HEISENBERG'S UNCERTAINTY PRINCIPLE IN THE SENSE OF BEURLING

By

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In memory of Boris Korenblum

Abstract. We shed new light on Heisenberg's uncertainty principle in the sense of Beurling, by offering a fundamentally different proof which allows us to weaken the assumptions rather substantially. The new formulation is pretty much optimal, as can be seen from examples. Our arguments involve Fourier and Mellin transforms. We also introduce a version which applies to two given functions. Finally, we show how our approach applies in the higher dimensional setting.

1 Introduction

1.1 Heisenberg's uncertainty principle. In general terms, Heisenberg's uncertainty principle asserts that a function and its Fourier transform cannot both be too concentrated. See, e.g., the book of Havin and Jöricke [8] for a recent development connected with partial differential equations and dynamical systems, also [9], [5]. As for notation, we write

$$\hat{f}(y) := \lim_{T\to+\infty} \int_{-T}^{T} \mathrm{e}^{-\mathrm{i}2\pi yt} f(t)\mathrm{d}t, \quad y \in \mathbb{R},$$

for the Fourier transform of the function f, whenever the limit exists. For $f \in L^1(\mathbb{R})$, the integral converges absolutely and $\hat{f}$ is continuous on $\mathbb{R}$ with limit 0 at infinity (the Riemann-Lebesgue Lemma); writing $\mathrm{C}_0(\mathbb{R})$ for the Banach space of all such functions, we are merely saying that $\hat{f} \in \mathrm{C}_0(\mathbb{R})$ whenever $f \in L^1(\mathbb{R})$.

1.2 Beurling's version of the uncertainty principle. Building on work of Hardy [7], Beurling (see [3, p. 372]) found a version of Heisenberg's uniqueness principle which is attractive for its simplicity and beauty.

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Theorem 1.1 (Beurling)**.** *If* $f \in L^1(\mathbb{R})$ *and*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)\hat{f}(y)|\mathrm{e}^{2\pi|xy|}\mathrm{d}x\mathrm{d}y < +\infty, \tag{1.1}$$

then $f = 0$ *a.e. on* $\mathbb{R}$.

Trivially, $1 \le \mathrm{e}^{2\pi|xy|}$, so that if $f \in L^1(\mathbb{R})$ satisfies (1.1), we must also have

$$\|f\|_{L^1(\mathbb{R})}\|\hat{f}\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)\hat{f}(y)|\mathrm{d}x\mathrm{d}y < +\infty.$$

Thus, the assumption (1.1) implies that f and $\hat{f}$ are both in $L^1(\mathbb{R})$. As a result, f is in the space $L^1(\mathbb{R})\cap C_0(\mathbb{R})$, which is contained in $L^p(\mathbb{R})$ for all p with $1 \le p \le +\infty$.

The statement in [3, p. 372] was published without proof. Subsequently, it transpired that Hörmander retained a copy of Beurling's original proof. Hörmander writes [10, p. 237] "The editors state that no proof has been preserved. However, in my files I have notes which I made when Arne Beurling explained this result to me during a private conversation some time during the years 1964-1968 when we were colleagues at the Institute for Advanced Study."

Here we indicate how to reduce the assumption (1.1) of Theorem 1.1 while preserving the conclusion that $f = 0$ almost everywhere. In so doing, we present a fundamentally different proof of Beurling's theorem.

1.3 Statement of the generalization of Beurling's theorem. Our analysis of Beurling's theorem (Theorem 1.1) is based on the observation that under (1.1), the function

$$F(\lambda) := \int_{\mathbb{R}}\int_{\mathbb{R}} \bar{f}(x)\hat{f}(y)\mathrm{e}^{\mathrm{i}2\pi\lambda xy}\mathrm{d}x\mathrm{d}y \tag{1.2}$$

defines a bounded holomorphic function in the strip $\mathbb{S} := \{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < 1\}$ which extends continuously to the closed strip $\bar{\mathbb{S}}$. Indeed, the complex exponentials $\mathrm{e}^{\mathrm{i}2\pi\lambda xy}$ are holomorphic in λ, and we have

$$|F(\lambda)| \le \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)\hat{f}(y)|\mathrm{e}^{-2\pi xy\mathrm{Im}\lambda}\mathrm{d}x\mathrm{d}y \le \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)\hat{f}(y)|\mathrm{e}^{2\pi|xy|}\mathrm{d}x\mathrm{d}y, \qquad \lambda \in \bar{\mathbb{S}},$$

from which the claim is immediate, by, e.g., uniform convergence. Next, in view of the Fourier Inversion Theorem, $\int_{\mathbb{R}} \hat{f}(y)\mathrm{e}^{\mathrm{i}2\pi\lambda xy}\mathrm{d}y = f(\lambda x)$ for $x, \lambda \in \mathbb{R}$, so the function $F(\lambda)$ given by (1.2) may be expressed in the form

$$F(\lambda) = \int_{\mathbb{R}} \bar{f}(x)f(\lambda x)\,\mathrm{d}x, \qquad \lambda \in \mathbb{R}. \tag{1.3}$$

It is also easy to see that $F(\lambda)$ is continuous on $\mathbb{R}^\times$ since $f \in L^2(\mathbb{R})$. Here, $\mathbb{R}^\times$ is shorthand for $\mathbb{R} \setminus \{0\}$.

