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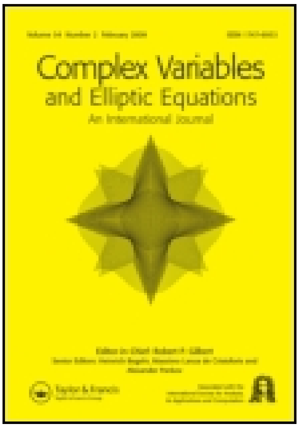

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Håkan Hedenmalm

[a] Department of Mathematics, The Royal Institute of Technology, S-10044 Stockholm, Sweden

[b] Remembering Matts

[c] Communicated by B.Gustafsson

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A note on Poincaré metric related quantities in conformal mapping

HÅKAN HEDENMALM*

Department of Mathematics, The Royal Institute of Technology, S–10044 Stockholm, Sweden

Remembering Matts

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1. INTRODUCTION

A univalent function f on the unit disk $\mathbb{D}$ is said to be in the class $\mathcal{S}$ if it maps $\mathbb{D}$ onto some simply connected domain, and is subject to the normalizations $f(0) = 0$ and $f'(0) = 1$. It is well known from the classical works of Köbe and Bieberbach [4,1] (see any book on Conformal Mapping) that for functions $f \in \mathcal{S}$, we have the following estimate:

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\bar{z}}{1-|z|^2}\right| \leq \frac{4}{1-|z|^2}, \qquad z \in \mathbb{D}. \tag{1.1}$$

This estimate is a consequence of an invariance property of the class $\mathcal{S}$ and of Bieberbach's theorem $|a_2| \leq 2$, where a_2 stands for the second (Taylor) coefficient (at the origin) of a function in $\mathcal{S}$. Here, we shall take a second look at the above estimate. Let $\Omega = f(\mathbb{D})$ be the image domain under f, and let $F = f^{-1}$ be the conformal map $\Omega \to \mathbb{D}$. Then (1.1) can be written in the form

$$\left|\frac{F''(z)}{F'(z)} + 2\,\frac{F'(z)\,\bar{F}(z)}{1-|F(z)|^2}\right| \leq 4\,\frac{|F'(z)|}{1-|F(z)|^2}, \qquad z \in \Omega. \tag{1.2}$$

*E-mail: haakanh@math.kth.se

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The gradient of a real-valued expression Φ defined on some open set in the plane is as usual the vector

$$\nabla\Phi(z) = \big(\partial_x\Phi(z), \partial_y\Phi(z)\big).$$

We shall use the notation ∂_z, $\bar\partial_z$ for the standard Wirtinger differentiation operators. We see that for real-valued Φ,

$$|\nabla\Phi(z)|^2 = 4\,|\partial_z\Phi(z)|^2 = 4\,|\bar\partial_z\Phi(z)|^2.$$

From the identity

$$\partial_z \log\frac{|F'(z)|}{1-|F(z)|^2} = \frac{1}{2}\frac{F''(z)}{F'(z)} + \frac{F'(z)\,\bar F(z)}{1-|F(z)|^2}$$

and the real-valuedness of the function

$$\log\frac{|F'(z)|}{1-|F(z)|^2}$$

we see that (1.2) states that

$$\left|\nabla\log\frac{|F'(z)|}{1-|F(z)|^2}\right| \le 4\,\frac{|F'(z)|}{1-|F(z)|^2}, \qquad z\in\Omega. \tag{1.3}$$

The *hyperbolic metric* in Ω is given by the expression

$$ds_\Omega(z) = \frac{|F'(z)|}{1-|F(z)|^2}\,|dz|, \qquad z\in\Omega;$$

let

$$\Phi_F(z) = \frac{|F'(z)|}{1-|F(z)|^2}, \qquad z\in\Omega,$$

be the associated density function. It is well known that the curvature of the hyperbolic metric is given by

$$\kappa = -\frac{4}{\Phi_F(z)^2}\,\Delta\log\Phi_F(z) = -4,$$

where

$$\Delta = \partial_z\bar\partial_z = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

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is the normalized Laplacian. In terms of Φ_F, the estimate (1.3) is expressed rather conveniently:

$$\frac{1}{\Phi_F(z)}\left|\nabla \log \Phi_F(z)\right| \le 4, \qquad z \in \Omega. \tag{1.4}$$

The left hand side vaguely resembles the expression for the curvature; we refer the reader to Osgood's paper [6] for related matters.

