# A FACTORING THEOREM FOR THE BERGMAN SPACE

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## 0. *Introduction*

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T}$ the unit circle, and $L^2_a(\mathbb{D})$ the Bergman space, consisting of those analytic functions on $\mathbb{D}$ that are square integrable on $\mathbb{D}$ with respect to area measure. The Bergman space is a closed subspace of the Hilbert space $L^2(\mathbb{D})$ of all square area-integrable complex-valued functions on $\mathbb{D}$. The inner product in $L^2(\mathbb{D})$, and hence in $L^2_a(\mathbb{D})$, is given by the formula

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{D}} f(z)\,\bar{g}(z)\,dA(z), \quad f, g \in L^2(\mathbb{D}),$$

where $dA$ denotes planar area measure, normalized so that $\mathbb{D}$ has total mass 1. The associated norm is denoted by $\|\cdot\|_{L^2}$. The Hardy space $H^2(\mathbb{D})$ consists of all functions $f$ holomorphic on $\mathbb{D}$ satisfying

$$\|f\|^2_{H^2} = \sup_{0<r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta/(2\pi) < \infty;$$

it is also a Hilbert space with the above norm $\|\cdot\|_{H^2}$, and it has a well-known factoring theory, the basic features of which we shall indicate briefly in the following.

Suppose that $\mathbf{a} = \{a_j\}_j$ is a finite or infinite sequence of points in $\mathbb{D}\backslash\{0\}$, and let

$$B_{\mathbf{a}}(z) = \prod_j \frac{\bar{a}_j}{|a_j|}\frac{a_j - z}{1-\bar{a}_j z}, \quad z \in \mathbb{D},$$

be the Blaschke product associated with $\mathbf{a}$. This Blaschke product converges to a nonidentically vanishing holomorphic function on the unit disk precisely when the sequence $\mathbf{a}$ satisfies the Blaschke condition:

$$\sum_j (1-|a_j|) < \infty.$$

Suppose, for the moment, that $f$ is an analytic function on $\mathbb{D}$ which vanishes on $\mathbf{a}$, by which we mean that we count multiplicities should the same point occur more than once in the sequence $\mathbf{a}$. If $f$ belongs to the Hardy space $H^2(\mathbb{D})$, and if $f$ does not vanish identically on $\mathbb{D}$, then $\mathbf{a}$ must be a Blaschke sequence, so that the Blaschke product $B_{\mathbf{a}}$ does not collapse to 0, and, moreover, $f$ has a factoring

$$f(z) = B_{\mathbf{a}}(z)\cdot g(z), \quad z \in \mathbb{D},$$

where $g \in H^2(\mathbb{D})$ and $\|g\|_{H^2} = \|f\|_{H^2}$.

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Blaschke products seem intrinsically connected with the Hardy spaces $H^p(\mathbb{D})$, $0 < p \leqslant \infty$, or, more generally, the Nevanlinna class of functions of bounded characteristic, and do not work very well to produce a satisfying factoring theory for other spaces like the Bergman space $L^2_a(\mathbb{D})$. For instance, it is well known [3, 5] that there are zero sequences for $L^2_a(\mathbb{D})$ which do not meet the Blaschke condition; in fact, no concrete condition that characterizes the Bergman space zero sequences is known. Some authors have approached this obstacle by producing modified Blaschke factors, which converge under some milder conditions on the zero sequence [5, 6]. For instance, if the sequence $\mathbf{a}$ has

$$\sum_j (1-|a_j|)^2 < \infty,$$

a condition which is certainly met by all Bergman space zero sequences, Charles Horowitz [5] introduced a product

$$P_{\mathbf{a}}(z) = \prod_j (b(z,a_j)(2-b(z,a_j))), \quad z \in \mathbb{D},$$

where

$$b(z,\alpha) = \frac{\bar{\alpha}}{|\alpha|}\frac{\alpha - z}{1-\bar{\alpha}z}, \quad \alpha, z \in \mathbb{D},$$

denotes a single Blaschke factor. It has some good properties, for instance, it is a contractive divisor, that is, if $f \in L^2_a(\mathbb{D})$ vanishes on $\mathbf{a}$, then $f/P_{\mathbf{a}} \in L^2_a(\mathbb{D})$, and $\|f/P_{\mathbf{a}}\|_{L^2} \leqslant \|f\|_{L^2}$; this fact does not follow directly from Horowitz' work, but was pointed out to me by Boris Korenblum and Richard O'Neill in the summer of 1989. One difficulty with the Horowitz product is that it may grow wildly towards the boundary $\mathbb{T}$, and, in general, it itself will not belong to the Bergman space. Emile LeBlanc [7], in his 1990 Berkeley dissertation, studied when the Horowitz product is likely to have bounded Bergman space norm, and thus obtained a sufficient condition on an increasing sequence $\{r_j\}_0^\infty$ of numbers in $]0,1[$ tending to 1, which ensures that the probabilistic sequence $\{r_j e^{i\theta_j}\}_0^\infty$, with the $\theta_j$ being uniformly distributed independent random variables on the interval $[0,2\pi]$, almost surely is a Bergman space zero sequence. Later, Gregory Bomash [1] showed that a much better result could not be obtained using Horowitz factors, but found a different collection of factors, inspired by the author's work in [4], and was able to narrow down substantially the gap between necessary and sufficient conditions for a random sequence to almost surely be a zero set.

To find better zero divisors for the Bergman space, let us take a fresh look at Blaschke products for the space $H^2(\mathbb{D})$, forgetting about their product representations. It is not difficult to check that for a Blaschke sequence $\mathbf{a}$ contained in $\mathbb{D}\backslash\{0\}$, $B_{\mathbf{a}}$ is in fact the unique extremal function for the problem

$$\sup\{\mathrm{Re} f(0) : f \in H^2(\mathbb{D}),\ f = 0 \text{ on } \mathbf{a},\ \|f\|_{H^2} \leqslant 1\}.$$

This suggests that for Bergman space zero sequences $\mathbf{a}$ in $\mathbb{D}\backslash\{0\}$, we should study the extremal functions for the problem

$$\sup\{\mathrm{Re} f(0) : f \in L^2_a(\mathbb{D}),\ f = 0 \text{ on } \mathbf{a},\ \|f\|_{L^2} \leqslant 1\}.$$

By standard Hilbert space theory, this problem has a unique extremal function, which we shall call $G_{\mathbf{a}}$. Observe that the function $G_{\mathbf{a}}$ clearly has the following properties: $G_{\mathbf{a}}(0) > 0$, $\|G_{\mathbf{a}}\|_{L^2} = 1$, and $G_{\mathbf{a}}$ vanishes on $\mathbf{a}$. In case the sequence $\mathbf{a} = \{a_j\}_1^N$ is finite, and all $a_j$ are distinct, it turns out that $G_{\mathbf{a}}$ is a finite linear combination of the functions $1, (1-\bar{a}_1 z)^{-2}, \ldots, (1-\bar{a}_N z)^{-2}$, uniquely determined by the conditions $G_{\mathbf{a}}(0) > 0$, $G_{\mathbf{a}}(a_j) = 0$ for $j = 1, \ldots, N$, and $G_{\mathbf{a}}(0)\, G_{\mathbf{a}}(\infty) = 1$. The main result of this paper is the following.