Let $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $\bar{\mathbb{D}}$ denote its closure (the closed unit disk). We denote by $\mathrm{d}A$ the area element in $\mathbb{C}$.

Theorem 1.2. *Suppose $f \in L^2(\mathbb{R})$, and let $F(\lambda)$ be given by* (1.3) *for $\lambda \in \mathbb{R}^\times$. Suppose that $F(\lambda)$ has a holomorphic extension to a neighborhood of $\bar{\mathbb{D}} \setminus \{\pm\mathrm{i}\}$, such that*

$$\int_{\mathbb{D}} |F(\lambda)|^2 |\lambda^2 + 1| \,\mathrm{d}A(\lambda) < +\infty. \tag{1.4}$$

Then

(a) $F(\lambda) \equiv c_0(1+\lambda^2)^{-1/2}$ *for some constant* $c_0 \geq 0$, *and*

(b) *if, in addition,* $\inf_{\mathbb{D}} |F(\lambda)|^2 |1+\lambda^2| = 0$, *then* $F(\lambda) \equiv 0$, *and consequently* $f = 0$ *almost everywhere.*

In comparison with Theorem 1.1, Theorem 1.2 assumes analytic continuation of $F(\lambda)$ to a much smaller set, and the a priori assumption that $f \in L^2(\mathbb{R})$ is weaker. Also, in Beurling's setting, the weighted square integrability condition (1.4) is trivially fulfilled because the function $F(\lambda)$ is then bounded on the strip $\mathbb{S}$, which also shows that $\inf_{\mathbb{D}} |F(\lambda)|^2 |1+\lambda^2| = 0$. Part (b) then gives that $f = 0$ almost everywhere.

1.4 Applications of the Mellin transforms.

To see what Theorem 1.2(a) means for the function f, we introduce the Mellin transforms $\mathbf{M}_0$ and $\mathbf{M}_1$ as follows:

$$\mathbf{M}_0[f](\tau) := \int_{\mathbb{R}^\times} |x|^{-1/2+\mathrm{i}\tau} f(x)\mathrm{d}x, \qquad \tau \in \mathbb{R}, \tag{1.5}$$

$$\mathbf{M}_1[f](\tau) := \int_{\mathbb{R}^\times} |x|^{-\frac{1}{2}+\mathrm{i}\tau} \operatorname{sgn}(x) f(x)\mathrm{d}x, \tag{1.6}$$

where $\operatorname{sgn}(x) = x/|x|$. When the above integrals fail to be absolutely convergent, they should be understood as the limits of integrals over the set $\epsilon < |x| < 1/\epsilon$, as $\epsilon \to 0^+$. The L^2 theory for the Mellin transform is analogous to that of the Fourier transform (the Mellin transform is associated with the multiplicative structure, while the Fourier transform is related with the additive structure). We remark that the multiplicative group $\mathbb{R}^\times$ is isomorphic to the additive group $\mathbb{R} \times \mathbb{Z}_2$, where

$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. We see that

$$\mathbf{M}_0[f](\tau) = \int_{\mathbb{R}_+} x^{-\frac{1}{2}+\mathrm{i}\tau}\{f(x)+f(-x)\}\mathrm{d}x$$

$$(1.7) \qquad = 2\pi \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi t\tau}\{f(\mathrm{e}^{2\pi t})+f(-\mathrm{e}^{2\pi t})\}\mathrm{e}^{\pi t}\mathrm{d}t$$

$$\mathbf{M}_1[f](\tau) = \int_{\mathbb{R}_+} x^{-\frac{1}{2}+\mathrm{i}\tau}\{f(x)-f(-x)\}\mathrm{d}x$$

$$(1.8) \qquad = 2\pi \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi t\tau}\{f(\mathrm{e}^{2\pi t})-f(-\mathrm{e}^{2\pi t})\}\mathrm{e}^{\pi t}\mathrm{d}t,$$

which explains how the well-known L^2 theory for the Fourier transform carries over to the Mellin transforms. Here, $\mathbb{R}_+ :=]0, +\infty[$ is the positive semi-axis. In particular, the Plancherel identity reads as follows:

$$(1.9) \qquad \frac{1}{4\pi}\int_{\mathbb{R}} \{|\mathbf{M}_0[f](\tau)|^2 + |\mathbf{M}_1[f](\tau)|^2\}\mathrm{d}\tau = \int_{\mathbb{R}} |f(x)|^2\mathrm{d}x.$$

