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Superalgebras and closed ideals

Abstract. Let A be a Banach algebra (complex, commutative, unital) which is equipped with a colletion $\mathscr{J}$ of closed ideals whose intersection is $\{0\}$. For Banach superalgebras B containing A as a dense subalgebra, we define what it should mean that B inherits $\mathscr{J}$ from A. The main result is that there exists a pseudo-Banach superalgebra $\mathscr{A}(\mathscr{J})$ of A such that B inherits $\mathscr{J}$ from A if and only if the injection mapping $A \to \mathscr{A}(\mathscr{J})$ extends to a bounded monomorphism $B \to \mathscr{A}(\mathscr{J})$.

Introduction. All algebras we will consider are assumed complex, commutative, and unital. For a Banach algebra, an *ideal theory* is a characterization of its closed ideals and the corresponding quotient algebras. In Hedenmalm [2], [3], the ideal theories of closely related Banach algebras were compared in some typical situations. Here, we will explore the same problem from a different angle, namely when one of the algebras is a dense subalgebra of the other.

1. Notation and basic concepts. An *epimorphism* is a surjective homomorphism, and a *monomorphism* is an injective homomorphism.

Let A be a Banach algebra. We will denote by $\mathscr{M}(A)$ the space of complex homomorphisms on A, endowed with the weak $*$ topology induced by the (topological) dual space A^*; this is the Gelfand space or maximal ideal space of A. Recall that a complex homomorphism is a nonzero homomorphism $A \to \boldsymbol{C}$, where $\boldsymbol{C}$ denotes the complex field.

We will denote by EUN(A) the set of all equivalent submultiplicative unital norms, that is, those equivalent norms p which satisfy

$$p(x \cdot y) \leqslant p(x)\,p(y), \quad x, y \in A, \quad \text{and} \quad p(1) = 1.$$

It is well known that this set is never empty.

Bornological algebras will appear in this paper. Good references are Allan, Dales, and McClure [1] and Waelbroeck [4], [5].

The so-called *pseudo-Banach algebras* (Allan, Dales, McClure [1]) constitute a particularly interesting subclass — they are roughly speaking unions of Banach algebras that are directed with respect to inclusion, endowed with the natural inductive limit bornology.

A linear mapping between two bornological algebras is called *bounded* if it maps bounded sets onto bounded sets.

A subalgebra A of a pseudo-Banach algebra $B = \bigcup_{\alpha \in I} B_\alpha$ (where B_α is a Banach algebra for every α in the index set I) is said to be a *Banach subalgebra* if it is equipped with a norm that makes A a Banach algebra and the injection mapping $A \to B$ is bounded. By the way the bornology on B is defined, a Banach subalgebra A of B must be contained in one of the Banach algebras B_α, and by the closed graph theorem, its norm is determined within equivalence. We speak of B as a *pseudo-Banach superalgebra* of A, or in case B is a Banach algebra, it is a *Banach superalgebra* of A. B is a *minimal* Banach superalgebra of A if it is a Banach superalgebra of A and A is dense in B.

Let $\mathscr{J}$ be any family of ideals in an algebra A. For ease of notation, we will use the convention of writing

$$\operatorname{rad}(\mathscr{J}) = \bigcap_{I \in \mathscr{J}} I.$$

2. Preliminaries. Let A be an arbitrary Banach algebra. The following lemma will prove useful.

LEMMA 2.1. *Suppose* $p_1, p_2 \in \mathrm{EUN}(A)$. *Then there exists a* $p \in \mathrm{EUN}(A)$ *such that* $p \leqslant \min(p_1, p_2)$.

Proof. Put

$$\mathscr{B}_1 = p_1^{-1}([0, 1]) \quad \text{and} \quad \mathscr{B}_2 = p_2^{-1}([0, 1]),$$

the respective closed unit balls. Since p_1 and p_2 are equivalent norms, there exists a $\lambda \geqslant 1$ such that

$$\lambda^{-1}\mathscr{B}_1 \subset \mathscr{B}_2 \subset \lambda\mathscr{B}_1.$$

Let $\mathscr{B}$ be the closed convex hull of $\mathscr{B}_1 \cdot \mathscr{B}_2$, which is a subset of $\lambda\mathscr{B}_1$ containing $\mathscr{B}_1 \cup \mathscr{B}_2$. It is easily checked that $\mathscr{B}$ is a convex balanced neighborhood of 0 such that

$$\mathscr{B} \cdot \mathscr{B} = \mathscr{B}. \tag{2.1}$$

Let p be the Minkowski functional of $\mathscr{B}$, which is an equivalent norm on A. Then $\mathscr{B} = p^{-1}([0, 1])$, and, by (2.1), p is submultiplicative. Hence $p(1) \geqslant 1$, but since $1 \in \mathscr{B}_1 \cap \mathscr{B}_2$, $p(1)$ must equal 1. We conclude that $p \in \mathrm{EUN}(A)$; that $p \leqslant \min(p_1, p_2)$ is obvious.