Now, suppose the simply connected domain Ω is contained in some other simply connected domain Ω', which is not the whole plane, and that G maps Ω' conformally onto $\mathbb{D}$, normalized so that $G(0) = 0$ and $G'(0) > 0$. *The remark we wish to make here is that we can find an estimate which generalizes* (1.4) *but involves both F and G*:

$$\left|\nabla \log \Phi_F(z) - \nabla \log \Phi_G(z)\right| \le 4\big(\Phi_F(z) - \Phi_G(z)\big), \qquad z \in \Omega. \tag{1.5}$$

It is easy to see how (1.5) results from (1.4) by letting Ω' grow so that it eventually covers the entire plane. As we write out (1.5) in terms of derivatives, we obtain

$$\left|\frac{F''(z)}{F'(z)} - \frac{G''(z)}{G'(z)} + 2\,\frac{F'(z)\bar{F}(z)}{1-|F(z)|^2} - 2\,\frac{G'(z)\bar{G}(z)}{1-|G(z)|^2}\right| \le 4\left(\frac{|F'(z)|}{1-|F(z)|^2} - \frac{|G'(z)|}{1-|G(z)|^2}\right), \quad z \in \Omega. \tag{1.6}$$

Professor Pommerenke has informed me that (1.6) can be deduced from the classical coefficient inequality

$$|a_2| \le 2\,|a_1|\,(1-|a_1|)$$

for conformal maps $\varphi(z) = a_1 z + a_2 z^2 + \cdots$ that map the unit disk $\mathbb{D}$ into itself. So, in a sense, the result presented here is not new. However, it could be argued that the formulation is of some interest, as well as the shape of certain expressions which appear in the proof.

2. THE PROOF

We use the Löwner–Kufarev method (see, for instance, Löwner [5], and Goluzin's book [3] as well as that of Duren [2]) which involves finding intermediate domains $\Omega(t)$, such that $\Omega(t)$ grows with t and has $\Omega(0) = \Omega$ and $\Omega(T) = \Omega'$, for some positive parameter value T. With each $\Omega(t)$, we associate a conformal mapping $f_t : \mathbb{D} \to \Omega(t)$, normalized so that $f_t(0) = 0$ and $f_t'(0) > 0$. It is well known that we may normalize the chain of domains so that $f_t'(0) = e^t$ [7]. Then $f_0 = f$ and $f_T = g$, and the inverse mappings are $f_0^{-1} = F$ and $f_T^{-1} = G$. The Löwner–Kufarev evolution equation then connects these mappings:

$$\frac{d}{dt} f_t(z) = z\,H(z,t) f_t'(z), \qquad z \in \mathbb{D}, \quad t \in [0,T], \tag{2.1}$$

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where $H(z,t)$ is analytic in z and Borel measurable in t, and for each $t \in [0,T]$ and $z \in \mathbb{D}$,

$$0 < \operatorname{Re} H(z,t), \qquad \operatorname{Re} H(0,t) = 1.$$

This means that we can write an integral representation formula

$$H(z,t) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\mu_t(\xi) + i\gamma(t), \tag{2.2}$$

where $d\mu_t$ is a Borel probability measure on $\mathbb{T}$ that varies measurably with t, and $\gamma(t)$ is a (Borel) measurable real-valued function of t. The function $\gamma(t)$ corresponds to introducing a rotational element in the evolution, which is not really needed; without loss of generality, we may assume $\gamma(t) \equiv 0$. This means getting rid of a rotation in the end point functions f and g, which is expressed by the requirements $0 < f'(0)$ and $0 < g'(0)$.