Theorem 1.3. *Suppose $f \in L^2(\mathbb{R})$, and let $F(\lambda)$ be given by* (1.3) *for $\lambda \in \mathbb{R}^\times$. Then $F(\lambda) \equiv c_0(1+\lambda^2)^{-1/2}$ holds for some constant $c_0 \geq 0$ if and only if f is even (i.e., $f(-x) = f(x)$ holds a.e.), and*

$$|\mathbf{M}_0[f](\tau)| = \frac{\sqrt{c_0}}{\pi^{1/4}}\left|\Gamma\left(\tfrac{1}{4}+\tfrac{\mathrm{i}}{2}\tau\right)\right|, \qquad \tau \in \mathbb{R}.$$

Remark 1.4. To appreciate how much weaker the assumptions of Theorem 1.2 are compared with those of Beurling's Theorem 1.1, we may consider the assertion (a) of Theorem 1.2 combined with Theorem 1.3. We then know the modulus of the Mellin transform $\mathbf{M}_0[f]$, while it is clear from the L^2 theory of the Mellin transform that the argument of $\mathbf{M}_0[f]$ may be an arbitrary measurable function. So we get plenty of functions f which solve (1.3) for the given $F(\lambda) = c_0(1+\lambda^2)^{-1/2}$. One of these is of course the Gaussian $f(x) = c_1\mathrm{e}^{-\pi\alpha x^2}$, where $\alpha > 0$ and $|c_1|^2 = c_0\alpha^{1/2}$. This contrasts with the analogues of Beurling's theorem where the constant (or polynomial) multiples of a Gaussian $\mathrm{e}^{-\pi\alpha x^2}$ are the only solutions [4].

1.5 Analysis of the sharpness of the results. It is of interest to analyze the sharpness of Theorems 1.2 and 1.3. We look at the example $f(x) = \mathrm{e}^{-\pi\beta x^2}$, where $\operatorname{Re}\beta > 0$. Then $f \in L^2(\mathbb{R})$, and the associated function $F(\lambda)$ is

$$F(\lambda) = \int_{\mathbb{R}} \bar{f}(x)f(\lambda x)\mathrm{d}x = \bar{\beta}^{-1/2}\left(1+\frac{\beta}{\bar{\beta}}\lambda^2\right)^{-1/2}.$$

This $F(\lambda)$ is holomorphic in $\mathbb{D}$ but possesses two square root branch points at the roots of $\lambda^2 = -\bar{\beta}/\beta$. These roots lie on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. This means that permitting just two such square root branch points along $\mathbb{T}$ in the formulation of Theorem 1.2 already falsifies the assertion of the theorem. Suppose $\mathcal{D} \subset \mathbb{D}$ is a proper convex subset which is symmetric under reflexion in the origin ($\lambda \mapsto -\lambda$). Then if, in the formulation of Theorem 1.2, the unit disk $\mathbb{D}$ is replaced with $\mathcal{D}$, the conclusion of the theorem would fail.

1.6 A variation on Beurling's theme. It seems natural to ask what would happen in Beurling's Theorem 1.1 if the condition (1.1) were replaced with the rather similar-looking condition

$$\int_{\mathbb{R}} |f(x)\hat{f}(x)| e^{2\pi x^2} dx < +\infty. \tag{1.10}$$

Clearly, the choice of the Gaussian $f(x) = e^{-\pi x^2}$ makes (1.10) false, which would lead us to hope that (1.10) together with, say, $f, \hat{f} \in L^1(\mathbb{R})$, would guarantee that $f = 0$ almost everywhere. However, this is not the case. Actually, it is rather easy to construct a nontrivial Fourier pair $f, \hat{f} \in L^1(\mathbb{R})$ with $f(x)\hat{f}(x) \equiv 0$, so that $\int_{\mathbb{R}} |f(x)\hat{f}(x)| e^{2\pi x^2} dx = 0$. The following construction is from [2]; compare with [1]. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be C^∞-smooth even function, whose support is contained in the interval $[-1/8, 1/8]$. If we put $f(x) := \sum_{n\in\mathbb{Z}} e^{i\pi n}\hat{\varphi}(n)\varphi(x-n)$, then Poisson summation gives

$$\hat{f}(y) = \hat{\varphi}(y) \sum_{m\in\mathbb{Z}} \varphi\Big(y - m + \frac{1}{2}\Big).$$

It is clear that the above f and $\hat{f}$ have disjoint supports, and as a consequence, $f(x)\hat{f}(x) \equiv 0$.