THEOREM. *Suppose that* $\mathbf{a}$ *is a Bergman space zero sequence contained in* $\mathbb{D}\backslash\{0\}$. *Then* $G_{\mathbf{a}}$ *vanishes precisely on* $\mathbf{a}$ *in* $\mathbb{D}$, *and every function* $f \in L_a^2(\mathbb{D})$ *which vanishes on* $\mathbf{a}$ *has a factoring* $f = G_{\mathbf{a}} \cdot g$, *where* $g \in L_a^2(\mathbb{D})$, *and* $\|g\|_{L^2} \leqslant \|f\|_{L^2}$. *Moreover, up to a unimodular constant multiple,* $G_{\mathbf{a}}$ *is unique among all functions* $G \in L_a^2(\mathbb{D})$ *of norm* 1, *which vanish precisely on* $\mathbf{a}$, *and permit every* $f \in L_a^2(\mathbb{D})$ *vanishing on* $\mathbf{a}$ *to be factored* $f = G \cdot g$, *with* $g \in L_a^2(\mathbb{D})$, *and* $\|g\|_{L^2} \leqslant \|f\|_{L^2}$.

Given a general Bergman space zero sequence $\mathbf{a}$ contained in $\mathbb{D}\backslash\{0\}$, one may wonder for which $f \in L_a^2(\mathbb{D})$ we have $G_{\mathbf{a}} f \in L_a^2(\mathbb{D})$. It turns out that at least this holds for all $f \in H^2(\mathbb{D})$; in fact, $G_{\mathbf{a}}$ is contractive as a multiplier $H^2(\mathbb{D}) \to L_a^2(\mathbb{D})$, that is,

$$\|G_{\mathbf{a}} f\|_{L^2} \leqslant \|f\|_{H^2}, \quad f \in H^2(\mathbb{D}).$$

It follows from [9, p. 232] that $G_{\mathbf{a}}$ enjoys the estimate

$$|G_{\mathbf{a}}(z)| \leqslant (1-|z|^2)^{-1/2}, \quad z \in \mathbb{D},$$

so that $G_{\mathbf{a}}$ is somewhat better than an arbitrary Bergman space function. This is analogous to the Hardy space situation, where the extremal functions are Blaschke products, which are uniformly bounded on $\mathbb{D}$, and hence multipliers $H^2(\mathbb{D}) \to H^2(\mathbb{D})$.

An invariant subspace of $L_a^2(\mathbb{D})$ is a closed subspace which is invariant under multiplication by the coordinate function $z$. If $I$ is an invariant subspace, and $Z(I) \subset \mathbb{D}\backslash\{0\}$, where

$$Z(I) = \{z \in \mathbb{D} : f(z) = 0 \text{ for all } f \in I\},$$

let $G_I$ be the unique extremal function for the problem

$$\sup\{\operatorname{Re} f(0) : f \in I, \|f\|_{L^2} \leqslant 1\}.$$

We shall call $G_I$ the extremal function for $I$; the functions $G_I$ correspond to inner functions in the Hardy space case. If $\mathbf{a}$ is a Bergman space zero sequence, and

$$\mathcal{I}(\mathbf{a}) = \{f \in L_a^2(\mathbb{D}) : f = 0 \text{ on } \mathbf{a}\}$$

its associated invariant subspace, then $G_{\mathcal{I}(\mathbf{a})} = G_{\mathbf{a}}$. One easily checks that if $I$ is an invariant subspace, then its associated extremal function $G_I$ has the property that $|G_I|^2\, dA$ is a representing measure for 0, that is,

$$h(0) = \int_{\mathbb{D}} h(z) |G_I(z)|^2\, dA(z)$$

holds for all bounded harmonic functions $h$ on $\mathbb{D}$. In fact, this property completely singles out extremal functions for invariant subspaces, for if $G \in L_a^2(\mathbb{D})$ is such that $G(0) > 0$ and $|G|^2\, dA$ is a representing measure for 0, then $G$ coincides with the extremal function for the invariant subspace $I(G)$ generated by $G$.

We shall prove that if $I$ is an invariant subspace with $Z(I) \subset \mathbb{D}\backslash\{0\}$, then

$$\|f\|_{L^2} \leqslant \|G_I f\|_{L^2} \leqslant \|f\|_{H^2}$$

holds for all $f \in H^2(\mathbb{D})$; as a consequence, $G_I$ enjoys the estimate

$$|G_I(z)| \leqslant (1-|z|^2)^{-1/2}, \quad z \in \mathbb{D}.$$

The present paper is a condensed version of my 1990 paper [4], with the shorter proofs that are now available due to the efforts of Boris Korenblum, Peter Duren, Dmitry Khavinson and Harold Shapiro. The main tool, as suggested to me first by Harold Shapiro in June 1990, who was at the time collaborating with Duren and Khavinson, is the fact that the biharmonic Green function is positive. It should be mentioned here that Duren, Khavinson, Shapiro and Carl Sundberg have written a joint paper [2] on factoring in $L^p_a(\mathbb{D})$, with $1 \leqslant p < \infty$.

## 1. *Preliminaries*

Let $H^\infty(\mathbb{D})$ denote the space of bounded analytic functions on $\mathbb{D}$, supplied with the uniform norm. Also, let us introduce Boris Korenblum's partial order $\prec$: given two functions $f, g \in L^2_a(\mathbb{D})$, we say that $g \prec h$ if

$$\int_{\mathbb{D}} |gf|^2 \, dA \leqslant \int_{\mathbb{D}} |hf|^2 \, dA$$

holds for all $f \in H^\infty(\mathbb{D})$. In particular, $h \succ 1$ if $\|hf\|_{L^2} \geqslant \|f\|_{L^2}$ for all $f \in H^\infty(\mathbb{D})$.

PROPOSITION 1.1. *Let $G$ be an analytic function on $\mathbb{D}$ which extends continuously to $\bar{\mathbb{D}}$, and suppose $G \succ 1$. Then $|G| \geqslant 1$ on $\mathbb{T}$.*

*Proof.* Fix a point $z_0 \in \mathbb{T}$, and let $p$ be the analytic function

$$p(z) = (z + z_0)/2, \quad z \in \bar{\mathbb{D}},$$

which peaks at $z_0$. For $n = 1, 2, 3, \ldots$, consider the functions

$$\phi_n(z) = (p(z))^n / \|p^n\|_{L^2}, \quad z \in \bar{\mathbb{D}};$$

they have $\|\phi_n\|_{L^2} = 1$ and converge to 0 uniformly on compact subsets of $\bar{\mathbb{D}} \backslash \{z_0\}$ as $n \to \infty$. Since $G$ is continuous at $z_0$,

$$\int_{\mathbb{D}} |G(z)\, \phi_n(z)|^2 \, dA(z) \longrightarrow |G(z_0)|^2$$

as $n \to \infty$. On the other hand, we have

$$\int_{\mathbb{D}} |G(z)\, \phi_n(z)|^2 \, dA(z) \geqslant \int_{\mathbb{D}} |\phi_n(z)|^2 \, dA(z) = 1$$

because $G \succ 1$. We conclude that $|G(z_0)| \geqslant 1$, and the assertion follows.

