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Superalgebras and closed ideals

Abstract. Let $A$ be a Banach algebra (complex, commutative, unital) which is equipped with a collection $\mathcal{F}$ of closed ideals whose intersection is $\{0\}$. For Banach superalgebras $B$ containing $A$ as a dense subalgebra, we define what it should mean that $B$ inherits $\mathcal{F}$ from $A$. The main result is that there exists a pseudo-Banach superalgebra $\mathcal{A}(\mathcal{F})$ of $A$ such that $B$ inherits $\mathcal{F}$ from $A$ if and only if the injection mapping $A \to \mathcal{A}(\mathcal{F})$ extends to a bounded monomorphism $B \to \mathcal{A}(\mathcal{F})$.

Introduction. All algebras we will consider are assumed complex, commutative, and unital. For a Banach algebra, an ideal theory is a characterization of its closed ideals and the corresponding quotient algebras. In Hedenmalm [2], [3], the ideal theories of closely related Banach algebras were compared in some typical situations. Here, we will explore the same problem from a different angle, namely when one of the algebras is a dense subalgebra of the other.

1. Notation and basic concepts. An epimorphism is a surjective homomorphism, and a monomorphism is an injective homomorphism. Let $A$ be a Banach algebra. We will denote by $\mathcal{M}(A)$ the space of complex homomorphisms on $A$, endowed with the weak * topology induced by the (topological) dual space $A^*$; this is the Gelfand space or maximal ideal space of $A$. Recall that a complex homomorphism is a nonzero homomorphism $A \to C$, where $C$ denotes the complex field. We will denote by $\text{EUN}(A)$ the set of all equivalent submultiplicative unital norms, that is, those equivalent norms $p$ which satisfy

$$p(x \cdot y) \leq p(x)p(y), \quad x, y \in A, \quad \text{and} \quad p(1) = 1.$$  

It is well known that this set is never empty.

Bornological algebras will appear in this paper. Good references are Allan, Dales, and McClure [1] and Waelbroeck [4], [5].

The so-called pseudo-Banach algebras (Allan, Dales, McClure [1]) constitute a particularly interesting subclass — they are roughly speaking unions of Banach algebras that are directed with respect to inclusion, endowed with the natural inductive limit bornology.
A linear mapping between two bornological algebras is called bounded if it maps bounded sets onto bounded sets.

A subalgebra $A$ of a pseudo-Banach algebra $B = \bigcup_{\alpha \in I} B_{\alpha}$ (where $B_{\alpha}$ is a Banach algebra for every $\alpha$ in the index set $I$) is said to be a Banach subalgebra if it is equipped with a norm that makes $A$ a Banach algebra and the injection mapping $A \to B$ is bounded. By the way the bornology on $B$ is defined, a Banach subalgebra $A$ of $B$ must be contained in one of the Banach algebras $B_{\alpha}$, and by the closed graph theorem, its norm is determined within equivalence. We speak of $B$ as a pseudo-Banach superalgebra of $A$, or in case $B$ is a Banach algebra, it is a Banach superalgebra of $A$. $B$ is a minimal Banach superalgebra of $A$ if it is a Banach superalgebra of $A$ and $A$ is dense in $B$.

Let $\mathcal{I}$ be any family of ideals in an algebra $A$. For ease of notation, we will use the convention of writing

$$\text{rad}(\mathcal{I}) = \bigcap_{I \in \mathcal{I}} I.$$  

2. Preliminaries. Let $A$ be an arbitrary Banach algebra. The following lemma will prove useful.

**Lemma 2.1.** Suppose $p_1, p_2 \in \text{EUN}(A)$. Then there exists a $p \in \text{EUN}(A)$ such that $p \leq \min(p_1, p_2)$.

**Proof.** Put

$$B_1 = p_1^{-1}([0, 1]) \quad \text{and} \quad B_2 = p_2^{-1}([0, 1]),$$

the respective closed unit balls. Since $p_1$ and $p_2$ are equivalent norms, there exists a $\lambda \geq 1$ such that

$$\lambda^{-1} B_1 \subset B_2 \subset \lambda B_1.$$

Let $B$ be the closed convex hull of $B_1 \cdot B_2$, which is a subset of $\lambda B_1$ containing $B_1 \cup B_2$. It is easily checked that $B$ is a convex balanced neighborhood of 0 such that

$$B \cdot B = B.$$  

Let $p$ be the Minkowski functional of $B$, which is an equivalent norm on $A$. Then $B = p^{-1}([0, 1])$, and, by (2.1), $p$ is submultiplicative. Hence $p(1) \geq 1$, but since $1 \in B_1 \cap B_2$, $p(1)$ must equal 1. We conclude that $p \in \text{EUN}(A)$; that $p \leq \min(p_1, p_2)$ is obvious.

