CLOSED IDEALS IN THE BALL ALGEBRA

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Section 0

Let $\boldsymbol{B}_n$ be the open unit ball of $\boldsymbol{C}^n$, and let $H^\infty(\boldsymbol{B}_n)$ be the algebra of bounded analytic functions on $\boldsymbol{B}_n$, endowed with the uniform norm on $\boldsymbol{B}_n$. The ball algebra is the space $A(\boldsymbol{B}_n) = C(\bar{\boldsymbol{B}}_n) \cap H^\infty(\boldsymbol{B}_n)$, also given the uniform norm on $\boldsymbol{B}_n$. We will write $\boldsymbol{S}_n = \partial \boldsymbol{B}_n$, $\boldsymbol{D} = \boldsymbol{B}_1$, and $\boldsymbol{T} = \boldsymbol{S}_1$. For a collection $\mathscr{F}$ of functions in $A(\boldsymbol{B}_n)$, associate the zero set

$$Z(\mathscr{F}) = \{z \in \bar{\boldsymbol{B}}_n : f(z) = 0 \quad \text{for all} \quad f \in \mathscr{F}\},$$

and if $E \subset \bar{\boldsymbol{B}}_n$, introduce the closed ideal

$$\mathscr{I}(E) = \{f \in A(\boldsymbol{B}_n) : f = 0 \quad \text{on} \quad E\}.$$

In Section 2, we will show every closed ideal I in $A(\boldsymbol{B}_n)$ has the form

$$I = \mathscr{I}(Z(I) \cap \boldsymbol{S}_n) \cap [I]_{w*},$$

where $[I]_{w*}$ denotes the weak-star closure of I in $H^\infty(\boldsymbol{B}_n)$. However, we have not yet said what the weak-star topology on $H^\infty(\boldsymbol{B}_n)$ is. The space $L^\infty(\boldsymbol{S}_n)$ is the dual space of $L^1(\boldsymbol{S}_n)$, so it has a weak-star topology. One can think of $H^\infty(\boldsymbol{B}_n)$ as a subspace of $L^\infty(\boldsymbol{S}_n)$ via radial limits, and it turns out that as such it is weak-star closed. We define the weak-star topology on $H^\infty(\boldsymbol{B}_n)$ by saying that a set $U \subset H^\infty(\boldsymbol{B}_n)$ is open if there is a weak-star open set $V \subset L^\infty(\boldsymbol{S}_n)$ with $U = V \cap H^\infty(\boldsymbol{B}_n)$. It is not hard to check that $[I]_{w*}$ is a (weak-star closed) ideal in $H^\infty(\boldsymbol{B}_n)$.

Let us look more closely at the case $n = 1$. Every weak-star closed ideal in $H^\infty(\boldsymbol{D})$ has the form $uH^\infty(\boldsymbol{D})$, for some inner function u. It follows that every closed ideal I in $A(\boldsymbol{D})$ has the form

$$I = \mathscr{I}(Z(I) \cap \boldsymbol{T}) \cap uH^\infty(\boldsymbol{D}),$$

where u is the inner function determined (up to a constant multiple) by $[I]_{w*} = uH^\infty(\boldsymbol{D})$. This result is known as the Beurling–Rudin theorem [**3**, pp. 82–89].

Unfortunately, when $n > 1$, no concrete description of the weak-star closed ideals in $H^\infty(\boldsymbol{B}_n)$ is known. Indeed, it is rather unlikely that one will ever be found. This of course limits the usefulness of Theorem 2.1.

For a function $f \in A(\boldsymbol{B}_n)$, let $I(f)$ be the closure of the principal ideal generated by f, and put

$$Z(f) = \{z \in \bar{\boldsymbol{B}}_n : f(z) = 0\}.$$

Say that f is BR-outer if $Z(f) \subset \boldsymbol{S}_n$ and $I(f) = \mathscr{I}(Z(f))$. In Section 3, we will show the following. If $f \in A(\boldsymbol{B}_n)$ is BR-outer, then

$$\log|f(0)| = \int_{S_n} \log|f|\, d\rho, \tag{0.1}$$

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for all Jensen measures ρ for 0. If, on the other hand, (0.1) holds for all representing measures ρ for 0, it follows that f is BR-outer. If $n = 1$, there is only one representing measure, namely normalized Lebesgue measure on $\boldsymbol{T}$. This yields the well-known result that a function $f \in A(\boldsymbol{D})$ is BR-outer if and only if it is outer in the sense that

$$\log|f(0)| = \int_{-\pi}^{\pi} \log|f(e^{i\theta})|\, d\theta/2\pi.$$

If $n > 1$, not every representing measure for the origin is a Jensen measure.