REMARK. Using a more sophisticated peaking function, one can show that if $f, g \in H^\infty(\mathbb{D})$ and $f \prec g$, then $|f| \leqslant |g|$ almost everywhere on $\mathbb{T}$.

## 2. *Extremal functions for general invariant subspaces*

If $J$ is a subspace of $L^2_a(\mathbb{D})$, its annihilator is the set

$$J^\perp = \{f \in L^2_a(\mathbb{D}) \colon \langle f, g \rangle_{L^2} = 0 \text{ for all } g \in J\}.$$

We shall need the following general result.

LEMMA 2.1. *Let $I$ be an invariant subspace of $L_a^2(\mathbb{D})$ with $0 \notin Z(I)$, and let $G_I$ be the extremal function for the problem*

$$\sup\{\operatorname{Re} f(0): f \in I, \|f\|_{L^2} \leqslant 1\}.$$

*Write $\mathscr{I}_0 = \{f \in L_a^2(\mathbb{D}): f(0) = 0\}$. Then $G_I \in I \cap (I \cap \mathscr{I}_0)^\perp$, and $I \cap (I \cap \mathscr{I}_0)^\perp$ is a one-dimensional subspace of $L_a^2(\mathbb{D})$, so that $G_I$ is uniquely determined by this condition and the conditions $\|G_I\|_{L^2} = 1$ and $G_I(0) > 0$.*

*Proof.* A moment's thought reveals that $\|G_I\|_{L^2} = 1$ and $G_I(0) > 0$ hold. For $\varepsilon \in \mathbb{C}$ and $h \in I \cap \mathscr{I}_0$, introduce the functions

$$F_\varepsilon = \frac{G_I + \varepsilon h}{\|G_I + \varepsilon h\|_{L^2}},$$

and observe that $F_\varepsilon \in I$, $\|F_\varepsilon\|_{L^2} = 1$ and

$$F_\varepsilon(0) = G_I(0)/\|G_I + \varepsilon h\|_{L^2}.$$

Since $G_I$ is extremal, we must have $F_\varepsilon(0) \leqslant G_I(0)$, and consequently $\|G_I + \varepsilon h\|_{L^2} \geqslant 1$ for all $\varepsilon \in \mathbb{C}$. Now

$$\|G_I + \varepsilon h\|_{L^2}^2 = \|G_I\|_{L^2}^2 + |\varepsilon|^2 \|h\|_{L^2}^2 + 2\operatorname{Re}\langle G_I, \varepsilon h\rangle_{L^2},$$

so by varying $\varepsilon$ and observing that $\|G_I\|_{L^2} = 1$, we obtain $\langle G_I, h\rangle_{L^2} = 0$, and the assertion $G_I \in I \cap (I \cap \mathscr{I}_0)^\perp$ follows.

We shall now demonstrate that $I \cap (I \cap \mathscr{I}_0)^\perp$ is one-dimensional. Since $0 \notin Z(I)$, we can find a $\psi \in I$ with $\psi(0) = 1$. If $Q: L_a^2(\mathbb{D}) \to I \cap \mathscr{I}_0$ is the orthogonal projection, then $\phi = \psi - Q\psi \in I \cap (I \cap \mathscr{I}_0)^\perp$ and $\phi(0) = 1$. Now if $f \in I \cap (I \cap \mathscr{I}_0)^\perp$ is arbitrary, we have

$$f - f(0)\phi \in (I \cap \mathscr{I}_0) \cap (I \cap \mathscr{I}_0)^\perp = \{0\},$$

so that $f = f(0)\phi$. We conclude that $I \cap (I \cap \mathscr{I}_0)^\perp$ is spanned by the vector $\phi$, which completes the proof.

PROPOSITION 2.2. *Suppose that $G \in L_a^2(\mathbb{D})$ is such that $|G|^2\, dA$ is a representing measure for 0, that is,*

$$h(0) = \int_{\mathbb{D}} h(z)|G(z)|^2\, dA(z)$$

*holds for all bounded harmonic functions $h$ on $\mathbb{D}$. Then, if $G(0) > 0$, $G$ equals the extremal function for the invariant subspace $I(G)$ generated by $G$. On the other hand, if $I$ is an invariant subspace in $L_a^2(\mathbb{D})$ with $Z(I) \subset \mathbb{D}\backslash\{0\}$, and $G_I$ its associated extremal function, then $G_I(0) > 0$, and $|G_I|^2\, dA$ is a representing measure for 0.*

*Proof.* If $G$ is an element of $L_a^2(\mathbb{D})$ such that $|G|^2\, dA$ is a representing measure for 0, having the additional property $G(0) > 0$, then, clearly, we have

$$\langle z^n G, G\rangle_{L^2} = \int_{\mathbb{D}} z^n |G(z)|^2\, dA(z) = 0, \quad n = 1, 2, 3, \ldots,$$

so that $G$ is perpendicular to the invariant subspace $zI(G)$ in $L_a^2(\mathbb{D})$. By [**8**, p. 596], every singly generated invariant subspace has the codimension 1 property, and in particular $I(G)$, so that $zI(G) = I(G) \cap \mathscr{I}_0$, where $\mathscr{I}_0$ is as in Lemma 2.1. The assertion, that $G$ is the extremal function associated with $I(G)$, is now a simple consequence of Lemma 2.1.

If, on the other hand, $I$ is an invariant subspace with $Z(I) \subset \mathbb{D}\backslash\{0\}$, then $G_I$ has norm 1 in $L^2_a(\mathbb{D})$, and $G_I(0) > 0$. Moreover, we have, by Lemma 2.1,

$$\int_{\mathbb{D}} z^n |G_I(z)|^2 \, dA(z) = \langle z^n G_I, G_I \rangle_{L^2} = 0, \quad n = 1, 2, 3, \ldots,$$

so that since $G_I$ has norm 1, we have

$$\int_{\mathbb{D}} f(z) |G_I(z)|^2 \, dA(z) = f(0)$$

for analytic polynomials $f$. If we take complex conjugates, note that every harmonic polynomial is the sum of an analytic and an antianalytic polynomial, and apply an approximation argument, the desired assertion that $|G_I|^2 \, dA$ is a representing measure for 0 follows.

To simplify notation, let us write $\partial$ and $\bar\partial$ instead of $\partial/\partial z$ and $\partial/\partial \bar z$, respectively, and let us agree to write $\Delta = \partial\bar\partial$, so that $\Delta$ denotes a quarter of the usual Laplacian. For $(z, \zeta) \in \bar{\mathbb{D}}^2$, let $\Gamma(z, \zeta)$ denote the usual harmonic Green function

$$\Gamma(z, \zeta) = \log \left| \frac{z - \zeta}{1 - \bar\zeta z} \right|^2, \quad z \neq \zeta,$$

and let $U_\zeta(z) = U(z, \zeta)$ denote the so-called biharmonic Green function

$$U(z, \zeta) = |z - \zeta|^2 \Gamma(z, \zeta) + (1 - |z|^2)(1 - |\zeta|^2), \quad z \neq \zeta,$$

which for fixed $\zeta \in \mathbb{D}$ solves the partial differential equation $\Delta^2 U_\zeta = \delta_\zeta$ on $\mathbb{D}$, with boundary data $U_\zeta = \partial U_\zeta / \partial n = 0$ on $\mathbb{T}$; here, $\delta_\zeta$ denotes the Dirac mass at the arbitrary point $\zeta \in \mathbb{D}$, and $\partial/\partial n$ is differentiation in the outward normal direction. The function $U(z, \zeta)$ solves the above fourth-order partial differential equation in the usual distribution theory sense, with one small deviation, in that we use normalized area measure on $\mathbb{D}$:

$$\phi(\zeta) = \int_{\mathbb{D}} U(z, \zeta) \Delta^2 \phi(z) \, dA(z), \quad \zeta \in \mathbb{D},$$

holds for all compactly supported infinitely differentiable $\phi$ on $\mathbb{D}$.