3. The problem and its solution. From now on, $A$ is a fixed arbitrarily chosen Banach algebra and $\mathcal{I}$ is a family of closed $A$-ideals such that

$$\text{rad}(\mathcal{I}) = \bigcap_{I \in \mathcal{I}} I = \{0\}.$$
Let $B$ be a minimal Banach superalgebra of $A$. $\Phi_B$ denotes the closure operation in $B$. For every $I \in \mathcal{I}$, $\Phi_B(I)$ is a (closed) $B$-deal, since $A$ was dense in $B$. We write $\mathcal{J}(B)$ for the image of $\mathcal{J}$ under the mapping $\Phi_B$.

**Definition 3.1.** We say that $B$ inherits $\mathcal{J}$ from $A$ if
(a) $\Phi_B(I) \cap A = I$ for all $I \in \mathcal{I}$, which makes $\Phi_B$ a bijection $\mathcal{J} \to \mathcal{J}(B)$,
(b) $\text{rad}(\mathcal{J}(B)) = \bigcap_{J \in \mathcal{J}(B)} J = \{0\}$, and
(c) the quotient algebras $A/I$ and $B/\Phi_B(I)$ are canonically isomorphic for all $I \in \mathcal{I}$.

**Remark 3.2.** It should be observed that it follows from (c) of Definition 3.1 that $A + J = B$ for every $J \in \mathcal{J}(B)$, and since $(A + J)/J$ is canonically isomorphic to $A/(J \cap A)$, it follows that $\Phi_B(I) \cap A = I$ for all $I \in \mathcal{I}$. Hence (a) is a consequence of (c).

The object of this paper is to characterize those algebras $B$ which inherit $\mathcal{J}$ from $A$. Our main result, Theorem 3.8, states that there exists a pseudo-Banach superalgebra $\mathcal{A}(\mathcal{J})$ of $A$ such that $B$ inherits $\mathcal{J}$ from $A$ if and only if the canonical monomorphism $A \to \mathcal{A}(\mathcal{J})$ extends to a (unique) bounded monomorphism $B \to \mathcal{A}(\mathcal{J})$.

It is now our intention to introduce a family $\{\mathcal{A}_p\}_{p \in \mathcal{P}}$ of minimal Banach superalgebras of $A$ such that $\mathcal{A}_p$ inherits $\mathcal{J}$ from $A$. The main reason for doing so is that we will be able to show that every minimal Banach superalgebra $B$ of $A$ which inherits $\mathcal{J}$ from $A$ is a (dense) Banach subalgebra of some $\mathcal{A}_p$. The pseudo-Banach algebra $\mathcal{A}(\mathcal{J})$ will be the union of all $\mathcal{A}_p$.

For all $I \in \mathcal{I}$, we will consider the norms in $\text{EUN}(A/I)$ as seminorms on $A$. Let $\mathcal{P} = \mathcal{P}(\mathcal{J})$ be the family of all mappings $p : A \times \mathcal{J} \to [0, \infty)$ such that $p(\cdot, I) \in \text{EUN}(A/I)$ and

$$p(\cdot, I) \leq C \cdot \| \cdot \|_{A/I} \quad \text{for all } I \in \mathcal{I},$$

for some constant $C$ independent of $I$. One should observe that $\mathcal{P}$ is nonempty. By our condition $\text{rad}(\mathcal{J}) = \{0\}$, the expression

$$\|x\|_p = \sup_{I \in \mathcal{J}} p(x, I)$$

is an algebra norm on $A$ for every $p \in \mathcal{P}$. $A_p$ denotes the completion of $A$ in the $\| \cdot \|_p$-norm, which is a (minimal) Banach superalgebra of $A$. Put

$$\mathcal{A}_p = A_p/\text{rad}(\mathcal{J}(A_p)), \quad p \in \mathcal{P}.$$

If we can show that

$$\text{rad}(\mathcal{J}(A_p)) \cap A = \{0\},$$

the composition of the injection mapping $A \to A_p$ and the canonical epimorphism $A_p \to \mathcal{A}_p$ will be a bounded (= continuous) monomorphism
$A \rightarrow \mathcal{A}_p$. By the definition of the norm in $A_p$, the canonical epimorphism $A \rightarrow A/I$ has a unique bounded extension $L_I: A_p \rightarrow A/I$, which is also an epimorphism, for every $I \in \mathcal{J}$. Clearly, $\ker L_I \supseteq \Phi_{A_p}(I)$, and since $L_I$ is canonical on $A$, $\ker L_I \cap I$. The assertion follows, and hence we may regard $\mathcal{A}_p$ as a minimal Banach superalgebra of $A$.