2 A family of bilinear forms

2.1 The bilinear forms. Let us consider the bilinear forms

$$\mathbf{B}[f, g](\lambda) := \int_{\mathbb{R}} f(t)\, g(\lambda t) dt, \quad \lambda \in \mathbb{R}^\times, \tag{2.1}$$

for $f, g \in L^2(\mathbb{R})$. The function $\mathbf{B}[f, g]$ is then continuous on $\mathbb{R}^\times$. It has the symmetry property

$$\mathbf{B}[f, g](\lambda) = \frac{1}{|\lambda|}\mathbf{B}[g, f]\Big(\frac{1}{\lambda}\Big), \quad \lambda \in \mathbb{R}^\times, \tag{2.2}$$

as we see by an elementary change of variables. It also satisfies the complex conjugation symmetry relation

$$\overline{\mathbf{B}[f,g](\lambda)} = \mathbf{B}[\bar f,\bar g](\lambda), \quad \lambda\in\mathbb{R}^\times. \tag{2.3}$$

2.2 Relation to multiplicative convolution. It is well known that the multiplicative convolution

$$f_1 \circledast f_2(x) := \int_{\mathbb{R}^\times} f_1(t)\, f_2\Big(\frac{x}{t}\Big)\frac{\mathrm{d}t}{|t|},$$

understood in the sense of Lebesgue, is commutative (i.e., $f_1\circledast f_2 = f_2\circledast f_1$). The relationship with the above bilinear forms $\mathbf{B}[f,g](\lambda)$ is

$$\mathbf{B}[f,g](\lambda) = g\circledast\tilde f(\lambda) = \frac{1}{|\lambda|}\, f\circledast\tilde g\Big(\frac{1}{\lambda}\Big), \quad \lambda\in\mathbb{R}^\times,$$

where

$$\tilde f(t) := \frac{1}{|t|}\, f\Big(\frac1t\Big), \quad \tilde g(t) := \frac{1}{|t|}\, g\Big(\frac1t\Big).$$

3 The proofs of the first set of theorems

Proof of Theorem 1.2. A comparison of (1.3) and (2.1) reveals that $F(\lambda) = \mathbf{B}[\bar f, f](\lambda)$ for $\lambda\in\mathbb{R}^\times$. In view of (2.2) and (2.3), $F(\lambda)$ has the symmetry property

$$F(\lambda) = \frac{1}{|\lambda|}\bar F\Big(\frac{1}{\lambda}\Big), \qquad \lambda\in\mathbb{R}^\times. \tag{3.1}$$

Now $J(\lambda) := \sqrt{1+\lambda^2}$ defines a single-valued holomorphic function in the slit complex plane $\mathbb{C}\setminus \mathrm{i}(\mathbb{R}\setminus]-1,1[)$ with value 1 at $\lambda=0$. We consider the function $\Phi := FJ$, which is well defined and continuous along $\mathbb{R}$ and defines a holomorphic function in (a neighborhood of) $\bar{\mathbb{D}}\setminus\{\pm\mathrm{i}\}$. Along the real line, we have, in view of (3.1),

$$\begin{aligned}\Phi(\lambda) = F(\lambda)J(\lambda) &= \frac{1}{|\lambda|}J(\lambda)\bar F\Big(\frac1\lambda\Big) = \frac{\sqrt{1+\lambda^2}}{|\lambda|}\bar F\Big(\frac1\lambda\Big)\\ &= \sqrt{1+\frac{1}{\lambda^2}}\,\bar F\Big(\frac1\lambda\Big) = \bar\Phi\Big(\frac1\lambda\Big) = \bar\Phi\Big(\frac{1}{\bar\lambda}\Big), \quad \lambda\in\mathbb{R}^\times.\end{aligned} \tag{3.2}$$