It should be noted here that using the explicit formula for $U(z, \zeta)$ above, one can show that $U(z, \zeta) \geqslant 0$ for all $(z, \zeta) \in \bar{\mathbb{D}}^2$, with strict inequality in $\mathbb{D}^2$. In fact, by the elementary inequality

$$\log x > 1 - 1/x, \quad 0 < x < 1,$$

we have

$$\Gamma(z, \zeta) > -\frac{(1 - |z|^2)(1 - |\zeta|^2)}{|z - \zeta|^2}, \quad (z, \zeta) \in \mathbb{D}^2, \quad z \neq \zeta,$$

from which the assertion $U(z, \zeta) > 0$ immediately follows for $(z, \zeta) \in \mathbb{D}^2$. We now present a formula that will prove fundamental in the theory we are developing.

THEOREM 2.3. *Suppose, as in Proposition* 2.2, *that* $G \in L^2_a(\mathbb{D})$ *is such that* $|G|^2 \, dA$ *is a representing measure for* 0. *Then, for* $f \in H^2(\mathbb{D})$, *we have the formula*

$$\int_{\mathbb{D}} |G(z) f(z)|^2 \, dA(z) = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) + \int_{\mathbb{D}^2} U(z, \zeta) \, |f'(z)|^2 \, |G'(\zeta)|^2 \, dA(z) \, dA(\zeta).$$

*Proof.* If $u \in C^2(\bar{\mathbb{D}})$, then Green's theorem gives us the identity

$$\int_{\mathbb{D}} U(z,\zeta)\,\Delta u(z)\,dA(z) = \int_{\mathbb{D}} \Delta_z U(z,\zeta)\,u(z)\,dA(z),$$

if we use the boundary data of the biharmonic Green function $U(z,\zeta)$. If we apply this formula to the special case $u(z) = |g(z)|^2$, where $g$ is analytic on $\mathbb{D}$ and $C^2$ on $\bar{\mathbb{D}}$, and note that $\Delta u(z) = |g'(z)|^2$ and that

$$\Delta_z U(z,\zeta) = \Gamma(z,\zeta) + (1-|\zeta|^2)\operatorname{Re}\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}, \quad z,\zeta \in \mathbb{D},$$

we obtain

$$\int_{\mathbb{D}} U(z,\zeta)\,|g'(z)|^2\,dA(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\,|g(z)|^2\,dA(z) + (1-|\zeta|^2)\int_{\mathbb{D}} h_\zeta(z)\,|g(z)|^2\,dA(z), \tag{2.1}$$

if $h_\zeta$ denotes the function

$$h_\zeta(z) = \operatorname{Re}\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}.$$

Considering that for fixed $\zeta \in \mathbb{D}$, $h_\zeta$ is a bounded function on $\mathbb{D}$, and $\Gamma(z,\zeta) < 0$ on $\mathbb{D}$, we see that by an approximation argument, (2.1) generalizes to all $g \in L_a^2(\mathbb{D})$. In particular, (2.1) holds for the choice $g = G$, and since $|G|^2\,dA$ is a representing measure for 0, and $h_\zeta$ is a bounded harmonic function for fixed $\zeta \in \mathbb{D}$, we have

$$\int_{\mathbb{D}} U(z,\zeta)|G'(z)|^2\,dA(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)|G(z)|^2\,dA(z) + (1-|\zeta|^2). \tag{2.2}$$

If $v$ is a $C^2$ function on $\bar{\mathbb{D}}$, and $P[v]$ is the harmonic function on $\mathbb{D}$ with boundary values $v$, then

$$\int_{\mathbb{D}} \Gamma(z,\zeta)\,\Delta v(\zeta)\,dA(\zeta) = v(z) - P[v](z), \quad z \in \mathbb{D},$$

so that by (2.2) and Fubini's theorem,

$$\int_{\mathbb{D}^2} U(z,\zeta)\,|G'(z)|^2\,\Delta v(\zeta)\,dA(z)\,dA(\zeta) = \int_{\mathbb{D}} v(z)\,|G(z)|^2\,dA(z)$$
$$- \int_{\mathbb{D}} P[v](z)\,|G(z)|^2\,dA(z) + \int_{\mathbb{D}} (1-|z|^2)\,\Delta v(z)\,dA(z). \tag{2.3}$$

Now since $P[v]$ is bounded and harmonic on $\mathbb{D}$, and $|G|^2\,dA$ is a representing measure for 0, we have

$$\int_{\mathbb{D}} P[v](z)\,|G(z)|^2\,dA(z) = P[v](0) = \int_{-\pi}^{\pi} v(e^{i\theta})\,d\theta/2\pi.$$

On the other hand, by Green's theorem,

$$\int_{\mathbb{D}} (1-|z|^2)\,\Delta v(z)\,dA(z) = \int_{-\pi}^{\pi} v(e^{i\theta})\,d\theta/2\pi - \int_{\mathbb{D}} v(z)\,dA(z),$$

so that (2.3) reduces to

$$\int_{\mathbb{D}^2} U(z,\zeta)\,|G'(z)|^2\,\Delta v(\zeta)\,dA(z)\,dA(\zeta) = \int_{\mathbb{D}} v(z)\,|G(z)|^2\,dA(z) - \int_{\mathbb{D}} v(z)\,dA(z).$$

If we put $v = |f|^2$, where $f$ is a $C^2$ function on $\bar{\mathbb{D}}$, holomorphic on $\mathbb{D}$, this becomes

$$\int_{\mathbb{D}^2} U(z,\zeta)\,|G'(z)|^2\,|f'(\zeta)|^2\,dA(z)\,dA(\zeta) = \int_{\mathbb{D}} |G(z)f(z)|^2\,dA(z) - \int_{\mathbb{D}} |f(z)|^2\,dA(z),$$

or, written differently,

$$\int_{\mathbb{D}} |G(z)f(z)|^2\,dA(z) = \int_{\mathbb{D}} |f(z)|^2\,dA(z) + \int_{\mathbb{D}^2} U(z,\zeta)\,|G'(z)|^2\,|f'(\zeta)|^2\,dA(z)\,dA(\zeta). \tag{2.4}$$

By (2.2),

$$0 \leqslant \int_{\mathbb{D}} U(z,\zeta)\,|G'(z)|^2\,dA(z) \leqslant 1-|\zeta|^2, \quad \zeta\in\mathbb{D}, \tag{2.5}$$

so that since the norm in $H^2(\mathbb{D})$ can be written in the form

$$\|f\|^2_{H^2} = \|f\|^2_{L^2} + \int_{\mathbb{D}} (1-|z|^2)\,|f'(z)|^2\,dA(z), \tag{2.6}$$

we see by an approximation argument that (2.4) extends to all $f\in H^2(\mathbb{D})$.

