, Φ must equal L_I. The adjoint mapping L_I^*: $I^{\perp} \to B^*$ furnishes the functionals in $I^{\perp}(\subset B_1^*)$ with unique extensions in B^* which annihilate $I \cdot B$.

(c) It is not hard to check that Theorem 3.11 applies to the algebras $H_n^\infty(W) = \{f \in H^\infty(W): f^{(n)} \in H^\infty(W)\}$ and $A_n(W) = \{f \in A(W): f^{(n)} \in A(W)\}$. Since $A_n(\Omega)$ is isomorphic to $A_n(\mathbf{D})$ in the obvious sense if $\Omega \subset\subset \mathbf{C}$ is simply connected and has sufficiently smooth boundary, (a) shows that Boris Korenblum's [KOR] description of the closed ideals in $A_n(\mathbf{D})$ carries over to $A_n(W)$ for a class of finitely connected domains W.

(d) A suitable application of Theorem 3.11 is to analytic Beurling algebras on the integers

$$l^1(w, \mathbf{Z}) = \left\{(a_n)_{-\infty}^{\infty}: \sum_{-\infty}^{\infty} |a_n| w_n < \infty\right\},$$

supplied with convolution multiplication, where $w = (w_n)_{-\infty}^{\infty}$ is a positive submultiplicative weight (see [GRS], §19). The algebra $l^1(w, \mathbf{Z})$ is analytic if

$$\alpha = \lim_{n \to -\infty} w_n^{1/n} < \lim_{n \to \infty} w_n^{1/n} = \beta,$$

so that its maximal ideal space is homeomorphic to the annulus $\{z \in \mathbf{C}: \alpha \leqq |z| \leqq \beta\}$. The natural decomposition of $l^1(w, \mathbf{Z})$ is

$$l^1(w, \mathbf{Z}) = l^1(w, \mathbf{N}) \oplus l^1(w, \mathbf{Z}_-).$$

The SD condition of Theorem 3.11 is met if $w_n = \alpha^n v_n$ for $n < 0$ and $w_n = \beta^n \cdot v_n$ for $n \geqq 0$, where the sequence $\{v_n\}_{-\infty}^{\infty}$ is submultiplicative. It is possible to construct an analytic weight w such that $l^1(w, \mathbf{Z})$ does not meet the full SD condition.

For results on the structure of the closed ideals in $l^1(w,\mathbf{N})$, see Feldman [FEL], Gurariĭ [GUR], and Hedenmalm [HED]. For the specific weight $w_n \equiv 1$, Feldman characterized the closed primary ideals in terms of their annihilators, which consisted of a certain class of entire functions restricted to $\mathbf{N}$. By (b), the annihilator of the $l^1(w,\mathbf{Z})$-ideal generated by a primary $l^1(w,\mathbf{N})$-ideal I at some point $z_0 \in \mathbf{T}$ can be identified with $I^\perp$ in the sense that $(I * l^1(w,\mathbf{Z}))^\perp$ consists of the restriction to $\mathbf{Z}$ of the entire functions in $I^\perp$.

4. Infinitely connected domains

The title of this section might be slightly misleading since, strictly speaking, we did not make any assumptions concerning finite connectivity back in Section 3. But we did assume that the intersection of the boundaries of W_1 and W_2 was empty. We will not do so this time, and it will be necessary to make use of the estimate of $\|(z-\zeta+I)^{-1}\|$ obtained in Section 2 in a crucial way. We will study a fairly restricted subclass of those infinitely connected domains, or rather H^∞ on them, for which Michael Behrens [BEH] obtained the corona theorem.

Let $\{\varepsilon_j\}_0^\infty \subset (0,1]$ be some strictly decreasing sequence tending to 0, and let $\{D_j\}_0^\infty$ be a sequence of disjoint closed subdiscs of the open unit disc $\mathbf{D}$ such that D_j is centered at $1-\varepsilon_j$; we denote the radius of D_j by r_j.

Moreover, we will assume that there exists a hyperbolically-rare sequence $\{\Delta_j\}_0^\infty$ of disjoint closed subdiscs of $\mathbf{D}$, such that Δ_j has the same center as D_j, the radius R_j of Δ_j is $> r_j$, and $\Sigma_0^\infty r_j/R_j < \infty$. By Behrens' [BEH] definition, this means that there are disjoint closed discs Δ_j' with centers $1-\varepsilon_j$ such that $\Delta_j \subset \Delta_j' \subset \mathbf{D}$, and such that $\Sigma_0^\infty R_j/\operatorname{rad}\Delta_j' < \infty$. Then $\Sigma_0^\infty R_j/\varepsilon_j < \infty$, and $\{D_j\}_0^\infty$ is hyperbolically-rare in $\mathbf{D}$, too. Behrens obtained his results without the assumption that the sequence $\{\Delta_j\}_0^\infty$ be hyperbolically-rare, so our situation is slightly more restrictive than his.