**Proposition 3.3.** A minimal Banach superalgebra $B$ of $A$ inherits $\mathcal{J}$ from $A$ if and only if the injection mapping $A \rightarrow \mathcal{A}_p$ extends boundedly to a monomorphism $B \rightarrow \mathcal{A}_p$ for some $p \in \mathcal{P}$.

**Proof.** We may assume without loss of generality that the norms of $A$ and $B$ are chosen in $\operatorname{EUN}(A)$ and $\operatorname{EUN}(B)$, respectively.

Let us deal with the "only if" part of the assertion first. So, assume $B$ inherits $\mathcal{J}$ from $A$. Since the norm of $B$ belongs to $\operatorname{EUN}(B)$, the induced norm on $B/\Phi_B(I)$ belongs to $\operatorname{EUN}(B/\Phi_B(I))$, and by (c) of Definition 3.1, its restriction to $A$ is in $\operatorname{EUN}(A/I)$ for every $I \in \mathcal{J}$. Put

$$p(x, I) = \|x + \Phi_B(I)\|_{B/\Phi_B(I)}, \quad x \in A, \ I \in \mathcal{J}.$$  

In order to show that $p \in \mathcal{P}$ it only remains to check that

$$p(\cdot, I) \leq C \|\cdot\|_{A/I}$$  

for some constant $C$ independent of $I$. But evidently,

$$\|x\|_B \leq C \|x\|_A, \quad x \in A,$$

for some constant $C$, since $A$ is a Banach subalgebra of $B$, and consequently,

$$p(x, I) = \|x + \Phi_B(I)\|_{B/\Phi_B(I)} \leq C \|x + I\|_{A/I}, \quad x \in A, \ I \in \mathcal{J}.$$  

Hence $p \in \mathcal{P}$, and since $p(x, I) \leq \|x\|_B$,

$$\|x\|_p \leq \|x\|_B, \quad x \in A,$$

so the injection mapping $A \rightarrow A_p$ extends to a (unique) bounded homomorphism $j: B \rightarrow A_p$. Our next step is to show that $j$ is a monomorphism. Let $y \in \ker j$ be arbitrary. Then there exists a sequence $\{y_n\}^\infty_{n=0}$ in $A$ converging to $y$ in the norm of $B$. Since $y \in \ker j$, $\|y_n\|_p \rightarrow 0$ as $n \rightarrow \infty$, and consequently $\|y_n + J\|_{B/J} \rightarrow 0$ as $n \rightarrow \infty$ for all $J \in \mathcal{J}(B)$. Hence $y \in \operatorname{rad}(\mathcal{J}(B)) = \{0\}$, and the assertion follows.

We will now show that the composition $l: B \rightarrow \mathcal{A}_p$ of $j$ and the canonical epimorphism $A_p \rightarrow \mathcal{A}_p$ is a monomorphism, too. Let us for simplicity regard $B$ as a subalgebra of $A_p$. By the definition of the norm in $A_p$, the canonical epimorphism $B \rightarrow B/J$ extends to a (unique) bounded epimorphism $A_j: A_p \rightarrow B/J$ for every $J \in \mathcal{J}(B)$. Clearly, $\ker A_j \supseteq \Phi_{A_p}(J)$, and since $A_j$ is canonical on $B$, $\ker A_j \cap B = J$. The assertion follows, since