COROLLARY 2.4. *Suppose, as in Proposition* 2.2, *that the function* $G\in L^2_a(\mathbb{D})$ *is such that* $|G|^2\,dA$ *is a representing measure for* 0, *and let* $\Phi_G$ *be the function*

$$\Phi_G(z) = \int_{\mathbb{D}} U(z,\zeta)\,|G'(\zeta)|^2\,dA(\zeta), \quad z\in\bar{\mathbb{D}}.$$

*Then* $\Phi_G$ *is real analytic on* $\mathbb{D}$, *and continuous up to the boundary* $\partial\mathbb{D}$, *and we have*

$$0 \leqslant \Phi_G(z) \leqslant 1-|z|^2, \quad z\in\bar{\mathbb{D}}.$$

*Let* $\mathscr{A}(G)$ *be the space*

$$\mathscr{A}(G) = \left\{ f\in L^2_a(\mathbb{D}) : \int_{\mathbb{D}} \Phi_G |f'|^2\,dA < \infty \right\},$$

*supplied with the Hilbert space norm*

$$\|f\|^2_{\mathscr{A}(G)} = \|f\|^2_{L^2} + \int_{\mathbb{D}} \Phi_G |f'|^2\,dA.$$

*If* $\mathscr{A}_0(G)$ *denotes the closure of the analytic polynomials in* $\mathscr{A}(G)$, *then multiplication by G is an isometry* $\mathscr{A}_0(G)\to L^2_a(\mathbb{D})$, *with image* $I(G)$, *the invariant subspace generated by G.*

*Proof.* From the ellipticity of the operator $\Delta^2$, and the real analyticity of the function $\Delta^2\Phi_G = |G'|^2$ on $\mathbb{D}$, we see that $\Phi_G$ is real analytic on $\mathbb{D}$, and by (2.5), $\Phi_G$ has

$$0 \leqslant \Phi_G(z) \leqslant 1-|z|^2, \quad z\in\bar{\mathbb{D}},$$

which forces $\Phi_G$ to be continuous up to the boundary $\partial\mathbb{D}$. By Theorem 2.3, the isometry

$$\|Gf\|_{L^2} = \|f\|_{\mathscr{A}(G)}$$

holds for all $f\in H^2(\mathbb{D})$, and an approximation argument extends it to all $f\in\mathscr{A}_0(G)$. Moreover, the image $G\cdot\mathscr{A}_0(G)$ of $\mathscr{A}_0(G)$ under multiplication by $G$ is a closed subspace of $L^2_a(\mathbb{D})$, in which $G$ times the analytic polynomials is dense, so it must equal $I(G)$, that is, $I(G) = G\cdot\mathscr{A}_0(G)$.

The space $\mathscr{A}(G)$ introduced in Corollary 2.4 is contained in $L^2_a(\mathbb{D})$, and the injection mapping $\mathscr{A}(G) \to L^2_a(\mathbb{D})$ is contractive. On the other hand, the norm on $H^2(\mathbb{D})$ can be written in the form

$$\|f\|^2_{H^2} = \|f\|^2_{L^2} + \int_{\mathbb{D}} (1-|z|^2)\,|f'(z)|^2\, dA(z),$$

so by the inequality $0 \leqslant \Phi_G(z) \leqslant 1-|z|^2$ obtained in Corollary 2.4, and the fact that the analytic polynomials are dense in $H^2(\mathbb{D})$, $H^2(\mathbb{D})$ is contained within $\mathscr{A}_0(G)$, and the injection mapping $H^2(\mathbb{D}) \to \mathscr{A}_0(G)$ is contractive. We arrive at the following result.

COROLLARY 2.5. *Suppose, as in Proposition* 2.2, *that the function $G \in L^2_a(\mathbb{D})$ is such that $|G|^2\, dA$ is a representing measure for* 0, *and that the spaces $\mathscr{A}(G)$ and $\mathscr{A}_0(G)$ are as in Corollary* 2.4. *Then $H^2(\mathbb{D}) \subset \mathscr{A}_0(G) \subset \mathscr{A}(G) \subset L^2_a(\mathbb{D})$, and the injection mappings $\mathscr{A}(G) \to L^2_a(\mathbb{D})$ and $H^2(\mathbb{D}) \to \mathscr{A}_0(G)$ are contractive. If $I(G)$ denotes the invariant subspace in $L^2_a(\mathbb{D})$ generated by $G$, then since multiplication by $G$ is an isometry $\mathscr{A}_0(G) \to L^2_a(\mathbb{D})$ with image $I(G)$, $G$ is an isometric divisor $I(G) \to \mathscr{A}_0(G)$, and a contractive divisor $I(G) \to L^2_a(\mathbb{D})$:*

$$\|f/G\|_{L^2} \leqslant \|f/G\|_{\mathscr{A}(G)} = \|f\|_{L^2}, \quad f \in I(G).$$

COROLLARY 2.6. *Suppose, as in Proposition* 2.2, *that the function $G \in L^2_a(\mathbb{D})$ is such that $|G|^2\, dA$ is a representing measure for* 0. *Then*

$$|G(z)| \leqslant (1-|z|^2)^{-1/2}, \quad z \in \mathbb{D}.$$

*Proof.* This follows from [**9**, p. 232], since $G$ is a contractive multiplier $H^2(\mathbb{D}) \to L^2_a(\mathbb{D})$, by Corollaries 2.4 and 2.5.

QUESTION 2.7. Does there exist a function $G \in L^2_a(\mathbb{D})$ with $|G|^2\, dA$ a representing measure for 0, such that $\mathscr{A}_0(G)$ is not all of $\mathscr{A}(G)$?

No matter what the answer to Question 2.7 is, it is tempting to suggest that the following is true; the notation is as in Corollary 2.4.

CONJECTURE 2.8. *If the function $G \in L^2_a(\mathbb{D})$ is such that $|G|^2\, dA$ is a representing measure for* 0, *then multiplication by $G$ is an isometry $\mathscr{A}(G) \to L^2_a(\mathbb{D})$.*

THEOREM 2.9. *Suppose that $I$ is an invariant subspace in $L^2_a(\mathbb{D})$, with $Z(I) \subset \mathbb{D}\backslash\{0\}$, such that $zI$ has codimension* 1 *in $I$. Then $G_I$ has no more zeros than do the functions in $I$, in the sense that $f/G_I$ is holomorphic in $\mathbb{D}$ for all $f \in I$.*

*Proof.* We argue by contradiction. So, suppose $G_I$ actually has one zero too many; let us call it $\beta \in \mathbb{D}$. We shall need the following auxiliary function:

$$G_\beta(z) = \frac{|\beta|}{\sqrt{(2-|\beta|^2)}}\,\frac{(1-z/\beta)(2-|\beta|^2-\bar{\beta}z)}{(1-\bar{\beta}z)^2}, \quad z \in \mathbb{D}.$$