As a consequence, Φ is real-analytic on $\mathbb{R}$ and has two holomorphic extensions, one to (a neighborhood of) $\bar{\mathbb{D}}\setminus\{\pm\mathrm{i}\}$, and the other to (a neighborhood of) $\bar{\mathbb{D}}_e\setminus\{\pm\mathrm{i}\}$; here, $\bar{\mathbb{D}}_e := \mathbb{C}\setminus\mathbb{D}$ is the closed exterior disk. These two holomorphic continuations must then coincide. So, Φ extends to a holomorphic function in $\mathbb{C}\setminus\{\pm\mathrm{i}\}$, which is bounded in a neighborhood of infinity, by inspection of (3.2). The integrability assumption of the theorem says that $\int_{\mathbb{D}}|\Phi(\lambda)|^2\mathrm{d}A(\lambda) < +\infty$, and the symmetry property (3.2) gives the corresponding integrability in the exterior disk $\mathbb{D}_e = \mathbb{C}\setminus\bar{\mathbb{D}}$:

$$\int_{\mathbb{D}_e}|\Phi(\lambda)|^2\frac{\mathrm{d}A(\lambda)}{|\lambda|^4} < +\infty.$$

In particular, Φ is square area-integrable in a neighborhood of $\{\pm\mathrm{i}\}$. But then Φ extends holomorphically across $\pm\mathrm{i}$ (one explanation among many: a two-point set has logarithmic capacity 0; see [6]). Now Φ is entire and bounded, so by Liouville's theorem, Φ is constant: $\Phi(\lambda)\equiv c_0$. That $c_0\geq 0$ follows from

$$c_0 = \Phi(1) = J(1)F(1) = \sqrt{2}\int_{\mathbb{R}}\bar{f}(x)f(x)\mathrm{d}x = \sqrt{2}\int_{\mathbb{R}}|f(x)|^2\mathrm{d}x \geq 0.$$

This yields the first assertion and the second. □

Proof of Theorem 1.3. We need to show that if

$$F(\lambda) = \int_{\mathbb{R}}\bar{f}(x)f(\lambda x)\mathrm{d}x \equiv c_0(1+\lambda^2)^{-1/2},$$

on $\mathbb{R}^\times$, then the Mellin transform $\mathbf{M}_0[f]$ has the indicated form. By symmetry, we see that $\mathbf{M}_1[F](\tau)\equiv 0$, while a computation reveals that

$$\mathbf{M}_0[F](\tau) = c_0\int_{\mathbb{R}^\times}|\lambda|^{-1/2+\mathrm{i}\tau}(1+\lambda^2)^{-1/2}\mathrm{d}\lambda = \frac{c_0}{\sqrt{\pi}}\left|\Gamma\left(\tfrac{1}{4}+\tfrac{\mathrm{i}}{2}\tau\right)\right|^2,$$

Applying the Mellin transforms $\mathbf{M}_0, \mathbf{M}_1$ to (1.3), we find that $\mathbf{M}_1[f]\equiv 0$ and that $|\mathbf{M}_0[f](\tau)|^2 = \mathbf{M}_0[F](\tau)$. Here, the natural way to verify the right-hand side equality is to apply the inverse Mellin transform to the two sides. The assertion that $\mathbf{M}_1[f]\equiv 0$ holds if and only if f is an even function (cf. (1.8)). □

4 A generalization involving two functions

4.1 A problem involving two functions. We consider two functions $f, g \in L^2\mathbb{R})$, and introduce the functions

$$F_1(\lambda) := \int_{\mathbb{R}}\bar{f}(x)g(\lambda x)\mathrm{d}x, \quad F_2(\lambda) := \int_{\mathbb{R}}\bar{g}(x)f(\lambda x)\mathrm{d}x. \tag{4.1}$$

We quickly observe that if f is even and g is odd, then $F_1(\lambda)\equiv F_2(\lambda)\equiv 0$ on $\mathbb{R}^\times$. The same conclusion holds if f is odd and g is even. This means that we cannot hope to claim that one of the functions f, g must vanish from the knowledge that $F_1(\lambda)\equiv F_2(\lambda)\equiv 0$. But sometimes this combination of even and odd is the only obstruction.

Theorem 4.1. *Suppose $f, g \in L^2(\mathbb{R})$, and let $F_j(\lambda)$ be given by* (4.1) *for $\lambda \in \mathbb{R}^\times$ and $j = 1, 2$. Suppose that both $F_j(\lambda)$ have holomorphic extensions to $\mathbb{D}$ such that*

$$\int_{\mathbb{D}} |F_j(\lambda)|^2 |\lambda^2 + 1| \mathrm{d}A(\lambda) < +\infty, \quad j = 1, 2.$$

Suppose, moreover, that one of the functions, say F_1, has a holomorphic extension to a neighborhood of $\bar{\mathbb{D}} \setminus \{\pm\mathrm{i}\}$. Then

(a) *$F_j(\lambda) \equiv c_j(1 + \lambda^2)^{-1/2}$ for $j = 1, 2$, for some constants $c_1, c_2 \in \mathbb{C}$ with $c_2 = \bar{c}_1$; and*

(b) *if in* (a) *we have $c_1 = 0$, then $F_1(\lambda) \equiv F_2(\lambda) \equiv 0$.*

If we compare with Beurling's result (Theorem 1.1), it is clear that if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \big(|f(x)\hat{g}(y)| + |g(x)\hat{f}(y)|\big)\, \mathrm{e}^{2\pi|xy|} \mathrm{d}x\mathrm{d}y < +\infty, \tag{4.2}$$

we are in the setting of part (b) of Theorem 4.1.