Connected with the evolution equation (2.1), there is the flow of points

$$z(t) = f_t^{-1} \circ f_0(z),$$

starting from a given point $z \in \mathbb{D}$ and moving inward toward the origin as time t progresses. Let us introduce the more convenient notation

$$\varphi_t(z) = f_t^{-1} \circ f_0(z),$$

and note that the evolution equation (2.1) transforms into

$$\frac{d}{dt}\,\varphi_t(z) = -\varphi_t(z)\, H\big(\varphi_t(z), t\big), \qquad z \in \mathbb{D}, \quad t \in [0,T], \tag{2.3}$$

with the initial value

$$\varphi_0(z) = z, \qquad z \in \mathbb{D}.$$

We also consider the quantity

$$\Lambda_t(z) = \frac{|\varphi_t'(z)|}{1 - |\varphi_t(z)|^2},$$

which comes from the Poincaré metric: under the coordinate shift $w = \varphi_t(z)$,

$$\frac{|dw|}{1 - |w|^2} = \Lambda_t(z)\, |dz|.$$

A simple calculation yields

$$\frac{\partial}{\partial z} \log \Lambda_t(z) = \frac{1}{2}\,\frac{\varphi_t''(z)}{\varphi_t'(z)} + \frac{\varphi_t'(z)\bar{\varphi}_t(z)}{1 - |\varphi_t(z)|^2}. \tag{2.4}$$

From the Löwner–Kufarev equation (2.3) and the above relation (2.4), it follows that

$$\frac{d}{dt}\frac{\partial}{\partial z}\log\Lambda_t(z) = -2\int_{\mathbb{T}}\frac{1-|\varphi_t(z)|^2}{|1-\xi\,\varphi_t(z)|^2}\,\frac{\xi\,\varphi_t'(z)}{(1-\xi\,\varphi_t(z))^2}\,d\mu_t(\xi), \tag{2.5}$$

where $d\mu_t$ is the probability measure in (2.2) and we have assumed $\gamma(t)\equiv 0$ as indicated previously. By a similar calculation, we check that

$$\frac{d}{dt}\Lambda_t(z) = -|\varphi_t'(z)|(1-|\varphi_t(z)|^2)\int_{\mathbb{T}}\frac{d\mu_t(\xi)}{|1-\xi\varphi_t(z)|^4}. \tag{2.6}$$

This means that (2.5) and (2.6) combine to give

$$\left|\frac{d}{dt}\frac{\partial}{\partial z}\log\Lambda_t(z)\right| \le -2\frac{d}{dt}\Lambda_t(z). \tag{2.7}$$

We integrate with respect to t, and obtain

$$\left|\frac{\partial}{\partial z}\log\Lambda_t(z) - \frac{\partial}{\partial z}\log\Lambda_0(z)\right| \le 2\left(\Lambda_0(z)-\Lambda_t(z)\right), \qquad z\in\mathbb{D}, \quad t\in[0,T]. \tag{2.8}$$

We perform the coordinate shift $z=f_0^{-1}(w)=F(w)$, where $w\in\Omega$, and notice that

$$|F'(w)|\,\Lambda_t\big(F(w)\big) = \frac{|F_t'(w)|}{1-|F_t(w)|^2} = \Phi_{F_t}(w), \qquad w\in\Omega.$$

Inserting this into (2.8), we arrive at

$$\left|\frac{\partial}{\partial w}\log\Phi_{F_t}(w) - \frac{\partial}{\partial w}\log\Phi_{F_0}(w)\right| \le 2\left(\Phi_{F_0}(w)-\Phi_{F_t}(w)\right), \qquad w\in\Omega, \quad t\in[0,T]. \tag{2.9}$$

Plugging in $t=T$, recalling that $F_0=F$ and $F_T=G$, we obtain

$$\left|\frac{\partial}{\partial w}\log\Phi_G(w) - \frac{\partial}{\partial w}\log\Phi_F(w)\right| \le 2\left(\Phi_F(w)-\Phi_G(w)\right), \qquad w\in\Omega.$$

If we replace w by z, and take gradients instead of Wirtinger derivatives, the result is

$$\left|\nabla\log\Phi_G(z) - \nabla\log\Phi_F(z)\right| \le 4\left(\Phi_F(z)-\Phi_G(z)\right), \qquad z\in\Omega, \tag{2.10}$$

as desired. We are done.

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