Put

$$V = \mathbf{C} \setminus \bigcup_0^\infty D_j \setminus \{1\},$$

$$U = \mathbf{C} \setminus \bigcup_0^\infty \Delta_j \setminus \{1\}, \quad \text{and}$$

$$W = \mathbf{D} \cap V = \mathbf{D} \setminus \bigcup_0^\infty D_j.$$

The existence of nontangential boundary values of bounded analytic functions on this type of domains was settled by Zalcman [ZAL, §2]. For $f \in H^\infty(\mathbf{D} \cap U)$ and $j \in \mathbf{N} = \{0,1,2,\ldots\}$ we define

$$Q_j f(z) = \frac{1}{2\pi i} \int_{\partial \Delta_j} \frac{f(\zeta)}{z-\zeta}\, d\zeta, \qquad z \in \mathbf{C} \setminus \Delta_j.$$

Then Q_j is a continuous projection of $H^\infty(\mathbf{D}\cap U)$ onto $H_0^\infty(\mathbf{C}\setminus\Delta_j)$, the ideal in $H^\infty(\mathbf{C}\setminus\Delta_j)$ of all functions vanishing at ∞, and Behrens shows in [BEH, §2] that for $f\in H^\infty(\mathbf{D}\cap U)$, the expression

$$Qf(z)=\sum_0^\infty Q_jf(z),\qquad z\in U$$

converges, uniformly on compact subsets of U, to a function in $H_0^\infty(U)$; here we needed our assumption that $\{\Delta_j\}_0^\infty$ is hyperbolically-rare. And of course Q is a projection $H^\infty(\mathbf{D}\cap U)\to H_0^\infty(U)$. Again according to Behrens [BEH, §2], $P\equiv I-Q$ is a projection onto $H^\infty(\mathbf{D})$. Here I denotes the identity operator. It should be observed that by the generalized maximum modulus principle, $H^\infty(\mathbf{D})$ and $H^\infty(U)$ are closed subalgebras of $H^\infty(\mathbf{D}\cap U)$. The existence of P and Q shows that

$$H^\infty(\mathbf{D}\cap U)=H^\infty(\mathbf{D})\oplus H_0^\infty(U),$$

and the closed graph theorem tells us that P and Q are continuous.

All this could have been done with V instead of U. For instance, the restrictions to $H^\infty(W)$ of P and Q are continuous projections onto $H^\infty(\mathbf{D})$ and $H_0^\infty(V)$, respectively, making

$$H^\infty(W)=H^\infty(\mathbf{D})\oplus H_0^\infty(V).$$

For a closed $H^\infty(\mathbf{D})$-ideal I such that $h(I,H^\infty(\mathbf{D}))\subset\mathcal{M}_1(H^\infty(\mathbf{D}))$, denote by $\tilde{L}_I$ the linear mapping $H^\infty(\mathbf{D}\cap U)\to H^\infty(\mathbf{D})/I$ which coincides with the canonical epimorphism on $H^\infty(\mathbf{D})$ and is defined by the expression

$$\tilde{L}_If=\frac{1}{2\pi i}\sum_{j=0}^\infty\int_{\partial\Delta_j}(z-\zeta+I)^{-1}f(\zeta)d\zeta$$

for $f\in H_0^\infty(U)$; $\tilde{L}_I$ is well defined and continuous (by the closed graph theorem) if

$$\sum_{j=0}^\infty\int_{\partial\Delta_j}\|(z-\zeta+I)^{-1}\|\cdot|d\zeta|<\infty.$$

Let L_I be the restriction of $\tilde{L}_I$ to $H^\infty(W)$. Since L_I is given by the formula

$$L_If=\frac{1}{2\pi i}\sum_{j=0}^\infty\int_{\partial D_j}(z-\zeta+I)^{-1}f(\zeta)d\zeta$$

on $H_0^\infty(V)$, one only needs to assume that

$$\sum_{j=0}^\infty\int_{\partial D_j}\|(z-\zeta+I)^{-1}\|\cdot|d\zeta|<\infty$$

to ensure that L_I exists. For maximal ideals in $\mathcal{M}_1(H^\infty(\mathbf{D}))$, Behrens [BEH] showed that L_I is a (complex) homomorphism. Recall that $\alpha(\cdot)$ was introduced in Section 2.