4.2 Application of the Mellin transforms.

The application of the Mellin transforms leads to the following result.

Theorem 4.2. *Suppose $f, g \in L^2(\mathbb{R})$, and let $F_1(\lambda)$ be given by* (4.1) *for $\lambda \in \mathbb{R}^\times$. Then $F_1(\lambda) \equiv c_1(1 + \lambda^2)^{-1/2}$ holds for some constant $c_1 \in \mathbb{C}$ if and only if*

$$\overline{\mathbf{M}_1[f](\tau)}\mathbf{M}_1[g](\tau) = 0, \quad a.e.\ \tau \in \mathbb{R},$$
$$\overline{\mathbf{M}_0[f](\tau)}\mathbf{M}_0[g](\tau) = \frac{c_1}{\sqrt{\pi}} \big|\Gamma\big(\tfrac{1}{4} + \tfrac{\mathrm{i}}{2}\tau\big)\big|^2, \quad a.e.\ \tau \in \mathbb{R}.$$

The assertion of Theorem 4.2 gives a very precise answer as to what f, g can be in the setting of Theorem 4.1. It may, however, be difficult at times to see what the conditions actually say when f, g are explicitly given. So let us explain a couple of cases in which one can be more precise. The *support* of a function $f \in L^2(\mathbb{R})$ – written $\operatorname{supp} f$ – is the intersection of all closed sets $E \subset \mathbb{R}$ such that $f = 0$ a.e. on $\mathbb{R} \setminus E$. Let us agree to say that a function $f \in L^2(\mathbb{R})$ has *dilationally one-sided support* if

(i) $\operatorname{supp} f$ is bounded in $\mathbb{R}$, or if

(ii) $\operatorname{supp} f \subset \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.

Theorem 4.3. *Let $f, g \in L^2(\mathbb{R})$, and $F_1(\lambda)$ be given by* (4.1) *for $\lambda \in \mathbb{R}^\times$. Suppose that $F_1(\lambda) \equiv 0$. If one of f, g, say f, has dilationally one-sided support, then one of the following holds:*

1. *f is even and g is odd,*
2. *f is odd and g is even, or*
3. *$f = 0$ almost everywhere or $g = 0$ almost everyehere.*

4.3 A comparison with the Beurling-type condition (4.2)**.** We compare the assumptions of Theorem 4.3 with the Beurling-type condition (4.2). Clearly (4.2) is a very strong assumption, as it actually forces $f = 0$ a.e. or $g = 0$ almost everywhere. This can be shown by a suitable modification of the argument in the Appendix in [4].

This suggests that if we strengthen the assumptions in Theorem 4.3 slightly, we should be able to rule out alternatives (a)–(b). To this end, we consider the functions $f_*(x) := f(x)\,\mathrm{sgn}(x)$ and $g_*(x) := g(x)\,\mathrm{sgn}(x)$, and put

$$F_3(\lambda) := \int_{\mathbb{R}}\int_{\mathbb{R}} \bar{f}_*(x)\hat{g}(y)\mathrm{e}^{\mathrm{i}2\pi\lambda xy}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} \bar{f}_*(x)g(\lambda x)\mathrm{d}x, \quad \lambda \in \mathbb{R},$$
$$F_4(\lambda) := \int_{\mathbb{R}}\int_{\mathbb{R}} \bar{g}_*(x)\hat{f}(y)\mathrm{e}^{\mathrm{i}2\pi\lambda xy}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} \bar{g}_*(x)f(\lambda x)\mathrm{d}x, \quad \lambda \in \mathbb{R}.$$

