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ON THE f – AND K– PROPERTIES OF CERTAIN FUNCTION SPACES

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I will assume that the reader is familiar with the Hardy spaces H^p on the open unit disc **D** for $0 < p \le \infty$, the spaces $BMO(\mathbf{T})$ and $VMO(\mathbf{T})$ over the unit circle $\mathbf{T} = \partial\mathbf{D}$, and their analytic counterparts $BMOA = BMO(\mathbf{T}) \cap H^2$ and $VMOA = VMO(\mathbf{T}) \cap H^2$. Let P denote the (orthogonal) Riesz projection $L^2 = L^2(\mathbf{T}) \to H^2$. For $\varphi \in L^\infty$, introduce the associated Toeplitz operator $T_\varphi(f) = P(\varphi f)$, $f \in H^2$. The Riesz projection P has a natural extension to the space $M(\mathbf{T})$ of Borel measures on **T**, and the range $P(M(\mathbf{T}))$ is contained in $\cap\{H^p : p < 1\}$. This makes it possible to define $T_\varphi(f)$ for $f \in H^1$.

DEFINITION. A subspace X of H^1 is said to have the

(a) f–property if for any $f \in X$ and any inner function u such that $f/u \in H^1$, it follows that $f/u \in X$.

(b) K_i–property if $T_{\bar{u}}(f) \in X$ for any $f \in X$ and any inner function u,

(c) K–property if $T_{\bar{g}}(f) \in X$ for any $f \in X$ and any $g \in H^\infty$.

Observe that $K \implies K_i \implies f$, that is, if X has the K_i–property, it also enjoys the f–property, and if X has the K–property, it also has the K_i–property.

PROPOSITION 1. *The spaces VMOA and BMOA both have the K–property.*

PROOF. It is standard to make the identifications $VMOA^* = H^1$ and $(H^1)^* = BMOA$. Let $f \in BMOA$, $g \in H^\infty$, and $h \in H^2$. Then the formula

$$\langle h, T_{\bar{g}}(f)\rangle = \langle gh, f\rangle$$

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and the entailing estimate

$$|\langle h, T_{\bar{g}}(f)\rangle| \leq \|g\|_{H^\infty} \cdot \|h\|_{H^1} \cdot \|f\|_{\mathrm{BMOA}}$$

show that $T_{\bar{g}}(f) \in (H^1)^* = BMOA$. If f is a polynomial, $T_{\bar{g}}(f)$ is a polynomial, too, and since $VMOA$ equals the closure of the polynomials in $BMOA$, it follows that $T_{\bar{g}}(VMOA) \subseteq VMOA$. The proof is complete. □

In [1,2], J. M. Anderson posed the following question: Does $QA = VMOA \cap H^\infty$ have the f– or K–property? With what we have done so far, the answer to the first part of this question is trivial.

PROPOSITION 2. *QA has the f–property.*

PROOF. Let $f \in QA$ and let u be an inner function such that $f/u \in H^1$. It follows that $f/u \in H^\infty$ by the Hardy space theory, and by Proposition 1, $f/u = P(\bar{u}f) \in VMOA$, and so $f/u \in QA$. □

REMARK. It appears that Proposition 2 was first discovered by P. Gorkin [3], apparently unaware of Anderson's question.

The fact that no subspace of H^∞ containing the disc algebra A, in particular QA, has the K_i–property (and hence not the K–property either), is a consequence of the following deep result, due to S. V. Hruščev and S. A. Vinogradov [4]. An infinite Blaschke product is said to be a Frostman Blaschke product if its associated zero sequence $\{a_n\}_1^\infty$, counted with respect to multiplicity, satisfies

$$\sup_{w \in \mathbf{T}} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|w - a_n|} < +\infty$$

All finite Blaschke products are also included in the collection of Frostman Blaschke products.

THEOREM 1. *Let $u \in H^\infty$ be an inner function. Then $T_{\bar{u}}$ is a bounded operator on A or H^∞ if and only if u is a Frostman Blaschke product.*

COROLLARY. *If u is an inner function, but not a Frostman Blaschke product, then $T_{\bar{u}}(A) \not\subset H^\infty$.*

PROOF. Suppose $T_{\bar{u}}(A) \subseteq H^\infty$. We know $T_{\bar{u}}$ is a bounded operator $H^2 \to H^2$. Assume $f_n \to f$ in A and $T_{\bar{u}}(f_n) \to g$ in H^∞. Since $T_{\bar{u}}$ is continuous on H^2, $T_{\bar{u}}(f) = g$. By the closed graph theorem, $T_{\bar{u}}$ is bounded $A \to H^{\infty}$. Now since $T_{\bar{u}}$ maps polynomials into polynomials, and the closure in H^{∞} of the polynomials is A, $T_{\bar{u}}(A) \subseteq A$, so by Theorem 1, u must be a Frostman Blaschke product, which is a contradiction. □

COROLLARY. *No subspace of H^{∞} containing A has the K_i–property, and therefore not the K–property either.*

COROLLARY. *The spaces H^{∞}, A, and QA do not have the K_i–property.*

REMARK. Early in 1987 I wrote a letter to J. M. Anderson mentioning that I had solved his problems. He later informed me that K. Izuchi [5] had solved the problems independently, and that Izuchi's letter about his solution arrived at approximately the same time as mine. However, it appears that my proofs are a lot shorter, and since I obtain a stronger result, a separate publication seems motivated.

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