$$\ker l = \operatorname{rad}(\mathcal{J}(A_p)) \cap B = \bigcap_{J \in \mathcal{J}(B)} \Phi_{A_p}(J) \cap B = \bigcap_{J \in \mathcal{J}(B)} J = 0.$$
Let us turn to the “if” part of the assertion. So, assume $B$ is a Banach subalgebra of $\mathcal{A}_p$ for some $p \in \mathcal{P}$. We have mentioned before that the canonical epimorphism $A \to A/I$ has a unique bounded linear extension $L_I: \mathcal{A}_p \to A/I$, an epimorphism which is canonical on $A$. Denote by $\mathcal{L}_I$ the restriction to $B$ of $L_I$, which is bounded since $B$ is a Banach subalgebra of $\mathcal{A}_p$. We now intend to show that $\ker \mathcal{L}_I = \Phi_B(I)$ for all $I \in \mathcal{I}$. Obviously, $\Phi_B(I) \subseteq \ker \mathcal{L}_I$. Choose a $B$-Cauchy sequence $\{x_n\}_{n=1}^\infty \subseteq A$ converging to an arbitrary $x \in \ker \mathcal{L}_I$. Then $\|x_n - y_m\|_A \to 0$ as $n \to \infty$ for some sequence $\{y_m\}_{m=1}^\infty \subseteq I$ since $\|x_n + I\|_{A/I} \to 0$ as $n \to \infty$, and it follows that $\{y_m\}_{m=1}^\infty$ is another $B$-Cauchy sequence converging to $x$. We conclude that $\ker \mathcal{L}_I = \Phi_B(I)$. Since $\mathcal{L}_I$ is canonical on $A$, $\ker \mathcal{L}_I \cap A = I$, and $A/I$ and $B/\ker \mathcal{L}_I$ are canonically isomorphic. This shows that conditions (a) and (c) of Definition 3.1 are met. (b) follows trivially, since $\text{rad}(\mathcal{J}(\mathcal{A}_p)) = \{0\}$. The proof of the proposition is complete.

**Remark 3.4.** A consequence of Proposition 3.3 is the following. Let $\mathcal{J}$ be the set of all closed $A$-ideals which contain an ideal in $\mathcal{J}$. Then a minimal Banach superalgebra of $A$ inherits $\mathcal{J}$ from $A$ if and only if it inherits $\mathcal{J}$ from $A$.

Putting $B = \mathcal{A}_p$, Proposition 3.3 has the following consequence.

**Corollary 3.5.** For every $p \in \mathcal{P}$, $\mathcal{A}_p$ inherits $\mathcal{J}$ from $A$.

There is a natural order relation on the set $\{\mathcal{A}_p\}_{p \in \mathcal{P}}$: for $p, q \in \mathcal{P}$, write $\mathcal{A}_q \preceq \mathcal{A}_p$ if for some constant $C$,

$$\|x\|_p \leq C \|x\|_q, \quad x \in A.$$ 

Clearly, $\{\mathcal{A}_p\}_{p \in \mathcal{P}}$ is partially ordered by “$\preceq$”. We have the following result.

**Proposition 3.6.** For any two $p_1, p_2 \in \mathcal{P}$, there is a $p \in \mathcal{P}$ such that $\mathcal{A}_{p_1} \preceq \mathcal{A}_p$ and $\mathcal{A}_{p_2} \preceq \mathcal{A}_p$.

**Proof.** For every $I \in \mathcal{J}$, Lemma 2.1 tells us that there exists a $p(\cdot, I) \in \text{EUN}(A/I)$ such that

$$p(\cdot, I) \leq \min(p_1(\cdot, I), p_2(\cdot, I)).$$

Clearly this $p$ will do.

The following proposition tells us that we may regard $\mathcal{A}_q$ as a (dense Banach) subalgebra of $\mathcal{A}_p$ if $\mathcal{A}_q \preceq \mathcal{A}_p$, and therefore the order relation “$\preceq$” is just ordinary inclusion.

**Proposition 3.7.** If $\mathcal{A}_q \preceq \mathcal{A}_p$ for two $p, q \in \mathcal{P}$, the injection mapping $A \to \mathcal{A}_p$ has a (unique) bounded extension $\mathcal{A}_q \to \mathcal{A}_p$, which is a monomorphism.

**Proof.** By the assumptions on $p$ and $q$, the injection mapping $A \to \mathcal{A}_p$ extends uniquely to a bounded homomorphism $l: A_q \to A_p$. Clearly, this will
define a bounded monomorphism $\mathcal{A}_q \rightarrow \mathcal{A}_p$ which extends the canonical monomorphism $A \rightarrow \mathcal{A}_p$ if we can show that

\begin{align}
(3.1) & \quad l(\Phi_{A_q}(I)) \subset \Phi_{A_p}(I) \quad \text{and} \\
(3.2) & \quad l^{-1}(\Phi_{A_p}(I) \cap l(A_q)) \subset \Phi_{A_q}(I)
\end{align}

for all $I \in \mathcal{F}$, since then

\begin{align}
 l(\text{rad} \left( \mathcal{F} (A_q) \right)) \subset \text{rad} \left( \mathcal{F} (A_p) \right) \quad \text{and} \quad l^{-1}(\text{rad} \left( \mathcal{F} (A_p) \right) \cap l(A_q)) \subset \text{rad} \left( \mathcal{F} (A_q) \right).
\end{align}