Lemma 4.1. *Let I be a closed $H^\infty(\mathbf{D})$-ideal with $h(I, H^\infty(\mathbf{D})) \subset \mathcal{M}_1(H^\infty(\mathbf{D}))$. If*

$$\sum_{j=0}^{\infty} (r_j/\varepsilon_j)\exp(\alpha(I)(\varepsilon_j - r_j)^{-1}) < \infty,$$

which is automatically satisfied when $\alpha(I) = 0$,

$$\sum_{j=0}^{\infty} \int_{\partial D_j} \|(z - \zeta + I)^{-1}\| \cdot |d\zeta| < \infty.$$

If

$$\sum_{j=0}^{\infty} (R_j/\varepsilon_j)\exp(\alpha(I)(\varepsilon_j - R_j)^{-1}) < \infty,$$

which is automatically satisfied when $\alpha(I) = 0$,

$$\sum_{j=0}^{\infty} \int_{\partial \Delta_j} \|(z - \zeta + I)^{-1}\| \cdot |d\zeta| < \infty.$$

Proof. We will only show the first part of the lemma, since the verification process for the rest is identical. By Proposition 2.3,

$$\|(z - \zeta + I)^{-1}\| \leqq 4(1 - |\zeta|)^{-1}\exp(\alpha(I)/|1 - \zeta|), \qquad \zeta \in \mathbf{D}.$$

Since $r_j = o(\varepsilon_j)$ as $j \to \infty$,

$$\sum_{j=0}^{\infty} \int_{\partial D_j} \|(z - \zeta + I)^{-1}\| \cdot |d\zeta| \leqq 8\pi \sum_{j=0}^{\infty} r_j (\varepsilon_j - r_j)^{-1} \cdot \exp(\alpha(I)/(\varepsilon_j - r_j)) < \infty$$

as asserted.

Lemma 4.2. $H^\infty(V) \subset A(U)$.

Proof. This follows from Lemma 2.3 in [BEH].

Proposition 4.3. *Let I be a closed $H^\infty(\mathbf{D})$-ideal with $h(I, H^\infty(\mathbf{D})) \subset \mathcal{M}_1(H^\infty(\mathbf{D}))$. If*

$$\sum_{j=0}^{\infty} (R_j/\varepsilon_j)\exp(\alpha(I)/(\varepsilon_j - R_j)) < \infty,$$

which is automatically satisfied when $\alpha(I) = 0$, L_I is a continuous epimorphism $H^\infty(W) \to H^\infty(\mathbf{D})/I$.

Proof. Let us first remark that by Lemma 4.1, $\tilde{L}_I$ is well defined and continuous, and consequently L_I is, too. Since the surjectivity of L_I is obvious, it suffices to show that it is a homomorphism. By Theorem 10.5 [GAM 1], rational functions with poles off $\bar{U}$ are dense in $A(U)$, and, as a consequence, finite linear combinations of $(z-\zeta)^{-1}$, $\zeta \in \mathbf{C}\setminus \bar{U} = \bigcup_0^\infty \Delta_j^0$, form a dense subspace of $A_0(U)$. Computational verification shows that

$$\tilde{L}_I((z-\zeta)^{-1}) = (z-\zeta+I)^{-1}$$

for $\zeta \in \mathbf{C}\setminus \bar{U}$, and that

$$\tilde{L}_I((z-\zeta_1)^{-1}(z-\zeta_2)^{-1}) = (z-\zeta_1+I)^{-1}(z-\zeta_2+I)^{-1}$$

for $\zeta_1, \zeta_2 \in \mathbf{C}\setminus \bar{U}$. Hence $\tilde{L}_I$ is a homomorphism when restricted to $A_0(U)$.