This function has the properties $0 < G_\beta(0) < 1$, $|G(e^{i\theta})| \geqslant 1$ for all real $\theta$, $G_\beta(\beta) = 0$ and

$$\|G_\beta f\| \geqslant \|f\|, \quad f \in L^2_a(\mathbb{D});$$

this last property will be postponed until Corollary 3.3. In fact, this is the explicit formula giving the extremal function corresponding to the single zero $\beta$; see

Corollary 3.3. The observation that $G_\beta$ vanishes only at $\beta$ inside $\bar{\mathbb{D}}$ will prove vital to us. Since $G_I$ vanishes at $\beta$, $G_\beta$ is bounded away from 0 near $\mathbb{T}$, and $G_\beta$ vanishes only at $\beta$ in $\mathbb{D}$, the function

$$\tilde{G}_I(z) = G_I(z)/G_\beta(z), \quad z \in \mathbb{D},$$

belongs to $L^2_a(\mathbb{D})$. According to [**8**, p. 587], if $zI$ has codimension 1 in $I$, we can say that $(z-\beta)I$ also has codimension 1 in $I$, so that if the functions in $I$ vanish at $\beta$ with multiplicity $n_\beta$, then $(z-\beta)I$ consists of all functions in $I$ that vanish at $\beta$ with multiplicity $n_\beta+1$. It follows that $G_I$ belongs to $(z-\beta)I$, and consequently $\tilde{G}_I \in I$. Multiplication by $G_\beta$ being norm expanding, we have $\|\tilde{G}_I\|_{L^2} \leqslant \|G_I\|_{L^2}$, and because $0 < G_\beta(0) < 1$, we also have $\tilde{G}_I(0) > G_I(0)$. This, however, means that $\tilde{G}_I$ is more extremal than $G_I$, yielding our desired contradiction.

REMARKS. (a) Given an analytic function $G$ in $\mathbb{D}$ with the property that $|G|^2\,dA$ is a representing measure for 0, its associated space $\mathscr{A}_0(G)$, and consequently, $\mathscr{A}(G)$ will be strictly smaller than $L^2_a(\mathbb{D})$ in general; for if not, by the closed graph theorem, it would then be a bounded multiplier $L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$, and hence a bounded analytic function on $\mathbb{D}$. As a consequence, its zeros would satisfy the Blaschke condition, in clear violation of Theorem 2.9, because not all Bergman space zero sequences meet the Blaschke condition.

(b) It is possible to define extremal functions $G_I$ also for invariant subspaces $I$ with $0 \in Z(I)$. Say, for instance, that the functions in $I$ have a common zero of order $n$ at 0. We can define $G_I$ to be the extremal function for the problem

$$\sup\{\operatorname{Re} f^{(n)}(0) : f \in I, \|f\|_{L^2} \leqslant 1\}.$$

Then this $G_I$ will have the property that $|G_I|^2\,dA$ is a representing measure for 0, so that all results stated in this paper hold for these extremal functions as well, unless they are trivially incorrect.

## 3. *Extremal functions for finite zero sets*

We shall now compute the extremal function $G_{\mathbf{a}}$ for the problem

$$\sup\{\operatorname{Re} f(0) : f \in \mathscr{I}(\mathbf{a}), \|f\|_{L^2} \leqslant 1\},$$

where

$$\mathscr{I}(\mathbf{a}) = \{f \in L^2_a(\mathbb{D}) : f = 0 \text{ on } \mathbf{a}\}$$

and $\mathbf{a} = \{a_j\}_{j=1}^N$ is a finite sequence of points in $\mathbb{D}\backslash\{0\}$.

For $\alpha \in \mathbb{D}$, the annihilator of the invariant subspace $\{f \in L^2_a(\mathbb{D}) : f(\alpha) = 0\}$ is the linear space spanned by the kernel function

$$k_\alpha(z) = (1-\bar{\alpha}z)^{-2}, \quad z \in \mathbb{D}.$$

From this it is evident that if the points $a_1, \ldots, a_N$ are all distinct, the annihilator $(\mathscr{I}(\mathbf{a}) \cap \mathscr{I}_0)^\perp$ is the $(N+1)$-dimensional subspace of $L^2_a(\mathbb{D})$ spanned by the vectors 1, $k_{a_1}, \ldots, k_{a_N}$, so that in this case, Lemma 2.1 takes the following form.

COROLLARY 3.1. *Suppose that* $\mathbf{a} = \{a_j\}_{j=1}^N$ *is a finite sequence of distinct points in* $\mathbb{D}\backslash\{0\}$, *and that* $G_{\mathbf{a}}$ *and* $k_\alpha$ *are as above. Then* $G_{\mathbf{a}} \in L^2_a(\mathbb{D})$ *is uniquely determined by the following conditions:*

(a) $G_{\mathbf{a}}$ *vanishes on* $\mathbf{a}$;
(b) $G_{\mathbf{a}}$ *is a linear combination of the functions* $1, k_{a_1}, \ldots, k_{a_N}$;
(c) $\|G_{\mathbf{a}}\|_{L^2} = 1$;
(d) $G_{\mathbf{a}}(0) > 0$.

REMARK. If the points $a_1, \ldots, a_N$ are not distinct, the situation is a little more complicated. Say, for instance, that the point $\alpha \in \mathbb{D}$ occurs $k$ times in $\mathbf{a}$. Then the annihilator $(\mathscr{I}(\mathbf{a}) \cap \mathscr{I}_0)^\perp$ contains the functions

$$(1-\bar{\alpha}z)^{-2}, \quad z(1-\bar{\alpha}z)^{-3}, \quad \ldots, \quad z^{k-1}(1-\bar{\alpha}z)^{-k-1}.$$

If we do this for each point in $Z(\mathscr{I}(\mathbf{a}))$, and add the function 1 afterwards, we obtain a basis for $(\mathscr{I}(\mathbf{a}) \cap \mathscr{I}_0)^\perp$.

For singleton sequences $\{\beta\}$, with $\beta \in \mathbb{D}\setminus\{0\}$, let us write $G_\beta$ instead of $G_{\{\beta\}}$. In this case, Corollary 3.1 specializes to the following assertion, if we note that the $G_\beta$ given by the formula below is a linear combination of the functions 1 and $k_\beta$, has norm 1, and satisfies $G_\beta(0) > 0$, as prescribed by Corollary 3.1.

COROLLARY 3.2. *For* $\beta \in \mathbb{D}\setminus\{0\}$, *we have the formula*

$$G_\beta(z) = \frac{|\beta|}{\sqrt{(2-|\beta|^2)}} \frac{(1-z/\beta)(2-|\beta|^2-\bar{\beta}z)}{(1-\bar{\beta}z)^2}, \quad z \in \mathbb{D}.$$

Note that $G_\beta$ vanishes only at $\beta$ inside $\mathbb{D}$.