For $I \in \mathcal{F}$, denote by $L_{I,p}$ the bounded epimorphism $A_p \rightarrow A/I$ which extends the canonical epimorphism $A \rightarrow A/I$, and let $L_{I,q}: A_q \rightarrow A/I$ be defined analogously. It is easy to see that $L_{I,q} = L_{I,p} \circ l$; just check on $A$ and remember that $A$ is dense in $A_q$. Thus

\[ \ker L_{I,q} = l^{-1}(\ker L_{I,p} \cap l(A_q)), \]

and if we can show that $\ker L_{I,p} = \Phi_{A_p}(I)$ and $\ker L_{I,q} = \Phi_{A_q}(I)$, (3.1)–(3.2) will follow easily from this relation because $l \circ l^{-1}$ is the identity mapping on the set of subsets of $l(A_q)$. It suffices to verify the assertion for $L_{I,p}$ only, since the proof for $L_{I,q}$ would be identical. We will employ the same type of argument as we used in the proof of Proposition 3.3. Obviously, $\Phi_{A_p}(I) \subset \ker L_{I,p}$. Choose an $A_p$-Cauchy sequence $\{x_n\}_{0}^{\infty} \subset A$ converging to an arbitrary $x \in \ker L_{I,p}$. Then $\|x_n + I\|_{A/I} \rightarrow 0$ as $n \rightarrow \infty$, and hence there is a sequence $\{y_n\}_{0}^{\infty} \subset I$ such that $\|x_n - y_n\|_{A} \rightarrow 0$ as $n \rightarrow \infty$. It follows that $\{y_n\}_{0}^{\infty}$ is another $A_p$-Cauchy sequence converging to $x$, and we conclude that $\ker L_{I,p} = \Phi_{A_p}(I)$. The proof of the proposition is complete.

We will regard $\mathcal{A}_q$ as a subalgebra of $\mathcal{A}_p$ if $\mathcal{A}_q \leq \mathcal{A}_p$ ($p, q \in \mathcal{P}$). Let

\[ A(\mathcal{F}) = \bigcup_{p \in \mathcal{P}(\mathcal{F})} \mathcal{A}_p, \]

which is a pseudo-Banach algebra when endowed with its inductive limit bornology. We are now ready to formulate our main result.

**Theorem 3.8.** A minimal Banach superalgebra $B$ of $A$ inherits $\mathcal{F}$ from $A$ if and only if the injection mapping $A \rightarrow A(\mathcal{F})$ extends boundedly to a monomorphism $B \rightarrow A(\mathcal{F})$.

**Proof.** The "only if" part is clear by Proposition 3.3. On the other hand, if $B$ is a Banach subalgebra of $A(\mathcal{F})$, then $B$ must be contained in some $\mathcal{A}_p$, $p \in \mathcal{P}$, by the way the bornology on $A(\mathcal{F})$ is defined, and therefore Proposition 3.3 proves the other direction, too.

**Examples 3.9.** (a) Let $A$ be the disc algebra $A(D)$, which consists of those holomorphic functions on $D = \{ z \in \mathbb{C} : |z| < 1 \}$ that extend continuously to the boundary $\partial D$, and let $\mathcal{F}$ consist of the ideals $z^n \cdot A(D)$, $n \geq 0$, where $z$ is the coordinate function $z(\zeta) = \zeta$, $\zeta \in D$. Then $A(\mathcal{F}) = \mathbb{C}[[z]]$, the
algebra of formal power series at the origin, and the sets
\[ \{ \sum_{n=0}^{\infty} a_n z^n \in C[[z]] : |a_n| \leq M_n \}, \]
where \( \{M_n\}_0^\infty \) ranges over all positive sequences, form a base of the bornology on \( \mathcal{A}(\mathcal{J}) \).

(b) Assume \( A \) is semisimple, and let \( \mathcal{J} = \mathcal{M}(A) \), the set of maximal ideals. Regard \( A \) as a subalgebra of \( C(\mathcal{M}(A)) \). Then \( \mathcal{A}(\mathcal{J}) \) is the uniform closure of \( A \).

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References


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