Section 1

Let σ_n denote normalized Lebesgue measure on the sphere $\boldsymbol{S}_n$, and let $M(\boldsymbol{S}_n)$ be the space of finite Borel measures on $\boldsymbol{S}_n$. We may regard $M(\boldsymbol{S}_n)$ as the dual space of $C(\boldsymbol{S}_n)$ via the dual action

$$\langle f, u\rangle = \int_{S_n} f\, d\mu, \quad f \in C(\boldsymbol{S}_n), \quad \mu \in M(\boldsymbol{S}_n).$$

We will write $f\mu$ for the Borel measure with $d(f\mu) = f\, d\mu$.

A representing measure for 0 is a Borel probability measure on $\boldsymbol{S}_n$ such that

$$f(0) = \int_{S_n} f\, d\rho, \quad f \in A(\boldsymbol{B}_n).$$

We will denote by $M_0(\boldsymbol{S}_n)$ the convex set of representing measures for the origin. A measure $\rho \in M_0(\boldsymbol{S}_n)$ is a Jensen measure if

$$\log|f(0)| \leqslant \int_{S_n} \log|f|\, d\rho, \quad f \in A(\boldsymbol{B}_n).$$

Section 2

THEOREM 2.1. *Every closed ideal I in $A(\boldsymbol{B}_n)$ has the form*

$$I = \mathscr{I}(Z(I) \cap \boldsymbol{S}_n) \cap [I]_{w^*}.$$

Proof. Let $\phi \in A(\boldsymbol{B}_n)^*$ be such that $\phi \perp I$. By the Hahn–Banach theorem, there is a Borel measure $\mu \in M(\boldsymbol{S}_n)$ such that

$$\langle f, \phi\rangle = \langle f, \mu\rangle = \int_{S_n} f\, d\mu, \quad f \in A(\boldsymbol{B}_n);$$

then $\mu \perp I$. If we can show that $\mu \perp \mathscr{I}(Z(I) \cap \boldsymbol{S}_n) \cap [I]_{w^*}$, the assertion will follow, again by the Hahn–Banach theorem. By the Glicksberg–König–Seever decomposition theorem (see [5, p. 194]), $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous with respect to some representing measure $\rho_0 \in M_0(\boldsymbol{S}_n)$, and μ_s is totally singular, that is, μ_s is singular to every representing measure in $M_0(\boldsymbol{S}_n)$. If $f \in I$,

$$f\mu_a + f\mu_s = f\mu \perp A(\boldsymbol{B}_n),$$

so by the Cole–Range theorem [5, p. 198], it follows that $f\mu_s = 0$. This implies that μ_s is supported on $Z(f) \cap \boldsymbol{S}_n$. By varying $f \in I$, we conclude that $\operatorname{supp} \mu_s \subset Z(I) \cap \boldsymbol{S}_n$, and hence $\mu_s \perp \mathscr{I}(Z(I) \cap \boldsymbol{S}_n)$. By Henkin's theorem [5, p. 189], μ_a is a Henkin measure, so by Valskiĭ's theorem [5, p. 187], $\mu_a = \nu + g\sigma_n$, where $\nu \perp A(\boldsymbol{B}_n)$ and $g \in L^1(\sigma_n)$. Since $\mu_s \perp \mathscr{I}(Z(I) \cap \boldsymbol{S}_n)$, $\mu_a \perp I$, and therefore $g\sigma_n \perp I$. It is clear that $g\sigma_n \perp [I]_{w^*}$, so that $\mu_a \perp [I]_{w^*} \cap A(\boldsymbol{B}_n)$. The desired conclusion

$$\mu = \mu_a + \mu_s \perp \mathscr{I}(Z(I) \cap \boldsymbol{S}_n) \cap [I]_{w^*}$$

follows.

Section 3

Let us recall the definition of the BR-outer functions.

DEFINITION 3.1. A function $f \in A(\boldsymbol{B}_n)$ is said to be *BR-outer* if $Z(f) \subset \boldsymbol{S}_n$ and $I(f) = \mathscr{I}(Z(f))$.

In [4], Rubel and Shields introduced the following class of functions.

DEFINITION 3.2. A function $f \in H^\infty(\boldsymbol{B}_n)$ is *exterior* if $fH^\infty(\boldsymbol{B}_n)$ is weak-star dense in $H^\infty(\boldsymbol{B}_n)$.