COROLLARY 3.3. *Suppose that* $\mathbf{a} = \{a_j\}_{j=1}^N$ *is a finite sequence of points in* $\mathbb{D}\setminus\{0\}$, *counting multiplicities, and that* $G_{\mathbf{a}}$ *is the corresponding extremal function. Then, with the notation of Corollary* 2.4, $G_{\mathbf{a}}$ *has the property that* $\mathscr{A}_0(G_{\mathbf{a}}) = L^2_a(\mathbb{D})$, *isomorphically but not isometrically, and*

$$\|f\|_{L^2} \leqslant \|f\|_{\mathscr{A}(G_{\mathbf{a}})} = \|G_{\mathbf{a}} f\|_{L^2}, \quad f \in L^2_a(\mathbb{D}). \tag{3.1}$$

*Proof.* By Proposition 2.2 and Corollaries 2.4 and 2.5, the desired inequality (3.1) holds for all $f \in H^2(\mathbb{D})$. The function $G_{\mathbf{a}}$ being bounded on $\mathbb{D}$, by Corollary 3.1 and the remark thereafter, we see that the norms $\|\cdot\|_{L^2}$ and $\|f\|_{\mathscr{A}(G_{\mathbf{a}})}$ are equivalent on functions in $H^2(\mathbb{D})$, so that an approximation argument allows us to show that $\mathscr{A}_0(G_{\mathbf{a}}) = L^2_a(\mathbb{D})$, and extend (3.1) to all $f \in L^2_a(\mathbb{D})$.

COROLLARY 3.4. *Suppose that* $\mathbf{a} = \{a_j\}_{j=1}^N$ *is a finite sequence of points in* $\mathbb{D}\setminus\{0\}$, *counting multiplicities, and that* $G_{\mathbf{a}}$ *is the corresponding extremal function. Then* $G_{\mathbf{a}}$ *extends analytically to the set* $\mathbb{C} \cup \{\infty\} \setminus \bigcup_j \{1/\bar{a}_j\}$, *vanishes precisely on the sequence* $\mathbf{a}$ *inside* $\mathbb{D}$, *and on* $\mathbb{T}$, $|G_{\mathbf{a}}| \geqslant 1$. *Moreover, with the notation of Corollary* 2.4, $I(G_{\mathbf{a}}) = \mathscr{I}(\mathbf{a})$.

*Proof.* It is clear from Corollary 3.1 that $G_{\mathbf{a}}$ extends analytically to the set $\mathbb{C} \cup \{\infty\} \setminus \bigcup_j \{1/\bar{a}_j\}$. Also, $G_{\mathbf{a}}$ of course vanishes on $\mathbf{a}$, and by Theorem 2.9, $G_{\mathbf{a}}$ cannot vanish anywhere else inside the open unit disk $\mathbb{D}$. Moreover, by Proposition 1.1 and Corollary 3.3, $|G_{\mathbf{a}}| \geqslant 1$ on $\mathbb{T}$. From this we see that the invariant subspace $I(G_{\mathbf{a}})$ generated by $G_{\mathbf{a}}$ equals $\mathscr{I}(\mathbf{a})$, as asserted.

The following statement is now a consequence of Corollary 2.5.

COROLLARY 3.5. *Suppose that* $\mathbf{a} = \{a_j\}_{j=1}^N$ *is a finite sequence of points in* $\mathbb{D}\setminus\{0\}$. *Then the extremal function* $G_{\mathbf{a}}$ *is an isometric divisor* $\mathscr{I}(\mathbf{a}) \to (L_a^2(\mathbb{D}), \|\cdot\|_{\mathscr{A}(G_{\mathbf{a}})})$, *and a contractive divisor* $\mathscr{I}(\mathbf{a}) \to L_a^2(\mathbb{D})$, *that is,*

$$\|f/G_{\mathbf{a}}\|_{L^2} \leqslant \|f/G_{\mathbf{a}}\|_{\mathscr{A}(G_{\mathbf{a}})} = \|f\|_{L^2}, \quad f \in \mathscr{I}(\mathbf{a}).$$

*In other words, every* $f \in \mathscr{I}(\mathbf{a})$ *has a factoring* $f = G_{\mathbf{a}} \cdot g$, *where* $g \in L_a^2(\mathbb{D})$ *has* $\|g\|_{L^2} \leqslant \|g\|_{\mathscr{A}(G_{\mathbf{a}})} = \|f\|_{L^2}$.

## 4. *Extremal functions for infinite zero sets*

In what follows, $\mathbf{a} = \{a_j\}_{j=1}^\infty$ is an infinite sequence of points in $\mathbb{D}\setminus\{0\}$, and $\mathbf{a}_N$ is its finite subsequence $\{a_j\}_{j=1}^N$. As in the introduction, $G_{\mathbf{a}}$ is the extremal function for the problem

$$\sup\{\mathrm{Re} f(0) : f \in L_a^2(\mathbb{D}), f = 0 \text{ on } \mathbf{a}, \|f\|_{L^2} \leqslant 1\},$$

which is unique by the same argument which was applied to finite sequences in the introduction, or simply by Lemma 2.1. The sequence $\mathbf{a}$ is a Bergman space zero sequence if there exists a function $f \in L_a^2(\mathbb{D})$ which vanishes precisely on $\mathbf{a}$ in $\mathbb{D}$. If there is a function $f \in L_a^2(\mathbb{D})$, other than 0, which vanishes on $\mathbf{a}$, we say that $\mathbf{a}$ is a subsequence of a Bergman space zero sequence; observe that in that case, $G_{\mathbf{a}}(0) > 0$ and $\|G_{\mathbf{a}}\|_{L^2} = 1$. It is a consequence of the following result that every subsequence of a Bergman space zero sequence is in fact itself a Bergman space zero sequence; this fact has been noted earlier by Charles Horowitz [5].

PROPOSITION 4.1. *Let* $\mathbf{a}$ *be as above. If* $\mathbf{a}$ *is a subsequence of a Bergman space zero sequence,* $G_{\mathbf{a}}$ *vanishes precisely on* $\mathbf{a}$ *in* $\mathbb{D}$, *and* $G_{\mathbf{a}_N} \to G_{\mathbf{a}}$ *in* $L_a^2(\mathbb{D})$ *as* $N \to \infty$. *If* $\mathbf{a}$ *is not a subsequence of any Bergman space zero sequence,* $G_{\mathbf{a}_N} \to 0$ *uniformly on compact subsets of* $\mathbb{D}$.

*Proof.* Let us first deal with the case when $\mathbf{a}$ is not a subsequence of a Bergman space zero sequence. Let $G$ be a normal limit to the sequence of $G_{\mathbf{a}_N}$, which then has $\|G\|_{L^2} \leqslant 1$, because $\|G_{\mathbf{a}_N}\|_{L^2} = 1$ for all $N$, and since $G$ is analytic and vanishes on $\mathbf{a}$, $G$ must vanish identically. If every normal limit is 0, the sequence must converge to 0 uniformly on compact subsets of $\mathbb{D}$.