3. The problem and its solution. From now on, A is a fixed arbitrarily chosen Banach algebra and $\mathscr{J}$ is a family of closed A-ideals such that

$$\operatorname{rad}(\mathscr{J}) = \bigcap_{I \in \mathscr{J}} I = \{0\}.$$

Let B be a minimal Banach superalgebra of A. Φ_B denotes the closure operation in B. For every $I \in \mathscr{J}$, $\Phi_B(I)$ is a (closed) B-deal, since A was dense in B. We write $\mathscr{J}(B)$ for the image of $\mathscr{J}$ under the mapping Φ_B.

DEFINITION 3.1. We say that B *inherits* $\mathscr{J}$ *from* A if

(a) $\Phi_B(I) \cap A = I$ for all $I \in \mathscr{J}$, which makes Φ_B a bijection $\mathscr{J} \to \mathscr{J}(B)$,

(b) $\operatorname{rad}(\mathscr{J}(B)) = \bigcap_{J \in \mathscr{J}(B)} J = \{0\}$, and

(c) the quotient algebras A/I and $B/\Phi_B(I)$ are canonically isomorphic for all $I \in \mathscr{J}$.

Remark 3.2. It should be observed that it follows from (c) of Definition 3.1 that $A+J = B$ for every $J \in \mathscr{J}(B)$, and since $(A+J)/J$ is canonically isomorphic to $A/(J \cap A)$, it follows that $\Phi_B(I) \cap A = I$ for all $I \in \mathscr{J}$. Hence (a) is a consequence of (c).

The object of this paper is to characterize those algebras B which inherit $\mathscr{J}$ from A. Our main result, Theorem 3.8, states that there exists a pseudo-Banach superalgebras $\mathscr{A}(\mathscr{J})$ of A such that B inherits $\mathscr{J}$ from A if and only if the canonical monomorphism $A \to \mathscr{A}(\mathscr{J})$ extends to a (unique) bounded monomorphism $B \to \mathscr{A}(\mathscr{J})$.

It is now our intention to introduce a family $\{\mathscr{A}_p\}_{p \in \mathscr{P}}$ of minimal Banach superalgebras of A such that $\mathscr{A}_p$ inherits $\mathscr{J}$ from A. The main reason for doing so is that we will be able to show that every minimal Banach superalgebra B of A which inherits $\mathscr{J}$ from A is a (dense) Banach subalgebra of some $\mathscr{A}_p$. The pseudo-Banach algebra $\mathscr{A}(\mathscr{J})$ will be the union of all $\mathscr{A}_p$.

For all $I \in \mathscr{J}$, we will consider the norms in $\operatorname{EUN}(A/I)$ as seminorms on A. Let $\mathscr{P} = \mathscr{P}(\mathscr{J})$ be the family of all mappings $p\colon A \times \mathscr{J} \to [0, \infty)$ such that $p(\cdot, I) \in \operatorname{EUN}(A/I)$ and

$$p(\cdot, I) \leqslant C \cdot \|\cdot\|_{A/I} \quad \text{for all } I \in \mathscr{J},$$

for some constant C independent of I. One should observe that $\mathscr{P}$ is nonempty. By our condition $\operatorname{rad}(\mathscr{J}) = \{0\}$, the expression

$$\|x\|_p = \sup_{I \in \mathscr{J}} p(x, I)$$

is an algebra norm on A for every $p \in \mathscr{P}$. A_p denotes the completion of A in the $\|\cdot\|_p$-norm, which is a (minimal) Banach superalgebra of A. Put

$$\mathscr{A}_p = A_p/\operatorname{rad}(\mathscr{J}(A_p)), \qquad p \in \mathscr{P}.$$