We have the following result.

THEOREM 3.3. *The following are equivalent for $f \in A(\boldsymbol{B}_n)$:*

(a) *f is exterior;*
(b) *f is BR-outer;*
(c) *there are $g_j \in A(\boldsymbol{B}_n)$ such that $|g_j f| \leqslant 1$ and $g_j f \to 1$ normally on $\boldsymbol{B}_n$;*
(d) *there are $g_j \in H^\infty(\boldsymbol{B}_n)$ such that $|g_j f| \leqslant 1$ and $g_j f \to 1$ normally on $\boldsymbol{B}_n$.*

Proof. If f is exterior, we must have $Z(f) \subset \boldsymbol{S}_n$, so that by Theorem 2.1, f must be BR-outer. This shows that (a) implies (b).

If f is BR-outer, there is a function $p \in A(\boldsymbol{B}_n)$ which peaks at $Z(f)$ [5, p. 204], and a sequence $\{h_j\}_0^\infty$ in $A(\boldsymbol{B}_n)$ such that $fh_j \to 1-p$ uniformly on $\bar{\boldsymbol{B}}_n$ as $j \to +\infty$. Fix $\varepsilon > 0$. Then

$$f \cdot \frac{h_j}{1+\varepsilon-p} \to \frac{1-p}{1+\varepsilon-p}$$

uniformly on $\bar{\boldsymbol{B}}_n$ as $j \to +\infty$. The norm of $(1-p)/(1+\varepsilon-p)$ is < 1. This means that we can find a subsequence $\{h_{j(k)}\}_{k=1}^\infty$ such that the functions

$$g_k(z) = \frac{h_{j(k)}(z)}{1+1/k-p(z)}, \quad z \in \bar{\boldsymbol{B}}_n,$$

have the properties $\|fg_k\| \leqslant 1$ and $f(0)\,g_k(0) \to 1$ as $k \to \infty$. This shows that (b) implies (c). That (c) implies (d) is obvious.

If condition (d) is satisfied, then as $j \to +\infty$, $f(z)\,g_j(z) \to 1$ boundedly and normally on $\boldsymbol{B}_n$, so that $fg_j \to 1$ weak-star in $H^\infty(\boldsymbol{B}_n)$. It follows that f is exterior, so by Theorem 2.1, f must be BR-outer.

The following result shows that all representing measures behave like Jensen measures on the class of ball algebra functions that are zero-free on $\boldsymbol{B}_n$.

PROPOSITION 3.4. *If $f \in A(\boldsymbol{B}_n)$ has $Z(f) \subset \boldsymbol{S}_n$, then*

$$\log|f(0)| \leqslant \int \log|f|\, d\rho$$

for all representing measures ρ for 0.

Proof. Let $f_r(z) = f(rz)$. Then f_r has a logarithm in $A(\boldsymbol{B}_n)$. Thus

$$\log|f_r(0)| = \int_{S_n} \log|f_r|\, d\rho$$

for any such ρ. Now let r increase to 1 and apply Fatou's lemma.

THEOREM 3.5. *If $f \in A(\boldsymbol{B}_n)$ is exterior, then*

$$\log|f(0)| = \int_{S_n} \log|f|\, d\rho$$

for all Jensen measures for 0.

Proof. By Theorem 3.3, we can find functions $g_j \in A(\boldsymbol{B}_n)$ such that $|g_j f| \leqslant 1$ and $g_j f \to 1$ on $\boldsymbol{B}_n$. Then

$$\begin{aligned} \log|f(0)| &= -\lim_{j\to\infty} \log|g_j(0)| \\ &\geqslant -\lim_{j\to\infty} \int_{S_n} \log|g_j|\, d\rho \\ &\geqslant \int_{S_n} \log|f|\, d\rho \end{aligned}$$

because ρ was a Jensen measure and $\log|g_j f| \leqslant 0$. The reverse inequality is the defining inequality for Jensen measures.

THEOREM 3.6. *If $f \in A(\boldsymbol{B}_n)$ satisfies $f(0) \neq 0$ and*

$$\log|f(0)| = \int_{S_n} \log|f|\, d\rho$$

for all representing measures ρ for 0, *then f is exterior.*

Proof. Apply Edward's theorem [**1**, pp. 3–4] to the function $-\log|f|$, to obtain a sequence g_j in $A(\boldsymbol{B}_n)$ with

$$\operatorname{Re} g_j < -\log|f|, \quad \operatorname{Re} g_j(0) \to -\log|f(0)|.$$

Then $h_j = \lambda_j \exp(g_j)$, with $\lambda_j \in \boldsymbol{T}$ suitably chosen, satisfies $|h_j f| \leqslant 1$ and $h_j f \to 1$ normally.