Our last step will be to show that

$$\tilde{L}_I(fg) = \tilde{L}_I(f)\cdot \tilde{L}_I(g) \qquad \text{for } f \in H^\infty(\mathbf{D}) \quad \text{and} \quad g \in A_0(U).$$

The assertion will then follow because $H_0^\infty(V) \subset A_0(U)$, by Lemma 4.2. Since a dense subspace of $A_0(U)$ is spanned by the functions $(z-\zeta)^{-1}$, $\zeta \in \mathbf{C}\setminus \bar{U}$, it suffices to show that

$$\begin{aligned}\tilde{L}_I(z-\zeta)^{-1}\cdot f) &= \tilde{L}_I((z-\zeta)^{-1})L_I(f)\\ &= (z-\zeta+I)^{-1}(f+I),\end{aligned}$$

for all $f \in H^\infty(\mathbf{D})$ and $\zeta \in \mathbf{C}\setminus \bar{U}$. To do this, choose a function $f \in H^\infty(\mathbf{D})$ and a $\zeta \in \mathbf{C}\setminus \bar{U}$. Recall the notation

$$T_\zeta f(z) = (f(z)-f(\zeta))/(z-\zeta), \qquad z, \zeta \in \mathbf{D}, \quad z \neq \zeta.$$

This is a function in $H^\infty(\mathbf{D})$, so the relation

$$(z-\zeta)T_\zeta f = f - f(\zeta)$$

shows that

$$L_I(T_\zeta f) = (z-\zeta+I)^{-1}(f-f(\zeta)+I),$$

and since

$$(z-\zeta)^{-1}f = T_\zeta f + (z-\zeta)^{-1}f(\zeta)$$

in $H^\infty(W)$, we can conclude that

$$\tilde{L}_I((z-\zeta)^{-1}f) = (z-\zeta+I)^{-1}(f+I),$$

as desired.

Remark 4.4. Let I be a closed $H^\infty(\mathbf{D})$-ideal with $h(I, H^\infty(\mathbf{D})) \subset \mathcal{M}_1(H^\infty(\mathbf{D}))$ such that $\alpha(I)=0$. Then for every $f \in A_0(U)$, $\tilde{L}_I f = f(1)+I$. To see this, recall that by Proposition 2.1, I contains the closed ideal

$$I_0 = \{f \in H^\infty: \hat{f}(m) = 0 \text{ for all } m \in \mathcal{M}_1(H^\infty(\mathbf{D}))\}.$$

Since $z - 1 \in I_0$, $z - \zeta + I = 1 - \zeta + I$ for $\zeta \in \mathbf{C}$, and consequently,

$$\begin{aligned} \tilde{L}_I f &= \sum_0^\infty \frac{1}{2\pi i} \int_{\partial\Delta_j} (z - \zeta + I)^{-1} f(\zeta) d\zeta \\ &= \sum_0^\infty \frac{1}{2\pi i} \int_{\partial\Delta_j} ((1-\zeta)^{-1} + I) f(\zeta) d\zeta \\ &= \sum_0^\infty Q_j f(1) + I \\ &= f(1) + I \end{aligned}$$

for $f \in A_0(U)$ ($\supset H_0^\infty(V)$).

We shall use the following lemma, the simple proof of which was kindly communicated to me by Harold S. Shapiro.

Lemma 4.5. *Suppose we have N functions $f_1, \ldots, f_N \in H^\infty(\mathbf{D})$ such that $\sum_{n=1}^N |f_n(z)| \geqq 1$ on the sequence $\{1 - \varepsilon_j\}_0^\infty$. Then there exist functions $g_1, \ldots, g_N \in H^\infty(\mathbf{D})$ such that*

$$\left| \sum_{n=1}^N f_n(z) g_n(z) \right| \geqq 1 \qquad \text{on } \{1 - \varepsilon_j\}_0^\infty.$$

Proof. By the Schwarz inequality,

$$\sum_{n=1}^N f_n(z) \bar{f}_n(z) = \sum_{n=1}^N |f_n(z)|^2 \geqq 1/N$$

on $\{1 - \varepsilon_j\}_0^\infty$. Since the sequence $\{1 - \varepsilon_j\}_0^\infty$ is real, the choice

$$g_n(z) = N \cdot \bar{f}_n(\bar{z}), \qquad z \in \mathbf{D}, \quad n = 1, \ldots, N,$$

meets all the required conditions.

We now state the main result of this section.