We now look at the remaining case, when $\mathbf{a}$ is the zero sequence of a Bergman space function. Again, let $G$ be a normal limit to the sequence of $G_{\mathbf{a}_N}$, which has $\|G\|_{L^2} \leqslant 1$ and $G(0) \geqslant G_{\mathbf{a}}(0)$, because $G_{\mathbf{a}_N} \geqslant G_{\mathbf{a}}(0)$ for all $N$. By the extremality of $G_{\mathbf{a}}$, $G$ must coincide with $G_{\mathbf{a}}$, and we obtain that $G_{\mathbf{a}_N}$ converges to $G_{\mathbf{a}}$ uniformly on compact subsets of $\mathbb{D}$. It follows that for $r$ with $0 < r < 1$, we have

$$\int_{r\mathbb{D}} |G_{\mathbf{a}_N}|^2 \, dA \to \int_{r\mathbb{D}} |G_{\mathbf{a}}|^2 \, dA \quad \text{as } N \to \infty,$$

and since $\|G_{\mathbf{a}_N}\|_{L^2} = \|G_{\mathbf{a}}\|_{L^2} = 1$, we obtain

$$\lim_{N \to \infty} \int_{\mathbb{D}\setminus r\mathbb{D}} |G_{\mathbf{a}_N}|^2 \, dA \to 0 \quad \text{as } r \to 1,$$

from which the assertion $\|G_{\mathbf{a}_N} - G_{\mathbf{a}}\|_{L^2} \to 0$ as $N \to \infty$ easily follows. The fact that $G_{\mathbf{a}}$ vanishes precisely on $\mathbf{a}$ in $\mathbb{D}$ follows from Theorem 2.9.

We are now ready to present the general factoring theorem for Bergman space functions.

THEOREM 4.2. *Let* $\mathbf{a} = \{a_j\}_{j=1}^{\infty} \subset \mathbb{D}\backslash\{0\}$ *be a Bergman space zero sequence, and suppose that* $f \in L_a^2(\mathbb{D})$ *vanishes on* $\mathbf{a}$. *Then we have* $f = G_{\mathbf{a}} \cdot g$, *where* $g \in \mathscr{A}(G_{\mathbf{a}}) \subset L_a^2(\mathbb{D})$, *and* $\|g\|_{L^2} \leqslant \|g\|_{\mathscr{A}(G_{\mathbf{a}})} \leqslant \|f\|_{L^2}$.

*Proof.* Let $g = f/G_{\mathbf{a}}$, which is analytic on the disk $\mathbb{D}$, by Proposition 4.1, and let $\mathbf{a}_N$ be the cutoff sequence $\{a_j\}_1^N$, as above. By Corollary 3.2, $f$ has a factoring $f = G_{\mathbf{a}_N} \cdot g_N$, where $g_N \in L_a^2(\mathbb{D})$ has $\|g_N\|_{L^2} \leqslant \|g_N\|_{\mathscr{A}(G_{\mathbf{a}_N})} = \|f\|_{L^2}$. By Proposition 4.1, we must have $g_N \to g$ uniformly on compact subsets of $\mathbb{D}$ as $N \to \infty$, so by Fatou's lemma,

$$\|g\|_{L^2} \leqslant \|g\|_{\mathscr{A}(G_{\mathbf{a}})} \leqslant \liminf_{N\to\infty} \|g_N\|_{L^2} \leqslant \|f\|_{L^2}.$$

The proof is complete.

The following result emphasizes the uniqueness of the function $G_{\mathbf{a}}$ in the formulation of Theorem 4.2.

THEOREM 4.3. *Let* $\mathbf{a} = \{a_j\}_{j=1}^{\infty} \subset \mathbb{D}\backslash\{0\}$ *be a Bergman space zero sequence, and suppose that* $G \in L_a^2(\mathbb{D})$ *vanishes on* $\mathbf{a}$, $G(0) \geqslant 0$, $\|G\|_{L^2} = 1$, *and that every* $f \in L_a^2(\mathbb{D})$ *that vanishes on* $\mathbf{a}$ *factors* $f = G \cdot g$, *where* $g \in L_a^2(\mathbb{D})$ *has* $\|g\|_{L^2} \leqslant \|f\|_{L^2}$. *Then* $G = G_{\mathbf{a}}$.

*Proof.* The function $G_{\mathbf{a}} \in L_a^2(\mathbb{D})$ vanishes on $\mathbf{a}$, so it must have a factoring $G_{\mathbf{a}} = G \cdot g$, where $g \in L_a^2(\mathbb{D})$ has $\|g\|_{L^2} \leqslant \|G_{\mathbf{a}}\|_{L^2} = 1$. Since $G_{\mathbf{a}}(0) > 0$, we must have $G(0) > 0$ and therefore $g(0) > 0$. The estimate $|g(0)|^2 \leqslant \|g\|_{L^2}^2 \leqslant 1$ shows that $G(0) \geqslant G_{\mathbf{a}}(0)$. The extremality of $G_{\mathbf{a}}$ now forces $G$ to coincide with $G_{\mathbf{a}}$.

In Theorem 4.2 we obtain an inequality where we actually have an equality for finite sequences (Corollary 3.5). It seems reasonable to suspect that the equality should survive also for infinite sequences; let us write this as a conjecture.

CONJECTURE 4.4. *Let* $\mathbf{a} = \{a_j\}_{j=1}^{\infty} \subset \mathbb{D}\backslash\{0\}$ *be a Bergman space zero sequence, and suppose that* $f \in L_a^2(\mathbb{D})$ *vanishes on* $\mathbf{a}$. *Then we have* $f = G_{\mathbf{a}} \cdot g$, *where* $g \in \mathscr{A}(G_{\mathbf{a}}) \subset L_a^2(\mathbb{D})$, *and* $\|g\|_{L^2} \leqslant \|g\|_{\mathscr{A}(G_{\mathbf{a}})} = \|f\|_{L^2}$. *On the other hand, if* $g \in \mathscr{A}(G_{\mathbf{a}})$, *then we have* $G_{\mathbf{a}} g \in L_a^2(\mathbb{D})$.

Of course, by Corollary 2.5, this would hold if we knew that $G_{\mathbf{a}}$ generates the invariant subspace $\mathscr{I}(\mathbf{a})$; the trouble is that at least I have no idea of whether this is actually the case. However, Conjecture 4.4 does hold if we can prove the following elasticity theory assertion.

CONJECTURE 4.5. *Let* $G$ *be a continuous function on the closed unit disk* $\bar{\mathbb{D}}$ *which is analytic in the open disk* $\mathbb{D}$, *and suppose that* $|G|^2 \, dA$ *is a representing measure for* 0. *Let* $U_\zeta^G(z) = U^G(z, \zeta)$ *be the solution to the boundary value problem*

$$\begin{cases} \Delta |G|^{-2} \Delta U_\zeta^G = \delta_\zeta & \text{on } \mathbb{D}, \\ U_\zeta^G, \nabla U_\zeta^G = 0 & \text{on } \mathbb{T}, \end{cases}$$

*where* $\nabla$ *is the gradient operator, and* $\delta_\zeta$ *the unit Dirac mass at the arbitrary point* $\zeta \in \mathbb{D}$. *We then have* $U^G(z, \zeta) \geqslant 0$ *for all* $z, \zeta \in \mathbb{D}$.

In a previous version of this paper, it was suggested that the following might hold.

CONJECTURE 4.6. *Let* $\mathbf{a} = \{a_j\}_{j=1}^{\infty} \subset \mathbb{D}\setminus\{0\}$ *be a Bergman space zero sequence. Then* $G_{\mathbf{a}}$ *is bounded on the unit disk* $\mathbb{D}$ *if and only if* $\mathbf{a}$ *is a finite union of interpolating sequences.*

Since then, however, it has become clear that this is not so: a counterexample was shown to me by Carl Sundberg at the Joint Mathematics Meetings, Special Session on Bergman Spaces, in Baltimore, Maryland, USA, 8–11 January 1992.

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