THEOREM 3.7. *For the ball algebra $A(\boldsymbol{B}_n)$, there are representing measures for* 0 *which are not Jensen measures if $n > 1$.*

Proof. For simplicity, consider the case $n = 2$ only. Fix a point $\alpha \in \boldsymbol{D}\backslash\{0\}$, and let ν_1 be normalized arc length measure on the slice

$$\{\alpha\} \times (1-|\alpha|^2)^{\frac{1}{2}}\, \boldsymbol{T} = \{z \in \boldsymbol{S}_2 : z_1 = \alpha\},$$

and let ν_2 be given by the formula $d\nu_2 = P_\alpha(\theta_1)\, d\theta_1/2\pi$, where $d\theta_1$ is the arc length measure on $\boldsymbol{T} \times \{0\}$, and

$$P_\alpha(\theta_1) = \frac{1-|\alpha|^2}{|e^{i\theta_1}-\alpha|^2}$$

is the Poisson kernel. Then ν_1 and ν_2 are both representing measures for the point $(\alpha, 0)$. Consider the measure ρ given by the formula

$$d\rho = d\theta_1/2\pi + \varepsilon(d\nu_1 - d\nu_2),$$

which is a representing measure for 0 if $\varepsilon > 0$ is small. This representing measure cannot be a Jensen measure because the function $f(z_1, z_2) = z_1 - \alpha$ satisfies

$$\int_{S_n} \log|f|\, d\rho = -\infty,$$

and $\log|f(0,0)| = \log|\alpha| > -\infty$.

REMARK 3.8. There is a certain discrepancy between the necessary and sufficient conditions for a function f to be exterior, which we obtained in Theorems 3.5 and 3.6. The reason why this occurs is that the set $\log|A(\boldsymbol{B}_n)|$ does not form a cone.

QUESTION 3.9. (a) If $f \in A(\boldsymbol{S}_n)$ and

$$\log|f(0)| = \int_{S_n} \log|f|\, d\rho$$

for all Jensen measures ρ for 0, must then f be exterior?

(b) Is there an exterior function f in $A(\boldsymbol{B}_n)$ such that

$$\log|f(0)| < \int_{S_n} \log|f|\, d\rho$$

for some representing measure ρ for 0?

Section 4

In a concrete situation, it is hard to use Theorems 3.5 and 3.6 to determine whether a function is exterior, primarily because the structure of representing measures for 0 is rather poorly understood (for some examples of representing measures, see [**5**, p. 201]). In [**2**], the author establishes the following. Maybe it will lead to some better understanding of representing measures.

THEOREM 4.1. *Let $f \in A(\boldsymbol{B}_n)$ have $Z(f) = \{(1,0,\ldots,0)\}$. Then f is exterior if*

$$\log 1/|f(z)| = o(1/(1-|z_1|)) \quad \text{as} \quad z \to (1,0,\ldots,0).$$

It can be shown that if $f \in A(\boldsymbol{B}_n)$ is exterior and has $Z(f) = \{(1,0,\ldots,0)\}$, then

$$\log 1/|f(z)| = 0(1/(1-|z|)) \quad \text{as} \quad z \to (1,0,\ldots,0).$$

QUESTION 4.2. Is this necessary condition sufficient as well?

It is natural to ask whether it is possible to obtain an analog to Theorem 2.1 for the polydisc $\boldsymbol{D}^n$. The author will discuss this situation in another paper.

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References

1. T. W. GAMELIN, *Uniform algebras and Jensen measures*, London Math. Soc. Lecture Note Series 32 (Cambridge University Press, 1978).
2. H. HEDENMALM, 'Outer functions of several complex variables', *J. Funct. Anal.* 80 (1988) 9–15.
3. K. HOFFMAN, *Banach spaces of analytic functions* (Prentice-Hall, Engelwood Cliffs, N.J., 1962).
4. L. A. RUBEL and A. L. SHIELDS, 'The failure of interior–exterior factorization in the polydisc and the ball', *Tôhoku Math. J.* 24 (1972) 409–413.
5. W. RUDIN, *Function theory in the unit ball of* C^n (Springer, Berlin, 1980).

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