Theorem 4.6. *Let I be a closed $H^\infty(\mathbf{D})$-ideal with $h(I, H^\infty(\mathbf{D})) \subset \mathcal{M}_1(H^\infty(\mathbf{D}))$, and assume*

$$\sum_{j=0}^\infty (R_j/\varepsilon_j) \exp(\alpha(I)/(\varepsilon_j - R_j)) < \infty,$$

which is automatically satisfied when $\alpha(I) = 0$. Then

(a) *L_I is a continuous epimorphism $H^\infty(W) \to H^\infty(\mathbf{D})/I$.*

(b) *The quotient algebras* $H^\infty(\mathbf{D})/I$ *and* $H^\infty(W)/\ker L_I$ *are canonically isomorphic.*
(c) $\ker L_I \cap H^\infty(\mathbf{D}) = I$.
(d) *If* $h(I, H^\infty(\mathbf{D}))$ *and the closure of* $\{1-\varepsilon_j\}_0^\infty$ *in* $\mathcal{M}(H^\infty(\mathbf{D}))$ *are disjoint subsets of* $\mathcal{M}(H^\infty(\mathbf{D}))$, *then* $\alpha(I)=0$ *and* $\ker L_I = I\cdot H^\infty(W) = \overline{I\cdot H^\infty(W)}$; *consequently,* L_I *is the only homomorphism* $H^\infty(W)\to H^\infty(\mathbf{D})/I$ *whose restriction to* $H^\infty(\mathbf{D})$ *coincides with the canonical epimorphism* $H^\infty(\mathbf{D})\to H^\infty(\mathbf{D})/I$.

Proof. (a) is the statement of Proposition 4.3. To obtain (b), mimic the argument proving part (c) of Theorem 3.11. (c) is immediate since L_I is canonical on $H^\infty(\mathbf{D})$.

Finally, we turn our attention to (d). By a compactness argument, we obtain finitely many functions $f_1,\dots,f_N\in I$ such that

$$\sum_{n=1}^{N} |f_n(z)| \geqq 1 \qquad \text{on } \{1-\varepsilon_j\}_0^\infty.$$

But then at least one of $f_1,\dots,f_N$, say f_1, must satisfy

$$\alpha(\{f_1\}) = -\limsup_{t\to 1-}(1-t)\log|f_1(t)| = 0,$$

making $\alpha(I)=0$. By Lemma 4.5, there exists a function $\varphi\in I$ such that $|\varphi|\geqq 1$ on $\{1-\varepsilon_j\}_0^\infty$. According to Lemma 2.4 in [BEH], or even simpler, by the elementary estimate

$$|\varphi'(\zeta)| \leqq \|T_\zeta\varphi\| \leqq 2\|\varphi\|(1-|\zeta|)^{-1}, \qquad \zeta\in\mathbf{D},$$

which is a consequence of Schwarz' lemma, $|\varphi|\geqq \frac{1}{2}$ on $\bigcup_{j=k}^\infty \Delta_j$ if k is sufficiently large. So, φ may have finitely many zeros on $\bigcup_{j=0}^{k-1}\Delta_j$. Let b be a finite Blaschke product having those very zeros with the same multiplicity. Then $\varphi/b\in H^\infty(\mathbf{D})$ satisfies $|\varphi/b|\geqq\delta$ on $\bigcup_{j=0}^\infty\Delta_j$ for some $\delta>0$, and it is easily checked that $\varphi/b\in I$ (multiply by b and observe that b is invertible modulo I). Hence we may assume without loss of generality that our $\varphi\in I$ was chosen so that $|\varphi|\geqq 1$ on $\bigcup_{j=0}^\infty\Delta_j$.

It is easy to check that $-T_\zeta\varphi/\varphi(\zeta)$ belongs to the coset $(z-\zeta+I)^{-1}$ for $\zeta\in\mathbf{D}$ such that $\varphi(\zeta)\neq 0$, and in particular for $\zeta\in\bigcup_{j=0}^\infty\Delta_j$. Pick an arbitrary $f\in H^\infty(V)$. The estimate

$$\|T_\zeta\varphi\|\leqq 2\|\varphi\|(1-|\zeta|)^{-1}, \qquad \zeta\in\mathbf{D},$$

shows that the sum

$$L_I^0 f \equiv \frac{1}{2\pi i}\sum_{j=0}^{\infty}\int_{\partial D_j}(-T_\zeta\varphi/\varphi(\zeta))f(\zeta)\,d\zeta$$

converges in the norm of $H^\infty(\mathbf{D})$, remembering that $|\varphi|\geqq 1$ on $\bigcup_{j=0}^\infty\Delta_j$, and our assumption $\sum_{j=0}^\infty r_j/\varepsilon_j<\infty$. Modulo I, $L_I^0 f$ equals $L_I f$. A calculation shows that