If we can show that

$$\operatorname{rad}(\mathscr{J}(A_p)) \cap A = \{0\},$$

the composition of the injection mapping $A \to A_p$ and the canonical epimorphism $A_p \to \mathscr{A}_p$ will be a bounded (= continuous) monomorphism

$A \to \mathscr{A}_p$. By the definition of the norm in A_p, the canonical epimorphism $A \to A/I$ has a unique bounded extension L_I: $A_p \to A/I$, which is also an epimorphism, for every $I \in \mathscr{J}$. Clearly, $\ker L_I \supset \Phi_{A_p}(I)$, and since L_I is canonical on A, $\ker L_I \cap I$. The assertion follows, and hence we may regard $\mathscr{A}_p$ as a minimal Banach superalgebra of A.

PROPOSITION 3.3. *A minimal Banach superalgebra B of A inherits $\mathscr{J}$ from A if and only if the injection mapping $A \to \mathscr{A}_p$ extends boundedly to a monomorphism $B \to \mathscr{A}_p$ for some $p \in \mathscr{P}$.*

Proof. We may assume without loss of generality that the norms of A and B are chosen in $\mathrm{EUN}(A)$ and $\mathrm{EUN}(B)$, respectively.

Let us deal with the "only if" part of the assertion first. So, assume B inherits $\mathscr{J}$ from A. Since the norm of B belongs to $\mathrm{EUN}(B)$, the induced norm on $B/\Phi_B(I)$ belongs to $\mathrm{EUN}(B/\Phi_B(I))$, and by (c) of Definition 3.1, its restriction to A is in $\mathrm{EUN}(A/I)$ for every $I \in \mathscr{J}$. Put

$$p(x, I) = \|x+\Phi_B(I)\|_{B/\Phi_B(I)}, \qquad x \in A, \ I \in \mathscr{J}.$$

In order to show that $p \in \mathscr{P}$ it only remains to check that

$$p(\cdot, I) \leqslant C\|\cdot\|_{A/I} \quad \text{for all } I \in \mathscr{J}$$

for some constant C independent of I. But evidently,

$$\|x\|_B \leqslant C\|x\|_A, \qquad x \in A,$$

for some constant C, since A is a Banach subalgebra of B, and consequently,

$$p(x, I) = \|x+\Phi_B(I)\|_{B/\Phi_B(I)} \leqslant C \cdot \|x+I\|_{A/I}, \qquad x \in A, \ I \in \mathscr{J}.$$

Hence $p \in \mathscr{P}$, and since $p(x, I) \leqslant \|x\|_B$,

$$\|x\|_p \leqslant \|x\|_B, \qquad x \in A,$$

so the injection mapping $A \to A_p$ extends to a (unique) bounded homomorphism j: $B \to A_p$. Our next step is to show that j is a monomorphism. Let $y \in \ker j$ be arbitrary. Then there exists a sequence $\{y_n\}_0^\infty$ in A converging to y in the norm of B. Since $y \in \ker j$, $\|y_n\|_p \to 0$ as $n \to \infty$, and consequently $\|y_n+J\|_{B/J} \to 0$ as $n \to \infty$ for all $J \in \mathscr{J}(B)$. Hence $y \in \mathrm{rad}(\mathscr{J}(B)) = \{0\}$, and the assertion follows.

We will now show that the composition l: $B \to \mathscr{A}_p$ of j and the canonical epimorphism $A_p \to \mathscr{A}_p$ is a monomorphism, too. Let us for simplicity regard B as a subalgebra of A_p. By the definition of the norm in A_p, the canonical epimorphism $B \to B/J$ extends to a (unique) bounded epimorphism Λ_J: $A_p \to B/J$ for every $J \in \mathscr{J}(B)$. Clearly, $\ker \Lambda_J \supset \Phi_{A_p}(J)$, and since Λ_J is canonical on B, $\ker \Lambda_J \cap B = J$. The assertion follows, since

$$\ker\ l = \mathrm{rad}(\mathscr{J}(A_p)) \cap B = \bigcap_{J \in \mathscr{J}(B)} \Phi_{A_p}(J) \cap B = \bigcap_{J \in \mathscr{J}(B)} J = 0.$$