Were we to assume (4.2), we would know that F_3, F_4 both extend holomorphically and boundedly to the strip $\mathbb{S}$. We assume less, namely that both extend holomorphically to the open unit disk $\mathbb{D}$, are area-L^2 integrable on $\mathbb{D}$, and that one of them extends holomorphically to (a neighborhood of) $\bar{\mathbb{D}} \setminus \{\pm\mathrm{i}\}$. It is easy to verify that $F_3(\lambda) = \bar{F}_4(1/\lambda)/\lambda$ for $\lambda \in \mathbb{R}^\times$; so $F_3(\lambda)$ has a holomorphic extension to $\mathbb{C} \setminus \{\pm\mathrm{i}\}$ which is area-L^2 integrable near $\pm\mathrm{i}$ and vanishes at infinity. Then the singularities at $\pm\mathrm{i}$ are removable, and Liouville's theorem gives $F_3(\lambda) \equiv 0$. At the same time, we know that $F_1(\lambda) \equiv 0$ from Theorem 4.1, since we ask that $c_1 = 0$ as in Theorem 4.3. Theorem 4.2 and its proof tell us that for $j = 0, 1$,

$$\overline{\mathbf{M}_j[f](\tau)}\mathbf{M}_j[g](\tau) = 0, \quad a.e.\ \tau \in \mathbb{R},$$
$$\overline{\mathbf{M}_j[f_*](\tau)}\mathbf{M}_j[g](\tau) = 0, \quad a.e.\ \tau \in \mathbb{R}.$$

Next, as in Theorem 4.3, we assume that f or g has dilationally one-sided support. Then Theorem 4.3 and its proof show that the only possibilities are that $f = 0$ a.e. or $g = 0$ almost everywhere.

5 Proofs of the theorems involving two functions

Proof of Theorem 4.1. A comparison of (1.3) and (2.1) reveals that $F_1(\lambda) = \mathbf{B}[\bar{f}, g](\lambda)$ and $F_2(\lambda) = \mathbf{B}[\bar{g}, f](\lambda)$ for $\lambda \in \mathbb{R}^\times$. In view of (2.2) and (2.3), we have the symmetry property

$$F_1(\lambda) = \frac{1}{|\lambda|}\bar{F}_2\left(\frac{1}{\lambda}\right), \quad \lambda \in \mathbb{R}^\times. \tag{5.1}$$

If we put $\Phi_j := F_jJ$, where, as before, $J(\lambda) = (1+\lambda^2)^{1/2}$, then (5.1) implies that $\Phi_1(\lambda) = \bar{\Phi}_2(1/\lambda)$ for $\lambda \in \mathbb{R}^\times$. The given assumptions on F_1, F_2 show that Φ_1 has a holomorphic extension to $\mathbb{C} \setminus \{\pm\mathrm{i}\}$, which is area-$L^2$ integrable locally around $\{\pm\mathrm{i}\}$. As a consequence, the singularities at $\{\pm\mathrm{i}\}$ are removable (see, e.g., [6]), and Liouville's theorem tells us that Φ_1 is constant. The remaining assertions are easy consequences of this.

Proof of Theorem 4.2. The proof is immediate by taking the Mellin transforms, as in the proof of Theorem 1.3. We omit the details. □

Proof of Theorem 4.3. In view of Theorem 4.2, we have that

$$\overline{\mathbf{M}_0[f](\tau)}\,\mathbf{M}_0[g](\tau) = \overline{\mathbf{M}_1[f](\tau)}\mathbf{M}_1[g](\tau) = 0, \quad \text{a.e. } \tau \in \mathbb{R}. \tag{5.2}$$

The assumption that f has dilationally one-sided support means in terms of Mellin transforms that up to a complex exponential factor, the functions $\mathbf{M}_j[f]$, $j = 0, 1$, both extend to a function in H^2 of either the upper or the lower half-plane. In any case, Privalov's theorem guarantees that for a given $j \in \{0, 1\}$, either $\mathbf{M}_j[f] = 0$ a.e. on $\mathbb{R}$, or $\mathbf{M}_j[f] \neq 0$ a.e. on $\mathbb{R}$. This leaves us with four distinct possibilities.

CASE 1. $\mathbf{M}_0[f] = 0$ a.e. and $\mathbf{M}_1[f] = 0$ almost everywhere. Then $f = 0$ a.e. is immediate, so we find ourselves in the setting of (c).

CASE 2. $\mathbf{M}_0[f] = 0$ a.e. and $\mathbf{M}_1[f] \neq 0$ almost everywhere. Then (5.2) gives that $\mathbf{M}_1[g] = 0$ a.e., so that f is odd and g is even (cf. (1.7)–(1.8)), and we are in the setting of (b).

CASE 3. $\mathbf{M}_0[f] \neq 0$ a.e. and $\mathbf{M}_1[f] = 0$ almost everywhere. Then (5.2) gives that $\mathbf{M}_0[g] = 0$ a.e., and we conclude that f is even and g is odd (cf. (1.7)–(1.8)), and we are in the setting of (a).