$$\frac{1}{2\pi i}\int_{\partial D_j}(-T_\zeta\varphi(z)/\varphi(\zeta))f(\zeta)d\zeta = Q_jf(z)-\varphi(z)Q_j(f/\varphi)(z)$$

for $z\in\mathbf{D}\setminus D_j$, where, perhaps somewhat incorrectly, $Q_j(f/\varphi)$ denotes the $H_0^\infty(\mathbf{C}\setminus D_j)$ function

$$Q_j(f/\varphi)(z)=\frac{1}{2\pi i}\int_{\partial D_j}\frac{f(\zeta)}{\varphi(\zeta)(z-\zeta)}\,d\zeta,\qquad z\in\mathbf{C}\setminus D_j.$$

For $z\in\Delta_j^0\setminus D_j$,

$$f(z)/\varphi(z)=\frac{1}{2\pi i}\int_{\partial(\Delta_j\setminus D_j)}\frac{f(\zeta)}{\varphi(\zeta)(\zeta-z)}\,d\zeta,$$

and consequently,

$$\begin{aligned}\|Q_j(f/\varphi)\| &= \sup_{z\in\partial D_j}|Q_j(f/\varphi)(z)|\\ &\leqq \|f\|\cdot(1+R_j/(R_j-r_j)),\end{aligned}$$

which has a bound independent of j since $r_j=o(R_j)$ as $j\to\infty$. Applying Lemma 2.1 in [BEH], we find that

$$\sum_{j=0}^{\infty}Q_j(f/\varphi)(z),\qquad z\in V,$$

converges, uniformly on compact subsets of V, to a function in $H_0^\infty(V)$. Summing up, we have shown that

$$L_I^0f(z)=f(z)-\varphi\sum_{j=0}^{\infty}Q_j(f/\varphi)(z)$$

for $z\in W$, and consequently,

$$f-L_I^0f\in I\cdot H_0^\infty(V).$$

Extending L_T^0 linearly to $H^\infty(W)$ by defining it to be the identity on $H^\infty(\mathbf{D})$, we obtain

$$f-L_I^0f\in I\cdot H_0^\infty(V)$$

for every $f\in H^\infty(W)$. Since $f\in\ker L_I$ if and only if $L_I^0f\in I$, it follows that

$$\ker L_I=I\cdot H^\infty(W).$$

The uniqueness of L_I follows by copying the arguments used in Remark 3.12(b). The proof of the theorem is finished.

Remarks 4.7. (a) It is not hard to modify the proof of Theorem 4.6 so as not to require Lemma 4.5. That makes it possible to obtain a Theorem 4.6(d) also for discs not having centers on the real axis.

(b) Denote by $\mathscr{K}$ the family of all closed $H^\infty(\mathbf{D})$-ideals I with $h(I, H^\infty(\mathbf{D})) \subset \mathcal{M}_1(H^\infty(\mathbf{D}))$ such that $h(I, H^\infty(\mathbf{D}))$ and the $\mathcal{M}(H^\infty(\mathbf{D}))$-closure of $\{1-\varepsilon_j\}_0^\infty$ are disjoint. Using Theorem 4.6, it is not hard to show that the image of $\mathscr{K}$ under the mapping $I \to I \cdot H^\infty(W)$ consists of those (closed) $H^\infty(W)$-ideals J for which $J \cap H^\infty(\mathbf{D}) \in \mathscr{K}$. It would be nice to have a characterization of the image in terms of the hulls $h(J, H^\infty(W))$.

(c) For every $m_0 \in \mathcal{M}_1(H^\infty(\mathbf{D}))$ Behrens [BEH] described the sets ("fibers")

$$\{m \in \mathcal{M}(H^\infty(W)) \colon m\,|_{H^\infty(\mathbf{D})} = m_0\}.$$

It would be nice to have the same analysis carried through for non-maximal closed ideals in $H^\infty(\mathbf{D}))$ with hulls in $\mathcal{M}_1(H^\infty(\mathbf{D}))$, too, that is, given any closed $H^\infty(\mathbf{D}))$-ideal with $h(I, H^\infty(\mathbf{D})) \subset \mathcal{M}_1(H^\infty(\mathbf{D}))$, characterize the set of (continuous if you like) epimorphisms $H^\infty(W) \to H^\infty(\mathbf{D})/I$, the restrictions to $H^\infty(\mathbf{D})$ of which coincide with the canonical epimorphism $H^\infty(\mathbf{D}) \to H^\infty(\mathbf{D})/I$.

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