Let us turn to the "if" part of the assertion. So, assume B is a Banach subalgebra of $\mathscr{A}_p$ for some $p \in \mathscr{P}$. We have mentioned before that the canonical epimorphism $A \to A/I$ has a unique bounded linear extension L_I: $\mathscr{A}_p \to A/I$, an epimorphism which is canonical on A. Denote by $\mathscr{L}_I$ the restriction to B of L_I, which is bounded since B is a Banach subalgebra of $\mathscr{A}_p$. We now intend to show that $\ker \mathscr{L}_I = \Phi_B(I)$ for all $I \in \mathscr{J}$. Obviously, $\Phi_B(I) \subset \ker \mathscr{L}_I$. Choose a B-Cauchy sequence $\{x_n\}_0^\infty \subset A$ converging to an arbitrary $x \in \ker \mathscr{L}_I$. Then $\|x_n - y_n\|_A \to 0$ as $n \to \infty$ for some sequence $\{y_n\}_0^\infty \subset I$ since $\|x_n + I\|_{A/I} \to 0$ as $n \to \infty$, and it follows that $\{y_n\}_0^\infty$ is another B-Cauchy sequence converging to x. We conclude that $\ker \mathscr{L}_I = \Phi_B(I)$. Since $\mathscr{L}_I$ is canonical on A, $\ker \mathscr{L}_I \cap A = I$, and A/I and $B/\ker \mathscr{L}_I$ are canonically isomorphic. This shows that conditions (a) and (c) of Definition 3.1 are met. (b) follows trivially, since $\operatorname{rad}(\mathscr{J}(\mathscr{A}_p)) = \{0\}$. The proof of the proposition is complete.

Remark 3.4. A consequence of Proposition 3.3 is the following. Let $\hat{\mathscr{J}}$ be the set of all closed A-ideals which contain an ideal in $\mathscr{J}$. Then a minimal Banach superalgebra of A inherits $\hat{\mathscr{J}}$ from A if and only if it inherits $\mathscr{J}$ from A.

Putting $B = \mathscr{A}_p$, Proposition 3.3 has the following consequence.

COROLLARY 3.5. *For every $p \in \mathscr{P}$, $\mathscr{A}_p$ inherits $\mathscr{J}$ from A.*

There is a natural order relation on the set $\{\mathscr{A}_p\}_{p \in \mathscr{P}}$: for $p, q \in \mathscr{P}$, write $\mathscr{A}_q \preccurlyeq \mathscr{A}_p$ if for some constant C,

$$\|x\|_p \leqslant C\|x\|_q, \qquad x \in A.$$

Clearly, $\{\mathscr{A}_p\}_{p \in \mathscr{P}}$ is partially ordered by "$\preccurlyeq$". We have the following result.

PROPOSITION 3.6. *For any two $p_1, p_2 \in \mathscr{P}$, there is a $p \in \mathscr{P}$ such that $\mathscr{A}_{p_1} \preccurlyeq \mathscr{A}_p$ and $\mathscr{A}_{p_2} \preccurlyeq \mathscr{A}_p$.*

Proof. For every $I \in \mathscr{J}$, Lemma 2.1 tells us that there exists a $p(\cdot, I) \in \mathrm{EUN}(A/I)$ such that

$$p(\cdot, I) \leqslant \min\big(p_1(\cdot, I), p_2(\cdot, I)\big).$$

Clearly this p will do.

The following proposition tells us that we may regard $\mathscr{A}_q$ as a (dense Banach) subalgebra of $\mathscr{A}_p$ if $\mathscr{A}_q \preccurlyeq \mathscr{A}_p$, and therefore the order relation "$\preccurlyeq$" is just ordinary inclusion.

PROPOSITION 3.7. *If $\mathscr{A}_q \preccurlyeq \mathscr{A}_p$ for two $p, q \in \mathscr{P}$, the injection mapping $A \to \mathscr{A}_p$ has a (unique) bounded extension $\mathscr{A}_q \to \mathscr{A}_p$, which is a monomorphism.*

Proof. By the assumptions on p and q, the injection mapping $A \to A_p$ extends uniquely to a bounded homomorphism l: $A_q \to A_p$. Clearly, this will define a bounded monomorphism $\mathscr{A}_q \to \mathscr{A}_p$ which extends the canonical monomorphism $A \to \mathscr{A}_p$ if we can show that