CASE 4. $\mathbf{M}_0[f] \neq 0$ a.e. and $\mathbf{M}_1[f] \neq 0$ almost everywhere. Then (5.2) shows that $\mathbf{M}_0[g] = \mathbf{M}_1[g] = 0$ a.e., so that $g = 0$ a.e., and we are in the setting of (c). □

6 A higher dimensional analogue

6.1 The bilinear form in higher dimensions. We present an analogue of Theorem 1.2 for $\mathbb{R}^n$, $n = 1, 2, 3, \ldots$; dvol_n is volume measure in $\mathbb{R}^n$. For $f \in L^2(\mathbb{R}^n)$, let $F(\lambda)$ be the function

$$F(\lambda) = \int_{\mathbb{R}^n} \bar{f}(x) f(\lambda x)\mathrm{dvol}_n(x), \quad \lambda \in \mathbb{R}^\times. \tag{6.1}$$

This function arises if we write

$$F(\lambda) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \bar{f}(x)\hat{f}(y)\mathrm{e}^{\mathrm{i}2\pi\lambda\langle x,y\rangle}\mathrm{dvol}_n(x)\mathrm{dvol}_n(y),$$

where $\hat{f}(y) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi\langle x,y\rangle} f(x)\mathrm{dvol}_n(x)$ is the usual Fourier transform and

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n, \quad x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n),$$

is the usual inner product in $\mathbb{R}^n$.

6.2 The higher-dimensional extension of the generalized Beurling theorem. We can now formulate the extension of Theorem 1.2 to the setting of $\mathbb{R}^n$.

Theorem 6.1. *Suppose $f \in L^2(\mathbb{R}^n)$, and let $F(\lambda)$ be given by* (6.1) *for $\lambda \in \mathbb{R}^\times$. Suppose that $F(\lambda)$ has a holomorphic extension to a neighborhood of $\bar{\mathbb{D}} \setminus \{\pm\mathrm{i}\}$ such that $\int_{\mathbb{D}} |F(\lambda)|^2 |\lambda^2+1|^n \,\mathrm{d}A(\lambda) < +\infty$. Then*

(a) *$F(\lambda) \equiv c_0(1+\lambda^2)^{-n/2}$ for some constant $c_0 \geq 0$; and*

(b) *if, in addition, $\inf_{\mathbb{D}} |F(\lambda)|^2 |1+\lambda^2|^n = 0$, then $F(\lambda) \equiv 0$, and consequently $f = 0$ almost everywhere.*

7 Proof of the higher dimensional analogue

Proof of Theorem 6.1. We indicate what differs from the case $n = 1$, which is covered by the proof of Theorem 1.2. An exercise involving a change of variables shows that $F(\lambda)$ has the symmetry property

$$F(\lambda) = \frac{1}{|\lambda|^n} \bar{F}\left(\frac{1}{\lambda}\right), \quad \lambda \in \mathbb{R}^\times. \tag{7.1}$$

Let $J_n(\lambda) := (1+\lambda^2)^{n/2}$. We consider the function $\Phi := F J_n$, which is well defined and continuous along $\mathbb{R}$ and defines a holomorphic function in (a neighborhood of) $\bar{\mathbb{D}} \setminus \{\pm\mathrm{i}\}$. Along the real line, we have, in view of (7.1),

$$\begin{aligned} \Phi(\lambda) = F(\lambda)J_n(\lambda) &= \frac{1}{|\lambda|^n} J_n(\lambda)\bar{F}\left(\frac{1}{\lambda}\right) = \frac{(1+\lambda^2)^{n/2}}{|\lambda|^n} \bar{F}\left(\frac{1}{\lambda}\right) \\ &= \left(1+\frac{1}{\lambda^2}\right)^{n/2} \bar{F}\left(\frac{1}{\lambda}\right) = \bar{\Phi}\left(\frac{1}{\lambda}\right) = \bar{\Phi}\left(\frac{1}{\bar{\lambda}}\right), \quad \lambda \in \mathbb{R}^\times. \end{aligned} \tag{7.2}$$

As a consequence of the assumptions, Φ extends to a holomorphic function in $\mathbb{C} \setminus \{\pm\mathrm{i}\}$, which is bounded in a neighborhood of infinity, by inspection of (7.2). The integrability assumption of the theorem says that Φ is area-L^2 integrable near $\{\pm\mathrm{i}\}$, so that the singularities at $\pm\mathrm{i}$ are removable. Liouville's theorem tells us that Φ is constant: $\Phi(\lambda) \equiv c_0$. That $c_0 \geq 0$ follows from

$$c_0 = \Phi(1) = J_n(1)F(1) = 2^{n/2} \int_{\mathbb{R}^n} \bar{f}(x) f(x) \mathrm{dvol}_n(x) = 2^{n/2} \int_{\mathbb{R}^n} |f(x)|^2 \mathrm{dvol}_n(x) \geq 0.$$

This yields the first assertion as well as the second. □

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