(3.1) $$l(\Phi_{A_q}(I)) \subset \Phi_{A_p}(I) \quad \text{and}$$

(3.2) $$l^{-1}(\Phi_{A_p}(I) \cap l(A_q)) \subset \Phi_{A_q}(I)$$

for all $I \in \mathscr{J}$, since then

$$l(\mathrm{rad}(\mathscr{J}(A_q))) \subset \mathrm{rad}(\mathscr{J}(A_p)) \quad \text{and} \quad l^{-1}(\mathrm{rad}(\mathscr{J}(A_p)) \cap l(A_q)) \subset \mathrm{rad}(\mathscr{J}(A_q)).$$

For $I \in \mathscr{J}$, denote by $L_{I,p}$ the bounded epimorphism $A_p \to A/I$ which extends the canonical epimorphism $A \to A/I$, and let $L_{I,q}$: $A_q \to A/I$ be defined analogously. It is easy to see that $L_{I,q} = L_{I,p} \circ l$; just check on A and remember that A is dense in A_q. Thus

$$\ker L_{I,q} = l^{-1}(\ker L_{I,p} \cap l(A_q)),$$

and if we can show that $\ker L_{I,p} = \Phi_{A_p}(I)$ and $\ker L_{I,q} = \Phi_{A_q}(I)$, (3.1)–(3.2) will follow easily from this relation because $l \circ l^{-1}$ is the identity mapping on the set of subsets of $l(A_q)$. It suffices to verify the assertion for $L_{I,p}$ only, since the proof for $L_{I,q}$ would be identical. We will employ the same type of argument as we used in the proof of Proposition 3.3. Obviously, $\Phi_{A_p}(I) \subset \ker L_{I,p}$. Choose an A_p-Cauchy sequence $\{x_n\}_0^\infty \subset A$ converging to an arbitrary $x \in \ker L_{I,p}$. Then $\|x_n + I\|_{A/I} \to 0$ as $n \to \infty$, and hence there is a sequence $\{y_n\}_0^\infty \subset I$ such that $\|x_n - y_n\|_A \to 0$ as $n \to \infty$. It follows that $\{y_n\}_0^\infty$ is another A_p-Cauchy sequence converging to x, and we conclude that $\ker L_{I,p} = \Phi_{A_p}(I)$. The proof of the proposition is complete.

We will regard $\mathscr{A}_q$ as a subalgebra of $\mathscr{A}_p$ if $\mathscr{A}_q \preccurlyeq \mathscr{A}_p$ ($p, q \in \mathscr{P}$). Let

$$A(\mathscr{J}) = \bigcup_{p \in \mathscr{P}(\mathscr{J})} \mathscr{A}_p,$$

which is a pseudo-Banach algebra when endowed with its inductive limit bornology. We are now ready to formulate our main result.

THEOREM 3.8. *A minimal Banach superalgebra B of A inherits $\mathscr{J}$ from A if and only if the injection mapping $A \to \mathscr{A}(\mathscr{J})$ extends boundedly to a monomorphism $B \to \mathscr{A}(\mathscr{J})$.*

Proof. The "only if" part is clear by Proposition 3.3. On the other hand, if B is a Banach subalgebra of $\mathscr{A}(\mathscr{J})$, then B must be contained in some $\mathscr{A}_p$, $p \in \mathscr{P}$, by the way the bornology on $\mathscr{A}(\mathscr{J})$ is defined, and therefore Proposition 3.3 proves the other direction, too.

EXAMPLES 3.9. (a) Let A be the disc algebra $A(\mathbf{D})$, which consists of those holomorphic functions on $\mathbf{D} = \{z \in \mathbf{C}: |z| < 1\}$ that extend continuously to the boundary $\partial\mathbf{D}$, and let $\mathscr{J}$ consist of the ideals $z^n \cdot A(\mathbf{D})$, $n \geq 0$, where z is the coordinate function $z(\zeta) = \zeta$, $\zeta \in \mathbf{D}$. Then $\mathscr{A}(\mathscr{J}) = \mathbf{C}[[z]]$, the algebra of formal power series at the origin, and the sets

$$\left\{\sum_{n=0}^{\infty} a_n z^n \in \mathbf{C}[[z]] : |a_n| \leqslant M_n\right\},$$

where $\{M_n\}_0^\infty$ ranges over all positive sequences, form a base of the bornology on $\mathscr{A}(\mathscr{J})$.

(b) Assume A is semisimple, and let $\mathscr{J} = \mathscr{M}(A)$, the set of maximal ideals. Regard A as a subalgebra of $C(\mathscr{M}(A))$. Then $\mathscr{A}(\mathscr{J})$ is the uniform closure of A.

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